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On Fourier Transforms of Radial Functions and Distributions

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Abstract We find a formula that relates the Fourier transform of a radial function on $\mathbf{R}^n$ with the Fourier transform of the same function defined on $\mathbf{R}^{n+2}$. This formula enables one to explicitly calculate the Fourier transform of any radial function $f(r)$ in any dimension, provided one knows the Fourier transform of the one-dimensional function $t \mapsto f(|t|)$ and the two-dimensional function $(x_1, x_2) \mapsto f(|(x_1, x_2)|)$. We prove analogous results for radial tempered distributions.

Keywords Radial Fourier transform · Hankel transform

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1 Introduction

The Fourier transform of a function Φ in $L^1(\mathbf{R}^n)$ is defined by the convergent integral

$$F_n(\Phi)(\xi) = \int_{\mathbf{R}^n} \Phi(x) e^{-2\pi i x\cdot\xi}\, dx\,.$$

If the function Φ is radial, i.e., $\Phi(x) = \varphi(|x|)$ for some function φ on the line, then its Fourier transform is also radial and we use the notation

$$F_n(\Phi)(\xi) = \mathcal{F}_n(\varphi)(r),$$

where $r = |\xi|$. In this article, we will show that there is a relationship between $\mathcal{F}_n(\varphi)(r)$ and $\mathcal{F}_{n+2}(\varphi)(r)$ as functions of the positive real variable r.

We have the following result.

Theorem 1.1 *Let $n \geq 1$. Suppose that f is a function on the real line such that the functions $f(|\cdot|)$ are in $L^1(\mathbf{R}^{n+2})$ and also in $L^1(\mathbf{R}^n)$. Then we have*

$$\mathcal{F}_{n+2}(f)(r) = -\frac{1}{2\pi}\frac{1}{r}\frac{d}{dr}\mathcal{F}_n(f)(r), \quad r > 0. \tag{1}$$

Moreover, the following formula is valid for all even Schwartz functions φ on the real line:

$$\mathcal{F}_{n+2}(\varphi)(r) = \frac{1}{2\pi}\frac{1}{r^2}\mathcal{F}_n\left(s^{-n+1}\frac{d}{ds}\big(\varphi(s)s^n\big)\right)(r), \quad r > 0. \tag{2}$$

Using the fact that the Fourier transform is a unitary operator on $L^2(\mathbf{R}^n)$ we may extend (1) to the case where the functions $f(|\cdot|)$ are in $L^2(\mathbf{R}^{n+2})$ and in $L^2(\mathbf{R}^n)$. Moreover, in Sect. 4 we extend (1) to tempered distributions. Applications are given in the last section.

Corollary 1.2 *Let $f(r)$ be a function on $[0,\infty)$ and k some positive integer such the functions $x \to f(|x|)$ are absolutely integrable over $\mathbf{R}^n$ for all n with $1 \leq n \leq 2k+2$. Then we have*

$$\mathcal{F}_{2k+1}(f)(\rho) = \frac{1}{(2\pi)^k}\sum_{\ell=1}^{k}\frac{(-1)^\ell(2k-\ell-1)!}{2^{k-\ell}(k-\ell)!(\ell-1)!}\frac{1}{\rho^{2k-\ell}}\left(\frac{d}{d\rho}\right)^\ell \mathcal{F}_1(f)(\rho)$$

and

$$\mathcal{F}_{2k+2}(f)(\rho) = \frac{1}{(2\pi)^k}\sum_{\ell=1}^{k}\frac{(-1)^\ell(2k-\ell-1)!}{2^{k-\ell}(k-\ell)!(\ell-1)!}\frac{1}{\rho^{2k-\ell}}\left(\frac{d}{d\rho}\right)^\ell \mathcal{F}_2(f)(\rho).$$

The corollary can be obtained using (1) by induction on k. The simple details are omitted. Again, absolute integrability can be replaced by square integrability.

