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Sampling of Operators

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Abstract Sampling and reconstruction of functions is a fundamental tool in science. We develop an analogous sampling theory for operators whose Kohn-Nirenberg symbols are bandlimited. We prove sampling theorems for in this sense bandlimited operators and show that our results generalize both, the classical sampling theorem, and the fact that a time-invariant operator is fully determined by its impulse response.

Keywords Generalized sampling · Operator identification · Channel measurement · Paley-Wiener spaces · Modulation spaces

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1 Introduction

The classical sampling theorem for bandlimited functions states that a function whose Fourier transform is supported on an interval of length Ω is completely characterized by samples taken at rate at least $1/\Omega$ per unit interval. That is, with $\mathcal{F}$ denoting the Fourier transform[1] we have the following:

Theorem 1.1 *For* $f \in L^2(\mathbb{R})$ *with* $\operatorname{supp} \mathcal{F} f \subseteq [-\Omega/2, \Omega/2]$, *choose* T *with* $T\Omega \leq 1$. *Then*

$$\big\| \{ f(nT) \} \big\|_{l^2(\mathbb{Z})} = T \, \| f \|_{L^2(\mathbb{R})}.$$

[1] See Sect. 2 for basic notation used throughout this paper.

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Moreover, f can be reconstructed by means of the uniformly and L^2-converging series

$$f(x)=\sum_{n\in\mathbb{Z}} f(nT)\frac{\sin(\pi T(x-n))}{\pi T(x-n)}.$$

Theorem 1.2 below describes sampling of operators in its simplest setting. We choose a Hilbert–Schmidt operator H on $L^2(\mathbb{R})$ with kernel κ_H and Kohn-Nirenberg symbol σ_H, that is $\sigma_H(x,D)=H$ in pseudodifferential operator notation [31, 62]. Recall that a Hilbert–Schmidt operator H on $L^2(\mathbb{R})$ is a bounded operator with Hilbert–Schmidt norm $\|H\|_{HS}=\|\kappa_H\|_{L^2}<\infty$. Let $\mathcal{F}^s$ denote the so-called symplectic Fourier transform on $L^2(\mathbb{R}^{2d})$.

Theorem 1.2 *For $H: L^2(\mathbb{R})\longrightarrow L^2(\mathbb{R})$ Hilbert–Schmidt with $\operatorname{supp}\mathcal{F}^s\sigma_H\subseteq[0,T]\times[-\Omega/2,\Omega/2]$ and $T\Omega\le 1$, we have*

$$\left\| H\sum_{k\in\mathbb{Z}}\delta_{kT}\right\|_{L^2(\mathbb{R})}=T\|H\|_{HS},$$

and H can be reconstructed by means of

$$\kappa_H(x+t,x)=\chi_{[0,T]}(t)\sum_{n\in\mathbb{Z}}\left(H\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)\frac{\sin(\pi T(x-n))}{\pi T(x-n)}$$

where $\chi_{[0,T]}(t)=1$ for $t\in[0,T]$ and 0 else and with convergence in Hilbert–Schmidt norm.

As shown in Sect. 4, Theorems 1.1 and 1.2 are special cases of Theorem 4.4, one of the key results presented in this paper.

The appearance of the sampling rate T in the description of the bandlimitation of the operator's Kohn–Nirenberg symbol reflects a fundamental difference between sampling of operators and sampling of functions. This fact is illuminated in terms of operator identification in [41, Theorem 3.6] and [53, Theorem 1.1], results which are extended in Theorems 5.6 and 5.7. In fact, in classical sampling theory, the bandlimitation of a function to a large interval can be compensated by choosing a sufficiently high sampling rate. In the here developed sampling theory for operators though, only bandlimitations to sets of area less than or equal to one permit sampling and reconstruction. The bandlimitation to, for example, a rectangle of area 2 cannot be compensated by increasing the sampling rate, and, in fact, operators characterized by such a bandlimitation cannot be determined in a stable manner by the application of the operator to a single function or distribution, regardless of whether it is supported on a discrete set as in our operator sampling results or not.

For illustrative reasons again, we state our key results Theorems 5.6 and 5.7 here as Theorem 1.3, statements 1 and 2, in terms of Hilbert–Schmidt operators. In this simple form, the result could also be derived from the previously mentioned operator identification results in [46, 54].

It is customary to define Paley–Wiener spaces

$$PW(M)=\left\{f\in L^2\left(\mathbb{R}^d\right):\operatorname{supp}\mathcal{F}f\subseteq M\right\}$$

to describe spaces of functions bandlimited to $M \subseteq \mathbb{R}^d$. Analogously, we define operator Paley–Wiener spaces by

$$OPW(M) = \{H \in HS(L^2(\mathbb{R}^d)) : \operatorname{supp} \mathcal{F}^s \sigma_H \subseteq M\}$$

to describe operators bandlimited to $M \subseteq \mathbb{R}^{2d}$. In short, the spaces $PW(M) \subseteq L^2(\mathbb{R}^{2d})$ and $OPW(M) \subseteq HS(L^2(\mathbb{R}^d))$ are linked by the Kohn-Nirenberg correspondence [16, 39].

Theorem 1.3 *Let $\mu(M)$ denote the Lebesgue measure of the set $M \subseteq \mathbb{R}^2$.*

1. *For M compact with $\mu(M) < 1$ exists $T > 0$, a periodic sequence $\{c_k\}$, and $A, B > 0$ with*

$$A\|H\|_{HS} \leq \Big\| H \sum_{k\in\mathbb{Z}} c_k \delta_{kT} \Big\|_{L^2(\mathbb{R})} \leq B\|H\|_{HS}, \quad H \in OPW(M).$$

2. *Let M be open with $\mu(M) > 1$. Then exists for any $g \in \mathcal{S}'(\mathbb{R})$ and $\epsilon > 0$ an operator $H \in OPW(M)$ with*

$$\|Hg\|_{L^2(\mathbb{R})} \leq \epsilon \|H\|_{HS}.$$

The sampling theory developed here has roots in [41] and [53] and in the seminal work of Kailath [37] and Bello [3]. The referenced papers address the identifiability of slowly time–varying operators, that is, of so-called underspread operators. Measurability or identifiability of a given operator class describes the property that all operators of that class can be distinguished by their action on a well chosen single function or distribution. The importance of operator identification and, therefore, operator sampling in engineering and science is illustrated by the following two examples.

In case of information transmission, complete knowledge of the communications channel operator at hand allows the transmitter to optimize its transmission strategy in order to transmit information close to channel capacity (see, for example, [21] and references therein). Ideally, knowledge of the channel operator would also allow for the inversion of the channel operator at the receiver so that the channel input signal and the embedded information can be completely recovered.

In radar a signal is send out and the goal is to determine the nature of reflecting objects from the received echo, that is, from the response to the radar channels input signal [41, 59].

A classical operator identification result states that time-invariant operators are fully characterized by their response to a Dirac impulse. Kailath [37] and Bello [3] investigated the identifiability of slowly time varying channels (operators) which are defined by the support size of the operators' spreading functions (the symplectic Fourier transforms of the operators' Kohn–Nirenberg symbols). In both papers, support size criteria were described that were then established for some families of trace class operators in [41], respectively [53]. In this paper, we build on these results to develop a widely applicable operator sampling theory. The generality of our operator

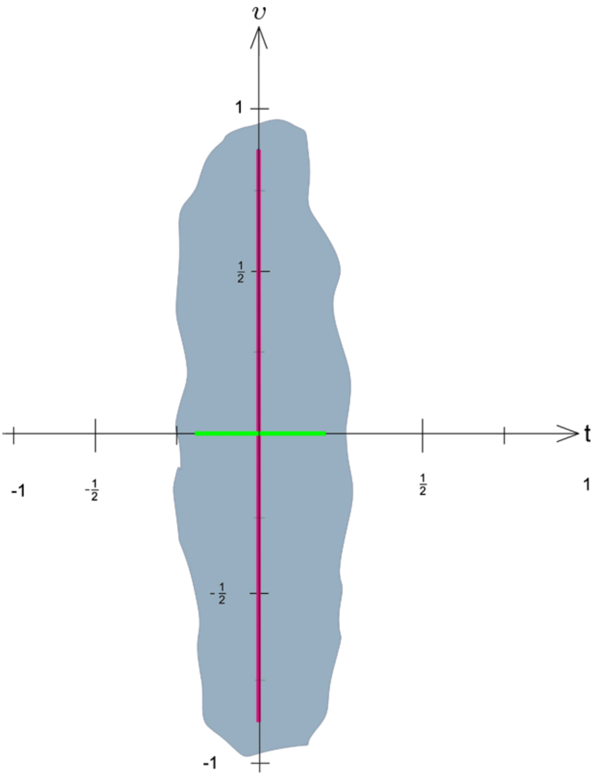

Fig. 1 In the one-dimensional case, the herein developed sampling theory for operators applies to any pseudodifferential operator whose Kohn–Nirenberg symbol is bandlimited to a compact set of Lebesgue measure less than one (for example, the *blue region* above). The results extend the classical sampling theorem described in Theorem 1.1 which is equivalent to the identifiability of operators whose Kohn-Nirenberg symbol is bandlimited to a segment of the frequency shift axis (*red*). Also, the fact that time-invariant operators with compactly supported impulse response can be identified from their action on the Dirac impulse is a special case of our results since the Kohn–Nirenberg symbols of time-invariant operators are bandlimited to the time shift axis (*green*) (Color figure online)

sampling results, Theorems 5.6 and 5.7, allows us not only to deduce the main results in [41, 53], but also the classical sampling theorem for functions, as well as the fact that time-invariant operators are identifiable by their impulse response (see Fig. 1).

The paper is structured as follows. Section 2 provides background on time–frequency analysis of functions and distributions, in particular on modulation spaces (Sect. 2.1), as well as on time–frequency analysis of pseudodifferential operators (Sect. 2.2). In Sect. 3 we establish novel bounds on the operator norms for classes of pseudodifferential operators. The results obtained in this section provide the upper bound B in Theorem 1.3 and respective upper bounds in Theorems 5.6 and 5.7. In Sects. 4 and 5 we state and prove our main results. Section 6 contains references to recent progress and open questions in the sampling theory for operators.

2 Background

The Hilbert space of complex valued, Lebesgue measurable functions on Euclidean space $\mathbb{R}^d$ is denoted by $L^2(\mathbb{R}^d)$ [17]. The *Fourier transformation* $\mathcal{F}$ and the *symplectic Fourier transformation* $\mathcal{F}^s$ are the unitary operators $\mathcal{F}: L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$, $f \mapsto \widehat{f} = \mathcal{F}f$, densely defined by

$$\widehat{f}(\gamma) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \gamma x}\, dx, \quad f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d),$$

respectively $\mathcal{F}^s : L^2(\mathbb{R}^{2d}) \longrightarrow L^2(\mathbb{R}^{2d})$ with

$$\begin{aligned}\mathcal{F}^s F(t,\nu) &= \iint_{\mathbb{R}^{2d}} F(x,\xi) e^{-2\pi i[(t,\nu),(x,\xi)]}\,dx\,d\xi \\ &= \iint_{\mathbb{R}^{2d}} F(x,\xi) e^{-2\pi i(\nu x - \xi t)}\,dx\,d\xi \\ &= \mathcal{F}F(\nu,-t), \quad F \in L^1(\mathbb{R}^{2d}) \cap L^2(\mathbb{R}^{2d}),\end{aligned}$$

where $[\cdot,\cdot]$ denotes the usual symplectic form on $\mathbb{R}^{2d}$. Throughout this paper, integration is with respect to Lebesgue measure which we denote by μ.

The Fourier transform defines isomorphisms on the Frechet space of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ and on its dual $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions (equipped with the weak-$*$ topology). Note that $\mathcal{S}'(\mathbb{R}^d)$ contains constant functions, *Dirac's delta* impulse $\delta : f \mapsto f(0)$, and weighted *Shah distributions* $\sum_{k\in\mathbb{Z}^d} c_k \delta_{kT}$, $T \in (\mathbb{R}^+)^d$, if $\{c_k\}$ has at most polynomial growth.

Similarly to the *Fourier transformation*, the *time shift operator* T_t, $t \in \mathbb{R}^d$, given by $T_t f(x) = f(x-t)$ and the *modulation operator* M_w, $w \in \mathbb{R}^d$, $M_w f(x) = e^{2\pi i w\cdot x} f(x)$, act as unitary operators on $L^2(\mathbb{R}^d)$ and isomorphically on $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$. Note that M_w is also called *frequency shift operator* since $\widehat{M_w f} = T_w \widehat{f}$. Further, we refer to $\pi(\lambda) = \pi(t,\nu) = M_\nu T_t$, $\lambda = (t,\nu) \in \mathbb{R}^{2d}$, as *time–frequency shift operator*. Note that $\mathcal{F} \circ \pi(t,\nu) = e^{2\pi i t\nu} \pi(\nu,-t) \circ \mathcal{F}$, that is, $\mathcal{F}\pi(t,\nu) f = e^{2\pi i t\nu} \pi(\nu,-t) \widehat{f}$ for $f \in \mathcal{S}'(\mathbb{R}^d)$.

The goal of *operator identification* is to select, for given spaces X and Y of functions or distributions on $\mathbb{R}^d$ and a given topological space of linear operators $\mathcal{Z}$ mapping X to Y, an element $g \in X$ which induces a continuous, open, and injective map $\Phi_g : \mathcal{Z} \longrightarrow Y(\mathbb{R}^d)$, $H \mapsto Hg$ (see Fig. 2).

Definition 2.1 Let X be a set, Y a topological vector space, and $\mathcal{Z}$ a topological vector space of operators mapping X to Y. The space $\mathcal{Z}$ is *identifiable by* $g \in X$ if $\Phi_g : \mathcal{Z} \longrightarrow Y$, $H \mapsto Hg$, is continuous, open and injective. In the case that Y and $\mathcal{H}$

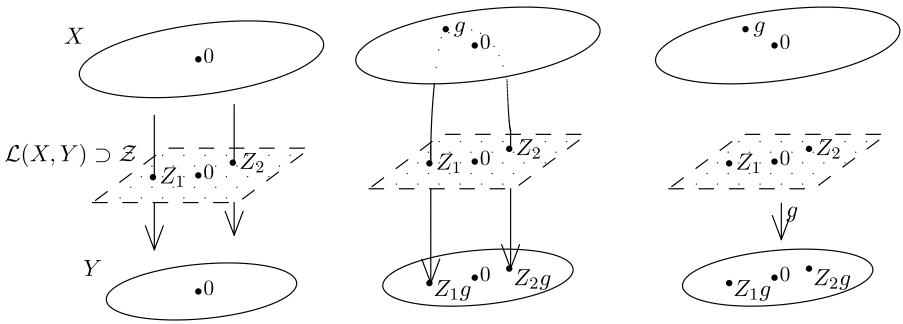

Fig. 2 Illustration of the operator identification and sampling problem. We seek an element $g \in X$ in the domain of the operator class $\mathcal{Z}$ which induces a map from $\mathcal{Z}$ into the range space Y which is continuous, open, and injective. If we can choose $g = \sum_j c_j \delta_{x_j}$, then $\mathcal{Z}$ permits operator sampling

are normed spaces, this reads: there exist $A, B > 0$ with

$$A\|H\|_{\mathcal{Z}} \leq \|Hg\|_Y \leq B\|H\|_{\mathcal{Z}}, \quad H \in \mathcal{Z}. \tag{1}$$

If we can choose $g \in X = X(\mathbb{R}^d)$ of the form $g = \sum_j c_j \delta_{x_j}$, $x_j \in \mathbb{R}^d$ and $c_j \in \mathbb{C}$ for $j \in \mathbb{Z}^d$, as identifier, then we say that $\mathcal{Z}$ mapping X to Y permits operator sampling and we call $\{x_j\}$ a set of sampling for $\mathcal{Z}$ with respective sampling weights $\{c_j\}$. Such g is referred to as a sampling function for the operator class $\mathcal{Z}$.

In the following, we abbreviate norm equivalences as the one given in (1) using the symbol $\asymp$. For example, (1) is

$$\|H\|_{\mathcal{Z}} \asymp \|Hg\|_Y, \quad H \in \mathcal{Z}.$$

In Sect. 2.1 we will describe the distribution spaces and in Sect. 2.2 the pseudodifferential operator spaces considered in this paper. Section 3 discusses boundedness of the respective pseudodifferential operators on the considered distribution spaces, namely, on modulation spaces.

2.1 Modulation Spaces

To describe the full scope of operator sampling, we need to employ recent results in time–frequency analysis, in particular, we have to enter the realm of so-called modulation spaces. As Theorems 1.2 and 1.3 indicate, the results presented here include the special case of Hilbert–Schmidt operators and the Hilbert space of square integrable functions as range space, and we advise readers without significant expertise in time–frequency analysis to focus on this case during a first reading.

Feichtinger introduced modulation spaces in [11]. Modulation space theory was then further developed by Feichtinger and Gröchenig as special case of their coorbit theory [13]: for ρ being a square integrable unitary and irreducible representation of a locally compact group G on a Hilbert space H and Y being a Banach space of functions on G, we consider for appropriate $\varphi \in H$ the so-called *voice transform* $V_\varphi : H \longrightarrow Y$ given by $V_\varphi f(x) = \langle f, \rho(x)\varphi \rangle$, $x \in G$. Given an appropriate Banach space Gelfand triple $X \subseteq H \subseteq X'$, the *coorbit space* M_Y consists of those $f \in X'$ with $\|f\|_{M_Y} = \|V_\varphi f\|_Y < \infty$ [15].

The special case of modulation spaces is based on the Schrödinger representation of the reduced Weyl–Heisenberg group. The corresponding voice transform simplifies to the short-time Fourier transform, that is, for any Schwartz class function $\varphi \neq 0$ we consider

$$V_\varphi f(\lambda) = \big\langle f, \pi(\lambda)\varphi \big\rangle = \mathcal{F}(f T_t \overline{\varphi})(\nu), \quad \lambda = (t, \nu) \in \mathbb{R}^{2d},$$

which is well defined for any $f \in \mathcal{S}'(\mathbb{R}^d)$ [24]. (Throughout this paper, dual pairings $\langle \cdot, \cdot \rangle$ are linear in the first component and antilinear in the second.) Any choice of $\varphi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ can be used to define modulation spaces (with equivalent norms), but, as is customary, we will choose a normalized Gaussian, namely $\varphi(x) = \mathfrak{g}(x) = 2^{\frac{d}{4}} e^{-\pi\|x\|_2^2}$, $x \in \mathbb{R}^d$.

The role of the Banach space Y in coorbit space theory is attained in modulation space theory by weighted mixed L^p spaces: for a measurable function f on $\mathbb{R}^d$ and $p = (p_1, \ldots, p_d)$, $1 \le p_1, \ldots, p_d \le \infty$, we define the mixed norm space$L^p(\mathbb{R}^d)$ by finiteness of

$$\|f\|_{L^p} = \left(\int \left(\cdots \left(\int \left(\int |f(x_1, \ldots, x_d)|^{p_1} dx_1\right)^{p_2/p_1} dx_2\right)^{p_3/p_2} \ldots dx_{d-1}\right)^{p_d/p_{d-1}} dx_d\right)^{1/p_d}, \qquad (2)$$

with the usual adjustments if some $p_k = \infty$ [5]. The mixed $l^p(\mathbb{Z}^d)$ spaces are defined accordingly.

Note that (2) is sensitive to the order of exponentiation and integration. For example, for $f(x, y) = 1$ if $|x - y| \le 1$ and $f(x, y) = 0$ else, we have $\sup_x \int |f(x, y)|\, dy = 2$ but $\int \sup_x |f(x, y)|\, dy = \infty$.

