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Directional and Anisotropic Regularity and Irregularity Criteria in Triebel Wavelet Bases

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Abstract Many natural mathematical objects, as well as many multi-dimensional signals and images from real physical problems, need to distinguish local directional behaviors (for tracking contours in image processing for example). Using some results of Jaffard and Triebel, we obtain criteria of directional and anisotropic regularities by decay conditions on Triebel anisotropic wavelet coefficients (resp. wavelet leaders).

Keywords Directional regularity · Anisotropic Hölder regularity · Anisotropic Triebel wavelet basis · Anisotropic wavelet coefficients · Anisotropic wavelet leaders

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1 Introduction

Let us first recall the classical notion of Hölder regularity.

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Definition 1 Let $h>0$, $y\in\mathbb{R}^m$ and $f:\mathbb{R}^m\to\mathbb{R}$ or $\mathbb{C}$ bounded in a neighborhood of y. We say that f belongs to $C^h(y)$ if there exist a constant $C>0$ and a polynomial P of degree less than h such that in a neighborhood of y we have

$$\left|f(x)-P(x-y)\right|\le C|x-y|^h, \tag{1}$$

where $|.|$ is the Euclidean norm.

The Hölder exponent (or regularity) of f at y is defined by

$$h_f(y)=\sup\left\{h:\ f\in C^h(y)\right\}.$$

We say that f belongs to $C^h(\mathbb{R}^m)$ if $f\in L^\infty(\mathbb{R}^m)$ and if (1) holds for any x and y in $\mathbb{R}^m$ with a uniform constant C.

If $m\ge 2$ then Definition 1 is uniform in all directions. However many images belong to classes of functions with various directional regularity behaviors. These behaviors are important for detection of edges, efficient image compression, ... (see for instance [1] and the references therein). In [3], it is shown that the directional regularity of any Gaussian random field with stationary increments is constant except maybe on a hyperplane of dimension at most $m-1$.

Standard isotropic multi-dimensional wavelets obtained as tensor product do not give a satisfactory algorithm to detect directional singularities. A wide range of directional transform ideas have been proposed. 'Steerable Pyramids' and 'Cortex Transforms' were developed in the 1980's by vision researchers (Adelson, Freeman, Heeger, and Simoncelli [19] and Watson [22]) to offer increased directional representativeness. Extensions of wavelet bases which can be elongated in particular directions were considered. They include the ridgelets of Candes and Donoho, see [6], or the bandelets of Mallat, see [18], but are efficient with singularities along lines, along hyperplanes, etc, for which wavelets do not deal with efficiently.

For pointwise singularities, it is natural to define the Hölder regularity at a point y in a direction $e\in\mathbb{R}^m$ with $|e|=1$ as the Hölder regularity at 0 of the one variable function $f_e: s\mapsto f(y+se)$. It seems that one cannot expect directional regularity to be characterized in terms of the size of the usual wavelet coefficients, because f_e is defined as the trace of f on a line, which is a set of vanishing measure and wavelets have a support of nonempty interior. Thus we should take into account the values of f around the line considered. Therefore the definition of directional smoothness should include such information. However, in the asymptotic of small scales, the values taken into account should be localized more and more sharply around this line. These considerations motivate the following definition and remark of Jaffard [15].

Definition 2 Let $f:\mathbb{R}^m\to\mathbb{R}$ or $\mathbb{C}$ be bounded in a neighborhood of y. Let $\overrightarrow{\alpha}=(\alpha_1,\dots,\alpha_m)$ where $\alpha_1\ge\cdots\ge\alpha_m>0$. Let $\mathcal{B}=(e_1,\dots,e_m)$ be an orthonormal basis of $\mathbb{R}^m$. We denote by $(x_1,\dots,x_m)$ the coordinates of x on the basis $\mathcal{B}$. We say that $f\in C^{\overrightarrow{\alpha}}(y,\mathcal{B})$ if there exist a constant $C>0$ and a polynomial $P(x)=\sum_{I=(i_1,\dots,i_m)\in\mathbb{N}^m}a_Ix^I=\sum_{I=(i_1,\dots,i_m)\in\mathbb{N}^m}a_Ix_1^{i_1}\cdots x_m^{i_m}$ of degree less than $\overrightarrow{\alpha}$ in the sense that

$$\max\left\{\sum_{n=1}^{m}\frac{i_n}{\alpha_n} : a_I \neq 0\right\} < 1$$

such that in a neighborhood of y we have

$$\left|f(x) - P(x-y)\right| \leq C\sum_{n=1}^{m}|x_n - y_n|^{\alpha_n}. \tag{2}$$

The following indices were given in [15] and are crucial for the rest of the paper.

Definition 3 Let $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_m)$ where $\alpha_1 \geq \cdots \geq \alpha_m > 0$.
The average regularity $\tilde{\alpha}$ is the harmonic mean of the α_n, i.e.,

$$\frac{1}{\tilde{\alpha}} = \frac{1}{m}\sum_{n=1}^{m}\frac{1}{\alpha_n}. \tag{3}$$

The anisotropy indices are

$$v_n = \frac{\tilde{\alpha}}{\alpha_n}. \tag{4}$$

Jaffard has obtained a necessary condition for the characterization of $C^{\overrightarrow{\alpha}}(y, \mathcal{B})$ based on the anisotropic Gabor-wavelet transform.

Remark 1 If (2) holds then for $1 \leq n \leq m$ the one dimensional function $f_{e_n} : s \mapsto f(y + se_n)$ belongs to $C^{\alpha_n}(0)$. So that, we will say that, in each direction e_n, the function f has Hölder regularity α_n at y.

Obviously, we have a partial ordering property: if $\alpha_1 \leq \beta_1, \ldots, \alpha_m \leq \beta_m$ then

$$f \in C^{\overrightarrow{\beta}}(y, \mathcal{B}) \quad \Rightarrow \quad f \in C^{\overrightarrow{\alpha}}(y, \mathcal{B}).$$

Therefore in [15] directional regularity exponents were defined in the following way.

Definition 4 Let $e \in \mathbb{R}^m$ with $|e| = 1$. The Hölder exponent of f in the direction e at y is

$$\alpha_f(y, e) = \sup\left\{\alpha_1 : \exists 0 < \varepsilon \leq \alpha_1 \ f \in C^{\overrightarrow{(\alpha_1, \varepsilon, \ldots, \varepsilon)}}(y, \mathcal{B})\right\}$$

where $\mathcal{B}$ is an orthonormal basis starting with the vector e.

