

## Fourier Transform and Convolutions on $L^p$ of a Vector Measure on a Compact Hausdorff Abelian Group

J.M. Calabuig · F. Galaz-Fontes · E.M. Navarrete ·  
E.A. Sánchez-Pérez

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**Abstract** Let  $\nu$  be a countably additive vector measure defined on the Borel subsets  $\mathcal{B}(G)$  of a compact Hausdorff abelian group  $G$ . In this paper we define and study a vector valued Fourier transform and a vector valued convolution for functions which are (weakly) integrable with respect to  $\nu$ . A form of the *Riemann Lebesgue Lemma* and a *Uniqueness Theorem* are established in this context. In order to study the vector valued convolution we discuss the invariance under reflection in  $G$  of these spaces of integrable functions. Finally we present a *Young's type inequality* in this setting and several relevant examples, namely related with the vector measure associated to different important classical operators coming from Harmonic Analysis.

**Keywords** Countably additive vector measure · Space of  $p$ -integrable functions · Fourier transform · Convolution · Pettis integrability

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J.M. Calabuig (✉) · E.A. Sánchez-Pérez  
Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, Camino de Vera s/n, 46022 Valencia, Spain  
e-mail: [jmcabalu@mat.upv.es](mailto:jmcabalu@mat.upv.es)

E.A. Sánchez-Pérez  
e-mail: [esasnce@mat.upv.es](mailto:esasnce@mat.upv.es)

F. Galaz-Fontes · E.M. Navarrete  
Centro de Investigaciones Matemáticas, A.C., Jalisco S/N, Col. Valenciana, 36240 Guanajuato, Gto., México

F. Galaz-Fontes  
e-mail: [galaz@ciamat.mx](mailto:galaz@ciamat.mx)

E.M. Navarrete  
e-mail: [edward@ciamat.mx](mailto:edward@ciamat.mx)

## 1 Introduction

Let  $G$  be a compact Hausdorff abelian group and  $m_G$  its normalized Haar measure, defined on the Borel subsets  $\mathcal{B}(G)$  of  $G$ . Take  $X$  to be a complex Banach space,  $\nu : \mathcal{B}(G) \rightarrow X$  a (countably additive) vector measure and consider  $L_w^1(\nu)$ , the space of weakly integrable functions with respect to  $\nu$ . The main goal of this article is to introduce and study a vector valued Fourier transform and a vector valued convolution for (weakly) integrable functions with respect to a vector measure.

The paper consists of four sections. After this preliminary first section, in the second one we start by introducing the notion of vector valued Fourier transform. First for integrable functions with respect to  $\nu$  (see Definition 2.1) and then for weakly integrable functions with respect to  $\nu$  (see Definition 2.6). After giving some important examples we prove that a natural version of the *Riemann-Lebesgue's Lemma* is not true in this setting. However, always assuming the vector measure  $\nu$  is absolutely continuous with respect to  $m_G$ , we characterize those measures for which this important lemma is satisfied (see Theorem 2.5). We finish this section with a version of a *Uniqueness Theorem* (see Theorem 2.7 and Corollary 2.8).

The definition of the convolution, that is the main goal of the last section, takes us naturally to discuss the effect in the space  $L_w^1(\nu)$  of the reflection in  $G$ , given by  $R(t) = -t$ . This is done in section three, following the work of O. Delgado and P.J. Miana in [3], where they introduced the property of norm integral translation invariance for a vector measure  $\nu : \mathcal{B}(G) \rightarrow X$ . Assuming  $\nu$  has this property, they established that if  $f \in L_w^1(\nu)$  then  $f_s \in L_w^1(\nu)$  and  $\|f_s\|_\nu = \|f\|_\nu$ ; here  $f_s = f(\cdot - s)$ ,  $s \in G$ . Actually, it turns out that the development of O. Delgado and P.J. Miana in [3], can be done when we do not have an algebraic structure on  $G$ , but only a homeomorphism  $h : G \rightarrow G$  with the corresponding property (see Definition 3.1 and Theorem 3.3). With our result we recover not only the translation invariant case but the reflection invariant case. In this section we also study the important case of vector measures associated to the convolution operator. Namely, it is well-known that the vector measure  $\nu_\mu(A) = \chi_A * \mu \in L^p(G)$ , where  $\mu$  is a complex regular measure defined on  $\mathcal{B}(G)$  and  $1 \leq p < \infty$ , is always norm integral translation invariant. However, this is not the case for reflection invariance (see Example 3.6). Concerning this special vector valued measure, we prove that  $\tilde{f} \in L_w^1(\nu_\mu)$  when  $f \in L_w^p(\nu_\mu)$  (have in mind that the vector measure  $\nu$  depends on both  $\mu$  and  $p$ ). This is done in Theorem 3.7. By using the notion of *vector measure scalarly  $k$ -dominated by  $m_G$*  (see Definition 3.8 and Theorem 3.9) we obtain a different proof of a well-known result that asserts that  $L^1(\nu_\mu) = L_w^1(\nu_\mu) = L^1(G)$ , with equivalent norms (see Corollary 3.10).

As we said before, in the last section we study a vector valued convolution for the case of weakly integrable functions (see Definition 4.1) and also for the case of integrable functions (see Definition 4.5). After some examples we establish several integration properties for both cases (see Theorem 4.8 and Theorem 4.10). In the last results of the paper we use the notions of the third section in order to prove the *Young's type inequality* in this context. This is done in Corollaries 4.11, 4.12 and 4.13.

### 1.1 Notation and Preliminaries

Let  $G$  be a compact Hausdorff abelian group, let  $\mathcal{B}(G)$  be the  $\sigma$ -algebra consisting of the Borel sets of  $G$  and let  $m_G$  be the normalized *Haar measure* of  $G$ , that is, the

unique non-negative regular and translation-invariant measure such that  $m_G(G) = 1$ . Recall that a complex continuous function  $\gamma$  defined on  $G$  is called a *character* if  $|\gamma(t)| = 1$  for all  $t \in G$  and  $\gamma(s+t) = \gamma(s)\gamma(t)$  for every  $s, t \in G$ . The set of all continuous characters of  $G$  is the *dual group of  $G$*  and will be denoted by  $\Gamma$ . Note that  $\Gamma$  is discrete since  $G$  is compact. The reader is referred to [7, 12] for additional information on this subject. We denote by  $L^0(G)$  the space of classes ( $m_G$ -a.e. equal functions being identified) of measurable functions defined on  $G$ , if  $1 \leq p < \infty$  then  $L^p(G)$  is the subspace of  $L^0(G)$  such that  $\int_G |f|^p dm_G < \infty$  and  $L^\infty(G)$  is the subspace of functions in  $L^0(G)$  which are  $m_G$ -a.e. bounded. Since the measure  $m_G$  is finite we have  $L^\infty(G) \subseteq L^p(G) \subseteq L^1(G)$  continuously. The *Fourier transform* of a function  $f \in L^1(G)$  is the function  $\hat{f}: \Gamma \rightarrow \mathbb{C}$  given by

$$\hat{f}(\gamma) = \int_G f(t) \overline{\gamma(t)} dm_G(t), \quad \gamma \in \Gamma.$$

Let  $s \in G$ . As usual, if  $f \in L^0(G)$  by  $f_s$  we denote its translation, that is  $f_s(t) = f(t-s)$  for each  $t \in G$ . Also by  $\tau_s$  we denote the map from  $L^0(G)$  into itself defined by  $\tau_s(f) = f_s$ .

Let  $X$  be a complex Banach space,  $B_X$  its closed unit ball and  $X'$  the topological dual of  $X$ . Let  $1 \leq p < \infty$ . A scalarly measurable function  $f: G \rightarrow X$  is said to be *Dunford  $p$ -integrable* (or simply *Dunford integrable* when  $p = 1$ ) if  $\langle f, x' \rangle \in L^p(G)$  for all  $x' \in X'$ . The set consisting of classes of these functions will be denoted by  $L_w^p(G, X)$ . For a function  $f \in L_w^p(G, X)$  the norm is defined by  $\|f\|_{L_w^p(G, X)} = \sup_{x' \in B_{X'}} (\int_G |\langle f, x' \rangle|^p dm_G)^{1/p}$ . A Dunford  $p$ -integrable function  $f: G \rightarrow X$  is said to be *Pettis  $p$ -integrable* (or simply *Pettis integrable* when  $p = 1$ ) if for each  $A \in \mathcal{B}(G)$  there is  $x_A \in X$  satisfying  $\int_A \langle f, x' \rangle dm_G = \langle x_A, x' \rangle$ , for all  $x' \in X'$ . The vector  $x_A$  is unique and will be denoted by  $(P) \int_A f dm_G$ . The set of classes of scalarly measurable functions which are Pettis  $p$ -integrable, endowed with the same norm as  $L_w^p(G, X)$ , will be denoted by  $\mathcal{P}^p(G, X)$ . For more information about Pettis integrability the reader is referred to [4, 10] and the nice survey [9, Chap. 12].

Let  $(\Omega, \Sigma, \mu)$  be a measure space and  $L^0(\mu)$  the set consisting of classes ( $\mu$ -a.e. equal functions being identified) of measurable functions. A Banach space  $E(\mu) \subseteq L^0(\mu)$  is called a *Banach function space related to  $\mu$*  if  $L^\infty(\mu) \subseteq E(\mu) \subseteq L^1(\mu)$  and if  $f \in E(\mu)$  and  $g \in L^0(\mu)$  satisfies  $|g| \leq |f|$   $\mu$ -a.e., then  $g \in E(\mu)$  and  $\|g\|_{E(\mu)} \leq \|f\|_{E(\mu)}$ . A Banach function space  $E(\mu)$  is called *order continuous* if every increasing order bounded sequence in  $E(\mu)$  converges in the norm of  $E(\mu)$ . The *Köthe dual* (or *associate space*) of  $E(\mu)$  is the Banach function space related to  $\mu$  defined by  $E(\mu)^\times = \{g \in L^0(\mu) : fg \in L^1(\mu), f \in E(\mu)\}$ , and endowed with the norm  $\|g\|_{E(\mu)^\times} = \sup_{f \in B_{E(\mu)}} \int_\Omega |fg| d\mu$ ,  $g \in E(\mu)^\times$ . Recall that  $E(\mu)$  is order continuous if and only if  $E(\mu)^\times$  coincides with  $E(\mu)'$ . In this case each  $x' \in E(\mu)'$  can be identified with a function  $g \in E(\mu)^\times$  via the correspondence  $\langle f, x' \rangle = \int_\Omega fg d\mu$  for all  $f \in E(\mu)$ . For questions related to Banach function spaces see [13, Chap. 2].