2 The Proof

The Fourier transform of an integrable radial function $f(|x|)$ on $\mathbf{R}^n$ is given by

$$\mathcal{F}_n(f)\big(|\xi|\big) = 2\pi \int_0^\infty f(s)\left(\frac{s}{|\xi|}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}\big(2\pi s|\xi|\big)\, s\, ds$$
$$= (2\pi)^{\frac{n}{2}} \int_0^\infty f(s)\widetilde{J}_{\frac{n}{2}-1}\big(2\pi s|\xi|\big)\, s^{n-1}\, ds,$$

where $\widetilde{J}_\nu(x) = x^{-\nu} J_\nu(x)$, and J_ν is the classical Bessel function of order ν. This formula can be found in many textbooks, and we refer to, e.g., [3, Sect. B.5] or [10, Sect. IV.1] for a proof. Moreover, this formula makes sense for all integers $n \ge 1$, even $n = 1$, in which case

$$J_{-1/2}(t) = \sqrt{\frac{2}{\pi}}\frac{\cos t}{\sqrt{t}}.$$

Let us set

$$\mathcal{H}_{\frac{n}{2}-1}(f)(r) = (2\pi)^{\frac{n}{2}} \int_0^\infty f(s)\widetilde{J}_{\frac{n}{2}-1}(2\pi sr)\, s^{n-1}\, ds.$$

Then we make use of B.2.(1) in [3], i.e., the identity

$$\frac{d}{dr}\widetilde{J}_\nu(r) = -r\widetilde{J}_{\nu+1}(r), \tag{3}$$

which is also valid when $\nu = -1/2$, since

$$J_{1/2}(t) = \sqrt{\frac{2}{\pi}}\frac{\sin t}{\sqrt{t}}.$$

In view of (3), it is straightforward to verify that

$$-\frac{1}{r}\frac{d}{dr}\mathcal{H}_{\frac{n}{2}-1}(f)(r) = 2\pi\mathcal{H}_{\frac{n}{2}}(f)(r) = 2\pi\mathcal{H}_{\frac{n+2}{2}-1}(f)(r),$$

provided f is such that interchanging differentiation with the integral defining $\mathcal{H}_{\frac{n}{2}-1}$ is permissible. For this to happen, we need to have that

$$\int_0^\infty \big|f(s)\big|\left|\frac{d}{dr}\big(\widetilde{J}_{\frac{n}{2}-1}(rs)\big)\right| s^{n-1}\, ds < \infty$$

and thus it will be sufficient to have

$$\int_0^\infty \big|f(s)\big| rs^2 \big|\widetilde{J}_{\frac{n}{2}}(rs)\big| s^{n-1}\, ds \le c\int_0^\infty \big|f(s)\big|\frac{rs^2}{(1+rs)^{\frac{n+1}{2}}} s^{n-1} ds < \infty \tag{4}$$

since $|\widetilde{J}_{\frac{n}{2}}(s)| \le c(1+s)^{-n/2-1/2}$. But since $f(|\cdot|)$ is in $L^1(\mathbf{R}^{n+2})$ we have

$$\int_0^{1/r} \big|f(s)\big| s^{n+1}\, ds + \int_{1/r}^\infty \big|f(s)\big| s^{\frac{n+1}{2}}\, ds < \infty \tag{5}$$

and this certainly implies (4) for all $r > 0$. We conclude (1) whenever (5) holds. We note that the appearance of condition (5) is natural as indicated in [8] (Lemma 25.1).

To prove (2) we argue as follows. We have

$$\mathcal{H}_{\frac{n}{2}-1}\left(r^{-n+1}\frac{d}{dr}\big(\varphi(r)r^n\big)\right)(r) = (2\pi)^{\frac{n}{2}}\int_0^\infty \frac{d}{ds}\big(\varphi(s)s^n\big)\widetilde{J}_{\frac{n}{2}-1}(2\pi sr)\,ds$$

and integrating by parts the preceding expression becomes

$$(2\pi)^{\frac{n}{2}+2}\int_0^\infty \varphi(s)s^n sr^2\widetilde{J}_{\frac{n+2}{2}-1}(2\pi sr)\,ds$$

which is equal to $2\pi r^2\mathcal{H}_{\frac{n+2}{2}-1}(\varphi)(r)$. This proves (2).

Remark 2.1 Note that we have

$$\mathcal{H}_\nu(f)(r) = \frac{2\pi}{r^\nu}H_\nu\big(f(s)s^\nu\big)(2\pi r),$$

where

$$H_\nu(f)(r) = \int_0^\infty f(s)J_\nu(rs)s\,ds, \quad \nu \geq -\frac{1}{2},$$

is the Hankel transform. This of course ties in with the fact that the Hankel transform also arises naturally as the spectral transformation associated with the radial part of the Laplacian $-\Delta$; we refer to [4, Sect. 5] and the references therein for further information. Moreover, note that [6] contains the associated recursion from Theorem 1.1 for the Hankel transform, but only for even Schwartz functions. This recursion was rediscovered in connection with the radial Fourier transform in [9] for the case of Schwartz functions. See also [5] for related results.