Clearly we can choose any orthonormal basis $\mathcal{B}$ starting with the vector e for two reasons: the first reason is the fact that the component $x_1 - y_1$ is the same in any $\mathcal{B}$ and is equal to the inner product of $x - y$ with e, and the second reason is the fact that $|x_2 - y_2|^{\varepsilon} + \cdots + |x_m - y_m|^{\varepsilon}$ is equivalent to $|(x_2 - y_2, \ldots, x_m - y_m)|^{\varepsilon}$, and all norms of $\mathbb{R}^{m-1}$ are equivalent.

Definition 2 can be seen as an extension of the notion of anisotropic regularity which was already introduced by Ben Slimane [2] in the case where $\mathcal{B}$ is the canonical basis of $\mathbb{R}^m$.

Note that in [7], Clausel and Vedel showed that Gaussian random fields are anisotropic generalizations of self-similar fields, and that the sharpest way of measuring smoothness is related to these anisotropies and thus to the geometry of these fields.

Let $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$ be such that

$$0 < u_1 \le \cdots \le u_m \quad \text{and} \quad \sum_{n=1}^{m} u_n = m. \tag{5}$$

For $I = (i_1, \dots, i_m) \in \mathbb{N}^m$, we set $d(I) = \sum_{l=1}^{m} \frac{u_l}{u_1} i_l = i_1 + \frac{u_2}{u_1} i_2 + \cdots + \frac{u_m}{u_1} i_m$ and $d_{\mathbf{u}}(I) = u_1 d(I) = \sum_{l=1}^{m} u_l\, i_l$. Thus $d(I)$ is the degree of homogeneity of the differential operator ∂^I, or, as we shall say, its homogeneous degree. If $P = \sum_{I \in \mathbb{N}^m} a_I x^I$, $a_I \in \mathbb{R}$ or $\mathbb{C}$ is a polynomial we define its homogeneous degree to be $d(P) := \max\{d(I) : a_I \neq 0\}$. We also define its $\mathbf{u}$-homogeneous degree to be

$$d_{\mathbf{u}}(P) := u_1 d(P) = \max\bigl\{d_{\mathbf{u}}(I) : a_I \neq 0\bigr\}.$$

Definition 5 Let $f : \mathbb{R}^m \to \mathbb{R}$ or $\mathbb{C}$ be bounded in a neighborhood of y. Let $h > 0$. Let $\mathcal{B} = (e_1, \dots, e_m)$ be an orthonormal basis of $\mathbb{R}^m$. We denote by $(x_1, \dots, x_m)$ the coordinates of x on the basis $\mathcal{B}$. We say that $f \in C^h_{\mathbf{u}}(y, \mathcal{B})$ if there exist a constant $C > 0$ and a polynomial P of $\mathbf{u}$-homogeneous degree less than h such that in a neighborhood of y we have

$$\bigl|f(x) - P(x-y)\bigr| \le C|x-y|^h_{\mathbf{u}}, \tag{6}$$

where

$$|x-y|_{\mathbf{u}} = \sum_{n=1}^{m} |x_n - y_n|^{1/u_n}. \tag{7}$$

The $\mathbf{u}$-Hölder exponent of f at y is defined by

$$h_{\mathbf{u},f}(y, \mathcal{B}) = \sup\bigl\{h : f \in C^h_{\mathbf{u}}(y, \mathcal{B})\bigr\}.$$

We say that f belongs to $C^h_{\mathbf{u}}(\mathbb{R}^m, \mathcal{B})$ if $f \in L^\infty(\mathbb{R}^m)$ and if (6) holds for any x and y in $\mathbb{R}^m$ with a uniform constant C.

In the next section we will give a criterion of directional Hölder regularities in terms of anisotropic regularities (see Theorem 1).

In the third section we will expose some materials which will be useful later, such as the homogeneous quasi-norm, anisotropic Taylor's theorem with remainder for this quasi-norm, and anisotropic Triebel wavelet bases.

A decomposition-recomposition numerical algorithm will be given in Sect. 4.

In the fifth section we characterize both uniform and pointwise $\mathbf{u}$-Hölder regularity by decay conditions of anisotropic Triebel wavelet coefficients (see Theorem 3).

In Sect. 6 we deduce the characterization of both uniform and pointwise $\mathbf{u}$-Hölder regularity by decay conditions of anisotropic Triebel wavelet leaders (see Theorem 4). We finally conclude with a numerical discussion and examples.

2 Directional Hölder Regularity Criterion

Our first main result gives a criterion of directional Hölder regularities in terms of a supremum on a wide range of orientations of anisotropic regularities:

Theorem 1 *Let $e \in \mathbb{R}^m$ with $|e| = 1$. Let $\mathcal{B}$ be an orthonormal basis starting with the vector e. Let E be the set of all $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$ satisfying both (5) and $u_2 = \dots = u_m$, i.e., $0 < u_1 \leq 1$ and $u_2 = \dots = u_m = \frac{m-u_1}{m-1}$. The Hölder exponent of f in the direction e at y is given by*

$$\alpha_f(y, e) = \sup_{\mathbf{u} \in E} \left(\frac{h_{\mathbf{u},f}(y, \mathcal{B})}{u_1} \right).$$

Proof of Theorem 1 Using the notations of (3) and (4), one easily checks that

$$\tilde{\alpha} = \frac{m \prod_{n=1}^m \alpha_n}{\sum_{n=1}^m (\prod_{l \neq n} \alpha_l)}, \qquad v_1 \leq \dots \leq v_m \quad \text{and} \quad \sum_{n=1}^m v_n = m.$$

Now if we put $\mathbf{v} = (v_1, \dots, v_m)$, then

$$\sum_{n=1}^m \frac{i_n}{\alpha_n} < 1 \quad \Leftrightarrow \quad \sum_{n=1}^m i_n v_n < \tilde{\alpha}$$

which implies that a polynomial P has degree less than $\overrightarrow{\alpha}$ if and only if its $\mathbf{v}$-homogeneous degree $d_{\mathbf{v}}(P)$ is less than $\tilde{\alpha}$.

