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# Heisenberg Uniqueness Pairs for the Parabola

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**Abstract** Let $\Gamma$ denote the parabola $y = x^2$ in the plane. For some simple sets $\Lambda$ in the plane we study the question whether $(\Gamma, \Lambda)$ is a Heisenberg uniqueness pair. For example we shall consider the cases where $\Lambda$ is a straight line or a union of two straight lines.

**Keywords** Fourier transforms · Heisenberg uniqueness pairs

**Mathematics Subject Classification** 42B10

## 1 Introduction

Let $\mu$ denote a finite complex-valued Borel measure in $\mathbb{R}^2$. The Fourier transform of $\mu$ is defined by

$$\hat{\mu}(x, y) = \int_{\mathbb{R}^2} e^{-i(x\xi + y\eta)} d\mu(\xi, \eta) \quad \text{for } (x, y) \in \mathbb{R}^2.$$

Let $\Gamma$ denote the parabola $y = x^2$ in $\mathbb{R}^2$. We assume that $\operatorname{supp} \mu \subset \Gamma$ and that $\mu$ is absolutely continuous with respect to the arc length measure on $\Gamma$. Also let $\Lambda$ be a subset of $\mathbb{R}^2$. Following Hedenmalm and Montes-Rodríguez [3] we say that $(\Gamma, \Lambda)$ is a Heisenberg uniqueness pair (or only uniqueness pair) if $\hat{\mu}(x, y) = 0$ for $(x, y) \in \Lambda$ implies that $\mu$ is the zero measure.

The case where $\Gamma$ is a hyperbola was discussed in [3], and Sjölin [5] and Lev [4] studied the case where $\Gamma$ is a circle. For further results see also Canto-Martin, Hedenmalm, and Montes-Rodríguez [1].

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We shall here let $\Gamma$ denote the parabola $y = x^2$.

If $\mu$ has the above properties it is clear that there exists a measurable function $f$ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} |f(t)|\sqrt{1+4t^2}dt < \infty$$

and $\int_{\mathbb{R}^2} h d\mu = \int_{\mathbb{R}} h(t,t^2) f(t)\sqrt{1+4t^2}dt$ if $h$ is continuous and bounded in $\mathbb{R}^2$. Thus we have

$$\hat{\mu}(x,y) = \int_{\mathbb{R}} e^{-i(xt+yt^2)} f(t)\sqrt{1+4t^2}dt, \quad (x,y) \in \mathbb{R}^2.$$

We shall prove the following theorem, where $\Gamma$ denotes the parabola $y = x^2$.

**Theorem 1**

(i) *Let $\Lambda = L$ where $L$ is a straight line. Then $(\Gamma, \Lambda)$ is a uniqueness pair if and only if $L$ is parallel to the $x$-axis.*
(ii) *Let $\Lambda = L_1 \cup L_2$, where $L_1$ and $L_2$ are different straight lines. Then $(\Gamma, \Lambda)$ is a uniqueness pair.*
(iii) *Assume that $L_1$ and $L_2$ are different straight lines, which are not parallel to the $x$-axis. Also assume that $E_1 \subset L_1$, and $E_2 \subset L_2$ and that $E_1$ and $E_2$ have positive one-dimensional Lebesgue measure. Set $\Lambda = E_1 \cup E_2$. Then $(\Gamma, \Lambda)$ is a uniqueness pair.*

*Remark* When we talk about the one-dimensional Lebesgue measure of a subset $E$ of a straight line $L$ in the plane, we identify $L$ with $\mathbb{R}$.

We also remark that Heisenberg uniqueness pairs are somewhat related to the notion of annihilating pairs (see Havin and Jöricke [2]). To give the definition of this concept we let $S$ and $\Sigma$ be subsets of $\mathbb{R}$. Following [2] we say that the pair $(S, \Sigma)$ is mutually annihilating if $\psi \in L^2(\mathbb{R})$, supp $\psi \subset S$, supp $\hat{\psi} \subset \Sigma$ implies that $\psi = 0$. Here $\hat{\psi}$ denotes the Fourier transform of $\psi$. We refer to [2] for results on annihilating pairs.