A transference theorem for radial multipliers which exploits the connection between the Fourier transform of radial functions on $\mathbf{R}^n$ and $\mathbf{R}^{n+2}$ was obtained in [1]. This multiplier theorem is based on an identity dual to (3).

3 Radial Distributions

We denote by $\mathcal{S}(\mathbf{R}^n)$ the space of Schwartz functions on $\mathbf{R}^n$ and by $\mathcal{S}'(\mathbf{R}^n)$ the space of tempered distributions on $\mathbf{R}^n$. A Schwartz function is called radial if for all orthogonal transformations $A \in O(n)$ (that is, for all rotations on $\mathbf{R}^n$) we have

$$\varphi = \varphi \circ A.$$

We denote the set of all radial Schwartz functions by $\mathcal{S}_{rad}(\mathbf{R}^n)$. For further background on radial distributions we refer to Treves [13, Lect. 5]. Observe that in the one-dimensional case the radial Schwartz functions are precisely the even Schwartz functions, that is:

$$\mathcal{S}_{rad}(\mathbf{R}) = \mathcal{S}_{even}(\mathbf{R}) = \big\{\varphi \in \mathcal{S}(\mathbf{R}) : \varphi(x) = \varphi(-x)\big\}.$$

Similarly, a distribution $u \in \mathcal{S}'(\mathbf{R}^n)$ is called radial if for all orthogonal transformations $A \in O(n)$ we have

$$u = u \circ A.$$

This means that

$$\langle u, \varphi \rangle = \langle u, \varphi \circ A \rangle$$

for all Schwartz functions φ on $\mathbf{R}^n$. We denote by $\mathcal{S}'_{rad}(\mathbf{R}^n)$ the space of all radial tempered distributions on $\mathbf{R}^n$. We also denote by $\mathbf{S}^{n-1}$ the $(n-1)$-dimensional unit sphere on $\mathbf{R}^n$ and by ω_{n-1} its surface area.

Given a general, non necessarily radial, Schwartz function there is a natural homomorphism

$$\mathcal{S}\big(\mathbf{R}^n\big) \to \mathcal{S}_{rad}(\mathbf{R}), \qquad \varphi(x) \mapsto \varphi^o(r) = \frac{1}{\omega_{n-1}} \int_{\mathbf{S}^{n-1}} \varphi(r\theta)\, d\theta$$

with the understanding that when $n = 1$, then $\varphi^o(x) = \frac{1}{2}(\varphi(x) + \varphi(-x))$. Conversely, given an even Schwartz function on $\mathbf{R}$ we can define a corresponding radial Schwartz function via

$$\mathcal{S}_{rad}(\mathbf{R}) \to \mathcal{S}_{rad}\big(\mathbf{R}^n\big), \qquad \varphi(r) \mapsto \varphi^O(x) = \varphi\big(|x|\big).$$

The map $\varphi \mapsto \varphi^O$ is a homomorphism; the proof of this fact is omitted since a stronger statement is proved at the end of this section. Both facts require the following lemma:

Lemma 3.1 *Suppose that f is a smooth even function on* $\mathbf{R}$. *Then there is a smooth function g on the real line such that*

$$f(x) = g\big(x^2\big)$$

for all $x \in \mathbf{R}$. Moreover, one has for $t \geq 0$

$$\big|g^{(k)}(t)\big| \leq C(k) \sup_{0 \leq s \leq \sqrt{t}} \big|f^{(2k)}(s)\big|. \tag{6}$$

Proof By Whitney's theorem [14], there is a smooth function g on the real line such that

$$f(t) = g\big(t^2\big)$$

for all real t.

To see the last assertion we use the following representation of the remainder in Taylor's theorem:

$$\begin{aligned}\frac{g^{(k)}(t^2)}{k!} &= (2t)^{-2k+1} k \binom{2k}{k} \int_0^t \big(t^2 - s^2\big)^{k-1} \frac{f^{(2k)}(s)}{(2k)!}\, ds \\ &= 2^{-2k} k \binom{2k}{k} \int_0^1 \big(1 - s^2\big)^{k-1} \frac{f^{(2k)}(st)}{(2k)!}\, ds\end{aligned}$$

from which one easily derives (6). This yields in particular that

$$\frac{g^{(k)}(0)}{k!} = \frac{f^{(2k)}(0)}{(2k)!}$$

since

$$2^{-2k} k \binom{2k}{k} \int_0^1 \left(1 - s^2\right)^{k-1} ds = 2^{-2k} k \binom{2k}{k} \frac{\Gamma(k)\Gamma(1/2)}{\Gamma(k+1/2)} = 1.$$