A locally integrable function $v : \mathbb{R}^d \longrightarrow \mathbb{R}_0^+$ with

$$v(x + y) \le v(x)v(y), \quad x, y \in \mathbb{R}^d,$$

is called a *submultiplicative weight function*. For example, $w_s(x) = (1 + \|x\|)^s$, $s \ge 0$, is a submultiplicative weight on $\mathbb{R}^d$. If v is a submultiplicative weight and the locally integrable function $w : \mathbb{R}^d \longrightarrow \mathbb{R}_0^+$ satisfies

$$w(x + y) \le C w(x)v(y), \quad x, y \in \mathbb{R}^d,$$

for some $C > 0$, then w is called v-*moderate weight function*. The class of v-moderate weight functions on $\mathbb{R}^d$ is denoted by $\mathcal{M}_v(\mathbb{R}^d)$. Note that for $s < 0$, for example, $1 \otimes w_s(x, \xi) = (1 + \|\xi\|)^s$ is not submultiplicative, but $1 \otimes w_s$ is $1 \otimes w_{-s}$-moderate. If w is a v-moderate weight function with respect to some submultiplicative weight, then we may simply say that w is *moderate*. Note that for any moderate weight function on $\mathbb{R}^d$ exists $\gamma, C > 0$ with $\frac{1}{C} e^{-\gamma\|x\|_\infty} \le w(x) \le C e^{\gamma\|x\|_\infty}$ [27, Lemma 4.2]. A moderate weight function w on $\mathbb{R}^d$ is a *subexponential weight function* if there exists $\gamma, C > 0$ and $0 < \beta < 1$ with

$$\frac{1}{C} e^{-\gamma\|x\|_\infty^\beta} \le w(x) \le C e^{\gamma\|x\|_\infty^\beta}.$$

Weight functions on discrete groups such as $\mathbb{Z}^d$ are defined accordingly. See [27] for a thorough discussion on the role of weight functions in time–frequency analysis.

Given a v-moderate weight function w, then the Banach space $L_w^p(\mathbb{R}^d)$ is defined through finiteness of the norm $\|f\|_{L_w^p} = \|wf\|_{L^p}$. The space $L_w^p(\mathbb{R}^d)$ is shift invariant and shift operators are bounded on $L_w^p(\mathbb{R}^d)$ but not isometric if w is not constant. Replacing $\mathbb{R}^d$ with $\mathbb{Z}^d$, or with a full rank lattice $\Lambda = A\mathbb{Z}^d$, $A \in \mathbb{R}^{d\times d}$ invertible, both equipped with the counting measure defines $l_w^p(\mathbb{Z}^d)$ and $l_w^p(\Lambda)$. If w is a moderate weight on $\mathbb{R}^d$, then its restriction to Λ, which we denote by $\widetilde{w}$, is moderate as well.

We are now well prepared to define modulation spaces.

Definition 2.2 Let $\mathfrak{g}(x) = 2^{\frac{d}{4}} e^{-\pi \|x\|_2^2}$. For $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$, $1 \leq p_k, q_k \leq \infty$, and w moderate on $\mathbb{R}^{2d}$, we define the modulation space $M_w^{p,q}(\mathbb{R}^d)$ by

$$M_w^{p,q}(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d) : V_{\mathfrak{g}} f \in L_w^{(p,q)}(\mathbb{R}^{2d}) \} \tag{3}$$

[11, 24]. The modulation space $M_w^{p,q}(\mathbb{R}^d)$ is a shift invariant Banach space with norm $\|f\|_{M_w^{p,q}} = \|w V_{\mathfrak{g}} f\|_{L^{p,q}}$. If $w \equiv 1$, then we write $M^{p,q}(\mathbb{R}^d) = M_w^{p,q}(\mathbb{R}^d)$. If $p_1 = \cdots = p_d$ and $q_1 = \cdots = q_d$ then we abbreviate $M_w^{p_1,q_1}(\mathbb{R}^d) = M_w^{(p_1,\ldots,p_d),(q_1,\ldots,q_d)}(\mathbb{R}^d)$.

Below we will use the fact that replacing the Gaussian $\mathfrak{g}$ with any other $\varphi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ in (3) defines the same space and an equivalent norm [24]. Note that if $p_1 \leq p_2, q_1 \leq q_2$, and $w_1 \geq c w_2$ for some $c > 0$, then $M_{w_1}^{p_1,q_1}$ embeds continuously in $M_{w_2}^{p_2,q_2}$, and consequently if $w_1 \asymp w_2$ then $M_{w_2}^{p,q}(\mathbb{R}^d) = M_{w_1}^{p,q}(\mathbb{R}^d)$ with equivalent norms.

The space $M^{1,1}(\mathbb{R}^d)$ is the Feichtinger algebra which is commonly denoted by $S_0(\mathbb{R}^d)$, and $M^{\infty,\infty}(\mathbb{R}^d)$ is its dual $S_0'(\mathbb{R}^d)$. In fact, in general we have $M_w^{p,q}(\mathbb{R}^d)' = M_{1/w}^{p',q'}(\mathbb{R}^d)$ for $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Note that $M_{1 \otimes w_s}^{2,2}(\mathbb{R}^d)$ is well known as Bessel potential spaces, in particular $L^2(\mathbb{R}^d) = M^{2,2}(\mathbb{R}^d)$.

To illustrate the chosen order of exponentiation and integration in the definition of the modulation space $M_w^{p,q}(\mathbb{R}^d)$ for $d > 1$ and $p \neq q$, we state exemplary that $f \in M_{1 \otimes w_s}^{(2,3),(4,5)}(\mathbb{R}^d)$ if and only if

$$\int \left(\int \left(\int \left(\int \left| \left(1 + \sqrt{\nu_1^2 + \nu_2^2}\right)^s V_{\mathfrak{g}} f(t_1, t_2, \nu_1, \nu_2) \right|^2 dt_1 \right)^{\frac{3}{2}} dt_2 \right)^{\frac{4}{3}} d\nu_1 \right)^{\frac{5}{4}} d\nu_2 < \infty.$$

Clearly, $f \otimes g \in M_{w_1 \otimes w_2}^{(p_1,p_2),(q_1,q_2)}(\mathbb{R}^{2d})$ if and only if $f \in M_{w_1}^{p_1,q_1}(\mathbb{R}^d)$ and $g \in M_{w_2}^{p_2,q_2}(\mathbb{R}^d)$. In this case, $\|f \otimes g\|_{M_{w_1 \otimes w_2}^{(p_1,p_2),(q_1,q_2)}} = \|f\|_{M_{w_1}^{p_1,q_1}} \|g\|_{M_{w_2}^{p_2,q_2}}$.

For compactly supported and bandlimited functions, modulation spaces reduce to weighted mixed $L^p(\mathbb{R}^d)$ spaces. The following is a simple generalization of results in [12, 45].

Lemma 2.3 *Let $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$ with $1 \leq p_k, q_k \leq \infty$, let $w = w_1 \otimes w_2$ be a moderate weight function on $\mathbb{R}^{2d}$, and suppose $M \subseteq \mathbb{R}^d$ compact. Then*

1. $\|f\|_{M_w^{p,q}} \asymp \|\widehat{f}\|_{L_{w_2}^q}$, $f \in \mathcal{S}'(\mathbb{R}^d)$, *with* $\operatorname{supp} f \subseteq M$;
2. $\|f\|_{M_w^{p,q}} \asymp \|f\|_{L_{w_1}^p}$, $f \in \mathcal{S}'(\mathbb{R}^d)$, *with* $\operatorname{supp} \widehat{f} \subseteq M$.

Modulation spaces can be described by growth conditions of so-called Gabor coefficients [24]. These descriptions rely on the following terminology.

Definition 2.4 Let X be a Banach space, $1 \leq p_1, \ldots, p_d \leq \infty$, and let w be moderate on the full rank lattice Λ.

1. $\{g_\lambda\}_{\lambda\in\Lambda} \subseteq X'$ is called l^p_w-frame for X if the analysis operator $C_{\{g_\lambda\}} : X \longrightarrow l^p_w(\Lambda)$, $f \mapsto \{\langle f, g_\lambda\rangle\}_{\lambda\in\Lambda}$, is well defined and

$$\|f\|_X \asymp \big\|\{\langle f, g_\lambda\rangle\}\big\|_{l^p_w}, \quad f \in X. \tag{4}$$

2. $\{g_\lambda\}_{\lambda\in\Lambda} \subseteq X$ is called l^p_w-Riesz basis in X if the synthesis operator $D_{\{g_\lambda\}} : l^p_w(\Lambda) \longrightarrow X$, $\{c_\lambda\}_{\lambda\in\Lambda} \mapsto \sum_\lambda c_\lambda g_\lambda$, is well defined and

$$\|\{c_\lambda\}\|_{l^p_w} \asymp \Big\|\sum_\lambda c_\lambda g_\lambda\Big\|_X, \quad \{c_\lambda\} \in l^p_w(\Lambda). \tag{5}$$

In the classical Hilbert space setting $X = X' = H$ and $l^p_w(\mathbb{Z}^{2d}) = l^2(\mathbb{Z}^{2d})$, the above is the definition of Hilbert space frames and Riesz basis sequences. In Hilbert space theory, condition (4) implies that $C_{\{g_\lambda\}}$ has a bounded left inverse, but in the general Banach space setting, (4) alone does not guarantee the existence of a left inverse. Therefore, the existence of a bounded left inverse $C_{\mathcal{F}}$ is frequently included in the definition of frames for Banach spaces [7, 18, 23].

Note that for any $1 \le p \le \infty$ and w moderate, l^p_w-Riesz bases form unconditional bases for their closed linear span. This follows directly from (5) and Definition 12.3.1 and Lemma 12.3.6 in [24].

For $g \in \mathcal{S}(\mathbb{R}^d)$ and Λ being a full rank lattice in $\mathbb{R}^{2d}$, we set $(g, \Lambda) = \{\pi(\lambda)g\}_{\lambda\in\Lambda}$. Theorem 2.5 is an important tool in modulation space theory, see for example Theorem 20 in [25] or Theorem 6.11 in [27].

Theorem 2.5 *Let Λ be a full rank lattice in $\mathbb{R}^{2d}$ and $g \in \mathcal{S}(\mathbb{R}^d)$. Let w be moderate on $\mathbb{R}^{2d}$ and set $\widetilde{w}_\lambda = w(\lambda)$.*

1. *If (g, Λ) is a frame for $L^2(\mathbb{R}^d)$, then (g, Λ) is an $l^{p,q}_{\widetilde{w}}$-frame for $M^{p,q}_w(\mathbb{R}^d)$ for all $1 \le p, q \le \infty$.*
2. *If (g, Λ) is a Riesz basis in $L^2(\mathbb{R}^d)$, then (g, Λ) is an $l^{p,q}_{\widetilde{w}}$-Riesz basis in $M^{p,q}_w(\mathbb{R}^d)$ for all $1 \le p, q \le \infty$.*

Proof 1. Assume (g, Λ), $g \in \mathcal{S}(\mathbb{R}^d)$, is a frame for $L^2(\mathbb{R}^d)$. Let $\widetilde{g}$ generate the canonical dual frame $(\widetilde{g}, \Lambda)$ of (g, Λ) [24]. We have $\widetilde{g} \in \mathcal{S}(\mathbb{R}^d)$ [36] and conclude that both, $C_{(g,\Lambda)} : M^{p,q}_w(\mathbb{R}^d) \longrightarrow l^{p,q}_{\widetilde{w}}(\Lambda)$ and $D_{(\widetilde{g},\Lambda)} : l^{p,q}_{\widetilde{w}}(\Lambda) \longrightarrow M^{p,q}_w(\mathbb{R}^d)$ are bounded operators. As $D_{(\widetilde{g},\Lambda)} \circ C_{(g,\Lambda)}$ is the identity on $L^2(\mathbb{R}^d)$, we can use a density argument to conclude that $D_{(\widetilde{g},\Lambda)} \circ C_{(g,\Lambda)}$ is the identity on $M^{p,q}_w(\mathbb{R}^d)$. Hence, $C_{(g,\Lambda)}$ is bounded below.

The proof of 2. follows similarly. □

2.2 Time–Frequency Analysis of Pseudodifferential Operators

The framework of Hilbert–Schmidt operators suffices to give a good idea of the key results in our sampling theory for operators. But important operators such as convolution operators, multiplication operators, and even the identity are not compact and thereby fall outside the realm of Hilbert–Schmidt operators. Rather than focusing only on operators with kernels in $L^2(\mathbb{R}^{2d})$, we will consider kernels and symbols in modulation spaces.

To formulate a widely applicable sampling theory for operators, we use the general correspondence of operators to distributional kernels given by the Schwartz kernel theorem (see, for example, [34]).

Theorem 2.6 *For any linear and continuous operator* $H : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d)$ *there exists a unique* $\kappa_H \in \mathcal{S}'(\mathbb{R}^{2d})$ *with* $\langle Hf, g\rangle = \langle \kappa_H, \overline{f} \otimes g\rangle$, $f, g \in \mathcal{S}(\mathbb{R}^d)$.

Alternatively to κ_H, we can consider the so-called time-varying impulse response $h_H \in S'(\mathbb{R}^{2d})$ of $H : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d)$ which is formally given by

$$h_H(x,t) = \kappa_H(x, x-t), \quad Hf(x) = \int h_H(x,t) f(x-t)\, dt.$$

The Kohn–Nirenberg symbol σ_H of an operator $H : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d)$ is densely defined by $\sigma_H = \mathcal{F}_{t\to\xi} h_H$, that is,

$$\sigma_H(x,\xi) = \int \kappa_H(x, x-t) e^{-2\pi i t\xi}\, dt, \quad Hf(x) = \int \sigma_H(x,\xi) \widehat{f}(\xi) e^{2\pi i x\xi}\, d\xi$$

[16, 39]. Note that the nth order linear differential operator $D: f \mapsto \sum_{n=0}^{N} a_n(x) f^{(n)}(x)$ has Kohn–Nirenberg symbol $\sigma_D(x,\xi) = \sum_{n=0}^{N} a_n(x)(2\pi i\xi)^n$ which is polynomial in ξ. Pseudodifferential operator classes, for example, those considered by Hörmander, have symbols σ_H which are not necessarily polynomial in ξ but which satisfy corresponding polynomial growth conditions [34].

In time–frequency analysis and in communications engineering, the spreading function η_H is commonly used to describe H. It is given by

$$\eta_H = \mathcal{F}^s \sigma_H, \quad Hf(x) = \iint \eta_H(t,\nu) M_\nu T_t f(x)\, dt\, d\nu. \tag{6}$$

Equation (6) can be validated weakly by first integrating with respect to x in

$$\langle Hf, \varphi\rangle = \iiint \eta_H(t,\nu)\pi(t,\nu) f(x)\overline{\varphi(x)}\, dt\, d\nu\, dx = \langle \eta_H, V_f\varphi\rangle, \quad f,\varphi \in \mathcal{S}\big(\mathbb{R}^d\big),$$

where $V_f\varphi(t,\nu) = \langle \varphi, \pi(t,\nu) f\rangle$ is the short-time Fourier transform defined above. Equation (6) illustrates that support restrictions on η_H reflect limitations on the maximal time and frequency shifts which the operator input signals undergo: Hf is a continuous superposition of time–frequency shifted versions of f with weight function η_H [22, 41, 53]. Moreover, as $h_H(x,t) = \int \eta_H(t,\nu) e^{2\pi i \nu x}\, d\nu$, the condition $\operatorname{supp} \eta_H(t,\cdot) \subseteq [-b/2, b/2]$, $t \in \mathbb{R}$, excludes high frequencies and therefore rapid change in the time-varying impulse response $h_H(x,t)$ considered as a function of x. In the time-invariant case, $\kappa_H(x, x-t) = h_H(x,t) = h_H(t)$ is, in fact, independent of x. These observations illuminate the role of support constraints on spreading functions in the analysis of *slowly time-varying* communications channels [2, 65]. Additional aspects on the use of pseudodifferential operator calculus in communications can be found in [60].

3 Operator Norm Bounds for Pseudodifferential Operators on Modulation Spaces

In this section we derive the functional analytic results necessary to obtain the right-hand inequality in (1) in the proof of identifiability of certain operator Paley-Wiener spaces, see Theorem 4.3 in Sect. 4.

Theorem 3.1 below generalizes Theorem 4.2 in [64] as well as results in [6, 8, 9, 19, 20, 61, 63] where, generally, the case $p_3 = q_3$ and $p_4 = q_4$ in the notation below is considered. Recall that p' denotes the conjugate exponent of $1 \leq p \leq \infty$, that is

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 3.1 *Assume* $1 \leq p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 \leq \infty$ *with,*

$$p_4 \leq q_3, q_4, \quad p_1', p_2 \leq p_3 \quad \textit{and} \quad q_1', q_2 \leq q_3, \tag{7}$$

as well as

$$1 + \frac{1}{p_2} \leq \frac{1}{p_1} + \frac{1}{p_3} + \frac{1}{p_4} \quad \textit{and} \quad 1 + \frac{1}{q_2} \leq \frac{1}{q_1} + \frac{1}{q_3} + \frac{1}{q_4}. \tag{8}$$

Let the moderate weight functions w, w_1, w_2 *satisfy*

$$w(x, \xi, \nu, t) \geq c \frac{w_2(x, \nu + \xi)}{w_1(t - x, \xi)}$$

with $c > 0$. *Then, for some* $C > 0$,

$$\begin{aligned} \|Hf\|_{M^{p_2,q_2}_{w_2}} &\leq C \|\sigma_H\|_{M^{(p_3,q_3),(q_4,p_4)}_{w}} \|f\|_{M^{p_1,q_1}_{w_1}}, \\ f &\in M^{p_1,q_1}_{w_1}\big(\mathbb{R}^d\big), \sigma_H \in M^{(p_3,q_3),(q_4,p_4)}_{w}\big(\mathbb{R}^{2d}\big), \end{aligned} \tag{9}$$

consequently, $H : M^{p_1,q_1}_{w_1}(\mathbb{R}^d) \longrightarrow M^{p_2,q_2}_{w_2}(\mathbb{R}^d)$ *is bounded for*

$$\sigma_H \in M^{(p_3,q_3),(q_4,p_4)}_{w}(\mathbb{R}^{2d}).$$

Theorem 3.1 is a consequence of the following lemma.

Lemma 3.2 *Assume* $1 \leq p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 \leq \infty$ *with* $p_3 \leq p_1, p_2, p_4$, $q_3 \leq q_1, q_2, q_4$,

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \quad \textit{and} \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}. \tag{10}$$

Let the moderate weight functions w, w_1, w_2 *satisfy*

$$w(x, t, \nu, \xi) \leq w_1(t - x, \xi) w_2(x, \nu + \xi). \tag{11}$$

Then for $\mathfrak{G}(x,t)=\mathfrak{g}(x)\mathfrak{g}(x-t)$, *we have*

$$\left(\int\left(\int\left(\int\left(\int\left|V_{\mathfrak{G}}\left(\overline{f}\otimes g\circ\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\right)(x,t,\nu,\xi)w(x,t,\nu,\xi)\right|^{p_3}dx\right)^{\frac{p_4}{p_3}}dt\right)^{\frac{q_3}{p_4}}d\xi\right)^{\frac{q_4}{q_3}}d\nu\right)^{\frac{1}{q_4}}$$

$$\le \|f\|_{M^{p_1,q_1}_{w_1}}\|g\|_{M^{p_2,q_2}_{w_2}},\qquad f\in M^{p_1,q_1}_{w_1}(\mathbb{R}^d),\ g\in M^{p_2,q_2}_{w_2}(\mathbb{R}^d), \tag{12}$$

where $\overline{f}\otimes g\circ\left(\begin{smallmatrix}1 & -1\\ 1 & 0\end{smallmatrix}\right)(x,t)^T=\overline{f}(x-t)g(x)$ *and where the usual adjustments are made to the left-hand side if some of the* p_3,p_4,q_3,q_4 *equal* ∞.