On the other hand there exist C_1 and C_2 depending on m and $\tilde{\alpha}$ such that

$$\forall t \quad C_1 |t|_{\mathbf{v}}^{\tilde{\alpha}} \leq \sum_{n=1}^m |t_n|^{\alpha_n} \leq C_2 |t|_{\mathbf{v}}^{\tilde{\alpha}}.$$

We deduce that

$$f \in C^{\overrightarrow{\alpha}}(y, \mathcal{B}) \quad \Leftrightarrow \quad f \in C_{\mathbf{v}}^{\tilde{\alpha}}(y, \mathcal{B}).$$

Therefore from Definition 4, if $\mathcal{B}$ is an orthonormal basis starting with the vector e, then

$$\begin{aligned} \alpha_f(y, e) &= \sup\left\{\alpha_1 : \exists 0 < \varepsilon \leq \alpha_1 \ f \in C^{\overrightarrow{(\alpha_1, \varepsilon, \dots, \varepsilon)}}(y, \mathcal{B})\right\} \\ &= \sup\left\{\alpha_1 : \exists 0 < \varepsilon \leq \alpha_1 \ f \in C_{\mathbf{v}}^{\tilde{\alpha}}(y, \mathcal{B})\right\} \end{aligned}$$

with

$$\tilde{\alpha} = \frac{m\alpha_1 \varepsilon}{\varepsilon + (m-1)\alpha_1}, \qquad v_1 = \frac{\tilde{\alpha}}{\alpha_1} \quad \text{and} \quad v_2 = \dots = v_m = \frac{\tilde{\alpha}}{\varepsilon}.$$

Therefore

$$\alpha_f(y,e) = \sup\left\{\frac{\tilde{\alpha}}{v_1} : \exists v_2 = \cdots = v_m \geq v_1 \ f \in C_{\mathbf{v}}^{\tilde{\alpha}}(y,\mathcal{B})\right\}.$$

Hence Theorem 1 holds. □

3 Some Mathematical Tools

Let $\mathbf{u} = (u_1, \ldots, u_m)$ be as in (5). For $r > 0$, we define the anisotropic dilation

$$r^{\mathbf{u}}x = \left(r^{u_1}x_1, \ldots, r^{u_m}x_m\right). \tag{8}$$

3.1 Homogeneous Quasi-norm

Recall that a quasi-norm on a vector space E satisfies the requirements of a norm except for the triangular inequality which is replaced by the weaker condition, i.e.,

$$\exists C > 0;\ \forall x, y \in E, \quad \|x+y\|_E \leq C \quad \left(\|x\|_E + \|y\|_E\right).$$

In [4, 5, 12] a quasi-norm $\rho_{\mathbf{u}}$ on $\mathbb{R}^m$ was used to develop a theory of anisotropic $\mathcal{H}^p(\mathbb{R}^m)$ spaces. It is defined by $\rho_{\mathbf{u}}(0) = 0$, and for all $x \neq 0$, $\rho_{\mathbf{u}}(x)$ is the unique value of r for which $|r^{-\mathbf{u}}x| = 1$.

The function $\rho_{\mathbf{u}}$ is continuous and homogeneous in the sense that

$$\rho_{\mathbf{u}}\left(r^{\mathbf{u}}x\right) = r\rho_{\mathbf{u}}(x). \tag{9}$$

Remark that the corresponding $\mathbf{u}$-ball $B_{\mathbf{u}}(x,r) := \{y \in \mathbb{R}^m\ ;\ \rho_{\mathbf{u}}(x-y) < r\}$, of $\rho_{\mathbf{u}}$-radius r centered on x, is an ellipse of axis of lengths $2r^{u_1}, \ldots, 2r^{u_m}$, centered on x. In the isotropic case ($u_i = 1$ for all $1 \leq i \leq m$), the homogeneous quasi-norm $\rho_{\mathbf{u}}$ coincides with the Euclidean norm on $\mathbb{R}^m$.

The homogeneous quasi-norm $\rho_{\mathbf{u}}$ satisfies the following properties:

$$\begin{aligned}
&\forall i \in \{1, \ldots, m\}; \quad |x_i|^{1/u_i} \leq \rho_{\mathbf{u}}(x), \\
&\exists \tilde{i} \in \{1, \ldots, m\}; \quad x_{\tilde{i}}^2 \geq \frac{1}{m}\,\rho_{\mathbf{u}}(x)^{2u_{\tilde{i}}}, \\
&|x|^{1/u_1} \leq \rho_{\mathbf{u}}(x) \leq |x|^{1/u_m}, \quad \text{if } \rho_{\mathbf{u}}(x) \leq 1
\end{aligned}$$

and

$$|x|^{1/u_m} \leq \rho_{\mathbf{u}}(x) \leq |x|^{1/u_1}, \quad \text{if } \rho_{\mathbf{u}}(x) \geq 1.$$

Then $|.|_{\mathbf{u}}$ defined in (7) is also an homogeneous quasi-norm and it is equivalent to $\rho_{\mathbf{u}}$ because

$$\frac{1}{m}\,|x|_{\mathbf{u}} \leq \rho_{\mathbf{u}}(x) \leq m^{1/2u_1}\,|x|_{\mathbf{u}}.$$

Note that the equivalence of general homogeneous norms is proved by Lemarié in [16].

3.2 The u-Taylor's Theorem

In [4, 5, 12], there are versions of Mean Value Theorem and Taylor's theorem with remainder for the homogeneous quasi-norm $\rho_{\mathbf{u}}$. Using the fact that $\rho_{\mathbf{u}}$ and $|.|_{\mathbf{u}}$ are equivalent we deduce the following results.

The u-Mean Value Theorem *There exist two positive constants C and ν such that for all functions f of class C^1 on $\mathbb{R}^m$ and all $x, y \in \mathbb{R}^m$,*

$$\left|f(y) - f(x)\right| \leq C \sum_{i=1}^{m} |y-x|_{\mathbf{u}}^{u_i} \sup_{|h|_{\mathbf{u}} \leq \nu |y-x|_{\mathbf{u}}} \left|\partial_{x_i} f(x+h)\right|.$$

We denote by Δ the additive sub-semigroup of $\mathbb{R}$ generated by $0, 1, \frac{u_2}{u_1}, \dots$ and $\frac{u_m}{u_1}$. In other words, Δ is the set of all numbers $d(I)$ as I ranges over $\mathbb{N}^m$. We observe that $\mathbb{N} \subset \Delta$.