□

The composition $\varphi \mapsto (\varphi^o)^O = \varphi^{rad}$ gives rise to a homomorphism from $\mathcal{S}(\mathbf{R}^n) \to \mathcal{S}_{rad}(\mathbf{R}^n)$ which reduces to the identity map on radial Schwarz functions. In particular, the map $\varphi \mapsto \varphi^o$ defines a one-to-one correspondence between radial Schwartz functions on $\mathbf{R}^n$ and even Schwartz functions on the real line. Moreover, φ is radial if and only if $\varphi = \varphi^{rad}$.

Proposition 3.2 *For $u \in \mathcal{S}'_{rad}(\mathbf{R}^n)$ and $\varphi \in \mathcal{S}(\mathbf{R}^n)$ we have*

$$\langle u, \varphi \rangle = \left\langle u, \varphi^{rad} \right\rangle.$$

Proof By a simple change of variables the formula holds for any u which is a polynomially bounded locally integrable function. Next we fix a tempered distribution u on $\mathbf{R}^n$ and we consider a radial Schwartz function ψ with integral 1 and we set $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$. Then we notice that the convolution of $\psi_\varepsilon * u$ converges to u in $\mathcal{S}'(\mathbf{R}^n)$ as $\varepsilon \to 0$. Hence, since the claim holds if u is replaced by $\psi_\varepsilon * u$ by the first observation, it remains true in the limit $\varepsilon \to 0$. □

In particular, note that a radial distribution is uniquely determined by its action on radial Schwartz functions. Furthermore, given a distribution $u \in \mathcal{S}'(\mathbf{R}^n)$ we can define a radial distribution $u^{rad} \in \mathcal{S}'_{rad}(\mathbf{R}^n)$ via

$$\left\langle u^{rad}, \varphi \right\rangle := \left\langle u, \varphi^{rad} \right\rangle.$$

Moreover, u is radial if and only if $u = u^{rad}$.

For $n \in \mathbf{Z}^+$ we denote by $\mathcal{R}_n = r^{n-1}\mathcal{S}_{even}(\mathbf{R})$ the space of functions of the form $\psi(r)r^{n-1}$, where ψ is an even Schwartz function on the line. This space inherits the topology of $S(\mathbf{R})$ and its dual space is denoted by $\mathcal{R}'_n$. Two distributions $w_1, w_2 \in \mathcal{S}'(\mathbf{R})$ are equal in the space $\mathcal{R}'_n$ if for all even Schwartz functions ψ on the line we have:

$$\left\langle w_1, r^{n-1}\psi(r) \right\rangle = \left\langle w_2, r^{n-1}\psi(r) \right\rangle.$$

Note that in dimension $n \geq 2$ we have that all distributions of order $n-2$ supported at the origin equal the zero distribution in the space $\mathcal{R}'_n$. Thus two radial distributions w_1 and w_2 are equal in $\mathcal{R}'_n$ whenever $w_1 - w_2$ is a sum of derivatives of the Dirac mass at the origin of order at most $n-2$.

One may build radial distributions on $\mathbf{R}^n$ starting from distributions in $\mathcal{R}'_n$. Indeed, given $u_\diamond$ in $\mathcal{R}'_n$ and φ in $\mathcal{S}(\mathbf{R}^n)$ we define a radial distribution u by setting

$$\langle u, \varphi\rangle := \frac{\omega_{n-1}}{2}\big\langle u_\diamond, \varphi^o(r)r^{n-1}\big\rangle.$$

The converse is the content of the following proposition.