Proof For $g,f\in\mathcal{S}(\mathbb{R}^d)$, we compute

$$\begin{aligned}
&V_{\mathfrak{G}}\left(\overline{f}\otimes g\circ\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\right)(x,t,\nu,\xi)\\
&\quad=\iint g(x')\overline{f}(x'-t')e^{-2\pi i(x'\nu+t'\xi)}\mathfrak{g}(x'-x)\mathfrak{g}\big(x'-x-(t'-t)\big)\,dt'\,dx'\\
&\quad=\int g(x')e^{-2\pi i x'\nu}\mathfrak{g}(x'-x)\int\overline{f}(s)e^{-2\pi i(x'-s)\xi}\mathfrak{g}\big(s-(x-t)\big)\,ds\,dx'\\
&\quad=\int g(x')e^{-2\pi i x'(\nu+\xi)}\mathfrak{g}(x'-x)\,dx'\overline{\int f(s)e^{-2\pi i s\xi}\mathfrak{g}\big(s-(x-t)\big)\,ds}\\
&\quad=V_{\mathfrak{g}}g(x,\nu+\xi)\overline{V_{\mathfrak{g}}f(x-t,\xi)}.
\end{aligned}$$

Inequality (12) will follow from applying twice Young's inequality

$$\|f*g\|_{L^u}\le\|f\|_{L^r}\|g\|_{L^s}\qquad\text{if } r,s,u\in[1,\infty]\text{ satisfy } 1/r+1/s=1+1/u. \tag{13}$$

Indeed, assuming for notational simplicity $1\le p_3,p_4,q_3,q_4<\infty$ and $w=w_1=w_2\equiv 1$, we obtain

$$\left(\int\left(\int\left(\int\left(\int\left|V_{\mathfrak{G}}\left(\overline{f}\otimes g\circ\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\right)(x,t,\nu,\xi)\right|^{p_3}dx\right)^{\frac{p_4}{p_3}}dt\right)^{\frac{q_3}{p_4}}d\xi\right)^{\frac{q_4}{q_3}}d\nu\right)^{\frac{1}{q_4}} \tag{14}$$

$$=\left(\int\left(\int\left(\int\left(\int|V_{\mathfrak{g}}f(x-t,\xi)V_{\mathfrak{g}}g(x,\nu+\xi)|^{p_3}\,dx\right)^{\frac{p_4}{p_3}}dt\right)^{\frac{q_3}{p_4}}d\xi\right)^{\frac{q_4}{q_3}}d\nu\right)^{\frac{1}{q_4}} \tag{15}$$

$$\le\left(\int\left(\int\left(\|V_{\mathfrak{g}}f(\cdot,\xi)^{p_3}\|_{L^{r_1}}\|V_{\mathfrak{g}}g(\cdot,\nu+\xi)^{p_3}\|_{L^{s_1}}\right)^{\frac{p_4}{p_3}\frac{q_3}{p_4}}d\xi\right)^{\frac{q_4}{q_3}}d\nu\right)^{\frac{1}{q_4}} \tag{16}$$

$$=\left(\int\left(\int\left(\|V_{\mathfrak{g}}f(\cdot,\xi)^{p_3}\|_{L^{r_1}}\|V_{\mathfrak{g}}g(\cdot,\nu+\xi)^{p_3}\|_{L^{s_1}}\right)^{\frac{q_3}{p_3}}d\xi\right)^{\frac{q_4}{q_3}}d\nu\right)^{\frac{q_3}{q_4}\frac{q_4}{q_3}\frac{1}{q_4}}$$

$$\leq \Big\| \big\| V_{\mathfrak{g}} f^{p_3} \big\|_{L^{r_1}}^{\frac{q_3}{p_3}} \Big\|_{L^{r_2}}^{\frac{1}{q_3}} \Big\| \big\| V_{\mathfrak{g}} g^{p_3} \big\|_{L^{s_1}}^{\frac{q_3}{p_3}} \Big\|_{L^{s_2}}^{\frac{1}{q_3}} \tag{17}$$

$$= \left(\int \left(\int |V_{\mathfrak{g}} f|^{p_3 r_1} \, dx \right)^{\frac{r_2}{r_1} \frac{q_3}{p_3}} d\xi \right)^{\frac{1}{r_2 q_3}} \left(\int \left(\int |V_{\mathfrak{g}} g^{p_3 s_1} \, dx \right)^{\frac{s_2}{s_1} \frac{q_3}{p_3}} d\xi \right)^{\frac{1}{s_2 q_3}}$$

$$= \| f \|_{M^{p_3 r_1, q_3 r_2}} \| g \|_{M^{p_3 s_1, q_3 s_2}}.$$

To use Young's inequality to obtain (16), we assume $p_4 \geq p_3$ and choose $r_1, s_1 \in [1, \infty)$ with

$$\frac{1}{r_1} + \frac{1}{s_1} = 1 + \frac{p_3}{p_4}. \tag{18}$$

Similarly, to obtain (17), we use $q_4 \geq q_3$ and choose $r_2, s_2 \geq 1$ with

$$\frac{1}{r_2} + \frac{1}{s_2} = 1 + \frac{q_3}{q_4}. \tag{19}$$

We now set $p_1 = p_3 r_1$, $q_1 = q_3 r_2$, $p_2 = p_3 s_1$, and $q_2 = q_3 s_2$. As all factors must be greater than or equal to one, we require $p_1, p_2 \geq p_3$ and $q_1, q_2 \geq q_3$. Moreover, (18) and (19) need to be satisfied, this holds if and only if (10) holds.

To conclude our proof of the unweighted case, we observe that the case that for some k, $p_k = \infty$ or $q_k = \infty$ differs only in notation since Young's inequality remains applicable. For illustrative purposes, observe that if $p_3 < p_4 = \infty$, then the inner integrals in (14) are replaced by

$$\left(\sup_t \int \left| V_{\mathfrak{G}} \left(\overline{f} \otimes g \circ \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right) (x, t, \nu, \xi) \right|^{p_3} dx \right)^{\frac{q_3}{p_3}}$$

Setting $p_1 = p_3 r_1$, $q_1 = q_3 r_2$, $p_2 = p_3 s_1$, and $q_2 = q_3 s_2$ validates (12) as above.

The weighted case follows by simply replacing $V_{\mathfrak{G}} G$ with $w V_{\mathfrak{G}} G$ in equations (15) till (16), and then replacing $V_{\mathfrak{g}} f$ and $V_{\mathfrak{g}} g$ by $w_1 V_{\mathfrak{g}} f$ and $w_2 V_{\mathfrak{g}} g$. This is justified by (11). □

Proof of Theorem 3.1 Let $f, g \in \mathcal{S}(\mathbb{R}^d)$ and H with $\sigma_H \in M^{(p_3,q_3),(q_4,p_4)}(\mathbb{R}^{2d})$. Then

$$|\langle H f, g \rangle| = \left| \int \int h_H(x,t) f(x-t) \, dt \, \overline{g}(x) \, dx \right| = \left| \left\langle h_H, \left(\overline{f} \otimes g \circ \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right) \right\rangle \right|$$

$$= \left| \left\langle \sigma_H, \mathcal{F}_{t \to \xi} \left(\overline{f} \otimes g \circ \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right) \right\rangle \right|$$

$$\leq \| \sigma_H \|_{M_w^{(p_3,q_3),(q_4,p_4)}} \left\| \mathcal{F}_{t \to \xi} \left(\overline{f} \otimes g \circ \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right) \right\|_{M_{1/w}^{(p_3',q_3'),(q_4',p_4')}}, \tag{20}$$

where we applied Hölder's inequality for weighted mixed L^p-spaces to obtain (20) [24].

Inequality (20) is valid whenever its left-hand and right-hand side are well defined. Observe that $\sup\{|\langle \cdot, g \rangle|, g \in M_{1/w_2}^{p_2', q_2'}\}$ defines a norm which is equivalent to $\| \cdot \|_{M_{w_2}^{p_2,q_2}}$

for $p_2, q_2 \in [1,\infty]$ (see for example Proposition 1.2(3) in [64]), hence, to obtain (9) it suffices to show $\mathcal{F}_{t\to\xi}\big(\overline{f}\otimes g\circ\big(\begin{smallmatrix}1 & -1\\ 1 & 0\end{smallmatrix}\big)\big) \in M_{1/w}^{(p_3',q_3'),(q_4',p_4')}$ for $f \in M_{w_1}^{p_1,q_1}$ and $g \in M_{1/w_2}^{p_2',q_2'}$. Note that replacing $\mathfrak{g}$ by any other test function in (3) leads to a norm equivalent to $\|\cdot\|_{M_w^{p,q}}$, and we choose to show that for $\Psi = \mathcal{F}_{t\to\xi}\mathfrak{G}$, $\mathfrak{G}(x,\xi) = \mathfrak{g}(x)\mathfrak{g}(x-t)$, we have that

$$\left\| \frac{1}{w} V_\Psi \mathcal{F}_{t\to\xi}\left(\overline{f}\otimes g\circ\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\right)\right\|_{L^{p_3',q_3',q_4',p_4'}} \tag{21}$$

is bounded by the left-hand side in (12) for $f \in M_{w_1}^{p_1,q_1}$ and $g \in M_{1/w_2}^{p_2',q_2'}$. Note that as

$$\begin{aligned}&\left|\frac{1}{w} V_\Psi \mathcal{F}_{t\to\xi}\left(\overline{f}\otimes g\circ\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\right)\right|(x,\xi,\nu,t)\\ &\quad = \left|\frac{1}{w} V_{\mathfrak{G}}\left(\overline{f}\otimes g\circ\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\right)\right|(x,t,\nu,\xi), \quad x,\xi,t,\nu\in\mathbb{R}^d,\end{aligned}$$

the boundedness follows from an adjustment of the order of exponentiation and integration in (12). Using Minkowski's integral inequality twice, namely,

$$\left\|\int |w(x,\cdot)F(x,\cdot)|^p\,dx\right\|_{L^{\frac{q}{p}}} \le \int\left(\int |w(x,y)F(x,y)|^q\,dy\right)^{\frac{p}{q}} dx \quad \text{if } p\le q<\infty$$

and similarly if $p<q=\infty$ or $p=q=\infty$, we can move the $L_t^{p_4'}$ integral first inside the $L_\nu^{q_4'}$ integral and secondly inside the $L_\xi^{q_3'}$ integral, obtaining the left-hand side of (12) with p_3, p_4, q_3, q_4 replaced by p_3', p_4', q_3', q_4', respectively.

We now prepare to apply Lemma 3.2. Observe that if we assume

$$p_4', p_1, p_2' \ge p_3', \qquad q_4', q_1, q_2' \ge q_3', \qquad p_4' \ge q_3', q_4', \tag{22}$$

and

$$\frac{1}{p_1}+\frac{1}{p_2'} = \frac{1}{p_3'}+\frac{1}{p_4'} \quad \text{and} \quad \frac{1}{q_1}+\frac{1}{q_2'} = \frac{1}{q_3'}+\frac{1}{q_4'},$$

then, $p_4, p_1', p_2 \le p_3$ and $q_4, q_1', q_2 \le q_3$ and $p_4 \le q_3, q_4$, and

$$\frac{1}{p_1}+1-\frac{1}{p_2} = 1-\frac{1}{p_3}+1-\frac{1}{p_4} \quad \text{and} \quad \frac{1}{q_1}+1-\frac{1}{q_2} = 1-\frac{1}{q_3}+1-\frac{1}{q_4},$$

the latter being

$$\frac{1}{p_1}-\frac{1}{p_2} = 1-\frac{1}{p_3}-\frac{1}{p_4} \quad \text{and} \quad \frac{1}{q_1}-\frac{1}{q_2} = 1-\frac{1}{q_3}-\frac{1}{q_4}.$$

Hence, we obtain (9) if (22) is satisfied. Note that for $\widetilde{p}\le p$ and $\widetilde{q}\le q$ we have $M_w^{\widetilde{p},\widetilde{q}}$ embeds continuously in $M_w^{p,q}$ [24, Theorem 12.2.2]). Hence, (9) remains true if we decrease p_1, p_3, p_4 and q_1, q_3, q_4, and/or increase p_2 and q_2. As (7) and (8) also imply (22), we conclude that (7) and (8) imply (9). □

Remark 3.3 Note that for Hilbert–Schmidt operators, we have

$$\|H\|_{HS} = \|\kappa_H\|_{L^2} = \|h_H\|_{L^2} = \|\sigma_H\|_{L^2} = \|\eta_H\|_{L^2}, \tag{23}$$

a fact which is helpful to obtain norm inequalities of the form (1). But when considering modulation space norms for operator symbols, the chain of equalities (23) fails to hold. For example, we have

$$\big|\langle h_H, \pi(x,t,\nu,\xi)\mathfrak{g}\rangle\big| = \big|\langle \sigma_H, \pi(x,\xi,\nu,t)\mathfrak{g}\rangle\big| = \big|\langle \eta_H, \pi(t,\nu,\xi,x)\mathfrak{g}\rangle\big|,$$

but due to the implicitly given order of exponentiation and integration,

$$\|h_H\|_{M^{(p_1,p_2),(q_1,q_2)}} \not\asymp \|\sigma_H\|_{M^{(p_1,q_2),(q_1,p_2)}} \not\asymp \|\eta_H\|_{M^{(p_2,q_1),(q_2,p_1)}},$$
$$H : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d).$$

Consequently, when defining a modulation space type norm on sets of pseudodifferential operators, the norm can be based on applying modulation space norms to either h_H, σ_H, or η_H, each choice leading to different operator spaces and norms. Lemma 3.2 gives a hint that it may be advantageous to define operator modulation spaces $OM^{p_1,p_2,q_1,q_2}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$ through finiteness of the norm

$$\|H\|_{OM_w^{p_1,p_2,q_1,q_2}} = \left(\int\left(\int\left(\int\left(\int |V_{\mathfrak{g}}^s\sigma_H(x,t,\xi,\nu)w(x,t,\xi,\nu)|^{p_1}\,dx\right)^{\frac{p_2}{p_1}} dt\right)^{\frac{q_1}{p_2}} d\xi\right)^{\frac{q_2}{q_1}} d\nu\right)^{\frac{1}{q_2}},$$

where the *symplectic short-time Fourier transform* V^s with respect to the window function $\mathfrak{g} \in \mathcal{S}(\mathbb{R}^{2d})$ is given by

$$V_{\mathfrak{g}}^s F(x,t,\xi,\nu) = \mathcal{F}^s(F \cdot \overline{T_{x,\xi}\mathfrak{g}})(t,\nu), \quad F \in \mathcal{S}'(\mathbb{R}^{2d}).$$

This choice of order of exponentiation and integration arranges the *time* variables ahead of the *frequency* variables, while listing first the *absolute time* variable x and then the *time-shift* variable t, and first list the *absolute frequency* variable ξ and then the *frequency-shift* variable ν. More importantly, with this choice, we have

$$\|Hf\|_{M^{p_2,q_2}} \le C\|H\|_{OM^{p_3,p_4,q_3,q_4}}\|f\|_{M^{p_1,q_1}}$$

for all $1 \le p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 \le \infty$ satisfying the last two inequalities of (7) and inequality (8). The spaces OM^{p_1,p_2,q_1,q_2} have been analyzed in detail in [44].

For simplicity of terminology, we avoid the use of operator modulation spaces and symplectic short-time Fourier transforms in the following. Lemma 2.3 implies that this omission does not lead to a loss of generality in case of the here considered operator Paley–Wiener spaces.

4 Sampling and Reconstruction in Operator Paley–Wiener Spaces

We introduce *operator Paley–Wiener spaces*.

Definition 4.1 For $1 \leq p, q \leq \infty$, a compact set M, and a moderate weight w on $\mathbb{R}^{2d}$, operator Paley–Wiener spaces are given by

$$OPW_w^{p,q}(M) = \big\{H : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d) : \operatorname{supp} \mathcal{F}^s \sigma_H \subseteq M \text{ and } \sigma_H \in L_w^{p,q}(\mathbb{R}^{2d})\big\}.$$

$OPW_w^{p,q}(M)$ is a Banach space with norm $\|H\|_{OPW_w^{p,q}} = \|\sigma_H\|_{L_w^{p,q}}$. If $w \equiv 1$ and $p = q = 2$ then we simply write $OPW(M) = \{H \in HS(L^2(\mathbb{R}^d)) : \operatorname{supp} \mathcal{F}_s \sigma_H \subseteq M\}$.

Note that, as illustrated in Corollary 4.6 and Example 4.7 below, it is appropriate to choose $OPW_w^{p,\infty}(M)$, respectively $OPW_w^{\infty,q}(M)$, when considering multiplication respectively convolution operators. Moreover, observe that $OPW_w^{\infty,\infty}(M)$ consists of all operators in the weighted Sjöstrand class with Kohn–Nirenberg symbol bandlimited to M [26, 57, 58, 60].

Remark 4.2 Hörmander considers pseudodifferential operators with Kohn–Nirenberg symbol in

$$S_{\rho,\delta}^m = \big\{\sigma_H \in C^\infty(\mathbb{R}^{2d}) : |\partial_\xi^\alpha \partial_x^\beta \sigma_H(x,\xi)| \leq C_{\alpha,\beta}(1 + \|\xi\|_2)^{m-\rho(\alpha_1+\cdots+\alpha_d)+\delta(\beta_1+\cdots+\beta_d)}, \\ \alpha, \beta \in \mathbb{N}_0^d\big\}$$

where $m \in \mathbb{R}$, $0 < \rho \leq 1$, and $0 \leq \delta < 1$ [34]. Clearly, if $\operatorname{supp} \mathcal{F}^s \sigma_H \subseteq M$ and $\sigma_H \in S_{\rho,\delta}^m$, then $H \in OPW_{1\otimes w_s}^{\infty,\infty}(M)$ if $s \leq -m$ and $H \in OPW_{1\otimes w_s}^{\infty,q}(M)$ if $(m+s)q < -d$.

Theorem 4.3 *Let $1 \leq p, q \leq \infty$ and w moderate on $\mathbb{R}^{2d}$. For M compact exists $C > 0$ with*

$$\|Hf\|_{M_w^{p,q}} \leq C \|\sigma_H\|_{L_w^{p,q}} \|f\|_{M^{\infty,\infty}}, \qquad H \in OPW_w^{p,q}(M),\ f \in M^{\infty,\infty}(\mathbb{R}^d).$$

Consequently, any $H \in OPW_w^{p,q}(M)$ extends to a bounded operator mapping $M^{\infty,\infty}(\mathbb{R}^d)$ to $M_w^{p,q}(\mathbb{R}^d)$.