## 2 Lemmas and Proofs

We let the function $f$ be defined as in the Introduction and set

$$g(t) = f(t)\sqrt{1+4t^2}, \quad t \in \mathbb{R}.$$

Then $g \in L^1(\mathbb{R})$ and

$$\hat{\mu}(x,y) = \int_{\mathbb{R}} e^{-i(xt+yt^2)} g(t)dt, \quad (x,y) \in \mathbb{R}^2. \tag{1}$$

We also set $Tg(x,y) = \hat{\mu}(x,y)$ so that $T$ is a mapping from $L^1(\mathbb{R})$ to $C(\mathbb{R}^2)$. It is then easy to see that if we set $U(x,y) = Tg(x,-y)$, then $U$ satisfies the Schrödinger equation

$$i\frac{\partial U}{\partial y} = \frac{\partial^2 U}{\partial x^2}.$$

We shall use the following theorem, which can be found in Havin and Jöricke [2], p. 36.

**Theorem A** *Assume that* $\varphi \in L^1(\mathbb{R})$ *and that* $\operatorname{supp}\varphi$ *is a subset of* $[0,\infty)$. *Also assume that*

$$\int_{\mathbb{R}} \log|\hat{\varphi}(x)| \frac{dx}{1+x^2} = -\infty.$$

*Then* $\varphi = 0$ *almost everywhere.*

Here $\hat{\varphi}$ denotes the Fourier transform of $\varphi$.

We shall then state and prove three lemmas. We let $|E|$ denote Lebesgue measure of a set $E$.

**Lemma 1** *Assume that* $g \in L^1(\mathbb{R})$ *and that* $E \subset \mathbb{R}$ *with* $|E| > 0$. *Then*

$$\int_{\mathbb{R}} e^{-iyt^2} g(t)dt = 0 \quad \textit{for every } y \in E \tag{2}$$

*if and only if* $g$ *is an odd function.*

*Proof* It is obvious that (2) holds if $g$ is odd. We have to prove the converse and assume that (2) holds. Denoting the integral in (2) by $I$ and performing a change of variable we obtain

$$\begin{aligned} I &= \int_{-\infty}^{0} e^{-iyt^2} g(t)dt + \int_0^\infty e^{-iyt^2} g(t)dt \\ &= \int_0^\infty e^{-iyt^2}\big(g(t)+g(-t)\big)dt \\ &= \int_0^\infty e^{-iyt^2} F(t)dt, \end{aligned}$$

where $F(t) = g(t) + g(-t)$ for $t \geq 0$. It is clear that $F \in L^1(0,\infty)$ and setting $u = t^2$ we obtain

$$I = \int_0^\infty e^{-iyu} F(\sqrt{u}) \frac{1}{2\sqrt{u}} du = 0 \quad \text{for } y \in E.$$

We have $F(\sqrt{u})/\sqrt{u} \in L^1(0,\infty)$ and we set $\varphi(u) = F(\sqrt{u})\frac{1}{2\sqrt{u}}$ for $u > 0$ and $\varphi(u) = 0$ for $u \leq 0$. It follows that $\varphi \in L^1(\mathbb{R})$ and $I = \hat{\varphi}(y) = 0$ for $y \in E$. Since

$|E| > 0$ we conclude that

$$\int_{\mathbb{R}} \log\left|\hat{\varphi}(x)\right| \frac{dx}{1+x^2} = -\infty$$

and Theorem A implies that $\varphi = 0$ almost everywhere. Hence $F(t) = 0$ almost everywhere on $[0, \infty)$. It follows that $g(-t) = -g(t)$ for almost every $t$, i.e. $g$ is odd. □

**Lemma 2** *Let $g \in L^1(\mathbb{R})$ and let $\gamma$ and $\delta$ denote different real numbers. Assume that $g(u-\gamma)$ and $g(u-\delta)$ are odd as functions of $u$. Then $g = 0$ almost everywhere.*

*Proof* Using first the fact that $g(u-\gamma)$ is odd and then the fact that $g(u-\delta)$ is odd we obtain

$$g(u-\gamma) = -g(-u-\gamma) = -g\big((-u+\delta-\gamma)-\delta\big) = g(u-\delta+\gamma-\delta) = g(u+\gamma-2\delta)$$

for almost every $u$. Hence $g(u) = g(u+2\gamma-2\delta)$, that is $g$ has period $2\gamma - 2\delta \neq 0$. Since $g \in L^1(\mathbb{R})$ it follows that $g = 0$ almost everywhere. □

We shall need one more lemma.