Proposition 3.3 *The map $\mathcal{R}_n \to \mathcal{S}_{rad}(\mathbf{R}^n)$, $\psi(r)r^{n-1} \mapsto \psi^O(x)$ is a homeomorphism and hence for every radial distribution u we can define $u_\diamond$ in $\mathcal{R}'_n$ via*

$$\big\langle u_\diamond, \psi(r)r^{n-1}\big\rangle := \frac{2}{\omega_{n-1}}\big\langle u, \psi^O\big\rangle.$$

Proof It suffices to show the first claim. To this end we will show that for all multi-indices α and β we have

$$\sup_{x\in\mathbf{R}^n}\big|x^\alpha\partial_x^\beta\big(\psi\big(|x|\big)\big)\big| \le \sum_{0\le\ell,m\le 4(|\beta|+|\alpha|+n)}\sup_{r>0}\left|r^m\left(\frac{d}{dr}\right)^\ell\big(r^{n-1}\psi(r)\big)\right|.$$

First we consider the case $|x|\le 1$. Setting $r=|x|\le 1$ we have

$$\begin{aligned}\big|x^\alpha\partial_x^\beta\big(\psi\big(|x|\big)\big)\big| &\le C_\beta|x|^{|\alpha|}\sum_{k=0}^{|\beta|}|x|^k\big|g^{(k)}\big(|x|^2\big)\big| = C_\beta\sum_{k=0}^{|\beta|}\big|r^{k+|\alpha|}g^{(k)}\big(r^2\big)\big|\\ &\le C_\beta\sum_{k=0}^{|\beta|}\big|g^{(k)}\big(r^2\big)\big| \le C_\beta\sum_{k=0}^{|\beta|}C(k)\sup_{0<s<r}\big|\psi^{(2k)}(s)\big|,\end{aligned}$$

using Lemma 3.1 with $\psi(t)=g(t^2)$.

We will make use of the inequality

$$\big|\psi(s)\big| \le \sup_{0<t<s}\left|\left(\frac{d}{dt}\right)^M\big(t^M\psi(t)\big)(s)\right| \tag{7}$$

which follows by applying the fundamental theorem of calculus M times and of the identity:

$$s^M\frac{d^m\psi}{ds^m}(s) = \sum_{\ell=0}^{m}(-1)^\ell\ell!\binom{m}{\ell}\binom{M}{\ell}\left(\frac{d}{ds}\right)^{m-\ell}\big(s^{M-\ell}\psi(s)\big) \tag{8}$$

which is valid for $M\ge m$ and is easily proved by induction.

Applying (7) to $\psi^{(2k)}(s)$ we obtain

$$\big|\psi^{(2k)}(s)\big| \le \sup_{0<t<s}\left|\left(\frac{d}{dt}\right)^M\big(t^M\psi^{(2k)}(t)\big)(s)\right| \tag{9}$$

and using (8) for $s^M\psi^{(2k)}(s)$ with $M = 2|\beta| + n - 1$ and $m = 2k$ we deduce that $|\psi^{(2k)}(s)|$ is pointwise bounded by a sum of derivatives of terms $s^{n-1}\psi(s)$ multiplied by powers of s. It follows that $\sup_{s>0}|\psi^{(2k)}(s)|$ is controlled by a finite sum of Schwartz seminorms of the function $s^{n-1}\psi(s)$.

The case $|x| \geq 1$ is easier since when $|\beta| \neq 0$

$$\left|\partial_x^\beta\big(\psi\big(|x|\big)\big)\right| \leq \sum_{j=1}^{|\beta|}\left|\psi(j)\big(|x|\big)\right|\frac{C_{j,\beta}}{|x|^{|\beta|-j}},$$

and taking $M = \max(|\alpha|, |\beta| + n - 1)$ we have

$$\sup_{|x|\geq 1}\left|x^\alpha\partial_x^\beta\big(\psi\big(|x|\big)\big)\right| \leq C_\beta\sum_{j=1}^{|\beta|}\sup_{s\geq 1}\left\{s^M\left|\psi^{(j)}(s)\right|\right\}, \tag{10}$$

which is certainly controlled by a finite sum of Schwartz seminorms of $s^{n-1}\psi(s)$ in view of (8). □

Note that if u is given by a function $f(x)$, then $u_\diamond$ is given by the function $f^o(x)$. We also remark that the map $\frac{1}{r}\frac{d}{dr}$ is a homomorphism from $\mathcal{R}'_n$ to $\mathcal{R}'_{n+1}$ defined as the dual map of $-\frac{d}{dr}\frac{1}{r}$.

A related approach defining $u_\diamond$ for a given distribution u supported in $\mathbf{R}^n \setminus \{0\}$ can be found in [11]. Our approach does not impose restrictions on the support of the distribution.

4 The Extension to Tempered Distributions

Let u be a radial distribution on $\mathbf{R}^k$ and let $F_k(u)$ be the k-dimensional Fourier transform of u.