Let  $\nu: \mathcal{B}(G) \rightarrow X$  be a countably additive vector measure. A set  $A \in \mathcal{B}(G)$  is called  $\nu$ -null if  $\nu(B) = 0$  for all  $B \in \mathcal{B}(G)$  such that  $B \subseteq A$ . Given  $x' \in X'$  let us denote by  $\langle \nu, x' \rangle$  the scalar measure defined by  $\langle \nu, x' \rangle(A) = \langle \nu(A), x' \rangle$  for each  $A \in$

$\mathcal{B}(G)$ . The *semivariation* of  $\nu$  is the non-negative set function  $\|\nu\| : \mathcal{B}(G) \rightarrow [0, \infty)$  defined by  $\|\nu\|(A) = \sup_{x' \in B_{X'}}$   $|\langle \nu, x' \rangle|(A)$ ,  $A \in \mathcal{B}(G)$ , where  $|\langle \nu, x' \rangle|$  is the variation of the measure  $\langle \nu, x' \rangle$ . Recall that a set  $A \in \mathcal{B}(G)$  is  $\nu$ -null if and only if  $\|\nu\|(A) = 0$ . The vector measure  $\nu$  is *absolutely continuous* with respect to a positive measure  $\lambda$  if  $\lambda(A) \rightarrow 0$  implies that  $\|\nu\|(A) \rightarrow 0$ . As usual in this situation we write  $\nu \ll \lambda$ . In the case when  $\lambda$  is finite we have that  $\nu \ll \lambda$  if and only if any  $\lambda$ -null set is  $\nu$ -null. A *Rybakov control measure* for  $\nu$  is a finite positive measure  $\mu$  satisfying  $\nu \ll \mu$  and  $\mu = |\langle \nu, x'_0 \rangle|$  for some  $x'_0 \in X'$ . This kind of measures always exist by the Rybakov Theorem (see [4, Theorem IX.2.2]). If  $\mu$  is a Rybakov control measure for  $\nu$ , note that  $\mu$  and  $\nu$  have the same null sets. Information on vector measures can be found in [4, Chaps. I and IX].

A measurable function  $f : G \rightarrow \mathbb{C}$  is said to be *weakly integrable with respect to the vector measure  $\nu$*  if  $f \in L^1(|\langle \nu, x' \rangle|)$ , for all  $x' \in X'$ . The subspace consisting of these functions will be denoted by  $L^1_w(\nu)$ ; this is a Banach space with the norm  $\|f\|_\nu = \sup_{x' \in B_{X'}}$   $\int_G |f|d|\langle \nu, x' \rangle|$ ,  $f \in L^1(\nu)$ . A weakly integrable function  $f : G \rightarrow \mathbb{C}$  is said to be *integrable with respect to  $\nu$*  if for each  $A \in \mathcal{B}(G)$  there is  $x_A \in X$  such that  $\int_A f d\langle \nu, x' \rangle = \langle x_A, x' \rangle$ , for all  $x' \in X'$ . Again the vector  $x_A$  is unique and it is denoted by  $\int_A f d\nu$ . The set consisting of classes ( $\nu$ -a.e. equal functions being identified) of these functions will be denoted by  $L^1(\nu)$ . The *integration operator*  $I_\nu : L^1(\nu) \rightarrow X$  is the linear and continuous function defined by  $I_\nu(f) = \int_G f d\nu$ , for  $f \in L^1(\nu)$ . Given  $1 \leq p < \infty$  we denote by  $L^p(\nu)$  (resp.  $L^p_w(\nu)$ ) the set of classes of measurable functions  $f : G \rightarrow \mathbb{C}$  satisfying that  $|f|^p \in L^1(\nu)$  (resp.  $|f|^p \in L^1_w(\nu)$ ). These vector spaces are complete when endowed with the norm  $\|f\|_{\nu,p} = \| |f|^p \|_\nu^{1/p} = \sup_{x' \in B_{X'}}$   $(\int_G |f|^p d|\langle \nu, x' \rangle|)^{1/p}$ . Note that the space  $L^\infty(\nu)$  consisting of classes of measurable functions which are  $\nu$ -a.e. bounded, coincides with the space  $L^\infty(\mu)$ , where  $\mu$  is a Rybakov control measure for  $\nu$ . For  $1 \leq p < \infty$  the spaces  $L^p(\nu)$  are order continuous Banach function spaces related to any Rybakov control measure for  $\nu$ . Hence the set of simple functions  $\mathcal{S}(G)$  is dense in  $L^p(\nu)$ . For more information about integration of scalar functions with respect to vector measures see [11] and the references therein.

## 2 The Fourier Transform Associated to $\nu$

Let  $G$  be a compact Hausdorff abelian group,  $X$  a complex Banach space and  $\nu : \mathcal{B}(G) \rightarrow X$  a countably additive vector measure. Recall that if  $f \in L^1(\nu)$  and  $\phi \in L^\infty(G)$  then  $f\phi \in L^1(\nu)$ . This allows us to give the following definition.

**Definition 2.1** The *Fourier Transform of a function  $f \in L^1(\nu)$  with respect to the vector measure  $\nu$*  is the vector valued function  $\widehat{f}^\nu : \Gamma \rightarrow X$  given by

$$\widehat{f}^\nu(\gamma) = \int_G f(t)\overline{\gamma(t)}d\nu(t), \quad \gamma \in \Gamma.$$

Let us start with a general example using vector measures associated to operators that will be useful in the rest of the paper.

*Example 2.2* Let  $E(m_G)$  be an order continuous Banach function space related to the Haar measure  $m_G$  on a compact Hausdorff abelian group  $G$ . Let  $T : E(m_G) \rightarrow X$  be a bounded linear map, where  $X$  is a Banach space and consider the vector measure  $\nu_T : \mathcal{B}(G) \rightarrow X$  associated to  $T$ , defined by  $\nu_T(A) = T(\chi_A)$ ,  $A \in \mathcal{B}(G)$ . Then it is well-known (see for instance [11, Proposition 4.4]) that every function  $f \in E(m_G)$  is integrable with respect to  $\nu_T$  and  $\int_A f d\nu_T = T(f\chi_A)$ ,  $A \in \mathcal{B}(G)$ . Hence, given  $f \in E(m_G)$  since  $f\bar{\gamma} \in E(m_G)$  for all  $\gamma \in \Gamma$ , we have

$$\widehat{f}^{\nu_T}(\gamma) = T(f\bar{\gamma}). \tag{2.1}$$

In particular throughout this paper we work with the following important cases:

- (1) Given a complex regular measure  $\mu$  defined on  $\mathcal{B}(G)$  and  $1 \leq p < \infty$  we consider the operator  $T_\mu : L^p(G) \rightarrow L^p(G)$  given by  $T_\mu(f) = f * \mu$  for each  $f \in L^p(G)$ , where

$$f * \mu(t) = \int_G f(t - s) d\mu(s), \quad t \in G.$$

Writing  $\nu_\mu$  for  $\nu_{T_\mu}$ , the vector measure associated to  $T_\mu$ , formula (2.1) gives  $\widehat{f}^{\nu_\mu}(\gamma) = f\bar{\gamma} * \mu$ ,  $\gamma \in \Gamma$ . For future references, note that  $\nu_\mu \ll m_G$ . In the particular case given by the compact Hausdorff abelian group  $\mathbb{T} = \{x \in \mathbb{C} : |x| = 1\}$  let us take  $x_0 \in \mathbb{T}$  and define  $\mu = \delta_{x_0}$ , the delta function centered at  $x_0$ . Notice that  $\mu$  is a positive finite regular measure defined on  $\mathcal{B}(\mathbb{T})$ . In accordance, we will change our notation to that of multiplication. Hence we have  $\widehat{f}^{\nu_\mu}(n)(t) = nf(t - x_0)$ ,  $n \in \mathbb{Z}, t \in \mathbb{T}$ .

- (2) Let us consider  $T = \text{id}$ , the identity map from  $E(m_G)$  into itself. Then  $\widehat{f}^\nu(\gamma) = f\bar{\gamma}$  for all  $\gamma \in \Gamma$ .
- (3) Let  $1 \leq p \leq 2$  and  $p'$  be its conjugate exponent, that is  $1/p + 1/p' = 1$ . Let us consider now the vector measure associated to the Fourier transform  $\mathcal{F}$  which, by the *Hausdorff-Young inequality*, satisfies  $\mathcal{F} : L^p(G) \rightarrow \ell^{p'}(\Gamma)$ . Hence  $\nu_{\mathcal{F}} : \mathcal{B}(G) \rightarrow \ell^{p'}(\Gamma)$  is given by

$$\nu_{\mathcal{F}}(A)(\gamma) = \mathcal{F}(\chi_A)(\gamma) = \widehat{\chi_A}(\gamma) = \int_A \overline{\gamma(t)} dm_G(t), \quad A \in \mathcal{B}(G), \gamma \in \Gamma.$$

Therefore  $\widehat{f}^{\nu_{\mathcal{F}}}(\gamma) = (\mathcal{F}(f\bar{\gamma}))(\xi)_{\xi \in \Gamma} = (\widehat{f}(\gamma\xi))_{\xi \in \Gamma}$ ,  $\gamma \in \Gamma$ .

- (4) Let  $1 \leq p, q, r \leq \infty$  and  $g \in L^q(G)$  be such that  $1/p + 1/q = 1/r$ . By *Hölder's inequality* we can consider the multiplication operator  $M_g : L^p(G) \rightarrow L^r(G)$  given by  $M_g(f) = fg$ . Hence for the vector measure associated to  $T_g$  we have that  $\widehat{f}^{\nu_{M_g}}(\gamma) = fg\bar{\gamma}$ ,  $\gamma \in \Gamma$ . Note that the same result for  $1 < p < \infty$  can be obtained by using [11, Proposition 3.43 and Remark 3.44] (see also [2, 5]) and either  $M_g : L^p(v) \rightarrow L^1(v)$  for  $g \in L^{p'}_w(v)$ , or  $M_g : L^p_w(v) \rightarrow L^1(v)$  for  $g \in L^{p'}(v)$ , or  $M_g : L^p_w(v) \rightarrow L^1_w(v)$  for  $g \in L^{p'}_w(v)$ .