The u-Taylor Inequality *Suppose $\delta \in \Delta$ ($\delta > 0$), and $k = [\delta]$. There are two constants $C_\delta > 0$ and $\nu > 0$ such that for all functions f of class $C^{(k+1)}$ on $\mathbb{R}^m$ and all $x, y \in \mathbb{R}^m$,*

$$\left|f(y) - P(y-x)\right| \leq C_\delta \sum_{|I| \leq k+1,\, d(I) > \delta} |y-x|_{\mathbf{u}}^{d_{\mathbf{u}}(I)} \sup_{|h|_{\mathbf{u}} \leq \nu^{k+1} |y-x|_{\mathbf{u}}} \left|\partial^I f(x+h)\right|$$

where P is the Taylor polynomial of f at x of homogeneous degree δ

$$P(y-x) = \sum_{I : d(I) \leq \delta} \frac{\partial^I f(x)}{I!} (y-x)^I. \tag{10}$$

3.3 Anisotropic Triebel Wavelet Bases

The wavelet characterization of (isotropic) Besov spaces has important applications in data compression and nonlinear approximation (see [8, 9, 11]). In recent years, an increasing interest in non-isotropic models has turned attention to the more general class of anisotropic Besov spaces. To characterize these spaces by wavelets, Triebel has constructed in [20, 21] wavelet bases through anisotropic multiresolution analysis; If ψ_F and ψ_M are the Lemarié-Meyer [17] (resp. Daubechies [10]) father and mother wavelets in the Schwartz class (resp. arbitrarily smooth with a corresponding compact support) such that all moments (resp. a finite number of moments) of ψ_M vanish, $\int_{\mathbb{R}} \psi_F(x)dx = 1$ and the collection $(\psi_F(. - k))_{k \in \mathbb{Z}}$ and $(2^{j/2}\psi_M(2^j . - k))_{j \in \mathbb{N}, k \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$. Let $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$ and $r^{\mathbf{u}}x$ be as in (5) and (8). In [20, 21] Triebel has considered anisotropic multiresolution analysis; consider, for any $j \in \mathbb{Z}$, the closed subspace $V_{j,\mathbf{u}}$ of $L^2(\mathbb{R}^m)$ spanned by the orthonormal basis $(\sqrt{2}^{\sum_{i=1}^m [j u_i]} \Phi_{j,k,\mathbf{u}})_{k \in \mathbb{Z}^m}$, where

$$\Phi_{j,k,\mathbf{u}}(x) = \prod_{i=1}^{m} \psi_F\left(2^{[j u_i]} x_i - k_i\right).$$

The sequence $(V_{j,\mathbf{u}})_{j\in\mathbb{Z}}$ is an anisotropic multiresolution analysis of $L^2(\mathbb{R}^m)$ in the sense that

(i) $\forall j \in \mathbb{Z}$, $V_{j,\mathbf{u}} \subset V_{j+1,\mathbf{u}}$.
(ii) $f(x) \in V_{j,\mathbf{u}} \iff f(2^{\mathbf{u}}x) \in V_{j+1,\mathbf{u}}$.
(iii) $\overline{\bigcup_{j\in\mathbb{Z}} V_{j,\mathbf{u}}} = L^2(\mathbb{R}^m)$.

For $j \in \mathbb{Z}$, let $I_{j,\mathbf{u}}$ be the set of pairs $(G, \mathbf{l})$ where $G = (G_1, \dots, G_m) \in \{F, M\}^m$ such that at least one component G_i is M and $\mathbf{l} = (l_1, \dots, l_m) \in \mathbb{N}^m$ where

$$l_i = [ju_i] \quad \text{if } G_i = F, \tag{11}$$

$$[ju_i] \le l_i < \big[(j+1)u_i\big] \quad \text{if } G_i = M \text{ and } \big[(j+1)u_i\big] > [ju_i], \tag{12}$$

and

$$l_i = [ju_i] \quad \text{if } G_i = M \text{ and } \big[(j+1)u_i\big] = [ju_i]. \tag{13}$$

Clearly the cardinality $\sharp I_{j,\mathbf{u}}$ of $I_{j,\mathbf{u}}$ is bounded independently of j, more precisely

$$1 \le \sharp I_{j,\mathbf{u}} \le \big(2^m - 1\big) \prod_{i=1}^{m} (2 + u_i). \tag{14}$$

The following proposition is given in [20, 21] in the case where $\mathcal{B}$ is the canonical basis of $\mathbb{R}^m$. It remains valid in the case where $\mathcal{B}$ is any orthonormal basis of $\mathbb{R}^m$.

Proposition 1 *Let $\mathcal{B}$ be an orthonormal basis of $\mathbb{R}^m$*. Let $(x_1, \dots, x_m)$ be the coordinates of x in $\mathcal{B}$. *Set*

$$\begin{aligned} \Phi_{k,\mathcal{B}}(x) &:= \prod_{i=1}^{m} \psi_F(x_i - k_i), \\ \Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}(x) &= \prod_{i=1}^{m} \psi_{G_i}\big(2^{l_i} x_i - k_i\big) \quad \textit{and} \quad |\mathbf{l}| := \sum_{i=1}^{m} l_i. \end{aligned} \tag{15}$$

The collection of the union of $(\Phi_{k,\mathcal{B}})$ for $k \in \mathbb{Z}^m$ and $(2^{|\mathbf{l}|/2}\, \Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}})$ for $j \in \mathbb{N}$, $(G, \mathbf{l}) \in I_{j,\mathbf{u}}$ and $k \in \mathbb{Z}^m$, is then an orthonormal basis of $L^2(\mathbb{R}^m)$. Thus any function $f \in L^2(\mathbb{R}^m)$ can be written as

$$f(x) = \sum_{k\in\mathbb{Z}^m} C_{k,\mathcal{B}}\, \Phi_{k,\mathcal{B}}(x) + \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^m} \sum_{(G,\mathbf{l})\in I_{j,\mathbf{u}}} c^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}} \Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}(x), \tag{16}$$

with

$$C_{k,\mathcal{B}} = \int_{\mathbb{R}^m} f(x) \Phi_{k,\mathcal{B}}(x)\, dx$$

and

$$c^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}} = 2^{|\mathbf{l}|} \int_{\mathbb{R}^m} f(x) \Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}(x)\, dx.$$

Furthermore, a wavelet characterization of some functional spaces was established in [20, 21], in particular anisotropic Besov spaces $B^{s,q}_{p,\mathbf{u}}(\mathbb{R}^m)$, namely