Theorem 4.1 *Given an even tempered distribution v_0 on the real line, define radial distributions v_n on $\mathbf{R}^n$ and v_{n+2} on $\mathbf{R}^{n+2}$ via the identities*

$$\langle v_n, \varphi\rangle = \left\langle v_0, \frac{1}{2}\omega_{n-1}r^{n-1}\varphi^o\right\rangle \tag{11}$$

for all radial Schwartz functions $\varphi(x) = \varphi^o(|x|)$ on $\mathbf{R}^n$ and

$$\langle v_{n+2}, \varphi\rangle = \left\langle v_0, \frac{1}{2}\omega_{n+1}r^{n+1}\varphi^o\right\rangle$$

for all radial Schwartz functions $\varphi(x) = \varphi^o(|x|)$ on $\mathbf{R}^{n+2}$.

Let $u^n = F_n(v_n)$ and $u^{n+2} = F_{n+2}(v_{n+2})$. Then the identity

$$-\frac{1}{2\pi r}\frac{d}{dr}u^n_\diamond = u^{n+2}_\diamond \tag{12}$$

holds on $\mathcal{R}'_{n+2}$.

Proof We denote by $\langle\cdot,\cdot\rangle_n$ the action of the distribution on a function in dimension n. Let $\psi(r)$ be an even Schwartz function on the real line. Then we need to show that

$$\left\langle -\frac{1}{2\pi r}\frac{d}{dr}u_\diamond^n, \omega_{n+1}r^{n+1}\psi(r)\right\rangle_1 = \left\langle u_\diamond^{n+2}, \omega_{n+1}r^{n+1}\psi(r)\right\rangle_1. \tag{13}$$

This is equivalent to showing that

$$\frac{1}{2\pi}\left\langle u_\diamond^n, \omega_{n+1}\left(r^n\psi(r)\right)'\right\rangle_1 = \left\langle u_\diamond^{n+2}, \omega_{n+1}r^{n+1}\psi(r)\right\rangle_1. \tag{14}$$

We introduce the even Schwartz function $\eta(r) = r^{-n+1}(r^n\psi(r))' = n\psi(r) + r\psi'(r)$ on the real line and functions η^O on $\mathbf{R}^n$ and ψ^O on $\mathbf{R}^{n+2}$ by setting

$$\psi^O(x) = \psi\left(|x|\right), \qquad \eta^O(y) = \eta\left(|y|\right)$$

for $y \in \mathbf{R}^n$ and $x \in \mathbf{R}^{n+2}$. Then (14) is equivalent to

$$\frac{1}{2\pi}\frac{\omega_{n+1}}{\omega_{n-1}}\left\langle u_\diamond^n, \omega_{n-1}r^{n-1}\eta(r)\right\rangle_1 = \left\langle u_\diamond^{n+2}, \omega_{n+1}r^{n+1}\psi(r)\right\rangle_1 \tag{15}$$

which is in turn equivalent to

$$\frac{1}{2\pi}\frac{\omega_{n+1}}{\omega_{n-1}}\left\langle F_n(v_n), \eta^O\right\rangle_n = \left\langle F_{n+2}(v_{n+2}), \psi^O\right\rangle_{n+2} \tag{16}$$

and also to

$$\frac{1}{2\pi}\frac{\omega_{n+1}}{\omega_{n-1}}\left\langle v_n, F_n\left(\eta^O\right)\right\rangle_n = \left\langle v_{n+2}, F_{n+2}\left(\psi^O\right)\right\rangle_{n+2}. \tag{17}$$

We now switch to dimension one by writing (17) equivalently as

$$\frac{1}{2\pi}\frac{\omega_{n+1}}{\omega_{n-1}}\left\langle v_0, \omega_{n-1}r^{n-1}\mathcal{F}_n(\eta)(r)\right\rangle_1 = \left\langle v_0, \omega_{n+1}r^{n+1}\mathcal{F}_{n+2}(\psi)(r)\right\rangle_1. \tag{18}$$

But this identity holds if

$$\frac{1}{2\pi}\mathcal{F}_n(\eta)(r) = r^2\mathcal{F}_{n+2}(\psi)(r),$$

which is valid as a restatement of (2); recall that $\eta(r) = r^{-n+1}\frac{d}{dr}(r^n\psi(r))$. This proves (13). □

It is straightforward to check that for polynomially bounded smooth functions all operations coincide with the usual ones. We end this section with a few more illustrative examples. Let δ_n be the Dirac mass on $\mathbf{R}^n$.