The first classical and important result concerning the Fourier transform is *Riemann-Lebesgue’s Lemma*. Note that for a given  $\gamma \in \Gamma$  one has

$$\|\widehat{f}^\nu(\gamma)\|_X = \sup_{x' \in B_{X'}} |(\widehat{f}^\nu(\gamma), x')| = \sup_{x' \in B_{X'}} \left| \int_G f(t) \overline{\gamma(t)} d\langle \nu, x' \rangle(t) \right| \leq \|f\|_\nu.$$

Therefore

**Proposition 2.3** *If  $f \in L^1(\nu)$ , then  $\widehat{f}^\nu \in \ell^\infty(\Gamma, X)$  and  $\|\widehat{f}^\nu\|_{\ell^\infty(\Gamma, X)} \leq \|f\|_\nu$ . Hence the Fourier transform  $\mathcal{F}^\nu : L^1_\nu(\nu) \rightarrow \ell^\infty(\Gamma, X)$  given by  $\mathcal{F}^\nu(f) = \widehat{f}^\nu$ , is a bounded linear operator.*

However, the following simple example shows that  $\widehat{f}^\nu$  is not always in  $c_0(\Gamma, X)$ , as it happens in the classical *Riemann-Lebesgue’s Lemma*.

*Example 2.4* Let us consider the case (2) in Example 2.2. Recall that, in this case,  $E(m_G)$  is isometrically isomorphic to  $L^1(\nu)$  (see for instance [11, Proposition 3.9]). Hence, since  $f^\nu(\gamma) = f\overline{\gamma}$  for all  $\gamma \in \Gamma$ ,

$$\|\widehat{f}^\nu(\gamma)\|_{E(m_G)} = \|f\overline{\gamma}\|_{E(m_G)} = \|f\overline{\gamma}\|_\nu = \|f\|_\nu.$$

When  $\Gamma$  is an infinite set, this shows that  $\widehat{f}^\nu \notin c_0(\Gamma, X)$ .

If  $\nu \ll m_G$  then the trigonometric polynomials on  $G$  are dense in  $L^1(\nu)$  (see [3, Corollary 3.9]). This result allows us to characterize when a version of the *Riemann-Lebesgue Lemma* is satisfied in our setting.

**Theorem 2.5** *Assume that  $\nu \ll m_G$ . The following assertions are equivalent:*

- (a)  $\widehat{f}^\nu \in c_0(\Gamma, X)$  for all  $f \in L^1(\nu)$ ,
- (b)  $\widehat{\chi}_G^\nu \in c_0(\Gamma, X)$ .

*Proof* Trivially (a) implies (b) so assume that (b) is satisfied and let us see (a). Fix  $\varepsilon > 0$ . We have to show that the set  $\Gamma_0 = \{\gamma \in \Gamma : \|\widehat{f}^\nu(\gamma)\|_X > \varepsilon\}$  is finite. Given  $f \in L^1(\nu)$  let us take a trigonometric polynomial  $p = \sum_{j=1}^n \alpha_j \gamma_j$ , where  $\alpha_j \in \mathbb{C} \setminus \{0\}$ ,  $\gamma_j \in \Gamma$  and  $j = 1, \dots, n$  are such that  $\|f - p\|_\nu \leq \varepsilon/2$ . By Proposition 2.3 one has  $\|\widehat{f - p}^\nu(\gamma)\|_X \leq \|f - p\|_\nu \leq \varepsilon/2$ ,  $\gamma \in \Gamma$ . And by the assumption (b) for all  $j = 1, \dots, n$  the sets

$$\Gamma_j = \left\{ \gamma \in \Gamma : \|\widehat{\chi}_G^\nu(\overline{\gamma_j} \gamma)\|_X > \frac{\varepsilon}{2|\alpha_j|n} \right\},$$

are finite. Let us see that

$$\Gamma_0 \subseteq \left\{ \gamma \in \Gamma : \|\widehat{p}^\nu(\gamma)\|_X > \varepsilon/2 \right\} \subseteq \bigcup_{j=1}^n \Gamma_j.$$

So take  $\gamma \notin \bigcup_{j=1}^n \Gamma_j$ . Then

$$\|\widehat{p}^v(\gamma)\|_X \leq \sum_{j=1}^n |\alpha_j| \left\| \int_G \gamma_j(t) \overline{\gamma(t)} dv(t) \right\|_X = \sum_{j=1}^n |\alpha_j| \|\widehat{\chi}_G^v(\overline{\gamma_j}\gamma)\|_X \leq \frac{\varepsilon}{2},$$

and also  $\|\widehat{f}^v(\gamma)\|_X \leq \|\widehat{f - p}^v(\gamma)\|_X + \|\widehat{p}^v(\gamma)\|_X \leq \varepsilon$ . Consequently,  $\Gamma_0$  is finite.  $\square$

Concerning the previous result we have the following open problem.

*Open problem* Are there Banach spaces  $X$  for which the previous result holds for every  $X$ -valued measure? Are there natural subclasses of vector measures for which it holds? Are there classes of operators that transforms vector measures into vector measures for which the result holds?

We finish this section by defining a “Fourier transform” for  $f \in L^1_w(\nu)$ . We will be able to do it under the requirement that the vector measure  $\nu$  is absolutely continuous with respect to  $m_G$ . Then fixing  $x' \in X'$  we have that  $\langle \nu, x' \rangle \ll |\langle \nu, x' \rangle| \ll \nu \ll m_G$  and so we can consider the Radon-Nikodým derivative

$$h_{x'} = \frac{d\langle \nu, x' \rangle}{dm_G}. \tag{2.2}$$

Take now  $f \in L^1_w(\nu)$ . Since  $\int_G |f| d|\langle \nu, x' \rangle| = \int_G |f h_{x'}| dm_G$ , then  $f h_{x'} \in L^1(G)$ . Hence we can define a “weak” Fourier transform as follows.

**Definition 2.6** The *Fourier Transform of a function*  $f \in L^1_w(\nu)$  with respect to the vector measure  $\nu$  is

$$\mathcal{F}_\nu : L^1_w(\nu) \rightarrow L(X', \ell^\infty(\Gamma)), \quad \mathcal{F}_\nu[f](x') = \widehat{f h_{x'}}.$$

As usual,  $L(Y, Z)$  stands here for the Banach space of all bounded linear operators going from the Banach space  $Y$  into the Banach space  $Z$ .

This transform is well defined since for all  $x' \in X'$  one has

$$\|\mathcal{F}_\nu[f](x')\|_{\ell^\infty(\Gamma)} = \|\widehat{f h_{x'}}\|_{\ell^\infty(\Gamma)} \leq \|f h_{x'}\|_{L^1(G)} \leq \|f\|_\nu \|x'\|.$$

Clearly if  $f \in L^1(\nu)$  (and  $\nu \ll m_G$ ) then

$$\mathcal{F}_\nu[f](x') = \langle \widehat{f}^v, x' \rangle = \langle \mathcal{F}^v(f), x' \rangle, \quad x' \in X'. \tag{2.3}$$

Our next result is a uniqueness theorem.

**Theorem 2.7** Let  $\nu \ll m_G$ . If  $\mathcal{F}_\nu[f] = 0$  for  $f \in L^1_w(\nu)$  then  $f = 0$  in  $L^1_w(\nu)$ .

*Proof* Let  $x' \in X'$ . Since  $\mathcal{F}_\nu[f](x') = 0$  then for all  $\gamma \in \Gamma$ ,

$$\int_G f(t) \overline{\gamma(t)} h_{x'}(t) dm_G(t) = \int_G f(t) \overline{\gamma(t)} d\langle \nu, x' \rangle(t) = 0.$$

This means that  $\widehat{f h_{x'}}(\gamma) = 0$  for all  $\gamma \in \Gamma$  and so  $f h_{x'} = 0$ . Let  $\widetilde{G} = \{t \in G : h_{x'}(t) = 0\}$ . Note that

$$|\langle \nu, x' \rangle|(\widetilde{G}) = \int_{\widetilde{G}} |h_{x'}| dm_G = 0.$$

Therefore  $\widetilde{G}$  is  $\nu$ -null. But on the other hand  $f = 0$  on  $G \setminus \widetilde{G}$  and so  $\int_G |f| d|\langle \nu, x' \rangle| = 0$ , for each  $x' \in X'$ . Hence  $f = 0$ . □

Using (2.3) and Theorem 2.7 we obtain immediately the following.

**Corollary 2.8** *Let  $\nu \ll m_G$ . If  $\widehat{f}^\nu = 0$  for  $f \in L^1(\nu)$  then  $f = 0$  in  $L^1(\nu)$ .*

From the case (1) in Example 2.2 and the above corollary we obtain directly the next result.

*Example 2.9* Let  $\mu$  be a complex regular measure defined on  $\mathcal{B}(G)$ . If  $f \in L^p(G)$ ,  $1 \leq p < \infty$  and  $f \overline{\gamma} * \mu = 0$  for all  $\gamma \in \Gamma$ , then  $f = 0$ .

### 3 Homeomorphism Invariant Spaces

Let  $G$  be a compact Hausdorff abelian group and  $m_G$  its Haar measure. Then  $L^1(G)$  is *norm translation invariant*, that is if  $f \in L^1(G)$  then  $f_t \in L^1(G)$  and  $\|f\|_{L^1(G)} = \|f_t\|_{L^1(G)}$ ,  $f \in L^1(G)$ ,  $t \in G$ . O. Delgado and P.J. Miana introduced the following property for a countably additive vector measure defined on  $\mathcal{B}(G)$ :  $\nu$  is said to be *norm integral translation invariant* if  $\|I_\nu(\tau_t \phi)\|_X = \|I_\nu(\phi)\|_X$ , for every simple function  $\phi \in \mathcal{S}(G)$  and  $t \in G$ . When this condition holds the spaces  $L^1_w(\nu)$  and  $L^1(\nu)$  are norm translation invariant (see [3, Proposition 3.5]). Since we are dealing with  $G$  compact, its Haar measure is also reflection invariant, that is  $m_G(-A) = m_G(A)$ ,  $A \in \mathcal{B}(G)$ . This implies that  $L^1(G)$  is *norm reflection invariant*, that is if  $f \in L^1(G)$  then  $\widetilde{f} \in L^1(G)$  and  $\|f\|_{L^1(G)} = \|\widetilde{f}\|_{L^1(G)}$ , where  $\widetilde{f}$  is the function given by  $\widetilde{f}(t) = f(-t)$ ,  $t \in G$ . We are now interested in this property for the spaces  $L^1(\nu)$  and  $L^1_w(\nu)$ . It turns out that the development of O. Delgado and P.J. Miana can be readily adapted to provide a solution for this question and, moreover, it can be done in a more general situation than that of  $G$  being a group.