Proof Set $\omega(x,\xi,\nu,t) = w(x,\xi+\nu)$ and choose $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp} \widehat{\varphi} \subseteq [-1,1]^d$. Then we use Lemma 2.3 and $\operatorname{supp} V_{\varphi\otimes\varphi}\sigma_H \subseteq \mathbb{R}^{2d} \times M + [-1,1]^{2d}$, hence, $\omega \asymp w \otimes 1$ on $\operatorname{supp} V_{\varphi\otimes\varphi}\sigma_H$, to obtain

$$\begin{aligned}\|\sigma_H\|_{L_w^{p,q}} &\asymp \|\sigma_H\|_{M_{w\otimes 1}^{(p,q),(1,1)}} \asymp \|w \otimes 1 V_{\varphi\otimes\varphi}\sigma_H\|_{L^{(p,q),(1,1)}} \asymp \|\omega V_{\varphi\otimes\varphi}\sigma_H\|_{L^{(p,q),(1,1)}} \\ &\asymp \|\sigma_H\|_{M_\omega^{(p,q),(1,1)}}.\end{aligned}$$

An application of Theorem 3.1 with $p_1 = q_1 = \infty$, that is $p_1' = q_1' = 1$, $p_2 = p_3 = p$, $q_2 = q_3 = q$, and $p_4, q_4 = 1$ concludes the proof. □

In the following, we set $Q_T = [0, T_1) \times \cdots \times [0, T_d)$ for $T = (T_1, \ldots, T_d) \in (\mathbb{R}^+)^d$, and $R_\Omega = [-\frac{\Omega_1}{2}, \frac{\Omega_1}{2}) \times \cdots \times [-\frac{\Omega_d}{2}, \frac{\Omega_d}{2})$ for $\Omega = (\Omega_1, \ldots, \Omega_d) \in (\mathbb{R}^+)^d$.

Theorem 4.4 *Let $1 \le p, q \le \infty$ and let $w = w_1 \otimes w_2$ be moderate on $\mathbb{R}^{2d}$. Let $T, \Omega \in (\mathbb{R}^+)^d$ satisfy $T_m \Omega_m < 1, m = 1, \dots, d$. Let $\Lambda = T_1\mathbb{Z} \times \cdots \times T_d\mathbb{Z}$ and choose $s \in M^{1,1}(\mathbb{R}^d)$ with $\operatorname{supp} \widehat{s} \subseteq R_{1/T}$ and $\widehat{s} \equiv T_1 \cdot \cdots \cdot T_d$ on R_Ω. Then*

$$\Big\| H \sum_{\lambda \in \Lambda} \delta_\lambda \Big\|_{M^{p,q}_w} \asymp \|H\|_{OPW^{p,q}_w}, \quad H \in OPW^{p,q}_w(Q_T \times R_\Omega), \tag{24}$$

and any $H \in OPW^{p,q}_w(Q_T \times R_\Omega)$ can be reconstructed by means of

$$\kappa_H(x+t, x) = \chi_{Q_T}(t) \sum_{\lambda \in \Lambda} \Big(H \sum_{\lambda' \in \Lambda} \delta_{\lambda'} \Big)(t+\lambda) s(x-\lambda). \tag{25}$$

with convergence in $OPW^{p,q}_w(\mathbb{R}^{2d})$ for $1 \le p, q < \infty$ and weak-$*$ convergence else.

Proof We will show (25). The norm equivalence (24) can be shown by adapting the steps of the proof of Theorem 5.6.

For $\Lambda = T_1\mathbb{Z} \times \cdots \times T_d\mathbb{Z}$, we consider the Zak transform given by

$$Z_\Lambda f(t, \nu) = \sum_{\lambda \in \Lambda} f(t-\lambda) e^{2\pi i \lambda \nu}, \quad (t, \nu) \in Q_T \times R_{\frac{1}{T}}.$$

Note

$$\Big(H \sum_{\lambda' \in \Lambda} \delta_{\lambda'} \Big)(x) = \Big\langle \kappa_H(x, \cdot), \sum_{\lambda' \in \Lambda} \delta_{\lambda'} \Big\rangle = \sum_{\lambda' \in \Lambda} \kappa_H(x, \lambda') = \sum_{\lambda' \in \Lambda} h_H(x, x-\lambda').$$

We consider first $h_H \in M^{1,1}(\mathbb{R}^{2d})$ and use the Tonelli–Fubini Theorem and the Poisson Summation Formula [24, page 250], to obtain for $(t, \nu) \in Q_{T, \frac{1}{T}}$,

$$\begin{aligned}
Z_\Lambda \circ H \sum_{\lambda' \in \Lambda} \delta_{\lambda'}(t, \nu) &= \sum_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} h_H\big(t-\lambda, t-\lambda-\lambda'\big) e^{2\pi i \lambda \nu} \\
&= \sum_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} \int \eta_H\big(t-\lambda-\lambda', \nu'\big) e^{2\pi i (t-\lambda)\nu'} \, d\nu' e^{2\pi i \lambda \nu} \\
&= \sum_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} \int \eta_H\big(t-\lambda-\lambda', \nu''+\nu\big) e^{2\pi i ((t-\lambda)(\nu+\nu'')+\lambda\nu)} \, d\nu'' \\
&= e^{2\pi i t \nu} \sum_{\lambda'' \in \Lambda} \sum_{\lambda \in \Lambda} \int \eta_H\big(t-\lambda'', \nu+\nu''\big) e^{2\pi i t \nu''} e^{-2\pi i \lambda \nu''} \, d\nu'' \\
&= e^{2\pi i t \nu} D(\Lambda) \sum_{\lambda'' \in \Lambda} \sum_{\lambda \in \Lambda^\perp} \eta_H\big(t-\lambda'', \nu+\lambda\big) e^{2\pi i t \lambda} \\
&= D(\Lambda) \sum_{\lambda'' \in \Lambda} \sum_{\lambda \in \Lambda^\perp} \eta_H\big(t-\lambda'', \nu-\lambda\big) e^{2\pi i t (\nu-\lambda)},
\end{aligned}$$

where $\Lambda^\perp = \{\lambda \in \mathbb{R}^d : e^{2\pi i \lambda\lambda'} = 1 \text{ for all } \lambda' \in \Lambda\} = \frac{1}{T_1}\mathbb{Z} \times \cdots \times \frac{1}{T_d}\mathbb{Z}$ is the dual lattice of Λ and $D(\Lambda) = (T_1 \cdot \cdots \cdot T_d)^{-1} = \mu(Q_T)^{-1}$ is the density of the lattice Λ.

This leads directly to (25) since

$$\int \chi_{Q_T}(t) \sum_{\lambda\in\Lambda} \Big(H \sum_{\lambda'\in\Lambda} \delta_{\lambda'}\Big)(t+\lambda)s(x-\lambda)e^{-2\pi i\nu x}\,dx \tag{26}$$

$$= \chi_{Q_T}(t) \sum_{\lambda\in\Lambda} \Big(H \sum_{\lambda'\in\Lambda} \delta_{\lambda'}\Big)(t+\lambda) \int s(x-\lambda)e^{-2\pi i\nu x}\,dx \tag{27}$$

$$\begin{aligned}
&= \chi_{Q_T}(t) \sum_{\lambda\in\Lambda} \Big(H \sum_{\lambda'\in\Lambda} \delta_{\lambda'}\Big)(t+\lambda)e^{-2\pi i\lambda\nu}\widehat{s}(\nu) \\
&= \chi_{Q_T}(t)\widehat{s}(\nu)\Big(Z_\Lambda\Big(H \sum_{\lambda'\in\Lambda} \delta_{\lambda'}\Big)\Big)(t,\nu) \\
&= D(\Lambda)\cdot T_1 \cdot \cdots \cdot T_d \eta_H(t,\nu)e^{2\pi i\nu t} \\
&= \int h_H(x,t)e^{-2\pi i\nu(x-t)}\,dx = \int h_H(x+t,t)e^{-2\pi i\nu x}\,dx.
\end{aligned}$$

We can apply Lemma 2.3 to show that $\|H\|_{OPW^{p,q}_w} \asymp \|h_H\|_{M^{(p,1),(1,q)}_{\widetilde{w}}}$, $\widetilde{w}(x,t,\nu,\xi) = w(x,\xi)$, and validity of (25) for $h_H \in M^{(p,1),(1,q)}_{\widetilde{w}}(\mathbb{R}^{2d})$ follows then from the density of $M^{1,1}_{\widetilde{w}}(\mathbb{R}^{2d})$ in $M^{(p,1),(1,q)}_{\widetilde{w}}(\mathbb{R}^{2d})$. In case of $p=\infty$ or $q=\infty$ it follows from weak-$*$ density. □

Theorem 1.2 involves the sinc function $\sin(\pi Tx)/(\pi Tx)$ which is not in $M^{1,1}(\mathbb{R}^d)$. Hence, it is not a consequence of Theorem 4.4 but can be easily obtained by adjusting the proof of Theorem 4.4 as described below.

Proof of Theorem 1.2 If $p,q=2$ and $w\equiv 1$, then replacing the absolutely converging integrals in (26) and (27) with the Fourier transform on $L^2(\mathbb{R}^d)$ allows us to choose $\widehat{s} = T_1 \cdot \cdots \cdot T_d \chi_{R_\Omega} \in M^{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. Moreover, in this case we can replace the inequality $T_m\Omega_m < 1$ by $T_m\Omega_m \le 1$ in the hypothesis of Theorem 4.4; the reconstruction formula in Theorem 1.2 follows.

To obtain the correspondence of norms, we first assume $\kappa_H \in \mathcal{S}(\mathbb{R}^2)$ so that trivially $H\sum_{k\in\mathbb{Z}}\delta_{kT} = \sum_{k\in\mathbb{Z}}\kappa_H(x,kT)$ is continuous and $\{H\sum_{k\in\mathbb{Z}}\delta_{kT}(t+nT)\}_{n\in\mathbb{Z}}$ is absolutely summable for all $t\in\mathbb{R}$. We use Theorem 1.1 to compute

$$\begin{aligned}
T^2\|H\|_{HS}^2 &= T^2\|\kappa_H\|_{L^2}^2 = T^2 \iint |\kappa_H(x+t,x)|^2\,dx\,dt \\
&= T^2 \iint \left|\chi_{[0,T]}(t)\sum_{n\in\mathbb{Z}}\Big(H\sum_{k\in\mathbb{Z}}\delta_{kT}\Big)(t+nT)\frac{\sin(\pi T(x-n))}{\pi T(x-n)}\right|^2 dx\,dt \\
&= \int_0^T \sum_{n\in\mathbb{Z}}\left|\Big(H\sum_{k\in\mathbb{Z}}\delta_{kT}\Big)(t+nT)\right|^2 dt \\
&= \int \left|\Big(H\sum_{k\in\mathbb{Z}}\delta_{kT}\Big)(t)\right|^2 dt = \Big\|H\sum_{k\in\mathbb{Z}}\delta_{kT}\Big\|_{L^2}^2.
\end{aligned}$$

Density of $\mathcal{S}(\mathbb{R}^2)$ in $L^2(\mathbb{R}^2)$ guarantees the postulated scaled norm equality for all Hilbert-Schmidt operators with $\operatorname{supp} \mathcal{F}_s \sigma_H \subseteq [0,T] \times [-\Omega/2, \Omega/2]$. □

Note that Theorem 4.4 and its proof generalize trivially to the following setting.

Theorem 4.5 *Let* $1 \leq p, q \leq \infty$ *and* $w = w_1 \otimes w_2$ *be moderate on* $\mathbb{R}^{2d}$. *Let* $A, B \subseteq \mathbb{R}^d$ *be bounded, and let* Λ *be a lattice such that* A *is contained in a fundamental domain of* Λ *and for some* $\epsilon > 0$, $B + [-\epsilon, \epsilon)^d$ *is contained in a bounded fundamental domain of* $\Lambda^\perp = \{\lambda \in \mathbb{R}^d : e^{2\pi i \lambda \lambda'} = 1 \text{ for all } \lambda' \in \Lambda\}$. *Choose* $s \in M^{1,1}(\mathbb{R}^d)$ *with* $\operatorname{supp} \widehat{s} \subseteq B + [-\epsilon, \epsilon)^d$ *and* $\widehat{s} \equiv D(\Lambda)^{-1}$ *on* B *where* $D(\Lambda)$ *is the Beurling density of* Λ. *Then*

$$\Big\| H \sum_{\lambda \in \Lambda} \delta_\lambda \Big\|_{M^{p,q}_w} \asymp \|H\|_{OPW^{p,q}_w}, \qquad H \in OPW^{p,q}_w(A \times B),$$

and any $H \in OPW^{p,q}_w(A \times B)$ *can be reconstructed by means of*

$$\kappa_H(x+t, t) = \chi_A(t) \sum_{\lambda \in \Lambda} \Big(H \sum_{\lambda' \in \Lambda} \delta_{\lambda'} \Big)(t + \lambda) s(x - \lambda).$$

with convergence in $OPW^{p,q}_w(A \times B)$ *for* $1 \leq p, q < \infty$ *and weak-*$*$ *convergence else.*

Considering $OPW^{p,\infty}([0,T] \otimes [-\Omega/2, \Omega/2])$, we obtain the classical sampling theorem as corollary to Theorem 4.4.

Corollary 4.6 *For* $m \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, *with* $\operatorname{supp} \widehat{m} \subseteq [-\Omega/2, \Omega/2]$ *and* T *with* $T\Omega < 1$ *choose* $s \in M^{1,1}(\mathbb{R})$ *with* $\operatorname{supp} \widehat{s} \subseteq [-\Omega/2, \Omega/2]$ *and* $\widehat{s} \equiv T$ *on* $[-1/2T, 1/2T]$. *Then*

$$\|m\|_{L^p} \asymp \big\| \{ m(kT) \} \big\|_{l^p} \tag{28}$$

and

$$m(x) = \sum_{k \in \mathbb{Z}} m(kT) s(x - kT).$$

Proof For $m \in L^p(\mathbb{R})$ with $\operatorname{supp} \widehat{m} \subseteq [-\Omega/2, \Omega/2]$, we define the multiplication operator M formally by $M : f \longrightarrow m \cdot f$, $f \in \mathcal{S}(\mathbb{R})$. We have

$$\begin{aligned} Mf(x) = m(x) f(x) &= \int m(x) \delta_0(t) f(x-t) dt \\ &= \iint \delta_0(t) \widehat{m}(\nu) e^{2\pi i x \nu} f(x-t) \, dt \, d\nu. \end{aligned}$$

Hence, $\delta_0 \otimes \widehat{m} = \eta_M = \mathcal{F}^s \sigma_M$, and, picking any T with $T\Omega < 1$, we conclude $M \in OPW^{p,\infty}([0,T) \times [-\Omega/2, \Omega/2])$.

Theorem 4.4 implies that H and therefore m is fully recoverable from $M\sum_{k\in\mathbb{Z}}\delta_{kT} = \sum_{k\in\mathbb{Z}} m(kT)\delta_{kT}$, in fact, the reconstruction formula (25) reduces then to the classical reconstruction formula for functions:

$$\begin{aligned}
m(x)\delta_0(t) &= m(x+t)\delta_0(t) \\
&= \kappa_M(x+t,x) \overset{\text{Thm. 4.4}}{=} \chi_{[0,T)}(t)\sum_{n\in\mathbb{Z}}\Big(M\sum_{k\in\mathbb{Z}}\delta_{kT}\Big)(t+nT)s(x-nT) \\
&= \chi_{[0,T)}(t)\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} m(kT)\delta_{kT}(t+nT)s(x-nT) \\
&= \chi_{[0,T)}(t)\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} m(kT)\delta_0\big(t-(kT-nT)\big)s(x-nT) \\
&= \begin{cases} 0, & \text{if } t\notin[0,T) \text{ or } t\notin T\mathbb{Z}; \\ \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} m(kT)\delta_0((n-k)T)s(x-nT), & \\ \quad = \sum_{k\in\mathbb{Z}} m(kT)s(x-kT), & \text{if } t=0; \end{cases} \\
&= \delta_0(t)\sum_{k\in\mathbb{Z}} m(kT)s(x-kT).
\end{aligned}$$

The norm equivalence in (28) is obtained by verifying that

$$\begin{aligned}
\|m\|_{L^p} \asymp \|M\|_{OPW^{p,\infty}} &\asymp \Big\|M\sum_{n\in\mathbb{Z}}\delta_{nT}\Big\|_{M^{p,\infty}} = \Big\|\sum_{n\in\mathbb{Z}} m(nT)\delta_{nT}\Big\|_{M^{p,\infty}} \\
&\asymp \big\|\{m(nT)\}_n\big\|_p, \quad m\in L^p(\mathbb{R}), \operatorname{supp}\widehat{m}\subseteq\Big[-\frac{\Omega}{2},\frac{\Omega}{2}\Big]. \qquad \square
\end{aligned}$$

In addition to the application of Theorem 4.4 to multiplication operators, we consider now time-invariant operators in $OPW^{\infty,p}([0,T]\otimes[-\Omega/2,\Omega/2])$.

Example 4.7 The Schwartz kernel theorem implies that time-invariant operators mapping $\mathcal{S}(\mathbb{R})$ continuously into $\mathcal{S}'(\mathbb{R})$ are convolution operators with distributional impulse response. Indeed, time-invariance implies that $\kappa_H(x-t,y-t)=\kappa_H(x,y)$ as tempered distributions for all $t\in\mathbb{R}$ and, hence, $\kappa_H(x,y)=\kappa_H(x-y,0)=h(x-y)$ with

$$Hf(x) = h * f(x) = \int h(x-s)f(s)\,ds$$

and where $\kappa_H\in\mathcal{S}'(\mathbb{R}^2)$ implies $h\in\mathcal{S}'(\mathbb{R})$.

Such operators represent the classical example of operator identification/sampling namely, as formally $H\delta_0(x)=h(x)$, $H\delta_0$ determines h and therefore H completely. In the framework of operator sampling, we consider the case that $h\in L^p(\mathbb{R})$ with $\operatorname{supp} h\subseteq[0,T]$. We have $\eta_H(t,\nu)=\mathcal{F}^s\sigma_H(t,\nu)=h(t)\delta_0(\nu)$ and $H\in OPW^{\infty,p}([0,T]\times[-\frac{1}{4T},\frac{1}{4T}])$. Moreover, with appropriate s we may obtain $H\delta_0=h$

from (25), as

$$
\begin{aligned}
h(t) &= h_H(x,t) \\
&= h_H(x+t,t) \overset{\text{Thm 4.4}}{=} \chi_{[0,T)}(t) \sum_{n\in\mathbb{Z}} \Big(H \sum_{k\in\mathbb{Z}} \delta_{kT} \Big)(t+nT) s(x-nT) \\
&= \chi_{[0,T)}(t) \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} h\big(t-(kT-nT)\big) s(x-nT) \\
&= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \chi_{[0,T)}(t) h\big(t-(kT-nT)\big) s(x-nT) \\
&= \sum_{n\in\mathbb{Z}} H\delta_0(t) s(x-nT) = H\delta_0(t) \sum_{n\in\mathbb{Z}} s(x-nT) \\
&= H\delta_0(t) \sum_{\ell\in\mathbb{Z}} \widehat{s}\Big(\frac{\ell}{T}\Big) e^{2\pi i x\ell} = H\delta_0(t)\widehat{s}(0) = H\delta_0(t).
\end{aligned}
$$

The distributional spreading support of a time-invariant operator is also indicated in Fig. 1.

5 Necessary and Sufficient Conditions for Operator Sampling and Identification

The aim of this section is to show that the applicability of sampling methods to operators depends solemnly on the size of the spreading support set M, that is, on the Jordan content of M (see Definition 5.4 below). Our main result in this section, namely Theorem 5.6, though, only covers the case $d=1$, that is, operators $H:\mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R})$. Possible means for generalizing Theorem 5.6 to operators $H:\mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d)$ are briefly discussed in Sect. 6.

Before recalling the definition of Jordan domains and some of their basic properties, and before stating and proving Theorems 5.6 and 5.7, we will use a geometric approach to obtain a sufficient condition for the identifiability of $OPW_w^{p,p}(M)$ if $M = A(Q_T \times R_\Omega) + (t_0,\nu_0) \subseteq \mathbb{R}^{2d}$, $T,\Omega \in (\mathbb{R}^+)^d$, $T_m\Omega_m < 1$, $m=1,\dots,d$, and A is a so-called symplectic matrix. Theorem 5.2 below generalizes [41, Theorem 5.4].