Examples

(a) Let $v_n = \delta_n$. One can see that

$$v_0 = \frac{2(-1)^{n-1}}{\omega_{n-1}(n-1)!}\left(\frac{d}{dr}\right)^{(n-1)}(\delta_1)$$

satisfies (11). Acting v_0 on $r^{n+1}\varphi^o(r)$ yields that $v_{n+2} = 0$ and thus $u_\diamond^{n+2} = 0$. Also $u_\diamond^n = 1$; so both sides of (12) are equal to zero.

(b) Let $v_{n+2} = \delta_{n+2}$. Then

$$v_0 = \frac{2(-1)^{n+1}}{\omega_{n+1}(n+1)}\left(\frac{d}{dr}\right)^{(n+1)}(\delta_1).$$

Let $\Delta = \partial_1^2 + \cdots + \partial_n^2$ be the Laplacian. We claim that the distribution

$$v_n = \frac{\omega_{n-1}}{\omega_{n+1}}\frac{1}{n}\Delta(\delta_n) \tag{19}$$

satisfies (11). Then $u_\diamond^{n+2} = 1$ and also $u_\diamond^n = -r^2(2\pi)^2\omega_{n-1}/(2n\omega_{n+1})$. Thus (12) is valid since $2\pi\omega_{n-1} = n\omega_{n+1}$.

It remains to prove that the distribution v_n in (19) satisfies (11). For $\varphi(x) = \varphi^o(|x|)$ in $\mathcal{S}(\mathbf{R}^n)$ we have

$$\langle v_n, \varphi\rangle = \left\langle v_0, \omega_{n-1}r^{n-1}\varphi^o(r)\right\rangle = \frac{\omega_{n-1}}{\omega_{n+1}}\frac{2}{(n+1)!}\left\langle\delta_1, \left(r^{n-1}\varphi^o(r)\right)^{(n-1)}\right\rangle \tag{20}$$

and one notices that the $(n-1)$st derivative of $r^{n-1}\varphi^o(r)$ evaluated at zero is equal to $\frac{1}{2}(n+1)!(\varphi^o)''(0)$. To compute the value of this derivative we use Lemma 3.1 to write $\varphi(x) = \varphi^o(|x|) = g(|x|^2)$ where $g'(0) = \frac{1}{2}(\varphi^o)''(0)$. It follows that $g'(0) = \frac{1}{2n}\Delta(\varphi)(0)$. Combining these observations yields that the expression in (20) is equal to

$$\frac{\omega_{n-1}}{\omega_{n+1}}\frac{1}{n}\Delta(\varphi)(0) = \left\langle\frac{\omega_{n-1}}{\omega_{n+1}}\frac{1}{n}\Delta(\delta_n), \varphi\right\rangle,$$

which proves the claim.

Remark 4.2 As pointed out in Remark 2.1, the action of the Fourier transform on the associated function on the reals φ^o is given by the Hankel transform. In particular, the results in this section also give a natural extension of the Hankel transform (for half-integer order) to distributions. Of course this coincides with the usual approach, see [6, 15, 16] and the references therein. To this end observe that the space F used in [6] is precisely the set of functions on $[0, \infty)$ which extend to an even Schwartz function on $\mathbf{R}$.

5 Applications

We begin with a simple example. In dimension one we have that the Fourier transform of $\operatorname{sech}(\pi|x|)$ is $\operatorname{sech}(\pi|\xi|)$. It follows from (1) that in dimension three we have

$$F_3\big(\operatorname{sech}\big(\pi|x|\big)\big)(\xi) = \frac{1}{2|\xi|}\operatorname{sech}\big(\pi|\xi|\big)\tanh\big(\pi|\xi|\big).$$

since

$$\frac{d}{dr}\frac{2}{e^{\pi r}+e^{-\pi r}} = -2\pi\frac{e^{\pi r}-e^{-\pi r}}{(e^{\pi r}+e^{-\pi r})^2} = -2\pi\frac{1}{2}\operatorname{sech}(\pi r)\tanh(\pi r).$$

Continuing this process, one can explicitly calculate the Fourier transform of $\operatorname{sech}(\pi|x|)$ in all odd dimensions.