Let  $K$  be a nonempty compact Hausdorff space,  $\nu : \mathcal{B}(K) \rightarrow X$  be a (countably additive) vector measure and  $h : K \rightarrow K$  an homeomorphism. Given a function  $f : K \rightarrow \mathbb{C}$ , let  $f_h : K \rightarrow \mathbb{C}$  be defined by  $f_h = f \circ h^{-1}$ . If  $f$  is measurable, notice that  $f_h$  is also measurable.

**Definition 3.1** The vector measure  $\nu$  is said to be *norm integral  $h$ -invariant* if

$$\|I_\nu(\phi_h)\|_X = \|I_\nu(\phi)\|_X, \tag{3.1}$$

for all  $\phi \in \mathcal{S}(K)$ . The space  $L^1_w(\nu)$  (resp.  $L^1(\nu)$ ) is norm  $h$ -invariant if for each  $f \in L^1_w(\nu)$  (resp.  $f \in L^1(\nu)$ ) we have that  $f_h \in L^1_w(\nu)$  (resp.  $f_h \in L^1(\nu)$ ) and  $\|f_h\|_\nu = \|f\|_\nu$ .



**Lemma 3.2** *If  $\nu$  is norm integral  $h$ -invariant and  $\|\nu\|(A) = 0$  for  $A \in \mathcal{B}(K)$  then  $\|\nu\|(h(A)) = 0$ .*

*Proof* We will use the equivalent norm for  $L^1(\nu)$  given by

$$\|f\| = \sup_{B \in \mathcal{B}(K)} \left\| \int_B f d\nu \right\|_X. \tag{3.2}$$

First of all note that  $(\chi_D)_h = \chi_D \circ h^{-1} = \chi_{h(D)}$ ,  $D \in \mathcal{B}(K)$ . So, from (3.1) it follows that  $\|\nu(h(D))\|_X = \|\nu(D)\|_X$ . Let  $A \in \mathcal{B}(K)$  be such that  $\|\nu\|(A) = 0$ . Then

$$\|\chi_{h(A)}\| = \sup_{B \in \mathcal{B}(K)} \left\| \int_{h(B)} \chi_{h(A)} d\nu \right\|_X = \sup_{B \in \mathcal{B}(K)} \left\| \int_B \chi_A d\nu \right\|_X \leq \|\chi_A\|_\nu = 0.$$

This shows that  $\|\chi_{h(A)}\| = 0$  so  $\|\nu\|(h(A)) = \|\chi_{h(A)}\|_\nu = 0$ . □

Note that the lemma above establishes that

$$T_h : L^0(K) \rightarrow L^0(K), \quad T_h(f) = f_h \tag{3.3}$$

is a well defined linear operator. The next theorem generalizes [3, Lemma 3.4 and Proposition 3.5]. It additionally indicates that the norm  $\|\cdot\|$  is also preserved by the operator that carries  $f \rightarrow f_h$ . The first part of the proof is similar to that in [3], but we include it for the sake of completeness.

**Theorem 3.3** *If  $\nu$  is norm integral  $h$ -invariant, then the spaces  $L^1_w(\nu)$  and  $L^1(\nu)$  are norm  $h$ -invariant. Moreover,*

$$\|f_h\| = \|f\|, \quad \text{for all } f \in L^1(\nu). \tag{3.4}$$

*Proof* Take  $A \in \mathcal{B}(K)$ . Clearly  $T_h(\chi_A) = \chi_{h(A)}$ . Let  $X_\nu = I_\nu(\mathcal{S}(K)) \subseteq X$  and define  $S_h : X_\nu \rightarrow X_\nu$  by  $S_h(I_\nu(\phi)) = I_\nu(T_h\phi)$ , for  $\phi \in \mathcal{S}(K)$ . Since  $\nu$  is norm integral  $h$ -invariant, then  $S_h$  is a well-defined isometry and

$$S_h(\nu(A)) = \nu(h(A)), \quad A \in \mathcal{B}(K). \tag{3.5}$$

Fix  $x' \in X'$  and consider  $y'_h = x' \circ S_h \in (X_\nu)'$ . By using *Hahn-Banach's Theorem* we can find  $x'_h \in X'$  with  $\|x'_h\| = \|y'_h\| \leq \|x'\| \|S_h\| = \|x'\|$  such that  $x'_h = y'_h$  in  $X_\nu$ . From (3.5) we now obtain  $\langle \nu, x'_h \rangle(A) = \langle \nu, x' \rangle(h(A))$ ,  $A \in \mathcal{B}(K)$ , and from this it is easy to check that  $|\langle \nu, x'_h \rangle|(A) = |\langle \nu, x' \rangle|(h(A))$ ,  $A \in \mathcal{B}(K)$ . Therefore

$$\int_K T_h\phi d|\langle \nu, x' \rangle| = \int_K \phi d|\langle \nu, x'_h \rangle|, \quad \phi \in \mathcal{S}(K).$$

Take  $f \in L^1_w(\nu)$  and  $x' \in B_{X'}$ . Then there is  $(\phi_n)_n \subseteq \mathcal{S}(K)$  such that  $0 \leq \phi_n \nearrow |f|$  pointwise. Clearly  $0 \leq T_h\phi_n \nearrow |T_h f| = T_h|f|$ , so applying the *Monotone Conver-*

gence Theorem

$$\begin{aligned} \int_K |T_h f| d|\nu, x'| &= \lim_{n \rightarrow \infty} \int_K T_h \phi_n d|\nu, x'| = \lim_{n \rightarrow \infty} \int_K \phi_n d|\nu, x'_h| \\ &= \int_K |f| d|\nu, x'_h|. \end{aligned}$$

Since  $\|x'_h\| \leq \|x'\|$  we have that  $T_h f \in L^1_w(\nu)$  and  $\|T_h f\|_\nu \leq \|f\|_\nu$ . Taking  $\phi_h$  instead of  $\phi$  in (3.1) we find that the vector measure  $\nu$  is also norm integral  $h^{-1}$ -invariant. Hence we also have that  $\|f\|_\nu = \|T_{h^{-1}}(T_h f)\| \leq \|T_h f\|_\nu$ . Finally, take  $f \in L^1(\nu)$  and let  $(\phi_n)_n \subseteq \mathcal{S}(K)$  be a sequence converging to  $f$  in  $L^1(\nu)$ . Since  $(T_h(\phi_n))_n \subseteq \mathcal{S}(K) \subseteq L^1(\nu)$  and  $\|T_h \phi_n - T_h f\|_\nu = \|\phi_n - f\|_\nu$ , then  $T_h f \in L^1(\nu)$ . The continuity of  $I_\nu$  gives  $\|I_\nu(T_h f)\|_X = \|I_\nu(f)\|_X$ . We will now prove (3.4). Take  $\phi \in \mathcal{S}(K)$  and note that

$$\phi_h \chi_B = \phi_h(\chi_B \circ h) = (\phi \chi_{h^{-1}(B)})_h, \quad \text{for all } B \in \mathcal{B}(K).$$

Since  $\nu$  is  $h$ -invariant, we have

$$\begin{aligned} \|\phi_h\| &= \sup_{B \in \mathcal{B}(K)} \left\| \int_B \phi_h d\nu \right\|_X = \sup_{B \in \mathcal{B}(K)} \left\| \int_K (\phi \chi_{h^{-1}(B)})_h d\nu \right\|_X \\ &= \sup_{B \in \mathcal{B}(K)} \left\| \int_{h^{-1}(B)} \phi d\nu \right\|_X = \|\phi\|. \end{aligned}$$

Finally, the density of  $\mathcal{S}(K)$  in  $L^1(\nu)$  together with the continuity of the map  $T_h : L^1(\nu) \rightarrow L^1(\nu)$  gives (3.4). □

In a topological group, besides translations, the reflection  $R(t) = -t, t \in G$ , is also a canonical homeomorphism. We are now interested in applying our development to this case. More concretely, we will say that  $\nu$  is norm integral reflection invariant, if it is norm integral  $R$ -invariant. In the case when  $L^1_w(\nu)$  (resp.  $L^1(\nu)$ ) has an  $R$ -invariant norm we will say that  $L^1_w(\nu)$  (resp.  $L^1(\nu)$ ) is norm reflection invariant. Note that  $R^{-1} = R$  and if  $f : G \rightarrow \mathbb{C}$ , then  $\tilde{f}(t) = f \circ R^{-1}$ .

**Corollary 3.4** *Let  $G$  be a compact Hausdorff abelian group.*

- (a) *If  $\nu$  is a norm integral translation invariant vector measure, then the spaces  $L^1_w(\nu)$  and  $L^1(\nu)$  are norm translation invariant.*
- (b) *If  $\nu$  is norm integral reflection invariant, then the spaces  $L^1_w(\nu)$  and  $L^1(\nu)$  are norm reflection invariant.*

Next, we give some examples.