**Lemma 3** *Assume that $g \in L^1(\mathbb{R})$ and that $\hat{\mu}$ is given by (1). Also assume that $E \subset \mathbb{R}$ and $|E| > 0$. Then the following holds.*

(i) *Assume $x_0 \in \mathbb{R}$. Then $\hat{\mu}(x_0, y) = 0$ for $y \in E$ if and only if $e^{-ix_0t}g(t)$ is odd as a function of $t$.*
(ii) *Assume $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Then $\hat{\mu}(x, \alpha x) = 0$ for $x \in E$ if and only if $g(u - 1/2\alpha)$ is odd as a function of $u$.*
(iii) *Assume that $\alpha$ and $b$ are real numbers and $\alpha \neq 0$ and $b \neq 0$. Then $\hat{\mu}(x, \alpha x + b) = 0$ for $x \in E$ if and only if the function $h(t) = e^{-ibt^2}g(t)$ has the property that $h(u - 1/2\alpha)$ is odd as a function of $u$.*

*Proof* To prove (i) we invoke Lemma 1 and observe that

$$\hat{\mu}(x_0, y) = \int_{\mathbb{R}} e^{-iyt^2} e^{-ix_0t} g(t)\,dt = 0 \quad \text{for } y \in E$$

if and only if $e^{-ix_0t}g(t)$ is an odd function.

To obtain (ii) we write

$$\hat{\mu}(x, \alpha x) = \int_{\mathbb{R}} e^{-i(xt+\alpha x t^2)} g(t)\,dt = \int_{\mathbb{R}} e^{-ix(t+\alpha t^2)} g(t)\,dt.$$

However,

$$t + \alpha t^2 = \alpha\left(t^2 + \frac{1}{\alpha}t\right) = \alpha\left[\left(t + \frac{1}{2\alpha}\right)^2 - \frac{1}{4\alpha^2}\right] = \alpha\left(t + \frac{1}{2\alpha}\right)^2 - \frac{1}{4\alpha}$$

and hence

$$\hat{\mu}(x, \alpha x) = e^{ix/4\alpha} \int_{\mathbb{R}} e^{-ix\alpha(t+1/2\alpha)^2} g(t)dt.$$

Setting $u = t + 1/2\alpha$ we then obtain

$$\hat{\mu}(x, \alpha x) = e^{ix/4\alpha} \int_{\mathbb{R}} e^{-ix\alpha u^2} g(u - 1/2\alpha)du$$

and (ii) follows from an application of Lemma 1.

It remains to prove (iii). We have

$$\begin{aligned}\hat{\mu}(x, \alpha x + b) &= \int_{\mathbb{R}} e^{-i(xt+(\alpha x+b)t^2)} g(t)dt = \int_{\mathbb{R}} e^{-i(xt+\alpha x t^2)} e^{-ibt^2} g(t)dt \\ &= \int_{\mathbb{R}} e^{-i(xt+\alpha x t^2)} h(t)dt,\end{aligned}$$

where $h(t) = e^{-ibt^2} g(t)$, and (iii) follows from (ii). Thus Lemma 3 has been proved. □

Before proving Theorem 1 we mention the well-known fact that if $\Lambda$ is a subset of $\mathbb{R}^2$ and $\Lambda_1$ is a translate of $\Lambda$, then $(\Gamma, \Lambda)$ is a uniqueness pair if and only if $(\Gamma, \Lambda_1)$ is a uniqueness pair. This follows from elementary properties of the Fourier transform.

Finally we shall prove Theorem 1.

*Proof of Theorem 1* We first prove (i) and assume that $L$ is parallel to the $x$-axis. Using the above remark we may assume that $L$ is the $x$-axis. We assume that $g \in L^1(\mathbb{R})$ and that

$$\hat{\mu}(x, 0) = \int_{\mathbb{R}} e^{-ixt} g(t)dt = 0 \quad \text{for every } x.$$

Hence $\hat{g}(x) = 0$ everywhere and we conclude that $g = 0$ almost everywhere. It follows that $(\Gamma, L)$ is a uniqueness pair.

We then assume that $L$ is not parallel to the $x$-axis. It then follows directly from Lemma 3 that $(\Gamma, L)$ is not a uniqueness pair.

For example, if $L$ is also not parallel to the $y$-axis, we may assume that $L$ is the line $y = \alpha x$ where $\alpha \neq 0$. Then take $\varphi$ as a non-zero odd function in $L^1(\mathbb{R})$ and set $g(t) = \varphi(t + 1/2\alpha)$. It then follows from (ii) in Lemma 3 that $\hat{\mu} = Tg$ vanish on $L$ and thus $(\Gamma, L)$ is not a uniqueness pair.