Theorem 2 *Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. Let $\mathcal{B}$ be the canonical basis of $\mathbb{R}^m$. Then $f \in B^{s,q}_{p,\mathbf{u}}(\mathbb{R}^m)$ if and only if its norm*

$$\|f\|_{B^{s,q}_{p,\mathbf{u}}} := \Big(\sum_{k\in\mathbb{Z}^m} |C_{k,\mathcal{B}}|^p\Big)^{1/p} + \Big(\sum_{j\in\mathbb{N}}\Big(\sum_{k\in\mathbb{Z}^m}\sum_{(G,\mathbf{l})\in I_{j,\mathbf{u}}} \big|2^{(s-\frac{m}{p})j} c^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}\big|^p\Big)^{q/p}\Big)^{1/q} \tag{17}$$

is finite. With the usual modification if $p = \infty$ and/or $q = \infty$.

In Sect. 5, we will prove in particular that if $\mathcal{B}$ is the canonical basis of $\mathbb{R}^m$ then $C^s_{\mathbf{u}}(\mathbb{R}^m, \mathcal{B})$ coincides with $B^{s,\infty}_{\infty,\mathbf{u}}(\mathbb{R}^m)$.

4 Decomposition-Recomposition Numerical Algorithm

Without any loss of generality we can assume that $m = 2$. Let $(V_{j,\mathbf{u}})_{j\in\mathbb{Z}}$ be an anisotropic multiresolution analysis of $L^2(\mathbb{R}^2)$. Denote by $W_{j,\mathbf{u}}$ the closed subspace of $L^2(\mathbb{R}^2)$ such that $V_{j+1,\mathbf{u}} = V_{j,\mathbf{u}} \oplus W_{j,\mathbf{u}}$, for all $j \in \mathbb{Z}$. Then $W_{j,\mathbf{u}}$ is spanned by the orthonormal basis $(2^{|\mathbf{l}|/2}\, \Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}})$ given by (15). Note that for all $j' \le j$, both $V_{j',\mathbf{u}}$ and $W_{j',\mathbf{u}}$ are in $V_{j,\mathbf{u}}$.

Let $j \in \mathbb{Z}$ and f be a given discretized function in $V_{j,\mathbf{u}}$. We will first obtain a decomposition algorithm: consist to determinate the wavelet coefficients of f in $V_{j',\mathbf{u}}$ and $W_{j',\mathbf{u}}$, for all $j' \le j$, using the wavelet coefficients of f in $V_{j,\mathbf{u}}$. Without any loss of generality we can assume that $j = 0$. Let $\mathcal{B}$ be an orthonormal basis of $\mathbb{R}^m$ and denote $(x_1, \dots, x_m)$ the coordinates of x in $\mathcal{B}$, then

$$f(x) = \sum_{k=(k_1,k_2)\in\mathbb{Z}^2} C_{k,\mathcal{B}}\, \psi_F(x_1 - k_1)\psi_F(x_2 - k_2). \tag{18}$$

We distinguish then three cases:

- Assume that $W_{-1,\mathbf{u}}$ is spanned by

$$2^{([-u_1]+l_2)/2}\psi_F\big(2^{[-u_1]}x_1 - K_1\big)\psi_M\big(2^{l_2}x_2 - K_2\big), \quad \text{where } [-u_2] \le l_2 < 0.$$

Since $V_{0,\mathbf{u}} = V_{-1,\mathbf{u}} \oplus W_{-1,\mathbf{u}}$, then

$$f(x) = \begin{cases} \sum_{K=(K_1,K_2)\in\mathbb{Z}^2} C_{-1,K,\mathbf{u},\mathcal{B}}\, 2^{([-u_1]+[-u_2])/2}\psi_F(2^{[-u_1]}x_1 - K_1) \\ \quad \times \psi_F(2^{[-u_2]}x_2 - K_2) \\ \quad + \sum_{K=(K_1,K_2)\in\mathbb{Z}^2} c^{(G,\mathbf{l})}_{-1,K,\mathbf{u},\mathcal{B}}\, 2^{([-u_1]+l_2)/2}\psi_F(2^{[-u_1]}x_1 - K_1) \\ \quad \times \psi_M(2^{l_2}x_2 - K_2), \end{cases} \tag{19}$$

here $G=(F,M)$ and $\mathbf{l}=([-u_1],l_2)$. From (18) and (19), it follows that

$$C_{-1,K,\mathbf{u},\mathcal{B}}=\sum_{k=(k_1,k_2)\in\mathbb{Z}^2} C_{k,\mathcal{B}}\big\langle\psi_F(x_1-k_1)\psi_F(x_2-k_2),\\ 2^{([-u_1]+[-u_2])/2}\psi_F\big(2^{[-u_1]}x_1-K_1\big)\psi_F\big(2^{[-u_2]}x_2-K_2\big)\big\rangle$$

and

$$c^{(G,\mathbf{l})}_{-1,K,\mathbf{u},\mathcal{B}}=\sum_{k=(k_1,k_2)\in\mathbb{Z}^2} C_{k,\mathcal{B}}\big\langle\psi_F(x_1-k_1)\psi_F(x_2-k_2),\\ 2^{([-u_1]+l_2)/2}\psi_F\big(2^{[-u_1]}x_1-K_1\big)\psi_M\big(2^{l_2}x_2-K_2\big)\big\rangle.$$