Definition 5.1 The symplectic group $Sp(d,\mathbb{R})$ consists of those matrices $A \in SL(2d,\mathbb{R}) = \{A \in \mathbb{R}^{2d\times 2d} : \det A = 1\}$ with $A^T \big(\begin{smallmatrix} 0 & -I_d \\ I_d & 0 \end{smallmatrix}\big) A = \big(\begin{smallmatrix} 0 & -I_d \\ I_d & 0 \end{smallmatrix}\big)$, where I_d is the $d\times d$ identity matrix.

Note that $A \in Sp(d,\mathbb{R})$ if and only if $[A(x,\xi)^T, A(x',\xi')^T] = [(x,\xi),(x',\xi')]$ where $[\cdot,\cdot]$ is the symplectic form defined in Sect. 2.

Theorem 5.2 *Let* $A \in Sp(d,\mathbb{R})$, $(t_0,\nu_0) \in \mathbb{R}^{2d}$, $1 \le p \le \infty$, *and let* w *be a moderate weight on* $\mathbb{R}^{2d}$ *with* $w(A(x,\xi)^T) \le w(x,\xi)$. *Then*

1. *$OPW_w^{p,p}(M)$ mapping $M^{\infty,\infty}(\mathbb{R}^d)$ to $M_w^{p,p}(\mathbb{R}^d)$ is identifiable if and only if $OPW_w^{p,p}(AM+(t_0,\nu_0))$ mapping $M^{\infty,\infty}(\mathbb{R}^d)$ to $M_w^{p,p}(\mathbb{R}^d)$ is identifiable, and, consequently,*
2. *for $T,\Omega\in(\mathbb{R}^+)^d$ with $T_m\Omega_m<1$, $m=1,\dots,d$, we have $OPW_w^{p,p}(A(Q_T\times R_\Omega)+(t_0,\nu_0))$ mapping $M^{\infty,\infty}(\mathbb{R}^d)$ to $M_w^{p,p}(\mathbb{R}^d)$ is identifiable.*

The proof of Theorem 5.2 is based on the representation theory of the Weyl-Heisenberg group. Here, we only outline the proof, the interested reader can import details from [41, Sect. 5] or [16, 24].

Proof We will obtain the identifiability $OPW_w^{p,p}(AM)$ with $A\in Sp(2d,\mathbb{R})$ from the identifiability of $OPW_w^{p,p}(M)$ by using the canonical correspondence between elements in $OPW_w^{p,p}(AM)$ and elements in $OPW_w^{p,p}(M)$ which is given by a coordinate transformation in the spreading domain $M\subseteq\mathbb{R}^{2d}$, AM. In fact, Theorem 5.3 in [41] recalls that for $A\in Sp(d,\mathbb{R})$ there exists a unitary operators O_A on $L^2(\mathbb{R}^d)$ with $\pi(A(t,\nu))=O_A\pi(t,\nu){O_A}^*$, $t,\nu\in\mathbb{R}^d$. Such operators O_A, $A\in Sp(d,\mathbb{R})$ are called metaplectic operators, and they are intertwining operators for representations of the reduced Weyl–Heisenberg group that are unitarily equivalent to the Schrödinger representation [16, 24]. Metaplectic operators are finite compositions of the Fourier transform, multiplication operators with multiplier $e^{-\pi i x^TCx}$ with C selfadjoint, and normalized dilations $f\mapsto|\det D|^{\frac{1}{2}}f(Dx)$, D invertible. They extend, respectively restrict, to isomorphisms on $M_w^{p,p}(\mathbb{R}^d)$, $1\le p\le\infty$, if $w(A(x,\xi)^T)\le w(x,\xi)$ [14, Theorem 7.4].

The following formal calculations of operator valued integrals can be justified weakly for all $H\in OPW_w^{p,p}(AM)$. A similar computation can be made for $H\in OPW_w^{p,p}(M+(t_0,\nu_0))$, combining both leads Theorem 5.2. We compute

$$
\begin{aligned}
H&=\iint\eta_H(t,\nu)M_\nu T_t\,dt\,d\nu=\iint\eta_H(t,\nu)\pi(t,\nu)\,dt\,d\nu\\
&=\iint\eta_H\big(A(t,\nu)^T\big)\pi\big(A(t,\nu)^T\big)\,dt\,d\nu=\iint\eta_H\big(A(t,\nu)^T\big)O_A\pi(t,\nu){O_A}^*\,dt\,d\nu\\
&=O_A\iint\eta_{H_A}(t,\nu)\pi(t,\nu)\,dt\,d\nu{O_A}^*=O_AH_A{O_A}^*,
\end{aligned}
$$

where $\eta_{H_A}=\eta_H\circ A$ and $H_A\in OPW_w^{p,p}(M)$. The identifiability of $OPW_w^{p,p}(M)$ with identifier $f_M\in M^{\infty,\infty}(\mathbb{R}^d)$ leads therefore to the identifiability of $OPW_w^{p,p}(AM)$ with identifier $f_{AM}=O_Af_M\in M^{\infty,\infty}(\mathbb{R}^d)$. In fact, we have

$$
\begin{aligned}
\|Hf_{AM}\|_{M_w^{p,p}}&=\|HO_Af_M\|_{M_w^{p,p}}\asymp\|{O_A}^*HO_Af_M\|_{M_w^{p,p}}\\
&=\|H_Af_M\|_{M_w^{p,p}}\asymp\|\sigma_{H_A}\|_{L_w^{p,p}}\asymp\|\sigma_H\|_{L_w^{p,p}}=\|H\|_{OPW_w^{p,p}},\\
&\quad H\in OPW_w^{p,p}(AM). \qquad\square
\end{aligned}
$$

Remark 5.3 Theorem 5.2 is not an operator sampling result per se as not necessarily all O_A map discretely supported distributions to discretely supported distributions.

But Theorem 5.2 can be used to show that $OPW_w^{p,p}(M)$ permits operator sampling by showing that

1. $M \subseteq A\widetilde{M} + (t_0, \nu_0)$,
2. $A \in Sp(d, \mathbb{R})$,
3. $w(A(x,\xi)) \leq w(x,\xi)$,
4. $OPW_w^{p,p}(\widetilde{M})$ permits operator sampling with sampling set $\{x_j\}$ and weights $\{c_j\}$, and
5. $O_A^* \sum c_j \delta_{x_j}$ is discretely supported.

Note also that the restriction to $p = q$ in Theorem 5.2 is necessary, as, for example, the Fourier transform is not an isomorphism on $M^{p,q}$ whenever $p \neq q$.

Theorem 5.2 relies on arguments based on symplectic geometry on phase space. As discussed above, Theorems 5.6 and 5.7 give a characterization for the identifiability of operators $H : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R})$ which does not rely on any geometric properties.

Definition 5.4 For $K, L \in \mathbb{N}$ set $R_{K,L} = [0, \frac{1}{K}) \times [0, \frac{K}{L})$ and

$$\mathcal{U}_{K,L} = \left\{ \bigcup_{j=1}^{J} \left(R_{K,L} + \left(\frac{k_j}{K}, \frac{p_j K}{L} \right) \right) : k_j, p_j \in \mathbb{Z}, J \in \mathbb{N} \right\}.$$

The inner content, respectively outer content, of a bounded set $M \subseteq \mathbb{R}^2$ is

$$\mathrm{vol}^-(M) = \sup\left\{ \mu(U) : U \subseteq M \text{ and } U \in \mathcal{U}_{K,L} \text{ for some } K, L \in \mathbb{N} \right\}, \tag{29}$$

respectively

$$\mathrm{vol}^+(M) = \inf\left\{ \mu(U) : U \supseteq M \text{ and } U \in \mathcal{U}_{K,L} \text{ for some } K, L \in \mathbb{N} \right\}. \tag{30}$$

Clearly, we have $\mathrm{vol}^-(M) \leq \mathrm{vol}^+(M)$. If $\mathrm{vol}^-(M) = \mathrm{vol}^+(M)$, then we say that M is a Jordan domain with Jordan content $\mathrm{vol}(M) = \mathrm{vol}^-(M) = \mathrm{vol}^+(M)$.

We collect some well known and useful facts on Jordan domains to illustrate their generality [17].

Proposition 5.5 *Let $M \subseteq \mathbb{R}^2$.*

1. *If M is a Jordan domain, then M is Lebesgue measurable with $\mu(M) = \mathrm{vol}(M)$.*
2. *If M is Lebesgue measurable and bounded and its boundary ∂M is a Lebesgue zero set, that is, $\mu(\partial M) = 0$, then M is a Jordan domain.*
3. *If M is open, then $\mathrm{vol}^-(M) = \mu(M)$ and if M is compact, then $\mathrm{vol}^+(M) = \mu(M)$.*
4. *If $\mathcal{P} \subseteq \mathbb{N}$ is unbounded, then replacing the quantifier* "for some $L \in \mathbb{N}$" *with* "for some $L \in \mathcal{P}$" *in (29) and in (30) leads to equivalent definitions of inner and outer Jordan content.*

The second main result of this paper has been stated in simple form as Theorem 1.3, part 1, in Sect. 1. It also generalizes [53, Theorem 1.1].

Theorem 5.6 *For $1 \le p, q \le \infty$ and $w = w_1 \otimes w_2$ moderate, the class $OPW_w^{p,q}(M)$ mapping $M^{\infty,\infty}(\mathbb{R})$ to $M_w^{p,q}(\mathbb{R})$ permits operator sampling if* $\mathrm{vol}^+(M) < 1$. *In fact, if* $\mathrm{vol}^+(M) < 1$, *then there exists $L > 0$ and a periodic sequence $\{c_n\}$ such that*

$$\Big\| H \sum_{n\in\mathbb{Z}} c_n \delta_{\frac{n}{L}} \Big\|_{M_w^{p,q}} \asymp \|H\|_{OPW_w^{p,q}}, \qquad H \in OPW_w^{p,q}(M). \tag{31}$$

Theorem 5.6 is complemented by Theorem 5.7 which generalizes [53, Theorem 1.1] and [54, Theorem 5.2, statement 2].

Theorem 5.7 *Let $1 \le p, q \le \infty$ and w subexponential. The class $OPW_w^{p,q}(M)$ mapping $M^{\infty,\infty}(\mathbb{R})$ to $M_w^{p,q}(\mathbb{R})$ is not identifiable if* $\mathrm{vol}^-(M) > 1$, *that is, for all $f \in M^{\infty,\infty}$ we have*

$$\|Hf\|_{M_w^{p,q}} \not\asymp \|H\|_{OPW_w^{p,q}}, \qquad H \in OPW_w^{p,q}(M).$$

Theorem 5.6 is proven below. Subsequently, we outline the proof of Theorem 5.7 which employs elements of the proof of [53, Theorem 1.1] and [47, Theorem 3.13].

5.1 Proof of Theorem 5.6

The following observations are special cases of Theorem 5.2. They will be used in the following to reduce notational complexity.

Proposition 5.8 *Let $1 \le p, q \le \infty$ and let w be a moderate weight on $\mathbb{R}^2$.*

1. *$OPW_w^{p,q}(M)$ is identifiable by f if and only if $OPW_w^{p,q}\big(M\cdot\begin{pmatrix} 1/a & 0 \\ 0 & a\end{pmatrix}\big)$ is identifiable by $D_a f : x \mapsto f(ax)$.*
2. *$OPW_w^{p,q}(M)$ is identifiable by f if and only if $OPW_w^{p,q}(M+\lambda)$ is identifiable by $\pi(\lambda) f$.*

Proof We will proof 1., the proof of 2. follows similarly. For $H \in OPW_w^{p,q}\big(M\cdot\begin{pmatrix} 1/a & 0 \\ 0 & a\end{pmatrix}\big)$, define $H_a \in OPW_w^{p,q}(M)$ by $\eta_{H_a} = \eta_H \circ \begin{pmatrix} 1/a & 0 \\ 0 & a\end{pmatrix}$. Then $\sigma_{H_a} = \sigma_H \circ \begin{pmatrix} 1/a & 0 \\ 0 & a\end{pmatrix}$ as well. We compute formally

$$\begin{aligned}
(HD_a f)\Big(\frac{x}{a}\Big) &= \int \sigma_H\Big(\frac{x}{a}, \xi\Big) e^{2\pi i \frac{x}{a}\xi}\, \widehat{D_a f}(\xi)\, d\xi \\
&= \frac{1}{a}\int \sigma_H\Big(\frac{x}{a}, \xi\Big) e^{2\pi i x \frac{\xi}{a}}\, \widehat{f}\Big(\frac{\xi}{a}\Big)\, d\xi \\
&= \int \sigma_H\Big(\frac{x}{a}, a\xi\Big) e^{2\pi i x\xi}\, \widehat{f}(\xi)\, d\xi = H_a f(x).
\end{aligned}$$

Using standard density arguments, we conclude that

$$\begin{aligned}
\|HD_a f\|_{M_w^{p,q}} &\asymp \big\|D_{\frac{1}{a}} H D_a f\big\|_{M_w^{p,q}} = \|H_a f\|_{M_w^{p,q}} \\
&\asymp \|H_a\|_{OPW_w^{p,q}} \asymp \|H\|_{OPW_w^{p,q}}, \qquad H \in OPW_w^{p,q}\bigg(M\cdot\begin{pmatrix} 1/a & 0 \\ 0 & a\end{pmatrix}\bigg).
\end{aligned}$$

□

Assume now that $\mathrm{vol}^+(M) < 1$. Applying Proposition 5.8, we assume, without loss of generality, that for $\delta > 0$ sufficiently small, $M + [-\frac{\delta}{2}, \frac{\delta}{2})^2 \subseteq [0,1) \times \mathbb{R}^+$. We choose $K, L \in \mathbb{N}$ with L prime so that the following conditions are satisfied for some $0 < \epsilon < \delta$ and $M_\epsilon = M + [-\frac{\epsilon}{2}, \frac{\epsilon}{2})^2$

1. $\mathrm{vol}^+(M_\epsilon) < 1$,
2. $M_\epsilon \subseteq [0,1) \times [0, K)$,
3. $L \geq K$,
4. $M_\epsilon \subseteq U_M = \bigcup_{j=0}^{L-1} (R_{K,L} + (\frac{m_j}{K}, \frac{n_j K}{L})), m_j, n_j \in \mathbb{Z}$, where $R_{K,L} = [0, \frac{1}{K}) \times [0, \frac{K}{L})$ and $(m_j, n_j) \neq (m_{j'}, n_{j'})$ if $j \neq j'$.

Note that $1 = \mathrm{vol}(U_M)$.

The following result from [43] is a key component of our proof of Theorem 5.6. In fact, if the restriction to L prime below could be weakened, then we would obtain a generalization of Theorem 5.6 to higher dimensions, see Sect. 6 for details.

Theorem 5.9 *For $c \in \mathbb{C}^L$ define $\pi(k,\ell)c$ by $(\pi(k,\ell)c)_j = c_{j-k} e^{2\pi i \frac{j\ell}{L}}$, $k, \ell = 0, \ldots, L-1$, where $j-k$ is understood modulus L. If L is prime, then for almost every $c \in \mathbb{C}^L$, the vectors in $\mathcal{G}_c = \{\pi(k,\ell)c\}_{k,\ell=0,\ldots,L-1}$ are in general linear position, that is, any matrix composed of L vectors of $\mathcal{G}_c$ is invertible.*

Remark 5.10 Theorem 5.9 can be reformulated as a matrix identification result with identifier c [42]. The use of algorithms based on basis pursuit to determine a matrix M from Mc efficiently is discussed in [38, 49–51]. See also the overview article [48].

We now choose as $c \in l^\infty(\mathbb{Z})$ the periodic extension of a vector $(c_0, \ldots, c_{L-1})$ which satisfies the conclusions of Theorem 5.9. In the following, we will show that κ_H can be recovered from Hg with $g = \sum_{k \in \mathbb{Z}} c_k \delta_{\frac{k}{L}} \in M^{\infty,\infty}(\mathbb{R})$.

For simplicity, we will assume first that $H \in OPW^{1,1}(M)$, hence, $\sigma_H, \eta_H, \kappa_H \in M^{1,1}(\mathbb{R}^2)$ and for $g \in M^{\infty,\infty}(\mathbb{R})$ we have $Hg \in M^{1,1}(\mathbb{R})$ [54]. This enables us to switch the order of integration in many of the computations that follow. Using a standard density argument, we then obtain the result for general $H \in OPW^{p,q}$ if $p, q < \infty$. Replacing respective integrals with supremums in the case that $p = \infty$ and/or $q = \infty$ concludes the proof.

Choose nonnegative $\varphi \in C_c^\infty(\mathbb{R})$ with $\int \varphi(x)\,dx = 1$ and $\operatorname{supp} \varphi \subseteq [-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. We will consider the case $1 \leq p < \infty$ only, the case $p = \infty$ requires only the usual adjustments.

Note that

$$\begin{aligned}\left| V_\varphi Hg\left(t + \frac{n}{K}, \nu\right)\right| &= \left| \int \sum_{k\in\mathbb{Z}} c_k h\left(x, x + \frac{k}{K}\right) e^{-2\pi i \nu x} \varphi\left(x - \left(t + \frac{n}{K}\right)\right) dx \right| \\ &= \left| \int \sum_{k\in\mathbb{Z}} c_k h\left(x + \frac{n}{K}, x + \frac{n}{K} + \frac{k}{K}\right) e^{-2\pi i \nu \frac{n}{K}} \overline{\pi(t,\nu)\varphi(x)}\, dx \right| \\ &= \left| \int \sum_{k\in\mathbb{Z}} c_k h\left(x + \frac{n}{K}, x + \frac{n}{K} + \frac{k}{K}\right) \overline{\pi(t,\nu)\varphi(x)}\, dx \right|.\end{aligned}$$

Set $w=\sum_{m\in\mathbb{Z}} w_1(\frac{mL}{K})\chi_{[\frac{mL}{K},\frac{(m+1)L}{K})}$ and observe that $w\asymp w_1$. Then

$$\|Hg\|_{M^{p,q}_w}=\|V_\varphi Hg\|_{L^{p,q}_w}=\left\|\left(\sum_{n\in\mathbb{Z}}\int_{\frac{n}{K}}^{\frac{n+1}{K}}|V_\varphi Hg(t,\nu)w_1(t)|^p\right)^{\frac{1}{p}}dt\right\|_{L^q_{w_2}}$$

$$=\left\|\left(\sum_{n\in\mathbb{Z}}\int_0^{\frac{1}{K}}\left|V_\varphi Hg\left(t+\frac{n}{K},\nu\right)w_1\left(t+\frac{n}{K}\right)\right|^p\right)^{\frac{1}{p}}dt\right\|_{L^q_{w_2}}$$

$$=\left\|\left(\sum_{n\in\mathbb{Z}}\int_0^{\frac{1}{K}}\left|V_\varphi Hg\left(t+\frac{n}{K},\nu\right)w\left(t+\frac{n}{K}\right)\right|^p\right)^{\frac{1}{p}}dt\right\|_{L^q_{w_2}}.$$