Thus we have proved (i). We then observe that (ii) follows from (i) and (iii), and therefore it only remains to prove (iii). We suppose that $L_1, L_2, E_1, E_2$ and $\Lambda$ have the properties in the statement of (iii) and shall prove that $(\Gamma, \Lambda)$ is a uniqueness pair.

We first study the case where $L_1$ and $L_2$ intersect. Performing a translation we may assume that the point of intersection is the origin. First assume that $L_1$ and $L_2$ are the lines $y = \alpha_1 x$ and $y = \alpha_2 x$, where $\alpha_1$ and $\alpha_2 \neq 0$. We assume that $\hat{\mu}$ is given by (1) and that $\hat{\mu}$ vanishes on $\Lambda = E_1 \cup E_2$. It then follows from (ii) in Lemma 3 that

$g(u - 1/2\alpha_1)$ and $g(u - 1/2\alpha_2)$ are odd. Lemma 2 then implies that $g = 0$ almost everywhere.

We then treat the case where $L_1$ is the $y$-axis and $L_2$ is the line $y = \alpha x$ with $\alpha \neq 0$. Assuming that $\hat{\mu}$ vanishes on $\Lambda$ we conclude from (i) and (ii) in Lemma 3 that $g$ and $g(u - 1/2\alpha)$ are odd. We then invoke Lemma 2 to conclude that $g = 0$ almost everywhere.

It remains to study the case where $L_1$ and $L_2$ are parallel lines. First suppose that these lines are parallel to the $y$-axis. We may assume that $L_1$ is the line $x = 0$ and $L_2$ the line $x = x_0$ where $x_0 \neq 0$. We also assume that $\hat{\mu}$ is given by (1) and that $\hat{\mu}$ vanishes on $\Lambda = E_1 \cup E_2$. It follows from (i) in Lemma 3 that the functions $g$ and $e^{-ix_0 t} g(t)$ are odd. Hence

$$g(-t) = -g(t) \quad \text{and} \quad e^{ix_0 t} g(-t) = -e^{-ix_0 t} g(t).$$

We conclude that

$$-e^{ix_0 t} g(t) = -e^{-ix_0 t} g(t)$$

and

$$e^{i2x_0 t} g(t) = g(t).$$

We obtain

$$\bigl(e^{i2x_0 t} - 1\bigr) g(t) = 0$$

for almost every $t$. It is clear that $e^{i2x_0 t} = 1$ only for $t = \pi n / x_0$ where $n$ is an integer. We conclude that $g = 0$ almost everywhere.

We shall finally study the case where $L_1$ and $L_2$ are two parallel lines which are not parallel to the coordinate axes. We may assume that $L_1$ is the line $y = \alpha x$ and $L_2$ the line $y = \alpha x + b$ where $\alpha \neq 0$ and $b \neq 0$. We assume again that $\hat{\mu}$ is given by (1) and that $\hat{\mu}$ vanishes on $\Lambda$. According to (ii) in Lemma 3 it follows that $g(u - 1/2\alpha)$ is odd. Setting $h(t) = e^{-ibt^2} g(t)$ we also conclude from (iii) in Lemma 3 that $h(u - 1/2\alpha)$ is odd. Setting $\gamma = 1/2\alpha$ we then have $g(-u - \gamma) = -g(u - \gamma)$ and $h(-u - \gamma) = -h(u - \gamma)$ for almost every $u$.

The above equality for $h$ can be written

$$e^{-ib(u+\gamma)^2} g(-u - \gamma) = -e^{-ib(u-\gamma)^2} g(u - \gamma).$$

Using the fact that $g(u - \gamma)$ is odd we obtain

$$-e^{-ib(u+\gamma)^2} g(u - \gamma) = -e^{-ib(u-\gamma)^2} g(u - \gamma)$$

and

$$e^{ib(u-\gamma)^2 - ib(u+\gamma)^2} g(u - \gamma) = g(u - \gamma).$$

Hence

$$\bigl(e^{ib(u-\gamma)^2 - ib(u+\gamma)^2} - 1\bigr) g(u - \gamma) = 0$$

and

$$\big(e^{-ib4u\gamma} - 1\big)g(u-\gamma) = 0.$$

It is clear that $e^{-ib4u\gamma} = 1$ only for $u = 2\pi n/4b\gamma$, where $n$ is an integer, and we conclude that $g = 0$ almost everywhere. The proof of the theorem is complete. □

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