Note that

$$2^{([-u_1]+[-u_2])/2}\psi_F\big(2^{[-u_1]}x_1\big)\psi_F\big(2^{[-u_2]}x_2\big)\quad\text{and}\\ 2^{([-u_1]+l_2)/2}\psi_F\big(2^{[-u_1]}x_1\big)\psi_M\big(2^{l_2}x_2\big)$$

are in $V_{0,\mathbf{u}}$, then

$$2^{([-u_1]+[-u_2])/2}\psi_F\big(2^{[-u_1]}x_1\big)\psi_F\big(2^{[-u_2]}x_2\big)\\ =\sum_{k=(k_1,k_2)\in\mathbb{Z}^2}\alpha_{k,\mathbf{u},\mathcal{B}}\ \psi_F(x_1-k_1)\psi_F(x_2-k_2)$$

and

$$2^{([-u_1]+l_2)/2}\psi_F\big(2^{[-u_1]}x_1\big)\psi_M\big(2^{l_2}x_2\big)=\sum_{k=(k_1,k_2)\in\mathbb{Z}^2}\beta^{(G,\mathbf{l})}_{k,\mathbf{u},\mathcal{B}}\ \psi_F(x_1-k_1)\psi_F(x_2-k_2)$$

where

$$\alpha_{k,\mathbf{u},\mathcal{B}}=\big\langle\psi_F(x_1-k_1)\psi_F(x_2-k_2),\ 2^{([-u_1]+[-u_2])/2}\psi_F\big(2^{[-u_1]}x_1\big)\psi_F\big(2^{[-u_2]}x_2\big)\big\rangle \tag{20}$$

and

$$\beta^{(G,\mathbf{l})}_{k,\mathbf{u},\mathcal{B}}=\big\langle\psi_F(x_1-k_1)\psi_F(x_2-k_2),\ 2^{([-u_1]+l_2)/2}\psi_F\big(2^{[-u_1]}x_1\big)\psi_M\big(2^{l_2}x_2\big)\big\rangle. \tag{21}$$

Thus

$$C_{-1,K,\mathbf{u},\mathcal{B}}=\sum_{k=(k_1,k_2)\in\mathbb{Z}^2} C_{k,\mathcal{B}}\,\alpha_{k_1-2^{-[-u_1]}K_1,k_2-2^{-[-u_2]}K_2,\mathbf{u},\mathcal{B}} \tag{22}$$

and

$$c^{(G,\mathbf{l})}_{-1,K,\mathbf{u},\mathcal{B}}=\sum_{k=(k_1,k_2)\in\mathbb{Z}^2} C_{k,\mathcal{B}}\,\beta^{(G,\mathbf{l})}_{k_1-2^{-[-u_1]}K_1,k_2-2^{-l_2}K_2,\mathbf{u},\mathcal{B}}.$$

- Assume that $W_{-1,\mathbf{u}}$ is spanned by

$$2^{(l_1+[-u_2])/2}\psi_M\big(2^{l_1}x_1-K_1\big)\psi_F\big(2^{[-u_2]}x_2-K_2\big),\quad \text{where } [-u_1]\le l_1<0.$$

Arguing similarly as above, we get

$$f(x)=\begin{cases}\sum_{K=(K_1,K_2)\in\mathbb{Z}^2} C_{-1,K,\mathbf{u},\mathcal{B}}\, 2^{([-u_1]+[-u_2])/2}\psi_F(2^{[-u_1]}x_1-K_1)\\ \quad\times\,\psi_F(2^{[-u_2]}x_2-K_2)\\ \quad+\sum_{K=(K_1,K_2)\in\mathbb{Z}^2}\tilde{c}^{(G,\mathbf{l})}_{-1,K,\mathbf{u},\mathcal{B}}\, 2^{(l_1+[-u_2])/2}\psi_M(2^{l_1}x_1-K_1)\\ \quad\times\,\psi_F(2^{[-u_2]}x_2-K_2),\end{cases} \tag{23}$$

where $C_{-1,K,\mathbf{u},\mathcal{B}}$ is given by (22), $G=(M,F)$, $\mathbf{l}=(l_1,[-u_2])$,

$$\tilde{c}^{(G,\mathbf{l})}_{-1,K,\mathbf{u},\mathcal{B}}=\sum_{k=(k_1,k_2)\in\mathbb{Z}^2} C_{k,\mathcal{B}}\,\tilde{\beta}^{(G,\mathbf{l})}_{k_1-2^{-l_1}K_1,k_2-2^{-[-u_2]}K_2,\mathbf{u},\mathcal{B}}$$

and

$$\tilde{\beta}^{(G,\mathbf{l})}_{k,\mathbf{u},\mathcal{B}}=\big\langle\psi_F(x_1-k_1)\psi_F(x_2-k_2),\ 2^{(l_1+[-u_2])/2}\psi_M\big(2^{l_1}x_1\big)\psi_F\big(2^{[-u_2]}x_2\big)\big\rangle. \tag{24}$$

- Assume that $W_{-1,\mathbf{u}}$ is spanned by

$$2^{(l_1+l_2)/2}\psi_M\big(2^{l_1}x_1-K_1\big)\psi_M\big(2^{l_2}x_2-K_2\big),\quad \text{where } [-u_i]\le l_i<0.$$

As above

$$f(x)=\begin{cases}\sum_{K=(K_1,K_2)\in\mathbb{Z}^2} C_{-1,K,\mathbf{u},\mathcal{B}}\, 2^{([-u_1]+[-u_2])/2}\psi_F(2^{[-u_1]}x_1-K_1)\\ \quad\times\,\psi_F(2^{[-u_2]}x_2-K_2)\\ \quad+\sum_{K=(K_1,K_2)\in\mathbb{Z}^2}\breve{c}^{(G,\mathbf{l})}_{-1,K,\mathbf{u},\mathcal{B}}\, 2^{(l_1+l_2)/2}\psi_M(2^{l_1}x_1-K_1)\\ \quad\times\,\psi_M(2^{l_2}x_2-K_2),\end{cases} \tag{25}$$

here $C_{-1,K,\mathbf{u},\mathcal{B}}$ is given by (22), $G=(M,M)$, $\mathbf{l}=(l_1,l_2)$,

$$\breve{c}^{(G,\mathbf{l})}_{-1,K,\mathbf{u},\mathcal{B}}=\sum_{k=(k_1,k_2)\in\mathbb{Z}^2} C_{k,\mathcal{B}}\,\breve{\beta}^{(G,\mathbf{l})}_{k_1-2^{-l_1}K_1,k_2-2^{-l_2}K_2,\mathbf{u},\mathcal{B}}$$

and

$$\breve{\beta}^{(G,\mathbf{l})}_{k,\mathbf{u},\mathcal{B}}=\big\langle\psi_F(x_1-k_1)\psi_F(x_2-k_2),\ 2^{(l_1+l_2)/2}\psi_M\big(2^{l_1}x_1\big)\psi_M\big(2^{l_2}x_2\big)\big\rangle. \tag{26}$$