**Example 3.5** The following two vector measures are norm integral  $R$ -invariant.

- (a) Consider  $K = [-1, 1]$  and take  $\lambda$  to be Lebesgue’s measure on  $\mathcal{B}(K)$ . Given  $1 \leq p \leq \infty$ , let  $T : L^1(\lambda) \rightarrow L^p(\lambda)$  be Volterra’s operator, that is,  $Tf(x) =$

$\int_0^x f(s)ds$ . Define  $\nu$  to be  $\nu_T$ , the (countably additive) vector measure induced by  $T$  (recall that in this case  $I_\nu(f) = T(f)$ , for all  $f \in L^1(\nu)$ ). Therefore for a given  $\phi \in \mathcal{S}(K)$  we have

$$|I_\nu \phi_R(x)| = \left| \int_0^x \phi_R(s)ds \right| = \left| \int_0^x \phi(-s)ds \right| = \left| \int_0^{-x} \phi(s)ds \right| = |I_\nu \phi(-x)|. \tag{3.6}$$

Since  $\lambda$  is reflection invariant, this implies that  $\|I_\nu \phi_R\|_{L^p(K)} = \|I_\nu \phi\|_{L^p(K)}$  for  $1 \leq p \leq \infty$ .

- (b) Let us consider the vector measure  $\nu_{\mathcal{F}}$ , associated to the Fourier transform studied in the case (3) in Example 2.2. Take  $\phi \in \mathcal{S}(G)$  and  $\gamma \in \Gamma$ . By the reflection invariance of  $m_G$  we have

$$\begin{aligned} I_{\nu_{\mathcal{F}}} \tilde{\phi}(\gamma) &= \int_G \tilde{\phi}(t) \overline{\gamma(t)} dm_G(t) = \int_G \phi(-t) \overline{\gamma(t)} dm_G(t) \\ &= \int_G \phi(t) \overline{\gamma(-t)} dm_G(t) = I_{\nu_{\mathcal{F}}} \phi(\tilde{\gamma}). \end{aligned}$$

Since the correspondence  $\gamma \rightarrow \tilde{\gamma}$  is a bijection on  $\Gamma$ , it follows from above that  $\|I_{\nu_{\mathcal{F}}} \tilde{\phi}\|_{\ell^{p'}(\Gamma)} = \|I_{\nu_{\mathcal{F}}} \phi\|_{\ell^{p'}(\Gamma)}$ .

*Example 3.6* We will now discuss the case of the vector measure considered in the case (1) in Example 2.2; as before we will denote it by  $\nu_\mu$ .

- (a) As we said in the introduction, the vector measure  $\nu_\mu$  is always norm integral translation invariant. If, in addition,  $\mu$  is reflection invariant, we will next prove that  $\nu$  is also norm integral reflection invariant. Notice  $T_R(f) = \tilde{f}$  defines a linear isometry on  $L^p(G)$ . Let  $\phi \in \mathcal{S}(G)$  and  $t \in G$ . Then

$$g(t) = |I_{\nu_\mu} \tilde{\phi}(t)| = |\tilde{\phi} * \mu(t)| = \left| \int_G \phi(s - t) d\mu(s) \right|.$$

On other side, since  $\mu$  is reflection invariant, we have

$$h(t) = |I_{\nu_\mu} \phi(t)| = |\phi * \mu(t)| = \left| \int_G \phi(t - s) d\mu(s) \right| = \left| \int_G \phi(t + s) d\mu(s) \right| = \tilde{g}(t).$$

By the reflection invariance of the Haar measure, this implies that

$$\|I_{\nu_\mu} \tilde{\phi}\|_{L^p(G)} = \|g\|_{L^p(G)} = \|\tilde{g}\|_{L^p(G)} = \|h\|_{L^p(G)} = \|I_{\nu_\mu} \phi\|_{L^p(G)}.$$

- (b) However, the vector measure  $\nu_\mu$  could be norm integral reflection invariant although  $\mu$  is not reflection invariant. Namely, let us consider  $G = \mathbb{T}$ , take  $x_0 \in \mathbb{T}$  such that  $x_0 \neq \pm 1$  and define  $\mu = \delta_{x_0}$ , the delta function centered at  $x_0$ . Next, we choose  $A \in \mathcal{B}(\mathbb{T})$  satisfying  $x_0 \in A$  and  $x_0^{-1} \notin A$ . Since  $\mu(A) = 1$  and  $\mu(A^{-1}) = 0$ , the measure  $\mu$  is not reflection invariant. Let  $\phi \in \mathcal{S}(\mathbb{T})$ . Then

$$\begin{aligned} I_{\nu_\mu} \tilde{\phi}(t) &= \int_{\mathbb{T}} \phi(st^{-1}) d\mu(s) = \phi(x_0 t^{-1}), \\ I_{\nu_\mu} \phi(t) &= \int_{\mathbb{T}} \phi(ts^{-1}) d\mu(s) = \phi(tx_0^{-1}). \end{aligned}$$

Since the norm in  $L^p(\mathbb{T})$  is invariant under both reflection and translation, from what we just established above we obtain  $\|I_{\nu_\mu} \tilde{\phi}\|_{L^p(\mathbb{T})} = \|\phi\|_{L^p(\mathbb{T})} = \|I_{\nu_\mu} \phi\|_{L^p(\mathbb{T})}$ . This shows that  $\nu_\mu$  is norm integral reflection invariant.

- (c) We now present a vector measure  $\nu_\mu$  that is not norm integral reflection invariant. Let us consider the measure  $\mu(s) = sdm_{\mathbb{T}}(s)$  and notice it is regular. Take now as  $f : \mathbb{T} \rightarrow \mathbb{C}$  the continuous function defined by  $f(z) = z^{-1}$ . Then

$$I_{\nu_\mu} \tilde{f}(t) = \int_{\mathbb{T}} f(st^{-1})d\mu(s) = t \int_{\mathbb{T}} dm_{\mathbb{T}}(s) = t, \quad t \in \mathbb{T},$$

$$I_{\nu_\mu} f(t) = \int_{\mathbb{T}} f(ts^{-1})d\mu(s) = t^{-1} \int_{\mathbb{T}} s^2 dm_{\mathbb{T}}(s) = 0, \quad t \in \mathbb{T}.$$

Since  $f \in L^1(\nu_\mu)$ , it follows that  $\nu_\mu$  is not norm integral reflection invariant.

We have just seen that a vector measure  $\nu_\mu$  may not be norm integral reflection invariant. However, we have the following important result.

**Theorem 3.7** *Let  $1 \leq p < \infty$ . If  $f \in L^p_w(\nu_\mu)$ , then  $\tilde{f} \in L^1_w(\nu_\mu)$  and there exists  $k > 0$  such that*

$$\|\tilde{f}\|_{\nu_\mu} \leq k \|f\|_{\nu_\mu, p}, \quad f \in L^p_w(\nu). \tag{3.7}$$

Moreover, if  $f \in L^p(\nu_\mu)$ , then  $\tilde{f} \in L^1(\nu_\mu)$ .

*Proof* We will start by proving there is a positive real number  $C$  such that

$$\|I_{\nu_\mu} \tilde{\phi}\|_{L^p(G)} \leq C \|\phi\|_{\nu_\mu, p}, \quad \phi \in \mathfrak{S}(G). \tag{3.8}$$

Since the vector measure  $\nu_\mu$  is norm integral translation invariant, by Corollary 4.5 in [3] we have that  $L^p(\nu_\mu) \subseteq L^p(G)$  and  $\|f\|_{L^p(G)} \leq \|\nu_\mu\|(G)^{-1/p} \|f\|_{\nu_\mu, p}$ ,  $f \in L^p(\nu_\mu)$ . Take  $\phi \in \mathfrak{S}(G)$ . Then, by *Minkowski’s inequality* we find

$$\begin{aligned} \|I_{\nu_\mu} \tilde{\phi}\|_{L^p(G)} &\leq \left( \int_G \left( \int_G |\phi(s-t)|d|\mu|(s) \right)^p dm_G(t) \right)^{\frac{1}{p}} \\ &\leq \int_G \left( \int_G |\phi(s-t)|^p dm_G(t) \right)^{\frac{1}{p}} d|\mu|(s) = \|\phi\|_{L^p(G)} |\mu|(G) \\ &\leq C \|\phi\|_{\nu_\mu, p} \end{aligned}$$

with  $C = \|\nu_\mu\|(G)^{-1/p} |\mu|(G)$ . This establishes (3.8). Let us now consider the norm  $\|\cdot\|$  given in (3.2) and which is equivalent to that of  $L^1(\nu_\mu)$ . Then (3.8) gives

$$\begin{aligned} \|\tilde{\phi}\|_{\nu_\mu} \leq 4 \|\tilde{\phi}\| &= 4 \sup_{B \in \mathfrak{B}(G)} \left\| \int_B \tilde{\phi} d\nu_\mu \right\|_{L^p(G)} = 4 \sup_{B \in \mathfrak{B}(G)} \|I_{\nu_\mu}(\chi_B \tilde{\phi})\|_{L^p(G)} \\ &\leq 4C \|\chi_{-B} \phi\|_{\nu_\mu, p} \leq 4C \|\phi\|_{\nu_\mu, p}. \end{aligned}$$

Therefore

$$\|\tilde{\phi}\|_{\nu_\mu} \leq k \|\phi\|_{\nu_\mu, p}, \quad \phi \in \mathfrak{S}(G) \tag{3.9}$$

where  $k = 4C = 4\|v_\mu\|(G)^{-1/p}|\mu|(G)$ . Take  $B \in \mathcal{B}(G)$  such that  $\|v_\mu\|(B) = 0$ . Then, considering  $\phi = \chi_B$  in (3.9) it follows that  $\|v\|(-B) = 0$ . This shows that  $T_R f = \tilde{f}$  is a well defined linear operator on  $L^0(v_\mu)$ . Consider  $f \in L^p_w(v_\mu)$  and take  $(\phi_n)_n \subseteq \mathcal{S}(G)$  such that  $0 \leq \phi_n \nearrow |f|$ . Then  $0 \leq \tilde{\phi}_n \nearrow |\tilde{f}|$ . Since both  $L^p_w(v_\mu)$  and  $L^1_w(v_\mu)$  have the Fatou property, we have  $\|\phi_n\|_{v_\mu,p} \nearrow \|f\|_{v_\mu,p}$  and also  $\|\tilde{\phi}_n\|_{v_\mu} \nearrow \|\tilde{f}\|_{v_\mu}$ . On the other side, from (3.9) we obtain that  $\|\tilde{\phi}_n\|_{v_\mu} \leq k\|\phi_n\|_{v_\mu,p}$ ,  $n \in \mathbb{N}$ . Letting  $n \rightarrow \infty$ , we conclude that  $\tilde{f} \in L^1_w(v_\mu)$  and  $\|\tilde{f}\|_{v_\mu} \leq k\|f\|_{v_\mu,p}$ . Finally, let  $f \in L^p(v_\mu)$  and consider a sequence  $(\phi_n)_n \subseteq \mathcal{S}(G)$  such that  $\phi_n \rightarrow f$  in  $L^p(v_\mu)$ . From (3.7) it now follows that  $\tilde{\phi}_n \rightarrow \tilde{f}$  in  $L^1_w(v_\mu)$ . Since each  $\phi_n \in \mathcal{S}(G)$ , we have that  $\tilde{f} \in L^1(v_\mu)$ . □

The above result gives special relevance to the case  $p = 1$ . In this case we will obtain an alternative proof of the well-known result that  $L^1_w(v_\mu) = L^1(v_\mu) = L^1(G)$  (see [11, Proposition 7.35 and Remark 7.36]).