We set $\psi=\chi_{[-\frac{\epsilon}{2},\frac{K}{L}+\frac{\epsilon}{2})}*\varphi$ and observe that $\psi(\nu')T_\omega\varphi(\nu')=T_\omega\varphi(\nu')$ for $\omega\in[0,\frac{K}{L})$, a fact that will be used to drop ψ in (36) below. We compute for $t\in[0,\frac{1}{K})$

$$\sum_{n\in\mathbb{Z}}\left|w\left(t+\frac{n}{K}\right)\right|^p\left|V_\varphi Hg\left(t+\frac{n}{K},\nu\right)\right|^p$$

$$=\sum_{n\in\mathbb{Z}}\left|w\left(t+\frac{n}{K}\right)\right|^p\left|\int\sum_{k\in\mathbb{Z}}c_k h_H\left(x+\frac{n}{K},x+\frac{n}{K}+\frac{k}{K}\right)\overline{\pi(t,\nu)\varphi(x)}\,dx\right|^p$$

$$=\sum_{n\in\mathbb{Z}}\left|w\left(t+\frac{n}{K}\right)\right|^p\left|\int\sum_{k\in\mathbb{Z}}c_{n-k} h_H\left(x+\frac{n}{K},x+\frac{k}{K}\right)\overline{\pi(t,\nu)\varphi(x)}\,dx\right|^p$$

$$=\sum_{j=0}^{L-1}\sum_{m\in\mathbb{Z}}\left|w_1\left(\frac{mL}{K}\right)\right|^p\left|\int\sum_{k\in\mathbb{Z}}c_{j-k} h_H\left(x+\frac{mL+j}{K},x+\frac{k}{K}\right)\overline{\pi(t,\nu)\varphi(x)}\,dx\right|^p$$

$$=\sum_{j=0}^{L-1}\Bigg\|\sum_{m\in\mathbb{Z}}\int\sum_{k\in\mathbb{Z}}c_{j-k} h_H\left(x+\frac{mL+j}{K},x+\frac{k}{K}\right)\overline{\pi(t,\nu)\varphi(x)}\,dx$$
$$V_\varphi\left(M_{-\frac{mL+j}{K}}\psi\right)(x',\xi')\Bigg\|^p_{L^p_{1\otimes w}} \tag{32}$$

$$\asymp\sum_{j=0}^{L-1}\Bigg\|\iint\sum_{k\in\mathbb{Z}}c_{j-k}\left(\sum_{m\in\mathbb{Z}}h_H\left(x+\frac{mL+j}{K},x+\frac{k}{K}\right)e^{-2\pi i\nu'\frac{mL+j}{K}}\right)$$
$$\overline{\pi(t,\nu)\varphi(x)}\psi(\nu')\overline{\widehat{\pi}(x',\xi')\varphi(\nu')}\,d\nu'\,dx\Bigg\|^p_{L^p_{1\otimes w_1}}, \tag{33}$$

where we used that $M_{-\frac{mL+j}{K}}\psi$ is an $l^p_{\tilde{w}}$-Riesz basis in the Banach space $M^{p,p}_{1\otimes w_1}(\mathbb{R})$ to obtain (32), that is, we used

$$\left\|\sum_{m\in\mathbb{Z}}a_m\right\|_{l^p_{w_1}}\asymp\left\|\sum_{m\in\mathbb{Z}}a_m M_{-\frac{mL+j}{K}}\psi\right\|_{M^{p,p}_{w_1}}=\left\|\sum_{m\in\mathbb{Z}}a_m V_\varphi M_{-\frac{mL+j}{K}}\psi\right\|_{L^p_{w_1}}.$$

Note that

$$\sum_{m\in\mathbb{Z}} h_H\left(x+\frac{mL+j}{K}, x+\frac{k}{K}\right)e^{-2\pi i\nu'\frac{mL+j}{K}}$$

$$=\sum_{m\in\mathbb{Z}}\int \eta_H\left(x+\frac{k}{K},\xi'\right)e^{2\pi i(x+\frac{mL+j}{K})\xi'}\,d\xi' e^{-2\pi i\frac{mL+j}{K}\nu'}$$

$$=\sum_{m\in\mathbb{Z}}\int \eta_H\left(x+\frac{k}{K},\xi'\right)e^{2\pi i x\xi'}e^{2\pi i\frac{mL+j}{K}(\xi'-\nu')}\,d\xi'$$

$$=\sum_{m\in\mathbb{Z}}\int \eta_H\left(x+\frac{k}{K},\nu'+\xi'\right)e^{2\pi i x(\nu'+\xi')}e^{2\pi i\frac{j}{K}\xi'}e^{2\pi i\frac{mL}{K}\xi'}\,d\xi'$$

$$\asymp\sum_{\ell\in\mathbb{Z}}\eta_H\left(x+\frac{k}{K},\nu'+\frac{\ell K}{L}\right)e^{2\pi i x(\nu'+\frac{\ell K}{L})}e^{2\pi i\frac{j\ell}{L}} \tag{34}$$

$$=\sum_{\ell\in\mathbb{Z}}\eta_H\left(x+\frac{k}{K},\nu'+\frac{\ell K}{L}\right)e^{2\pi i(x+\frac{k}{K})(\nu'+\frac{\ell K}{L})}e^{-2\pi i(\frac{k}{K}\nu'+\frac{k\ell}{L})}e^{2\pi i\frac{j\ell}{L}}$$

$$=\sum_{\ell\in\mathbb{Z}}\widetilde{\eta_H}\left(x+\frac{k}{K},\nu'+\frac{\ell K}{L}\right)e^{2\pi i\frac{(j-k)\ell}{L}}e^{-2\pi i\frac{k}{K}\nu'}. \tag{35}$$

We have applied the Poisson Summation Formula to obtain (34). Moreover, we chose $\widetilde{\eta_H}(x',\nu')=\eta_H(x',\nu')e^{2\pi i x'\nu'}$ in (35).

After substituting (35) into (33), we integrate with respect to t on $[0,\frac{1}{K})$ to obtain

$$\int_0^{\frac{1}{K}}\sum_{n\in\mathbb{Z}}\left|w_1(t+\frac{n}{K},\nu)\right|^p\left|V_\varphi Hg\left(t+\frac{n}{K},\nu\right)\right|^p dt$$

$$\asymp\sum_{j=0}^{L-1}\int_0^{\frac{1}{K}}\Bigg\|\iint\sum_{k\in\mathbb{Z}}c_{j-k}\left(\sum_{\ell\in\mathbb{Z}}\widetilde{\eta}_H\left(x+\frac{k}{K},\nu'+\frac{\ell K}{L}\right)e^{2\pi i\frac{(j-k)\ell}{L}}e^{-2\pi i\frac{k}{K}\nu'}\right)$$

$$\times\overline{\pi(t,\nu)\varphi(x)}\psi(\nu')\overline{\pi(\xi',x')\varphi(\nu')}\,d\nu'\,dx\Bigg\|^p_{L^p_{1\otimes w_1}}dt$$

$$=\sum_{j=0}^{L-1}\int_0^{\frac{1}{K}}\iint\Bigg|\iint\sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}}c_{j-k}e^{2\pi i\frac{(j-k)\ell}{L}}\widetilde{\eta}_H\left(x+\frac{k}{K},\nu'+\frac{\ell K}{L}\right)$$

$$\times e^{-2\pi i\frac{k}{K}\nu'}\overline{\pi(t,\nu)\varphi(x)}\psi(\nu')\overline{\pi(\xi',x')\varphi(\nu')}\,d\nu'\,dx\,w_1(x')\Bigg|^p dx'\,d\xi'\,dt$$

$$\geq \sum_{j=0}^{L-1} \int_0^{\frac{1}{K}} \iint_0^{\frac{K}{L}} \bigg| \int \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} c_{j-k} e^{2\pi i \frac{(j-k)\ell}{L}} \int e^{-2\pi i \nu' \frac{k}{K}} \widetilde{\eta}_H \left(x + \frac{k}{K}, \nu' + \ell \frac{K}{L} \right)$$

$$\times \overline{\pi(t,\nu)\varphi(x)} \psi(\nu') \overline{\pi(\xi',x')\varphi(\nu')}\, d\nu'\, dx\, w_1(x') \bigg|^p d\xi'\, dx'\, dt$$

$$= \sum_{j=0}^{L-1} \int_0^{\frac{1}{K}} \iint_0^{\frac{K}{L}} \bigg| \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} c_{j-k} e^{2\pi i \frac{(j-k)\ell}{L}} \iint e^{-2\pi i \nu' \frac{k}{K}} \widetilde{\eta}_H \left(x + \frac{k}{K}, \nu' + \ell \frac{K}{L} \right)$$

$$\times \overline{\pi(t,\nu)\varphi(x)\pi(\xi',x')\varphi(\nu')}\, dx\, d\nu'\, w_1(x') \bigg|^p d\xi'\, dx'\, dt \tag{36}$$

$$= \sum_{j=0}^{L-1} \int_0^{\frac{1}{K}} \iint_0^{\frac{K}{L}} \bigg| \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} c_{j-k} e^{2\pi i \frac{(j-k)\ell}{L}} V_{\varphi\otimes\varphi} \widetilde{\eta}_H \left(t + \frac{k}{K}, \xi' + \frac{\ell K}{L}, \nu, x' + \frac{k}{K} \right)$$

$$\times e^{2\pi i \frac{k}{K}\nu} e^{2\pi i \frac{\ell k}{L}} e^{2\pi i x' \frac{\ell K}{L}} w_1(x') \bigg|^p d\xi'\, dx'\, dt$$

$$= \int_0^{\frac{1}{K}} \iint_0^{\frac{K}{L}} \sum_{j=0}^{L-1} \bigg| \sum_{j'=0}^{L} c_{j-k_{j'}} e^{2\pi i \frac{j\ell_{j'}}{L}} V_{\varphi\otimes\varphi} \widetilde{\eta}_H \left(t + \frac{k_{j'}}{K}, \xi' + \frac{\ell_{j'} K}{L}, \nu, x' + \frac{k_{j'}}{K} \right)$$

$$\times e^{2\pi i \frac{k_{j'}}{K}\nu} e^{2\pi i x' \frac{\ell_{j'} K}{L}} w_1(x') \bigg|^p d\xi'\, dx'\, dt \tag{37}$$

$$\asymp \int_0^{\frac{1}{K}} \iint_0^{\frac{K}{L}} \sum_{j'=0}^{L} \bigg| V_{\varphi\otimes\varphi} \widetilde{\eta}_H \left(t + \frac{k_{j'}}{K}, \xi' + \frac{\ell_{j'} K}{L}, \nu, x' + \frac{k_{j'}}{K} \right)$$

$$\times e^{2\pi i \frac{k_{j'}}{K}\nu} e^{2\pi i x' \frac{\ell_{j'} K}{L}} w_1 \left(x' + \frac{k_{j'}}{K} \right) \bigg|^p d\xi'\, dx'\, dt$$

$$= \sum_{j'=0}^{L} \int_0^{\frac{1}{K}} \iint_0^{\frac{K}{L}} \bigg| V_{\varphi\otimes\varphi} \widetilde{\eta}_H \left(t + \frac{k_{j'}}{K}, \xi' + \frac{\ell_{j'} K}{L}, \nu, x' \right) w_1(x') \bigg|^p d\xi'\, dx'\, dt$$

$$= \sum_{j'=0}^{L} \int_{\frac{k_{j'}}{K}}^{\frac{(k_{j'}+1)}{K}} \iint_{\frac{l_{j'} K}{L}}^{\frac{(l_{j'}+1)}{L}} \bigg| V_{\varphi\otimes\varphi} \widetilde{\eta}(t,\xi',\nu,x') w_1(x') \bigg|^p d\xi'\, dx'\, dt.$$

To obtain (37) we used that $V_{\varphi\otimes\varphi}\widetilde{\eta} \subseteq [0,1)\times[0,K)$. Moreover, we used

$$\iint e^{-2\pi i \nu' \frac{k}{K}} \widetilde{\eta}_H \left(x + \frac{k}{K}, \nu' + l \frac{K}{L} \right) \overline{\pi(t,\nu)\varphi(x)\pi(\xi',x')\varphi(\nu')}\, dx\, d\nu'$$

$$= \iint e^{-2\pi i \nu' \frac{k}{K}} \widetilde{\eta}_H \left(x + \frac{k}{K}, \nu' + l \frac{K}{L} \right) e^{-2\pi i x \nu} \varphi(x-t) e^{-2\pi i x' \nu'} \varphi(\nu' - \xi')\, dx\, d\nu'$$

$$= \iint \widetilde{\eta_H}(x,\nu')e^{-2\pi i\nu(x-\frac{k}{K})}\varphi\Big(x-\Big(t+\frac{k}{K}\Big)\Big)$$

$$\times e^{-2\pi i(\nu'-\frac{\ell K}{L})\frac{k}{K}}e^{-2\pi i x'(\nu'-\frac{\ell K}{L})}\varphi\Big(\nu'-\Big(\xi'+\frac{\ell K}{L}\Big)\Big)\,dx\,d\nu'$$

$$= V_{\varphi\otimes\varphi}\widetilde{\eta_H}\Big(t+\frac{k}{K},\xi'+\frac{\ell K}{L},\nu,x'+\frac{k}{K}\Big)e^{2\pi i\frac{k}{K}\nu}e^{2\pi i\frac{\ell k}{L}}e^{2\pi i x'\frac{\ell K}{L}}.$$

Replacing now $\varphi\otimes\varphi$ by any other test functions leads to equivalent norms of the modulation space at hand, we obtain for real valued $\varrho\in S(\mathbb{R}^2)$, $\varrho(t,\nu)=1$ for $[-1,2)\times[-K,2K)$,

$$\begin{aligned}
&\|Hg\|_{M^{p,q}_w}\\
&\quad\asymp \|V_\varphi Hg\|_{L^{p,q}}\\
&\quad= \bigg\|\bigg(\sum_{n\in\mathbb{Z}}\int_{\frac{n}{K}}^{\frac{n+1}{K}}|V_\varphi Hg(u,\nu)w_1(u)|^p\bigg)^{\frac{1}{p}}\bigg\|_{L^q_{w_2}}\\
&\quad= \bigg\|\bigg(\sum_{j'=0}^{L}\int_{\frac{k_{j'}}{K}}^{\frac{(k_{j'}+1)}{K}}\int_{\frac{l_{j'}K}{L}}^{\frac{(l_{j'}+1)}{L}}\int |V_{\varphi\otimes\varphi}\widetilde{\eta}(u,\xi,\nu,x)w_1(x)|^p\,d\xi\,dx\,du\bigg)^{\frac{1}{p}}\bigg\|_{L^q_{w_2}}\\
&\quad= \bigg\|\bigg(\int_0^1\int_0^K\int |V_{\varphi\otimes\varphi}\widetilde{\eta}(u,\xi,\nu,x)w_1(x)|^p\,d\xi\,dx\,du\bigg)^{\frac{1}{p}}\bigg\|_{L^q_{w_2}}\\
&\quad\asymp \bigg\|\bigg(\int_0^1\int_0^K\int |V_\varrho\widetilde{\eta}(u,\xi,\nu,x)w_1(x)|^p\,d\xi\,dx\,du\bigg)^{\frac{1}{p}}\bigg\|_{L^q_{w_2}}\\
&\quad\ge \bigg\|\bigg(\int_0^1\int_0^K\int |\chi_{[0,1)}(u)\chi_{[0,K)}(\xi)V_\varrho\widetilde{\eta}(u,\xi,\nu,x)w_1(x)|^p\,d\xi\,dx\,du\bigg)^{\frac{1}{p}}\bigg\|_{L^q(\nu)}\\
&\quad\ge \bigg\|\bigg(\int_0^1\int_0^K\int |\chi_{[0,1)}(u)\chi_{[0,K)}(\xi)\mathcal{F}^s\widetilde{\eta}(\nu,x)w_1(x)|^p\,d\xi\,dx\,du\bigg)^{\frac{1}{p}}\bigg\|_{L^q(\nu)}\\
&\quad= \bigg\|\bigg(K\int_0^K |\mathcal{F}^s\widetilde{\eta}(\nu,x)w_1(x)|^p\,dx\bigg)^{\frac{1}{p}}\bigg\|_{L^q_{w_2}}\\
&\quad= \bigg\|\bigg(K\int_0^K |\widetilde{\sigma}(x,\nu)w_1(x)|^p\,dx\bigg)^{\frac{1}{p}}\bigg\|_{L^q_{w_2}}\\
&\quad= \|\widetilde{\sigma}\|_{L^{p,q}_w}\asymp\|\sigma\|_{L^{p,q}_w}.
\end{aligned}\tag{38}$$

To obtain (38), we apply a mixed L^p-space version of Young's inequality for convolutions, namely, we use that for $\widetilde{\varrho}(x,\xi)=e^{2\pi i x\xi}\varrho(x,\xi)\in L^1(\mathbb{R})$, we have $\widetilde{\sigma}(x,\xi)=\widehat{\widetilde{\varrho}}*\sigma$ and $\sigma(x,\xi)=\widehat{\widehat{\widetilde{\varrho}}}*\widetilde{\sigma}$ [24, Theorem 11.1].

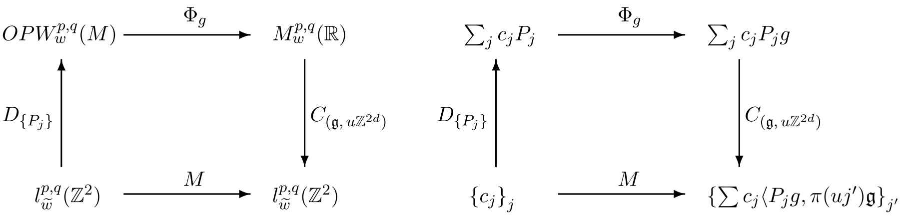

Fig. 3 *Sketch of the proof of Theorem* 5.7. We choose a structured operator family $\{P_j\} \subseteq OPW^{p,q}_w(M)$ so that the corresponding synthesis map $D_{\{P_j\}} : \{c_j\} \longrightarrow \sum c_j P_j$ has a bounded left inverse. Note that $C_{(\mathfrak{g},u\mathbb{Z}^{2d})}$ has a bounded left inverse for $u < 1$ as well. Theorem 5.13 shows that for any $g \in M^{\infty}(\mathbb{R})$, the composition $M = C_{(\mathfrak{g},u\mathbb{Z}^{2d})} \circ \phi_g \circ D_{\{P_j\}}$ has *no* bounded left inverses. This implies that $\phi_g : OPW^{p,q}_w(M) \longrightarrow M^{p,q}_w(\mathbb{R})$ also has *no* bounded left inverses

5.2 Outline of the Proof of Theorem 5.7

We omit detailed computations as they would closely resemble computations carried out in [41, 47, 53, 54]. For the interested reader, we suggest to use [53] as a companion when filling in detail.

We will show that for a measurable subset M with $\mathrm{vol}^-(M) > 1$, the operator class $OPW^{p,q}_w(M)$ is not identifiable. In detail, we will show that for *every* $g \in M^{\infty,\infty}(\mathbb{R})$, the operator

$$\Phi_g : OPW^{p,q}_w(M) \longrightarrow M^{p,q}_w(\mathbb{R}),\, H \mapsto Hg,$$

is *not* bounded below, that is, there exists *no* $c > 0$ for which we have $\|Hg\|_{M^{p,q}_w} \geq c\|\sigma_H\|_{L^{p,q}_w}$ for all $H \in OPW^{p,q}_w(M)$.

To this end, choose $K, L \in \mathbb{N}$ and $V_M = \bigcup_{j=0}^{L-1}(R_{K,L} + (\frac{m_j}{K}, \frac{n_j K}{L})), m_j, n_j \in \mathbb{Z}$, where $R_{K,L} = [0, \frac{1}{K}) \times [0, \frac{K}{L})$ and where $(m_j, n_j) \neq (m_{j'}, n_{j'})$ if $j \neq j'$, such that $V_M \subseteq M$ and $\mathrm{vol}(V_M) > 1$. It is sufficient to show that $OPW^{p,q}_w(V_M)$ is not identifiable as $OPW^{p,q}_w(V_M) \subseteq OPW^{p,q}_w(M)$.

The proof of Theorem 5.7 is also sketched in Fig. 3. The proof is based on extensions to results from [47, 53] which are stated below and which concern the construction of the operator family $\{P_j\}$ in Fig. 3.