In all these cases we obtain the decomposition of f on $V_{-1,\mathbf{u}}$ and $W_{-1,\mathbf{u}}$. The decomposition of f on $V_{-2,\mathbf{u}}$ and $W_{-2,\mathbf{u}}$ will be obtained from that on $V_{-1,\mathbf{u}}$ by iterating $C_{-1,K,\mathbf{u},\mathcal{B}}$, and so on. We stop with the lowest resolution $-J$ that we fix, then we get

$$f(x)=\sum_{j=1}^{J}\sum_{K\in\mathbb{Z}^2}\sum_{(G,\mathbf{l})\in I_{j,\mathbf{u}}} D^{(G,\mathbf{l})}_{-j,K,\mathbf{u},\mathcal{B}}\Psi^{(G,\mathbf{l})}_{-j,K,\mathbf{u},\mathcal{B}}(x)+\sum_{K\in\mathbb{Z}^2} C_{-J,K,\mathbf{u},\mathcal{B}}\Phi_{-J,K,\mathbf{u},\mathcal{B}}(x).$$

The "recomposition" algorithm of f from the above decomposition is similar. As above, we establish the passage algorithm from the decomposition of f on $V_{-1,\mathbf{u}}$ and $W_{-1,\mathbf{u}}$ to the decomposition on $V_{0,\mathbf{u}} = V_{-1,\mathbf{u}} \oplus W_{-1,\mathbf{u}}$. The complete algorithm will be obtained by reiterating this last J times. Let us explain the passage from $V_{-1,\mathbf{u}}$ and $W_{-1,\mathbf{u}}$ to $V_{0,\mathbf{u}}$. Let $f \in V_{-1,\mathbf{u}} \oplus W_{-1,\mathbf{u}}$.

- If $W_{-1,\mathbf{u}}$ is spanned by

$$2^{([-u_1]+l_2)/2}\psi_F\big(2^{[-u_1]}x_1 - K_1\big)\psi_M\big(2^{l_2}x_2 - K_2\big), \quad \text{where } [-u_2] \le l_2 < 0,$$

then f is given by (19). Since $f \in V_{0,\mathbf{u}}$, then f can be written as (18). It yields

$$\begin{aligned} C_{k,\mathcal{B}} = \sum_{K=(K_1,K_2)\in\mathbb{Z}^2} & C_{-1,K,\mathbf{u},\mathcal{B}}\, \alpha_{k_1-2^{-[-u_1]}K_1,k_2-2^{-[-u_2]}K_2,\mathbf{u},\mathcal{B}} \\ & + c^{(G,\mathbf{l})}_{-1,K,\mathbf{u},\mathcal{B}}\, \beta^{(G,\mathbf{l})}_{k_1-2^{-[-u_1]}K_1,k_2-2^{-l_2}K_2,\mathbf{u},\mathcal{B}} \end{aligned}$$

where $\alpha_{k,\mathbf{u},\mathcal{B}}$ and $\beta^{(G,\mathbf{l})}_{k,\mathbf{u},\mathcal{B}}$ are given respectively by (20) and (21).

- If $W_{-1,\mathbf{u}}$ is spanned by

$$2^{(l_1+[-u_2])/2}\psi_M\big(2^{l_1}x_1 - K_1\big)\psi_F\big(2^{[-u_2]}x_2 - K_2\big), \quad \text{where } [-u_1] \le l_1 < 0,$$

then f is given by (23). We get in this case that

$$\begin{aligned} C_{k,\mathcal{B}} = \sum_{K=(K_1,K_2)\in\mathbb{Z}^2} & C_{-1,K,\mathbf{u},\mathcal{B}}\, \alpha_{k_1-2^{-[-u_1]}K_1,k_2-2^{-[-u_2]}K_2,\mathbf{u},\mathcal{B}} \\ & + \tilde{c}^{(G,\mathbf{l})}_{-1,K,\mathbf{u},\mathcal{B}}\, \tilde{\beta}^{(G,\mathbf{l})}_{k_1-2^{-l_1}K_1,k_2-2^{-[-u_2]}K_2,\mathbf{u},\mathcal{B}} \end{aligned}$$

where $\tilde{\beta}^{(G,\mathbf{l})}_{k,\mathbf{u},\mathcal{B}}$ is given by (24).

- If $W_{-1,\mathbf{u}}$ is spanned by

$$2^{(l_1+l_2)/2}\psi_M\big(2^{l_1}x_1 - K_1\big)\psi_M\big(2^{l_2}x_2 - K_2\big), \quad \text{where } [-u_i] \le l_i < 0,$$

then f is given by (25) and we obtain

$$\begin{aligned} C_{k,\mathcal{B}} = \sum_{K=(K_1,K_2)\in\mathbb{Z}^2} & C_{-1,K,\mathbf{u},\mathcal{B}}\, \alpha_{k_1-2^{-[-u_1]}K_1,k_2-2^{-[-u_2]}K_2,\mathbf{u},\mathcal{B}} \\ & + \breve{c}^{(G,\mathbf{l})}_{-1,K,\mathbf{u},\mathcal{B}}\, \breve{\beta}^{(G,\mathbf{l})}_{k_1-2^{-l_1}K_1,k_2-2^{-l_2}K_2,\mathbf{u},\mathcal{B}} \end{aligned}$$

where $\breve{\beta}^{(G,\mathbf{l})}_{k,\mathbf{u},\mathcal{B}}$ is given by (26).

5 Wavelet Characterization of u-Regularity

The anisotropic wavelet transform characterizes the **u**-Hölder regularity by conditions analogous to those of the classic wavelet transform for the isotropic case. The following theorem is reminiscent of [13] where Jaffard proved similar results for the isotropic Hölder regularity.

Theorem 3 *Let $\mathcal{B}$ be an orthonormal basis of $\mathbb{R}^m$. Let $(y_1, \dots, y_m)$ be the coordinates of y in $\mathcal{B}$.*

If $f \in C_{\mathbf{u}}^{\beta}(\mathbb{R}^m, \mathcal{B})$ for $\beta > 0$, the $\mathbf{u}$-Hölder exponent of f can be expressed at every point by the formula

$$h_{\mathbf{u},f}(y, \mathcal{B}) = \liminf_{j\to\infty} \inf_{k\in\mathbb{Z}^m,(G,\mathbf{l})\in I_{j,\mathbf{u}}} \frac{\log(|c_{j,k,\mathbf{u},\mathcal{B}}^{(G,\mathbf{l})}|)}{\log(2^{-j} + |y - 2^{-\mathbf{l}}k|_{\mathbf{u}})},$$

where

$$2^{-\mathbf{l}}k := \left(\frac{k_1}{2^{l_1}}, \dots, \frac{k_m}{2^{l_m}}\right) \quad \text{the coordinates are in } \mathcal{B} \tag{27}$$

and

$$\left|y - 2^{-\mathbf{l}}k\right|_{\mathbf{u}} = \sum_{n=1}^{m} \left|y_n - \frac{k_n}{2^{l_n}}\right|^{1/u_n}.$$

Remark 2 If we use the Lemarié-Meyer wavelets then there is no added assumptions in the following results. However, if we use the Daubechies wavelets then we will not mention the regularity needed for them, which will be assumed to be smooth enough.