**Definition 3.8** Let  $k$  be a positive real number. A vector measure  $v$  is *k-scalarly bounded* by  $m_G$ , if  $|\langle v, x' \rangle|(A) \leq km_G(A)$  for all  $A \in \mathcal{B}(G)$  and  $x' \in B_{X'}$ .

In this situation, note that  $v \ll m_G$ .

**Theorem 3.9** Let  $v$  be a vector measure such that  $v \ll m_G$  and, given  $x' \in X'$ , take  $h_{x'}$  as in (2.2). Then, the following assertions are equivalent:

- (1)  $v$  is  $k$ -scalarly bounded by  $m_G$ ,
- (2)  $\|h_{x'}\|_{L^\infty(G)} \leq k$  for all  $x' \in B_{X'}$ ,
- (3)  $L^p(G) \subseteq L^p_w(v)$  with  $\|f\|_{v,p} \leq k^{1/p}\|f\|_{L^p(G)}$  for all  $f \in L^p(G)$  and  $1 \leq p < \infty$ .
- (4)  $L^1(G) \subseteq L^1_w(v)$  with  $\|f\|_v \leq k\|f\|_{L^1(G)}$  for all  $f \in L^1(G)$ .

*Proof* We start by proving that (1) implies (2). Given  $x' \in B_{X'}$  and  $A \in \mathcal{B}(G)$

$$\int_A |h_{x'}| dm_G = \int_A d|\langle v, x' \rangle| = |\langle v, x' \rangle|(A) \leq km_G(A).$$

It follows that  $h_{x'} \in L^\infty(G)$  and  $\|h_{x'}\|_{L^\infty(G)} \leq k$ . Assume now (2) and let us show (3). Given  $1 \leq p < \infty$  and  $f \in L^p(G)$  and  $x' \in B_{X'}$

$$\int_G |f|^p d|\langle v, x' \rangle| = \int_G |f|^p |h_{x'}| dm_G \leq k\|f\|_{L^p(G)}^p, \tag{3.10}$$

and (3) follows. Clearly (3) implies (4) so finally let us prove that (4) implies (1). Fixing  $x' \in B_{X'}$  one has

$$|\langle v, x' \rangle|(A) = \int_A d|\langle v, x' \rangle| \leq \|\chi_A\|_v \leq k\|\chi_A\|_{L^1(G)} = km_G(A),$$

so  $v$  is  $k$ -scalarly bounded by  $m_G$ . □

**Corollary 3.10**  $L_w^1(\nu_\mu) = L^1(\nu_\mu) = L^1(G)$ , with  $\|\cdot\|_{\nu_\mu}$  and  $\|\cdot\|_{L^1(G)}$  being equivalent norms.

*Proof* First, we remark that  $m_G$  and  $\|\nu_\mu\|$  have the same null sets ([3, Theorem 4.1]). Identifying  $x' \in B_{L^1(G)'} with  $g \in L^\infty(G)$ , it is easy to see that  $h_{x'} = \int_G g(\cdot + s)d\mu(s)$ . Therefore$

$$|h_{x'}| = \left| \int_G g(\cdot + s)d\mu(s) \right| \leq \|g\|_{L^\infty(G)}|\mu|(G) \leq |\mu|(G).$$

Hence  $\nu_\mu$  is  $k$ -scalarly bounded by  $m_G$ , with  $k = |\mu|(G)$ . It follows by Theorem 3.9(4) that  $L^1(G) \subseteq L_w^1(\nu_\mu)$  and  $\|f\|_{\nu_\mu} \leq |\mu|(G)\|f\|_{L^1(G)}$ ,  $f \in L^1(G)$ . On the other hand, [3, Corollary 4.5] indicates that  $L_w^1(\nu_\mu) \subseteq L^1(G)$  and  $\|f\|_{L^1(G)} \leq \|\nu_\mu\|(G)^{-1}\|f\|_{\nu_\mu}$ ,  $f \in L_w^1(\nu_\mu)$ . Finally, from  $L_w^1(\nu_\mu) = L^1(G)$  with equivalent norms, it follows that  $L_w^1(\nu_\mu)$  is order continuous. This implies that  $L_w^1(\nu_\mu) = L^1(\nu_\mu)$ .  $\square$

### 4 Convolution Associated to $\nu$

Let  $G$  be a compact Hausdorff abelian group,  $X$  a Banach space and  $\nu : \mathcal{B}(G) \rightarrow X$  a countably additive vector measure satisfying  $\nu \ll m_G$ . In this section we will then introduce a convolution operator and a vector-valued convolution function (both associated to  $\nu$ ) and relate them with the Fourier transform studied in Sect. 3. Given  $x' \in X'$  consider  $h_{x'}$  as in (2.2) and take  $f \in L^1(G)$  and  $g \in L_w^1(\nu)$ . Then  $gh_{x'} \in L^1(G)$  and so the classical *Young’s Inequality* implies that  $f * (gh_{x'}) \in L^1(G)$  and  $\|f * (gh_{x'})\|_{L^1(G)} \leq \|f\|_{L^1(G)}\|gh_{x'}\|_{L^1(G)} \leq \|f\|_{L^1(G)}\|g\|_\nu\|x'\|$ . This allows us to give the following definition.

**Definition 4.1** Let  $\nu$  be a vector measure such that  $\nu \ll m_G$ . For  $f \in L^1(G)$  and  $g \in L_w^1(\nu)$  we define the *convolution operator of  $f$  and  $g$  associated to  $\nu$* , as the linear and continuous map  $f *_\nu g : X' \rightarrow L^1(G)$  given by

$$f *_\nu g(x') = f * (gh_{x'}), \quad x' \in X'.$$

*Remark 4.2* Assume that the vector measure  $\nu$  is norm integral translation invariant and  $\nu \ll m_G$  and take  $1 \leq p < \infty$ . Then  $L_w^p(\nu) \subseteq L^1(G)$  (see [3, Sect. 3]). Therefore, in this situation Definition 4.1 makes also sense for  $f$  in  $L_w^p(\nu)$ .

**Theorem 4.3** Let  $\nu$  be a norm integral translation invariant vector measure such that  $\nu \ll m_G$  and  $f, g \in L^1(\nu)$ .

(a) Then for all  $\gamma \in \Gamma$  and  $x' \in X'$

$$f \widehat{*_\nu g}(x')(\gamma) = \langle \widehat{f}(\gamma)\widehat{g}^\nu(\gamma), x' \rangle. \tag{4.1}$$

(b) If, in addition,  $\nu$  is norm integral reflection invariant

$$\langle f *_\nu g(x'), \phi \rangle = \left\langle \int_G (\tilde{f} * \phi) g d\nu, x' \right\rangle, \quad \phi \in L^\infty(G), \quad x' \in X'. \quad (4.2)$$

*Proof*

(a) First of all note that for each  $x' \in X'$  and  $\gamma \in \Gamma$

$$\begin{aligned} \widehat{gh_{x'}}(\gamma) &= \int_G g(t) h_{x'}(t) \overline{\gamma(t)} dm_G(t) = \int_G g(t) \overline{\gamma(t)} d\langle \nu, x' \rangle(t) \\ &= \left\langle \int_G g(t) \overline{\gamma(t)} d\nu(t), x' \right\rangle = \langle \widehat{g}^\nu(\gamma), x' \rangle, \end{aligned}$$

hence

$$\begin{aligned} f *_\nu \widehat{g}(x')(\gamma) &= f * (\widehat{gh_{x'}})(\gamma) = \widehat{f}(\gamma) \widehat{gh_{x'}}(\gamma) = \widehat{f}(\gamma) \langle \widehat{g}^\nu(\gamma), x' \rangle \\ &= \langle \widehat{f}(\gamma) \widehat{g}^\nu(\gamma), x' \rangle. \end{aligned}$$

(b) Let us take  $x' \in X'$  and  $\phi \in L^\infty(G)$ . Using *Fubini's Theorem*

$$\begin{aligned} \langle f *_\nu g(x'), \phi \rangle &= \int_G f * (gh_{x'})(t) \phi(t) dm_G(t) \\ &= \int_G \left( \int_G f(t-s) g(s) d\langle \nu, x' \rangle(s) \right) \phi(t) dm_G(t) \\ &= \int_G \left( \int_G f(t-s) \phi(t) dm_G(t) \right) g(s) d\langle \nu, x' \rangle(s) \\ &= \int_G \tilde{f} * \phi(s) g(s) d\langle \nu, x' \rangle(s) = \left\langle \int_G (\tilde{f} * \phi) g d\nu, x' \right\rangle, \end{aligned}$$

where the last equality is a consequence of the following fact: since  $\tilde{f} \in L^1(\nu) \subseteq L^1(G)$  and  $\phi \in L^\infty(G)$  then the classical *Young's inequality* gives that  $\tilde{f} * \phi \in L^\infty(G)$ .  $\square$

**Corollary 4.4** *Let  $\nu$  be a norm integral translation and reflection invariant vector measure such that  $\nu \ll m_G$  and  $f, g \in L^1(\nu)$ . Then for all  $\gamma \in \Gamma$*

$$\int_G (\tilde{f} * \bar{\nu}) g d\nu = \widehat{f}(\gamma) \widehat{g}^\nu(\gamma).$$

*Proof* For  $\gamma \in \Gamma$  just take  $\phi = \bar{\nu}$  in (4.2) and use (4.1).  $\square$

Following the classical case one can define *the convolution of  $f$  and  $g$  with respect to  $\nu$*  in the following more natural way (for a similar definition in the setting of the bounded bilinear map see [1]).