Lemma 5.11 *Let* $P : \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$. *For* $p, r \in \mathbb{R}$ *and* $w, \xi \in \mathbb{R}$, *define* $\widetilde{P} = M_w T_{p-r} P T_r M_{\xi-w} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$. *Then* $\eta_{\widetilde{P}} = e^{2\pi i r\xi} M_{(w,r)} T_{(p,\xi)} \eta_P$.

Lemma 5.12 *Fix* $u > 1$ *with* $1 < u^4 < \frac{J}{L}$ *and* $0 < \epsilon < 1$. *Choose* $\eta_1, \eta_2 \in \mathcal{S}(\mathbb{R})$ *with values in* $[0, 1]$,

$$\eta_1(t) = \begin{cases} 1 & \text{for } t \in [\frac{u-1}{2uK}, \frac{u+1}{2uK}) \\ 0 & \text{for } t \notin [0, \frac{1}{K}) \end{cases} \quad \text{and} \quad \eta_2(\nu) = \begin{cases} 1 & \text{for } \nu \in [\frac{(u-1)K}{2uL}, \frac{(u+1)K}{2uL}) \\ 0 & \text{for } \nu \notin [0, \frac{K}{L}), \end{cases}$$

and $|\mathcal{F}\eta_1(\xi)| \leq Ce^{-\gamma|\xi|^{1-\epsilon}}$, $|\mathcal{F}^{-1}\eta_2(x)| \leq Ce^{-\gamma|x|^{1-\epsilon}}$. *Let* $\eta_P = \eta_1 \otimes \eta_2$. *Then* $\mathrm{supp}\,\eta_P \subseteq [0, \frac{1}{K}) \times [0, \frac{K}{L}) = R_{K,L}$ *and the operator* $P \in OPW^{1,1}(R_{K,L})$ *has the following properties*:

(a) *The operator family*

$$\{M_{uKk}T_{\frac{1}{K}m-\frac{uL}{K}l}PT_{\frac{uL}{K}l}M_{\frac{K}{L}n-uKk}\}_{k,l,m,n\in\mathbb{Z}}$$

is an $l^{p,q}_{\widetilde{w}}$*-Riesz basis sequence for* $OPW^{p,q}_{w}(\mathbb{R}^2)$.

(b) $P\in OPW^{1,1}(R_{K,L})$ *and there exist functions* $d_1,d_2:\mathbb{R}\to\mathbb{R}_0^+$ *with*

$$|Pf(x)|\le\|f\|_{M^{\infty,\infty}}\,d_1(x)\quad and\quad|\widehat{Pf}(\xi)|\le\|f\|_{M^{\infty,\infty}}\,d_2(\xi),\qquad f\in M^{\infty,\infty}(\mathbb{R}),$$

$$and\ d_1(x)\le\widetilde{C}e^{-\widetilde{\gamma}|x|^{1-\epsilon}},\ d_2(\xi)\le\widetilde{C}e^{-\widetilde{\gamma}|x|^{1-\epsilon}}\ for\ some\ \widetilde{C},\widetilde{\gamma}>0.$$

Proof The existence of η_1,η_2 satisfying the hypotheses stated above is established through mollifying characteristic functions. In fact, using constructions of Gevrey class functions, it has been shown that for $\epsilon,\delta>0$, there exists $\varphi:\mathbb{R}\to\mathbb{R}_0^+$ and $C,\gamma>0$ with $\operatorname{supp}\varphi\subseteq[-\delta,\delta]$, $\int\varphi=1$, $\widehat{\varphi}(\xi)\le Ce^{-\gamma|\xi|^{1-\epsilon}}$ [10, 32, 33]. Note that the restriction to w subexponential in Theorem 5.7 is a consequence to the fact that there exist no compactly supported functions whose Fourier transforms decay exponentially (see references in [22]).

(a) Due to Lemma 5.11,

$$\{M_{(uKk,\frac{uL}{K}l)}T_{(\frac{1}{K}m,\frac{K}{L}n)}\eta_P\}_{k,l,m,n\in\mathbb{Z}}$$

being an $l^{p,q}_{\widetilde{w}}$-Riesz basis for its closed linear span in $M^{(1,1),(p,q)}_{1\otimes w}(\mathbb{R}^2)$ implies that

$$\{M_{uKk}T_{\frac{1}{K}m-\frac{uL}{K}l}PT_{\frac{uL}{K}l}M_{\frac{K}{L}n-uKk}\}_{k,l,m,n\in\mathbb{Z}}$$

is an $l^{p,q}_{\widetilde{w}}$-Riesz basis for its closed linear span in $OPW^{p,q}_{w}(\mathbb{R}^2)$.

(b) As shown in the proof of Lemma 3.4 in [41], we have $|Pf(x)|\le|\widehat{\eta}_2(-x)|\times\|f\|_{M^{\infty,\infty}}\|\eta_1\|_{M^{1,1}}$, so we can choose $d_1(x)=|\widehat{\eta}_2(-x)|\|\eta_1\|_{M^{1,1}}$.

Similarly, we can compute $|\widehat{Pf}(\xi)|\le\|f\|_{M^{\infty,\infty}}\|\eta_2(\xi-\cdot)\widehat{\eta}_1(\cdot)\|_{M^{1,1}}$. We claim that $d_2(\xi)=\|\eta_2(\xi-\cdot)\widehat{\eta}_1(\cdot)\|_{M^{1,1}}$ has the postulated subexponential decay. Recall that for g supported on $[a,b]$, we have $\|g\|_{M^{1,1}}\le c\|\widehat{g}\|_{L^1}$ where c depends only on the support size $b-a$ (see Lemma 2.3 and [45]). As $\eta_2(\xi-\cdot)\widehat{\eta}_1(\cdot)$ is compactly supported with uniform support size, we can compute

$$\begin{aligned}d_2(\xi)&=\|\eta_2(\xi-\cdot)\widehat{\eta}_1(\cdot)\|_{M^{1,1}}\le c\left\|\mathcal{F}^{-1}\big(\eta_2(\xi-\cdot)\widehat{\eta}_1(\cdot)\big)\right\|_{L^1}\\&=c\int\left|\int\eta_2(\xi-\nu)\widehat{\eta}_1(\nu)e^{2\pi ix\nu}\,d\nu\right|dx=c\int|V_{\widetilde{\eta}_2}\widehat{\eta}_1(\xi,-x)|\,dx,\end{aligned}$$

where $\widetilde{\eta}_2(\xi)=\eta_2(-\xi)$. As the compact support of η_1,η_2 together with $|\mathcal{F}\eta_1(\xi)|\le Ce^{-\gamma|\xi|^{1-\epsilon}}$, $|\mathcal{F}^{-1}\eta_2(x)|\le Ce^{-\gamma|x|^{1-\epsilon}}$ imply that $\widehat{\eta}_1,\widetilde{\eta}_2$ are in the Gelfand–Shilov class $\mathcal{S}^{1-\epsilon}_{1-\epsilon}$ [28], we apply Proposition 3.12 in [29] to conclude that $V_{\widetilde{\eta}_2}\widehat{\eta}_1\in\mathcal{S}^{1-\epsilon}_{1-\epsilon}$, that is,

$$d_2(\xi)\le c\int\widetilde{C}^{-1}e^{-\widetilde{\gamma}\|(x,y)\|_\infty^{1-\epsilon}}\,dx\le\widetilde{\widetilde{C}}e^{-\widetilde{\gamma}|\xi|^{1-\epsilon}}.\qquad\square$$

Theorem 5.13 extends the main result in [47] to weighted and mixed l^p spaces with subexponential weights. Both results generalize to infinite dimensions the fact that $m \times n$ matrices with $m < n$ have a non-trivial kernel and, therefore, are not bounded below as operators acting on $\mathbb{C}^n$.

Theorem 5.13 *Let $1 \le p_1, p_2, q_1, q_2 \le \infty$, w_1, w_2 subexponential on $\mathbb{Z}^{2d}$, that is, for some $C, \gamma > 0$, $0 < \beta < 1$, we have*

$$C^{-1} e^{-\gamma \|n\|_\infty^\beta} \le w_1(n), \qquad w_2(n) \le C e^{\gamma \|n\|_\infty^\beta}.$$

If for $M = (m_{j'j}) : l_{w_1}^{p_1,q_1}(\mathbb{Z}^{2d}) \to l_{w_2}^{p_2,q_2}(\mathbb{Z}^{2d})$, exists $u > 1$, $K_0 > 0$, and

$$\rho : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+ \quad \text{with } \rho \le \widetilde{C} e^{-\widetilde{\gamma}\|n\|_\infty^{\widetilde{\beta}}}, \quad \widetilde{\beta} > \beta,$$

with

$$|m_{j'j}| \le \rho\big(u\|j'\|_\infty - \|j\|_\infty\big), \quad u\|j'\|_\infty - \|j\|_\infty > K_0,$$

then M has no bounded left inverses.

Proof Let $v(n) = C e^{\gamma \|n\|_\infty^\beta}$. Note that $l_{w_1}^{p_1,q_1}(\mathbb{Z}^{2d})$ embeds continuously in $l_{1/v}^{\infty,\infty}(\mathbb{Z}^{2d}) = l_{1/v}^{\infty}(\mathbb{Z}^{2d})$ and $l_v^{1,1}(\mathbb{Z}^{2d}) = l_v^1(\mathbb{Z}^{2d})$ embeds continuously in $l_{w_2}^{p_2,q_2}(\mathbb{Z}^{2d})$. Hence, it suffices to show that for all $\epsilon > 0$ exists $x \in l_{1/v}^\infty(\mathbb{Z}^{2d})$ with $\|x\|_{l_{1/v}^\infty} = 1$ and $\|Mx\|_{l_v^1} \le \epsilon$. For notational simplicity, we replace $2d$ by D in the following.

First, observe that

$$A_{K_1} = e^{\gamma K_1^\beta} \sum_{K \ge K_1} K^{D-1} e^{\gamma K^\beta} \sum_{k \ge K} k^{D-1} e^{-\widetilde{\gamma} k^{\widetilde{\beta}}} \to 0 \quad \text{as } K_1 \to \infty. \tag{39}$$

Applying the integral criterion for sums, this would follow from

$$e^{\gamma K_1^\beta} \int_{K_1}^\infty x^{D-1} e^{\gamma x^\beta} \int_x^\infty y^{D-1} e^{-\widetilde{\gamma} y^{\widetilde{\beta}}}\, dy\, dx \to 0 \quad \text{as } K_1 \to \infty. \tag{40}$$

For large x, a substitution yields

$$\begin{aligned}
\int_x^\infty y^{D-1} e^{-\widetilde{\gamma} y^{\widetilde{\beta}}}\, dy &= \frac{1}{\widetilde{\beta}\widetilde{\gamma}} \int_{\widetilde{\gamma} x^{\widetilde{\beta}}}^\infty \left(\frac{t}{\widetilde{\gamma}}\right)^{\frac{1}{\widetilde{\beta}}(D-1)} \left(\frac{t}{\widetilde{\gamma}}\right)^{\frac{1-\widetilde{\beta}}{\widetilde{\beta}}} e^{-t}\, dt \\
&= \frac{1}{\widetilde{\beta}\widetilde{\gamma}} \widetilde{\gamma}^{-\frac{D-\widetilde{\beta}}{\widetilde{\beta}}} \int_{\widetilde{\gamma} x^{\widetilde{\beta}}}^\infty t^{\frac{D-\widetilde{\beta}}{\widetilde{\beta}}} e^{-t}\, dt \\
&= \frac{1}{\widetilde{\beta}\widetilde{\gamma}} \widetilde{\gamma}^{-\frac{D-\widetilde{\beta}}{\widetilde{\beta}}} \Gamma\left(\frac{D-\widetilde{\beta}}{\widetilde{\beta}}, \widetilde{\gamma} x^{\widetilde{\beta}}\right) \le \frac{2}{\widetilde{\beta}\widetilde{\gamma}} \widetilde{\gamma}^{-\frac{D-\widetilde{\beta}}{\widetilde{\beta}}} \big(\widetilde{\gamma} x^{\widetilde{\beta}}\big)^{\frac{D-\widetilde{\beta}}{\widetilde{\beta}}-1} e^{-\widetilde{\gamma} x^{\widetilde{\beta}}} \\
&\le \frac{2}{\widetilde{\beta}} \widetilde{\gamma}^{-2} x^{D-2\widetilde{\beta}} e^{-\widetilde{\gamma} x^{\widetilde{\beta}}},
\end{aligned}$$

where Γ denotes the upper incomplete Gamma function, and we used $\frac{\Gamma(s,y)}{y^{s-1}e^{-y}} \to 1$ as $y \to \infty$ [1].

The fact that $\widetilde{\beta} > \beta$ allows us to estimate for large x

$$e^{\gamma x^{\beta}} e^{-\widetilde{\gamma} x^{\widetilde{\beta}}} = e^{-\widetilde{\gamma} x^{\widetilde{\beta}}(1-\frac{\gamma}{\widetilde{\gamma}} x^{\beta-\widetilde{\beta}})} \le e^{-\frac{1}{2}\widetilde{\gamma} x^{\widetilde{\beta}}}.$$

Hence, we can repeat the arguments above to the outer integral and the product with $e^{\gamma K_1^{\beta}}$ in (40) to obtain (40), and, hence, (39) holds.

To continue our proof, we fix $\epsilon > 0$ and note that (39) provides us with a $K_1 > K_0$ satisfying $A_{K_1} \le (2^{2D} D^2 \widetilde{C} C^2 e^{\gamma N^{\beta}})^{-1} \epsilon$.

As in [47], set $N = \lceil \frac{\lambda(K_1+1)}{\lambda-1} \rceil$ and $\widetilde{N} = \lceil \frac{N}{\lambda} \rceil + K_1$. Then $\frac{\lambda(K_1+1)}{\lambda-1} \le N \le \frac{\lambda(K_1+2)}{\lambda-1}$ implies $\lambda N \ge \lambda K_1 + \lambda + N$ and $N \ge K_1 + \frac{N}{\lambda} + 1 > K_1 + \lceil \frac{N}{\lambda} \rceil = \widetilde{N}$. Hence, $(2\widetilde{N} + 1)^D < (2N+1)^D$ so that the matrix $\widetilde{M} = (m_{j'j})_{\|j'\|_\infty \le \widetilde{N}, \|j\| \le N} : \mathbb{C}^{(2N+1)^D} \longrightarrow \mathbb{C}^{(2\widetilde{N}+1)^D}$ has a nontrivial kernel. We now choose $x \in l^{\infty}_{1/v}(\mathbb{Z}^D)$ with $\|x\|_{l^{\infty}_{1/v}} = 1$, $x_j = 0$ if $\|j\|_\infty > N$, and $\widetilde{M}\widetilde{x} = 0$ where $\widetilde{x}$ is x restricted to the set $\{j : \|j\|_\infty \le N\}$.

By construction we have $(Mx)_{j'} = 0$ for $\|j'\|_\infty \le \widetilde{N}$. To estimate $(Mx)_{j'}$ for $\|j'\|_\infty > \widetilde{N}$, we fix $K > K_1$ and one of the $2D(2(\lceil \frac{N}{\lambda} \rceil + K))^{D-1}$ indices $j' \in \mathbb{Z}^D$ with $\|j'\|_\infty = \lceil \frac{N}{\lambda} \rceil + K$. We have $\|\lambda j'\|_\infty \ge N + K\lambda$ and $\lambda \|j'\|_\infty - \|j\|_\infty \ge K\lambda \ge K$ for all $j \in \mathbb{Z}^D$ with $\|j\|_\infty \le N$. Therefore, using Hölder's inequality for weighted l^p-spaces, we obtain

$$\begin{aligned}
|(Mx)_{j'}| &= \Big| \sum_{\|j\|_\infty \le N} m_{j'j} x_j \Big| \le \|x\|_{l^{\infty}_{1/v}} \sum_{\|j\|_\infty \le N} v(j) |m_{j'j}| \\
&\le C e^{\gamma N^{\beta}} \sum_{\|j\|_\infty \le N} \rho\big(\lambda \|j'\|_\infty - \|j\|_\infty\big) \\
&\le C e^{\gamma N^{\beta}} \sum_{\|j\|_\infty \ge K} \rho(\|j\|_\infty) = 2^D D C e^{\gamma N^{\beta}} \sum_{k \ge K} k^{D-1} \rho(k).
\end{aligned}$$

Next, we compute

$$\begin{aligned}
\|Mx\|_{l^1_v} &= \sum_{j' \in \mathbb{Z}^D} v(j') |(Mx)_{j'}| = \sum_{\|j'\|_\infty \ge \lceil \frac{N}{\lambda} \rceil + K_1} v(j') |(Mx)_{j'}| \\
&\le 2^D D C e^{\gamma N^{\beta}} \sum_{\|j'\|_\infty \ge \lceil \frac{N}{\lambda} \rceil + K_1} v(j') \sum_{k \ge \|j'\|_\infty} k^{D-1} \rho(k) \\
&\le 2^D D C e^{\gamma N^{\beta}} \sum_{K \ge \lceil \frac{N}{\lambda} \rceil + K_1} 2D(2K)^{D-1} C e^{\gamma K^{\beta}} \sum_{k \ge K} k^{D-1} \rho(k) \\
&\le 2^{2D} D^2 \widetilde{C} C^2 e^{\gamma N^{\beta}} \sum_{K \ge \lceil \frac{N}{\lambda} \rceil + K_1} K^{D-1} e^{\gamma K^{\beta}} \sum_{k \ge K} k^{D-1} e^{-\widetilde{\gamma} k^{\widetilde{\beta}}} \le \epsilon.
\end{aligned}$$

□

Combining the results above, we can now proceed to prove Theorem 5.7.

Proof of Theorem 5.7 As w is subexponential, there exists $C,\gamma,\epsilon>0$ with $|w(x,\xi)|\le Ce^{\gamma\|(x,\xi)\|_\infty^{1-2\epsilon}}$. For this $\epsilon>0$ choose u, η_1, η_2, P, d_1, and d_2 as in Lemma 5.12.

Define the synthesis operator $E: l_w^{p,q}(\mathbb{Z}^2)\to OPW_w^{p,q}(V_M)\subseteq OPW_w^{p,q}(M)$ as follows. For $\sigma=\{\sigma_{k,p}\}\in l_{\widetilde{w}}^{p,q}(\mathbb{Z}^2)$ write $\sigma_{k,p}=\sigma_{k,lJ+j}$ for $l\in\mathbb{Z}$ and $0\le j<J$ and define

$$E(\sigma)=\sum_{k,l\in Z}\sum_{j=0}^{J-1}\sigma_{k,lJ+j}M_{uKk}T_{\frac{1}{K}k_j+\frac{uL}{K}l}PT_{-\frac{uL}{K}l}M_{\frac{K}{L}p_j-uKk}$$

with convergence in case $p,q\neq\infty$ and weak-$*$ convergence else. Since

$$\{M_{uKk}T_{\frac{1}{K}m-\frac{uL}{K}l}PT_{\frac{uL}{K}l}M_{\frac{K}{L}n-uKk}\}_{k,l,m,n\in\mathbb{Z}}$$

is an $l_{\widetilde{w}}^{p,q}$-Riesz basis for its closed linear span in $OPW_w^{p,q}(\mathbb{R}^2)$, so is its subset

$$\{M_{uKk}T_{\frac{1}{K}k_j+\frac{uL}{K}l}PT_{-\frac{uL}{K}l}M_{\frac{K}{L}p_j-uKk}\}_{k,l\in\mathbb{Z},0\le j<J}$$

and E is bounded and bounded below.