More sophisticated applications of our formulas appear in computations of functions of the Laplacian $-\Delta$, which arise in numerous applications. For example, in quantum mechanics the Laplacian $-\Delta$ arises as the free Schrödinger operator (cf., e.g., [7, 12]) and functions $f(-\Delta)$ are defined via the spectral theorem by

$$f(-\Delta)\varphi = K * \varphi, \quad \varphi \in \mathcal{S}\big(\mathbf{R}^n\big),$$

where K is the tempered distribution given by the inverse Fourier transform of the radial function $f(4\pi^2|\xi|^2)$, which is assumed polynomially bounded. Knowledge of the inverse Fourier transform of $f(4\pi^2|\xi|^2)$, for $\xi \in \mathbf{R}$ and $\xi \in \mathbf{R}^2$, yields explicit formulas for the kernel K of $f(-\Delta)$ in all dimensions.

An important application is the explicit calculation of the n-dimensional kernel $G_n(x)$ for the resolvent associated with the function $f(r) = (r-z)^{-1}$, $z \in \mathbf{C}\backslash[0,\infty)$. In the one-dimensional case, an easy computation shows that

$$G_1(x) = \frac{1}{2\sqrt{-z}}e^{-\sqrt{-z}|x|}.$$

Hence, by the L^2 version of Theorem 1.1 (cf. the discussion right after Theorem 1.1) the three-dimensional kernel is given by

$$G_3(x) = -\frac{1}{2\pi r}\frac{d}{dr}G_1(r)\bigg|_{r=|x|} = \frac{1}{4\pi|x|}e^{-\sqrt{-z}|x|}.$$

The computation of $G_5(x), G_7(x), \dots$ requires Theorem 4.1 since the assumptions of Theorem 1.1 are no longer satisfied. For instance, Theorem 4.1 gives

$$G_5(x) = \frac{1+|x|\sqrt{-z}}{8\pi^2|x|^3}e^{-\sqrt{-z}|x|}.$$

Another interesting situation where our theorem is useful are the spectral projections associated with the function $f(r) = \chi_{[0,E]}(r)$, $E > 0$. Again in the one-dimensional case the kernel for the resolvent can be easily computed and found to be

$$P_1(x) = \frac{\sin(x\sqrt{E})}{\pi x}.$$

Thus by Theorem 1.1 the three-dimensional kernel is given by

$$P_3(x) = -\frac{1}{2\pi r}\frac{d}{dr}P_1(r)\bigg|_{r=|x|} = \frac{\sin(|x|\sqrt{E}) - |x|\sqrt{E}\cos(|x|\sqrt{E})}{2\pi^2|x|^3}.$$

Finally, the Fourier transform is a crucial tool in solving constant coefficient linear partial differential equations (cf., e.g., [2]). Using the above trick one can of course derive the fundamental solution for the heat (or Schrödinger) equation in three dimensions from the one-dimensional one. However, since the three-dimensional case is no more difficult than the one-dimensional case we rather turn to the Cauchy problem for the wave equation

$$u_{tt} - \Delta u = 0, \qquad u(0,x) = \psi(x), \qquad u_t(0,x) = \varphi(x),$$

in $\mathbf{R}^n$, whose solution is given by

$$u(t,x) = \cos(t\sqrt{-\Delta})\psi(x) + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\varphi(x).$$

Since the first term can be obtained by differentiating the second (with respect to t) it suffices to look only at the second and assume $\psi = 0$. Moreover, since the Fourier transform of $f(x) = \frac{\sin(a\pi x)}{a\pi x}$ is $F_1(f)(\xi) = |a|^{-1}\chi_{[-1/2,1/2]}(\xi/a)$, we obtain

$$u(t,x) = \int_{\mathbf{R}} \frac{1}{2}\chi_{[-t,t]}(x-y)\varphi(y)\,dy,$$

which is of course just d'Alembert's formula. In order to apply Theorem 4.1 we use $v_0(r) = \frac{\sin(tr)}{r}$ such that $u^1 = F_1^{-1}(v_1)$ as well as $u^1_\diamond$ are associated with the function $\frac{1}{2}\chi_{[-t,t]}(x)$. Hence by Theorem 4.1

$$\left\langle F_3^{-1}(v_3), \varphi\right\rangle = \frac{\omega_2}{2}\left\langle -\frac{1}{2\pi r}\frac{d}{dr}\frac{1}{2}\chi_{[-t,t]}(r), r^2\varphi^o(r)\right\rangle = \frac{\omega_2}{4\pi}t\varphi^o(t)$$

and we obtain Kirchhoff's formula

$$u(t,x) = \frac{t}{4\pi}\int_{\mathbf{S}^2}\varphi(x - t\theta)\,d\theta.$$

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