**Definition 4.5** Let  $f, g \in L^0(G)$ . The convolution of  $f$  and  $g$  with respect to  $\nu$  is the vector-valued function  $f *^\nu g : G \rightarrow X$  given by

$$f *^\nu g(t) = \int_G f(t - s)g(s)d\nu(s), \quad t \in G,$$

provided that  $f(t - \cdot)g \in L^1(\nu)$  for  $m_G$ -almost all  $t \in G$ .

Clearly in the case when  $\nu \ll m_G$  we have that

$$f *^\nu g(x') = \langle f *^\nu g, x' \rangle, \tag{4.3}$$

for all  $x' \in X'$  and  $f \in L^1(G), g \in L^1_w(\nu)$  satisfying  $f(t - \cdot)g \in L^1(\nu)$  for almost all  $t \in G$ .

We give now some examples. Recall that given an order continuous Banach function space  $E(m_G)$  and a Banach space  $X$ , an operator  $T : E(m_G) \rightarrow X$  is said to be  $m_G$ -determined if the  $m_G$ -null sets and the  $\nu_T$ -null sets coincide (or equivalently if  $m_G$  is a control measure for  $\nu_T$  (see [11, Lemma 4.5])). This notion has been recently studied in [11, Chap. 4]. In particular multiplication operators, Fourier transform, translation operators and operators acting in  $L^p(G)$  via convolution with measures are  $m_G$ -determined (see respectively [11, Examples 4.7, 4.10, 4.17 and Remark 7.60]).

*Example 4.6* With the notation of Example 2.2 if we assume that  $T : E(m_G) \rightarrow X$  is  $m_G$ -determined, then we have that  $\nu_T \ll m_G$ ,

$$f *^{\nu_T} g(t) = \int_G \tau_t(\tilde{f})(s)g(s)d\nu_T(s) = T(\tau_t(\tilde{f})g), \quad t \in G, \tag{4.4}$$

and

$$f *_{\nu_T} g(x')(t) = \langle T(\tau_t(\tilde{f})g), x' \rangle, \quad x' \in X', t \in G, \tag{4.5}$$

provided that  $f \in L^1(G), g \in L^1_w(\nu_T)$  satisfy  $f(t - \cdot)g \in E(m_G) \subseteq L^1(\nu_T)$  for almost all  $t \in G$ . According to (4.4) and (4.5) we have the following particular cases.

- (a) Consider the vector measure  $\nu : \mathcal{B}(G) \rightarrow E(m_G)$  given by  $\nu(A) = \chi_A$  and introduced in (2) of Example 2.2. Then for all  $t \in G$  and  $x' \in E(m_G)'$  (identified with a function  $h \in E(m_G)^\times$ )

$$\begin{aligned} f *^\nu g(x')(t) &= f *^{\nu_{id}} g(x')(t) = \langle \tau_t(\tilde{f})g, x' \rangle \\ &= \int_G f(t - s)g(s)h(s)dm_G(s) = (f * (gh))(t), \end{aligned}$$

provided that  $f \in L^1(G), g \in L^1_w(\nu)$ . If in addition  $f(t - \cdot)g \in L^1(\nu)$  for almost all  $t \in G$  we obtain  $f *^\nu g(t) = f *^{\nu_{id}} g(t) = \tau_t(\tilde{f})g$ .



(b) If we consider the vector measure  $\nu_{\mathcal{F}}$  associated to the Fourier transform presented in (3) of Example 2.2 we have

$$f *^{\nu_{\mathcal{F}}} g(t) = \widehat{\tau_t(\tilde{f})}g \quad \text{and} \quad f *_{\nu_{\mathcal{F}}} g(x') = f * (g\widehat{y}),$$

for all  $t \in G$ ,  $f \in L^1(G)$  and  $g \in L^1_w(\nu_{\mathcal{F}})$  such that  $f(t - \cdot)g \in L^1(\nu_{\mathcal{F}})$  where  $x' \in (\ell^{p'}(\Gamma))'$  is identified with  $y \in \ell^p(\Gamma)$ .

(c) Take finally the vector measure  $\nu_{\mu} : \mathcal{B}(G) \rightarrow L^p(G)$  studied in (1) of Example 2.2. Then

$$f *^{\nu_{\mu}} g(t) = (\tau_t(\tilde{f})g) * \mu \quad \text{and} \quad f *_{\nu_{\mu}} g(x')(t) = \langle (\tau_t(\tilde{f})g) * \mu, h \rangle,$$

for all  $t \in G$ ,  $f \in L^1(G)$  and  $g \in L^1_w(\nu_{\mu})$  such that  $f(t - \cdot)g \in L^1(\nu_{\mu})$  and  $x' \in L^p(G)$  is identified with the function  $h \in L^{p'}(G)$ .

*Remark 4.7* In the previous example if the corresponding  $L^p(\nu)$  is both norm translation and reflection invariant, then given  $f \in L^p(\nu)$  we have that  $\tau_t(\tilde{f}) \in L^p(\nu)$  for all  $t \in G$ . Hence for all  $g \in L^{p'}(\nu)$  we obtain that  $f(t - \cdot)g \in L^1(\nu)$  for all  $t \in G$ .

In the following result we establish some integration properties for the vector-valued function  $f *^{\nu} g$ .

**Theorem 4.8** *Assume  $\nu \ll m_G$  and let  $f \in L^1(G)$ . If  $g \in L^1_w(\nu)$  is such that  $f(t - \cdot)g \in L^1(\nu)$  for almost all  $t \in G$ , then  $f *^{\nu} g$  is Dunford integrable and  $\|f *^{\nu} g\|_{L^1_w(G, X)} \leq \|f\|_{L^1(G)} \|g\|_{\nu}$ . Moreover, if  $g \in L^1(\nu)$  then  $f *^{\nu} g$  is Pettis integrable and (P)  $\int_G (f *^{\nu} g) dm_G = (\int_G f dm_G)(\int_G g d\nu)$ .*

*Proof* Take  $x' \in B_{X'}$  and let  $h_{x'}$  be as in (2.2). Since  $g \in L^1_w(\nu)$ , notice that  $gh_{x'} \in L^1(G)$ . Let  $\psi_{x'}$  be the ( $m_G$ -a.e. defined) function  $\psi_{x'}(t) = \langle f *^{\nu} g(t), x' \rangle, t \in G$ . Then

$$\psi_{x'}(t) = \int_G f(t - s)g(s)d\langle \nu, x' \rangle(s) = f * (gh_{x'})(t),$$

and so we conclude that  $\psi_{x'}$  is measurable. Next, using *Fubini's Theorem*

$$\begin{aligned} \int_G |\langle f *^{\nu} g(t), x' \rangle| dm_G(t) &= \int_G \left| \int_G f(t - s)g(s)d\langle \nu, x' \rangle(s) \right| dm_G(t) \\ &\leq \int_G \left( \int_G |f(t - s)g(s)| d|\langle \nu, x' \rangle|(s) \right) dm_G(t) \\ &= \int_G \left( \int_G |f(t - s)| dm_G(t) \right) |g(s)| d|\langle \nu, x' \rangle|(s) \\ &= \int_G \|\tau_s(f)\|_{L^1(G)} |g(s)| d|\langle \nu, x' \rangle|(s) \leq \|f\|_{L^1(G)} \|g\|_{\nu}. \end{aligned}$$

Hence,  $f *^{\nu} g$  is Dunford integrable and we obtain the inequality. For the second part fix  $x' \in X'$  and  $A \in \mathcal{B}(G)$ . Since the correspondence  $s \rightarrow \int_A f(t - s)dm_G(t)$  defines

a bounded measurable function, we have

$$x_A = \int_G \left( \int_A f(t-s) dm_G(t) \right) g(s) dv(s) \in X$$

and

$$\begin{aligned} \int_A \langle f *^v g, x' \rangle dm_G &= \int_A \left\langle \int_G f(t-s) g(s) dv(s), x' \right\rangle dm_G(t) \\ &= \int_G \left( \int_A f(t-s) dm_G(t) \right) g(s) d\langle v, x' \rangle(s) = \langle x_A, x' \rangle. \quad \square \end{aligned}$$

**Proposition 4.9** *Let  $\nu$  be a norm integral translation invariant vector measure such that  $\nu \ll m_G$  and  $f, g \in L^1(\nu)$ . If  $(\phi_n)_n \subseteq \mathfrak{S}(G)$  converges to  $f$  in  $L^1(\nu)$ , then for each  $x' \in X'$  the sequence  $(\langle \phi_n *^v g, x' \rangle)_n$  converges to  $f *^v g(x')$  in  $L^1(G)$ .*

*Proof* Fix  $x' \in X'$ . Using Fubini's Theorem

$$\begin{aligned} &\| \langle \phi_n *^v g, x' \rangle - f *^v g(x') \|_{L^1(G)} \\ &= \int_G | \langle \phi_n *^v g(t), x' \rangle - f *^v g(x')(t) | dm_G(t) \\ &= \int_G \left| \int_G (\phi_n(t-s) - f(t-s)) g(s) d\langle \nu, x' \rangle(s) \right| dm_G(t) \\ &= \int_G \left( \int_G | \phi_n(t-s) - f(t-s) | dm_G(t) \right) |g(s)| d|\langle \nu, x' \rangle|(s) \\ &\leq \| \phi_n - f \|_{L^1(G)} \|g\|_\nu. \quad \square \end{aligned}$$

In this last part of the paper we present various *Young's Theorems* in our setting.