By Theorem 2.5, the Gabor system $(\mathfrak{g},a'\mathbb{Z}\times b'\mathbb{Z})=\{M_{ka'}T_{lb'}\mathfrak{g}\}$ is an $l_{\widetilde{w}}^{p,q}$-frame for any $a',b'>0$ with $a'b'<1$, and we conclude that the analysis map given by

$$C_{\mathfrak{g}}: M_w^{p,q}(\mathbb{R})\to l_{\widetilde{w}}^{p,q}(\mathbb{Z}^2),\quad f\mapsto\big\{\langle f, M_{u^2Kk}T_{\frac{u^2L}{KJ}l}\mathfrak{g}\rangle\big\}_{k,l}$$

is bounded and bounded below since $u^2K\frac{u^2L}{KJ}=u^4\frac{L}{J}<1$.

For simplicity of notation, set $\alpha=K$ and $\beta=\frac{L}{KJ}$. Fix $f\in M^{\infty,\infty}(\mathbb{R})$ and consider the composition

$$\begin{array}{ccccccc} l_{\widetilde{w}}^{p,q}(\mathbb{Z}^2) & \overset{E}{\to} & OPW_w^{p,q}(M) & \overset{\Phi_g}{\to} & M_w^{p,q}(\mathbb{R}) & \overset{C_{\mathfrak{g}}}{\to} & l_{\widetilde{w}}^{p,q}(\mathbb{Z}^2)\\ \sigma & \mapsto & E\sigma & \mapsto & E\sigma g & \mapsto & \{\langle E\sigma g, M_{u^2\alpha k'}T_{u^2\beta l'}\mathfrak{g}\rangle\}_{k',l'}. \end{array}$$

It is easily computed that the operator $C_{\mathfrak{g}}\circ\Phi_f\circ E$ is represented—with respect to the canonical basis $\{\delta(\cdot-n)\}_n$ of $l_{\widetilde{w}}^{p,q}(\mathbb{Z}^2)$—by the bi-infinite matrix

$$\mathcal{M}=(m_{k',l',k,lJ+j})=\big(\langle M_{u\alpha k}T_{\frac{k_j}{\alpha}+u\beta lJ}PT_{-u\beta lJ}M_{\frac{p_j}{\beta J}-u\alpha k}f, M_{u^2\alpha k'}T_{u^2\beta l'}\mathfrak{g}\rangle\big).$$

Setting

$$\widetilde{d}_1=\max_{j=0,\dots,J-1}T_{\frac{k_j}{\alpha}-\lambda\beta j}d_1,$$

we observe

$$\begin{aligned}|m_{k',l',k,lJ+j}|&\le\big\langle T_{u\beta(lJ+j)}\big(T_{\frac{k_j}{\alpha}-u\beta j}|PT_{-u\beta lJ}M_{\frac{p_j}{\beta J}-u\alpha k}f|\big), T_{u^2\beta l'}\mathfrak{g}\big\rangle\\ &\le\|f\|_{M^{\infty,\infty}}\langle T_{u\beta(lJ+j)}T_{\frac{k_j}{\alpha}-u\beta j}d_1, T_{u^2\beta l'}\mathfrak{g}\rangle\end{aligned}$$

$$\leq \|f\|_{M^{\infty,\infty}} (\widetilde{d}_1 * \mathfrak{g})\big(u\beta\big(ul' - (lJ + j)\big)\big),$$

and

$$\begin{aligned} |m_{k',l',k,lJ+j}| &= \big|\big\langle T_{u\alpha k} M_{-\frac{k_j}{\alpha} - u\beta l J} (P T_{-u\beta l J} M_{\frac{p_j}{\beta J} - u\alpha k} f)\widehat{\ }, T_{u^2\alpha k'} M_{-u^2\beta l'} \mathfrak{g}\big\rangle\big| \\ &\leq \big\langle T_{u\alpha k} \big|(P T_{-u\beta l J} M_{\frac{p_j}{\beta J} - u\alpha k} f)\widehat{\ }\big|, T_{u^2\alpha k'} \mathfrak{g}\big\rangle \\ &\leq \|f\|_{M^{\infty,\infty}} (d_2 * \mathfrak{g})\big(u\alpha\big(uk' - k\big)\big) \end{aligned}$$

Observing that for appropriate $\widetilde{C}, \widetilde{\gamma}$, we have $\widetilde{d}_1 * \mathfrak{g}(x) d_2 * \mathfrak{g}(\xi) \leq \widetilde{C} e^{-\widetilde{\gamma}\|(x,\xi)\|_\infty^{1-\epsilon}}$ and $1 - 2\epsilon < 1 - \epsilon$, allows us to apply Theorem 5.13 to $\mathcal{M}$. This completes the proof. □

6 Outlook

A number of papers have been written in parallel and subsequently to the work presented here:

In [52], a reconstruction formula applicable to $OPW(M)$ with M not necessarily rectangular is presented. Also in this paper, necessary and sufficient criteria on the sampling rate—the density of the support of the discretely supported identifier, that is, the rate at which Dirac impulses are transmitted—for operator sampling are given.

The paper [35] considers irregular spacing of transmitted Dirac impulses and develops multi-channel sampling strategies for operators in $OPW(M)$ for the case that M has area larger than one. Similarly, identifiability results for Multiple-Input Multiple-Output (MIMO) channels are derived using ideas from operator sampling in [46].

In [40], coarse quantification schemes that give rise to local approximations of operators with bandlimited Kohn-Nirenberg symbols are derived, and operator approximation results based on replacing the Dirac comb with a truncated and mollified version thereof is studied, see also [4] and [30] for related work.

In [30], the question of identifying operators with small, but unknown spreading support set M is discussed using ideas from compressed sensing. Weaker results in this direction have been obtained independently in [52].

In [55] and [56], the sampling problem for stochastic operators is considered. Here, a stochastic operator H is given by a stochastic spreading function η_H and the goal of determining a deterministic H from the deterministic Hf is replaced in the most general setting with the goal of determining the autocorrelation of η_H from the autocorrelation of Hf.

Some very fundamental questions concerning sampling and identification of operator Paley-Wiener spaces are nonetheless still open. In the following we describe two such questions.

6.1 Unbounded Spreading Domains with Small Lebesgue Measure

The extension of Theorem 5.6 to $OPW_w^{p;q}(M)$ with M unbounded but with Lebesgue measure less than one remains open. The following observations encourage tackling this question:

1. Multiplication operators with not necessarily bandlimited symbol in $L^2(\mathbb{R}^2)$ are clearly identifiable with identifier $g = \chi_{\mathbb{R}} \in M^{\infty,\infty}(\mathbb{R})$. Note that the characteristic function $\chi_{\mathbb{R}}$ is the weak-$*$ limit of $T\sum_{n\in\mathbb{Z}} \delta_{nT}$ as $T \to 0$. Hence, the space $OPW^{\infty,\infty}(\{0\}\times\widehat{\mathbb{R}})$ is identifiable.
2. Time-invariant operators with not necessarily compactly supported $L^2(\mathbb{R}^2)$ impulse response are identifiable with identifier δ which is the weak-$*$ limit of $\sum_{n\in\mathbb{Z}} \delta_{nT}$ as $T \to \infty$. Consequently, the space $OPW^{\infty,\infty}(\mathbb{R}\times\{0\})$ is identifiable.
3. In [41] it is shown that $OPW(M)$ is identifiable if M is a possibly unbounded fundamental domain a lattice Λ in $\mathbb{R}^2$ with Λ having density less than or equal to one. This result covers, for example, $OPW(\{(t,\nu) : t\geq -1, \nu<1, 2^{-(t+1)}\leq\nu\leq 2^{-t}\})$ as the unbounded set $\{(t,\nu) : t\geq -1, \nu<1, 2^{-(t+1)}\leq\nu\leq 2^{-t}\}$ is a fundamental domain of $\mathbb{Z}^2$.

The natural approach to construct identifiers for $OPW(A)$ as weak-$*$ limit of identifiers g_N for $OPW(A\cap[-N,N]\times[-N,N])$ is difficult as the constants implied by $\asymp$ in (31) depend in a non-trivial matter on $g_N = \sum_n c_{n,N}\delta_{x_{n,N}}$, $N\in\mathbb{N}$, if the sequences $\{c_{n,N}\}$ are not constant.

6.2 Generalizations to Higher Dimensions

As mentioned in Sect. 5, our proof of Theorem 5.6 hinges on the existence of identifiers in an analogous setup where the locally compact Abelian group $\mathbb{R}$ is replaced by an appropriate finite cyclic group of prime order $\mathbb{Z}_p$ [42, 43]. In fact, generalizing Theorem 5.6, to operators acting on $L^2(\mathbb{R}^d)$ would be possible if the conclusions of Theorem 5.9 hold for sufficiently many composites n taking the place of prime p. Consequently, in [42, 48] we ask the following:

Question 6.1 Is it true that for all $L\in\mathbb{N}$ exists $c\in\mathbb{C}^L$ so that the vectors $\pi(k,\ell)c$, $k,\ell\in 0,\dots,L-1$, defined by $(\pi(k,\ell)c)_j = c_{j-k}e^{2\pi i\frac{j\ell}{L}}$, $k,\ell=0,\dots,L-1$, are in general linear position.

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References

1. Abramowitz, M., Stegun, I.A. (eds.): Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover, New York (1992). Reprint of the 1972 edition
2. Bello, P.A.: Time–frequency duality. IEEE Trans. Commun. **10**, 18–33 (1964)
3. Bello, P.A.: Measurement of random time-variant linear channels. IEEE Trans. Commun. **15**, 469–475 (1969)

4. Bajwa, W.U., Gedalyahu, K., Eldar, Y.C.: Identification of parametric underspread linear systems and super-resolution radar. IEEE Trans. Signal Process. **59**(6), 2548–2561 (2011)
5. Benedek, A., Panzone, R.: The space L^p, with mixed norm. Duke Math. J. **28**, 301–324 (1961)
6. Cordero, E., Gröchenig, K.: Time-frequency analysis of localization operators. J. Funct. Anal. **205**(1), 107–131 (2003)
7. Christensen, O.: An Introduction to Frames and Riesz Bases. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston (2003)
8. Cordero, E., Nicola, F.: Pseudodifferential operators on L^p, Wiener amalgam and modulation spaces. Int. Math. Res. Not. **10**, 1860–1893 (2010)
9. Czaja, W.: Boundedness of pseudodifferential operators on modulation spaces. J. Math. Anal. Appl. **284**(1), 389–396 (2003)
10. Dziubański, J., Hernández, E.: Band-limited wavelets with subexponential decay. Can. Math. Bull. **41**(4), 398–403 (1998)
11. Feichtinger, H.G.: On a new segal algebra. Monatshefte Math. **92**, 269–289 (1981)
12. Feichtinger, H.G.: Atomic characterizations of modulation spaces through Gabor-type representations. In: Proc. Conf. Constructive Function Theory, Edmonton, July, 1986, pp. 113–126 (1989)
13. Feichtinger, H.G., Gröchenig, K.: Banach spaces related to integrable group representations and their atomic decompositions. I. J. Funct. Anal. **86**, 307–340 (1989)
14. Feichtinger, H.G., Gröchenig, K.: Gabor wavelets and the Heisenberg group: Gabor expansions and short time Fourier transform from the group theoretical point of view. In: Wavelets. Wavelet Anal. Appl. vol. 2, pp. 359–397. Academic Press, Boston (1992)
15. Feichtinger, H.G., Kozek, W.: Quantization of TF–lattice invariant operators on elementary LCA groups. In: Feichtinger, H.G., Strohmer, T. (eds.) Gabor Analysis and Algorithms: Theory and Applications, pp. 233–266. Birkhäuser, Boston (1998)
16. Folland, G.B.: Harmonic Analysis in Phase Space. Annals of Math. Studies. Princeton Univ. Press, Princeton (1989)
17. Folland, G.B.: Real Analysis, 2nd edn. Modern Techniques and Their Applications. Wiley, New York (1999)
18. Feichtinger, H.G., Zimmermann, G.: A Banach space of test functions for Gabor analysis. In: Feichtinger, H.G., Strohmer, T. (eds.) Gabor Analysis and Algorithms: Theory and Applications, pp. 123–170. Birkhäuser, Boston (1998)
19. Gröchenig, K., Heil, C.: Modulation spaces and pseudodifferential operators. Integral Equ. Oper. Theory **34**(4), 439–457 (1999)
20. Gröchenig, K., Heil, C.: Counterexamples for boundedness of pseudodifferential operators. Osaka J. Math. **41**(3), 681–691 (2004)
21. Goldsmith, A.: Wireless Communications. Cambridge University Press, Cambridge (2005)
22. Grip, N., Pfander, G.E.: A discrete model for the efficient analysis of time–varying narrowband communications channels. Multidimens. Syst. Signal Process. **19**(1), 3–40 (2008)
23. Gröchenig, K.: Describing functions: atomic decompositions versus frames. Monatshefte Math. **112**(3), 1–42 (1991)
24. Gröchenig, K.: Foundations of Time-Frequency Analysis. Applied and Numerical Harmonic Analysis. Birkhäuser, Boston (2001)
25. Gröchenig, K.: Localization of frames, Banach frames, and the invertibility of the frame operator. J. Fourier Anal. Appl. **10**(2), 105–132 (2004)
26. Gröchenig, K.: Composition and spectral invariance of pseudodifferential operators on modulation spaces. J. Anal. Math. **98**, 65–82 (2006)
27. Gröchenig, K.: Weight functions in time-frequency analysis. In: Pseudo-Differential Operators: Partial Differential Equations and Time-Frequency Analysis. Fields Inst. Commun, vol. 52, pp. 343–366. Amer. Math. Soc., Providence (2007)
28. Gel'fand, I.M., Shilov, G.E.: Generalized Functions. Spaces of Fundamental and Generalized Functions, vol. 2. Academic Press, New York (1968). Translated from the Russian by M.D. Friedman, A. Feinstein and C.P. Peltzer
29. Gröchenig, K., Zimmermann, G.: Spaces of test functions via the STFT. J. Funct. Spaces Appl. **2**(1), 25–53 (2004)
30. Heckel, R., Boelcskei, H.: Identification of sparse linear operators. Preprint
31. Hörmander, L.: The Weyl calculus of pseudo-differential operators. Commun. Pure Appl. Math. **32**, 359–443 (1979)
32. Hörmander, L.: The Analysis of Linear Partial Differential Operators I. Classics in Mathematics. Springer, Berlin (2003). Distribution Theory and Fourier Analysis, Reprint of the 2nd edn. (1990)

33. Hörmander, L.: The Analysis of Linear Partial Differential Operators II. Classics in Mathematics. Springer, Berlin (2005). Differential Operators with Constant Coefficients, Reprint of the 1983 original
34. Hörmander, L.: The Analysis of Linear Partial Differential Operators III. Classics in Mathematics. Springer, Berlin (2007). Pseudo-Differential Operators, Reprint of the 1994 edn.
35. Hong, Y.M., Pfander, G.E.: Irregular and multi-channel sampling of operators. Appl. Comput. Harmon. Anal. **29**(2), 214–231 (2010)
36. Janssen, A.J.E.M.: Duality and biorthogonality for Weyl-Heisenberg frames. J. Fourier Anal. Appl. **1**(4), 403–436 (1995)
37. Kailath, T.: Measurements on time–variant communication channels. IEEE Trans. Inf. Theory **8**(5), 229–236 (1962)
38. Krahmer, F., Mendelson, S., Rauhut, H.: Suprema of chaos processes and the restricted isometry property. Preprint
39. Kohn, J.J., Nirenberg, L.: An algebra of pseudo–differential operators. Commun. Pure Appl. Math. **18**, 269–305 (1965)
40. Krahmer, F., Pfander, G.E.: Localized sampling and local approximation of operators with bandlimited Kohn-Nirenberg symbols. Preprint
41. Kozek, W., Pfander, G.E.: Identification of operators with bandlimited symbols. SIAM J. Math. Anal. **37**(3), 867–888 (2006)
42. Krahmer, F., Pfander, G.E., Rashkov, P.: Uncertainty in time-frequency representations on finite Abelian groups and applications. Appl. Comput. Harmon. Anal. **25**(2), 209–225 (2008)
43. Lawrence, J., Pfander, G.E., Walnut, D.F.: Linear independence of Gabor systems in finite dimensional vector spaces. J. Fourier Anal. Appl. **11**(6), 715–726 (2005)
44. Molahajloo, S., Pfander, G.E.: Boundedness of pseudo-differential operators on L^p, Sobolev and modulation spaces. Math. Model. Nat. Phenom. **8**, 175–192 (2013)
45. Okoudjou, K.A.: A Beurling-Helson type theorem for modulation spaces. J. Funct. Spaces Appl. **7**(1), 33–41 (2009)
46. Pfander, G.E.: Measurement of time–varying multiple-input multiple-output channels. Appl. Comput. Harmon. Anal. **24**, 393–401 (2008)
47. Pfander, G.E.: On the invertibility of "rectangular" bi-infinite matrices and applications in time-frequency analysis. Linear Algebra Appl. **429**(1), 331–345 (2008)
48. Pfander, G.E.: Gabor frames in finite dimensions. In: Casazza, P.G., Kutyniok, G. (eds.) Finite Frames, Applied and Numerical Harmonic Analysis. Birkhäuser, Boston (2012)
49. Pfander, G.E., Rauhut, H.: Sparsity in time–frequency representations. J. Fourier Anal. Appl. **16**(2), 233–260 (2010)
50. Pfander, GE., Rauhut, H., Tropp, JA.: The restricted isometry property for time-frequency structured random matrices. Probab. Theory Relat. Fields (2012). doi:10.1007/s00440-012-0441-4
51. Pfander, G.E., Rauhut, H., Tanner, J.: Identification of matrices having a sparse representation. IEEE Trans. Signal Process. **56**(11), 5376–5388 (2008)
52. Pfander, G.E., Walnut, D.: Sampling and reconstruction of operators. Preprint
53. Pfander, G.E., Walnut, D.: Measurement of time–variant channels. IEEE Trans. Inf. Theory **52**(11), 4808–4820 (2006)
54. Pfander, G.E., Walnut, D.: Operator identifcation and Feichtinger's algebra. Sampl. Theory Signal. Image Process **5**(2), 151–168 (2006)
55. Pfander, G.E., Zheltov, P.: Identification of stochastic operators. Preprint
56. Pfander, G.E., Zheltov, P.: Sampling of stochastic operators. Preprint
57. Sjöstrand, J.: An algebra of pseudodifferential operators. Math. Res. Lett. **1**(2), 185–192 (1994)
58. Sjöstrand, J.: Wiener type algebras of pseudodifferential operators. In: Séminaire sur les Équations aux Dérivées Partielles, 1994–1995, vol. IV, École Polytech, Palaiseau (1995), 19 pp.
59. Skolnik, M.I.: Introduction to Radar Systems. McGraw-Hill Book, New York (1980)
60. Strohmer, T.: Pseudodifferential operators and Banach algebras in mobile communications. Appl. Comput. Harmon. Anal. **20**(2), 237–249 (2006)
61. Tachizawa, K.: The boundedness of pseudodifferential operators on modulation spaces. Math. Nachr. **168**, 263–277 (1994)
62. Taylor, M.E.: Pseudodifferential Operators. Princeton Mathematical Series, vol. 34. Princeton University Press, Princeton (1981)
63. Toft, J.: Continuity properties for modulation spaces, with applications to pseudo-differential calculus. II. Ann. Glob. Anal. Geom. **26**(1), 73–106 (2004)

64. Toft, J.: Continuity and Schatten properties for pseudo-differential operators on modulation spaces. In: Modern Trends in Pseudo-Differential Operators. Oper. Theory Adv. Appl., vol. 172, pp. 173–206. Birkhäuser, Basel (2007)
65. Zadeh, L.A.: A general theory of linear signal transmission systems. J. Franklin Inst. **253**, 293–312 (1952)