**Theorem 4.10** *Let  $\nu$  be a vector measure such that  $\nu \ll m_G$ ,  $1 \leq p < \infty$  and  $g \in L^1_w(\nu)$ .*

- (a) *Then  $f *^v g \in L(X', L^p(G))$  and  $\|f *^v g\|_{L(X', L^p(G))} \leq \|f\|_{L^p(G)} \|g\|_\nu$ , for every  $f \in L^p(G)$ .*
- (b) *Assume that  $\nu$  is also norm integral translation invariant. If  $f \in L^p_w(\nu)$ , then  $f *^v g \in L(X', L^p_w(\nu))$  and  $\|f *^v g\|_{L(X', L^p_w(\nu))} \leq \|f\|_{L^p_w(\nu)} \|g\|_\nu$ . Moreover, if  $f \in L^p(\nu)$  and  $g \in L^1(\nu)$  then  $f *^v g \in L(X', L^p(\nu))$ .*

*Proof* (a) Take  $x' \in B_{X'}$ . Using Minkowski's Inequality we have that

$$\begin{aligned} &\left( \int_G |f *^v g(x')(t)|^p m_G(t) \right)^{1/p} \\ &= \left( \int_G \left| \int_G f(t-s) g(s) d\langle \nu, x' \rangle(s) \right|^p dm_G(t) \right)^{1/p} \end{aligned}$$

$$\begin{aligned} &\leq \left( \int_G \left( \int_G |f(t-s)| |g(s)| d\langle \nu, x' \rangle(s) \right)^p dm_G(t) \right)^{1/p} \\ &\leq \int_G \left( \int_G |f(t-s)|^p dm_G(t) \right)^{1/p} |g(s)| d\langle \nu, x' \rangle(s) \\ &= \int_G \|\tau_s(f)\|_{L^p(G)} |g(s)| d\langle \nu, x' \rangle(s) \leq \|f\|_{L^p(G)} \|g\|_\nu. \end{aligned}$$

Hence  $f *_\nu g(x') \in L^p(G)$  for all  $x' \in B_{X'}$  and the inequality follows.

(b) Fix now  $x' \in B_{X'}$  and let us show that  $f *_\nu g(x') \in L^p_w(\nu)$ . Take  $z' \in B_{X'}$ . As in the previous case we use *Minkowski's Inequality* together with the fact that  $\nu$  is norm integral translation invariant to obtain

$$\begin{aligned} &\left( \int_G |f *_\nu g(x')(s)|^p d\langle \nu, z' \rangle(s) \right)^{1/p} \\ &\leq \int_G \left( \int_G |f(t-s)|^p d\langle \nu, z' \rangle(t) \right)^{1/p} |g(s)| d\langle \nu, x' \rangle(s) \\ &\leq \int_G \|\tau_s(f)\|_{\nu,p} |g(s)| d\langle \nu, x' \rangle(s) \leq \|f\|_{\nu,p} \|g\|_\nu. \end{aligned}$$

Hence  $f *_\nu g(x') \in L^p_w(\nu)$  for all  $x' \in B_{X'}$  and the inequality follows. Assume now that  $f \in L^p(\nu)$  and  $g \in L^1(\nu)$  and take  $(\phi_n)_n$  and  $(\psi_n)_n$ , sequences of simple functions converging to  $f$  and  $g$  in  $L^p(\nu)$  and  $L^1(\nu)$  respectively. Clearly  $\phi_n \in L^\infty(G)$  and  $\psi_n h_{x'} \in L^1(G)$ , so  $\phi_n *_\nu \psi_n(x') = \phi_n * (\psi_n h_{x'})$  is a bounded function and, in particular, belongs to  $L^p(\nu)$ . On the other hand since  $f *_\nu g(x') \in L^p_w(\nu)$

$$\begin{aligned} &\|f *_\nu g(x') - \phi_n *_\nu \psi_n(x')\|_{\nu,p} \\ &\leq \|f *_\nu (g - \psi_n)(x')\|_{\nu,p} + \|(f - \phi_n) *_\nu \psi_n(x')\|_{\nu,p} \\ &\leq \|f\|_{\nu,p} \|g - \psi_n\|_\nu \|x'\| + \|(f - \phi_n)\|_{\nu,p} \|\psi_n\|_\nu \|x'\| \rightarrow 0, \end{aligned}$$

then  $f *_\nu g(x') \in L^p(\nu)$ , since  $L^p(\nu)$  is closed in  $L^p_w(\nu)$ . Hence  $f *_\nu g \in L(X', L^p(\nu))$ . □

Assuming that  $\nu$  is not only norm integral translation invariant but also reflection invariant we obtain a natural extension of Theorem 4.10(b).

**Corollary 4.11** *Let  $\nu$  be a norm integral translation and reflection invariant vector measure such that  $\nu \ll m_G$  and  $1 \leq p < \infty$ . If  $f \in L^p_w(\nu)$  and  $g \in L^q_w(\nu)$  with  $1 \leq q \leq p'$  then  $f *_\nu g \in L(X', L^r_w(\nu))$  with  $1/p + 1/q = 1 + 1/r$  and  $\|f *_\nu g\|_{L(X', L^r_w(\nu))} \leq \|f\|_{\nu,p} \|g\|_{\nu,q}$ . Moreover if  $f \in L^p(\nu)$  and  $g \in L^q(\nu)$  then  $f *_\nu g \in L(X', L^r(\nu))$ .*

*Proof* Let  $x' \in B_{X'}$ . By Theorem 4.10(b) the linear map  $T_{f,x'}(g) = f *_\nu g(x')$  is a bounded application from  $L^1_w(\nu)$  into  $L^p_w(\nu)$  and  $\|T_{f,x'}\|_{L(L^1_w(\nu), L^p_w(\nu))} \leq \|f\|_{\nu,p}$ . On the other hand, since  $\nu$  is norm integral translation and reflection invariant, given

$g \in L_w^{p'}(\nu)$  then Hölder’s Inequality gives that for all  $t \in G$

$$\begin{aligned} |T_{f,x'}(g)(t)| &\leq \int_G |\tau_t(\tilde{f})(s)| |g(s)| d|\langle \nu, x' \rangle|(s) \\ &\leq \left( \int_G |\tau_t(\tilde{f})(s)|^p d|\langle \nu, x' \rangle|(s) \right)^{1/p} \left( \int_G |g(s)|^{p'} d|\langle \nu, x' \rangle|(s) \right)^{1/p'} \\ &\leq \|\tau_t(\tilde{f})\|_{v,p} \|g\|_{v,p'} \leq \|f\|_{v,p} \|g\|_{v,p'}. \end{aligned}$$

Hence,  $T_{f,x'}$  is also a bounded linear map from  $L_w^{p'}(\nu)$  into  $L^\infty(\nu) = L_w^\infty(\nu)$  and  $\|T_{f,x'}\|_{L(L_w^{p'}(\nu), L^\infty(\nu))} \leq \|f\|_{v,p}$ . Then using a standard interpolation argument (see [6, Theorem 3.4]) the map  $T_{f,x'}$  is bounded from  $L_w^q(\nu)$  into  $L_w^r(\nu)$ , where  $1 \leq q \leq p'$  and  $1/p + 1/q = 1 + 1/r$ , and we have

$$\begin{aligned} \|T_{f,x'}\|_{L(L_w^q(\nu), L_w^r(\nu))} &\leq \|T_{f,x'}\|_{L(L_w^1(\nu), L_w^p(\nu))}^{1-p/r} \|T_{f,x'}\|_{L(L_w^{p'}(\nu), L^\infty(\nu))}^{p/r} \\ &\leq \|f\|_{v,p}^{1-p/r} \|f\|_{v,p}^{p/r} = \|f\|_{v,p}. \end{aligned}$$

Therefore  $\|f *_\nu g(x')\|_{L_w^r(\nu)} = \|T_{f,x'}(g)\|_{L_w^r(\nu)} \leq \|f\|_{v,p} \|g\|_{v,q}$ , so  $f *_\nu g(x') \in L_w^r(\nu)$  and the inequality follows. For the second part, again using Theorem 4.10(b) the map  $T_{f,x'}$  is now bounded from  $L^1(\nu)$  into  $L^p(\nu)$ . But since  $L^\infty(\nu) = L_w^\infty(\nu)$  then  $T_{f,x'}$  is also bounded from  $L^{p'}(\nu)$  into  $L^\infty(\nu)$  and the same interpolation theorem gives the result.  $\square$

We finish this paper by giving a generalization of Theorem 4.8.

**Corollary 4.12** *Let  $\nu$  be a norm integral translation invariant vector measure such that  $\nu \ll m_G$  and  $1 \leq p < \infty$ . Let  $f \in L_w^p(\nu)$  and  $g \in L_w^1(\nu)$  be such that  $f(t - \cdot)g \in L^1(\nu)$ . Then  $f *_\nu g$  is Dunford  $p$ -integrable and  $\|f *_\nu g\|_{L_w^p(G, X)} \leq \|\nu\|(G)^{-1/p} \|f\|_{v,p} \|g\|_\nu$ . Moreover if  $f \in L^p(\nu)$  and  $g \in L^1(\nu)$  then  $f *_\nu g$  is Pettis  $p$ -integrable.*

*Proof* For the first part just apply [3, Corollary 4.5], (4.3) and Theorem 4.10(b) to obtain, for each  $x' \in X'$

$$\begin{aligned} \|\langle f *_\nu g, x' \rangle\|_{L^p(G)} &\leq \|\nu\|(G)^{-1/p} \|\langle f *_\nu g, x' \rangle\|_{v,p} \leq \|\nu\|(G)^{-1/p} \|f *_\nu g(x')\|_{v,p} \\ &\leq \|\nu\|(G)^{-1/p} \|f\|_{v,p} \|g\|_\nu. \end{aligned}$$

Hence  $f *_\nu g$  is Dunford  $p$ -integrable and the inequality follows. The second part is also a direct consequence of Theorem 4.10(b).  $\square$

With the same proof but applying now Corollary 4.11 we have the following final result.

**Corollary 4.13** *Let  $\nu$  be a norm integral translation and reflection invariant vector measure such that  $\nu \ll m_G$  and  $1 \leq p < \infty$ . Let  $f \in L_w^p(\nu)$  and  $g \in L_w^q(\nu)$  with  $1 \leq q \leq p'$  such that  $f(t - \cdot)g \in L^1(\nu)$ . Then  $f *_\nu g$  is Dunford  $r$ -integrable with  $1/p + 1/q = 1 + 1/r$  and  $\|f *_\nu g\|_{L_w^r(G, X)} \leq \|\nu\|(G)^{-1/r} \|f\|_{\nu, p} \|g\|_{\nu, q}$ . Moreover if  $f \in L^p(\nu)$  and  $g \in L^q(\nu)$  then  $f *_\nu g$  is Pettis  $r$ -integrable.*

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