Clearly if $f \in C^{\varepsilon}(\mathbb{R}^m)$ for $\varepsilon > 0$, then for every $\mathbf{u}$ there exists $\beta > 0$ such that $f \in C_{\mathbf{u}}^{\beta}(\mathbb{R}^m, \mathcal{B})$. Hence Theorems 1 and 3 yield the following corollary.

Corollary 1 *Let $f \in C^{\varepsilon}(\mathbb{R}^m)$ for $\varepsilon > 0$. Let $e \in \mathbb{R}^m$ with $|e| = 1$. Let $\mathcal{B}$ be any orthonormal basis starting with the vector e. Let E be the set of all anisotropies $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$ satisfying $0 < u_1 \le 1$ and $u_2 = \dots = u_m = \frac{m-u_1}{m-1}$. The Hölder exponent of f in the direction e at y is given by*

$$\begin{aligned} &\alpha_f(y, e) \\ &\quad = \sup_{\mathbf{u}\in E}\left(\liminf_{j\to\infty} \inf_{k\in\mathbb{Z}^m,(G,\mathbf{l})\in I_{j,\mathbf{u}}} \frac{\log(|c_{j,k,\mathbf{u},\mathcal{B}}^{(G,\mathbf{l})}|)}{\log(2^{-ju_1} + |y_1 - \frac{k_1}{2^{l_1}}| + \sum_{i=2}^{m} |y_i - \frac{k_i}{2^{l_i}}|^{\frac{(m-1)u_1}{m-u_1}})}\right). \end{aligned}$$

Theorem 3 is a consequence of the following proposition.

Proposition 2

1. *$F \in C_{\mathbf{u}}^{s}(\mathbb{R}^m, \mathcal{B})$ if and only if there exists a constant $C > 0$ such that*

$$|C_{k,\mathcal{B}}| \le C \quad \forall k$$

and

$$\left|c_{j,k,\mathbf{u},\mathcal{B}}^{(G,\mathbf{l})}\right| \le C 2^{-js} \quad \forall j \in \mathbb{N},\ \forall k, (G, \mathbf{l}).$$

2. *If $F \in C^s_{\mathbf{u}}(y, \mathcal{B})$ then there exists a constant $C > 0$ such that*

$$\left|c^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}\right| \le C 2^{-js}\left(1 + 2^j|y - 2^{-\mathbf{l}}k|_{\mathbf{u}}\right)^s \quad \forall j \in \mathbb{N},\ \forall k, (G, \mathbf{l}). \tag{28}$$

3. *If (28) holds and if $F \in C^\beta_{\mathbf{u}}(\mathbb{R}^m, \mathcal{B})$ for $\beta > 0$, there exist a constant $C > 0$ and a polynomial P of* $\mathbf{u}$*-homogeneous degree smaller than s such that if $|x - y|_{\mathbf{u}} \le 1/2$,*

$$\left|F(x) - P(x - y)\right| \le C|x - y|^s_{\mathbf{u}} \log\left(\frac{1}{|x - y|_{\mathbf{u}}}\right). \tag{29}$$

4. *If there exist $s' < s$ and a constant $C > 0$ such that*

$$\left|c^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}\right| \le C 2^{-js}\left(1 + 2^j\left|y - 2^{-\mathbf{l}}k\right|_{\mathbf{u}}\right)^{s'} \quad \forall j \in \mathbb{N},\ \forall k, (G, \mathbf{l}) \tag{30}$$

   then $F \in C^s_{\mathbf{u}}(y, \mathcal{B})$.
5. *If $F \in C^\gamma_{\mathbf{u}}(\mathbb{R}^m, \mathcal{B})$ for $\gamma > 0$ and there exists $s' > s$ such that (30) holds, then $F \in C_{\mathbf{u}}^{\frac{\gamma s'}{\gamma - s + s'}}(y, \mathcal{B})$.*

Proof of Proposition 2 1. Suppose that $F \in C^s_{\mathbf{u}}(\mathbb{R}^m, \mathcal{B})$. Since $F \in L^\infty(\mathbb{R}^m)$, then

$$|C_{k,\mathcal{B}}| = \left|\int_{\mathbb{R}^m} F(x)\Phi_{k,\mathcal{B}}(x)\,dx\right| \le C\|F\|_{L^\infty}.$$

On the other hand since ψ_M has vanishing moments, then for $j \in \mathbb{N}$

$$\begin{aligned}
\left|c^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}\right| &= 2^{|\mathbf{l}|}\left|\int_{\mathbb{R}^m} F(x)\Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}(x)\,dx\right| \\
&= 2^{|\mathbf{l}|}\left|\int_{\mathbb{R}^m}\left(F(x) - P\left(x - 2^{-\mathbf{l}}k\right)\right)\Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}(x)\,dx\right| \\
&\le C 2^{|\mathbf{l}|}\int |x - 2^{-\mathbf{l}}k|^s_{\mathbf{u}}\left|\prod_{i=1}^m \psi_{G_i}\left(2^{l_i}x_i - k_i\right)\right| dx.
\end{aligned}$$

Put $x - 2^{-\mathbf{l}}k = 2^{-\mathbf{l}}t = 2^{-j\mathbf{u}}z$, thus

$$\begin{aligned}
\left|c^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}\right| &\le C\int |2^{-\mathbf{l}}t|^s_{\mathbf{u}}\left|\prod_{i=1}^m \psi_{G_i}(t_i)\right| dt \\
&\le C\prod_{i=1}^m 2^{l_i - ju_i}\int |2^{-j\mathbf{u}}z|^s_{\mathbf{u}}\left|\prod_{i=1}^m \psi_{G_i}\left(2^{l_i - ju_i}z_i\right)\right| dz \\
&\le C 2^{-js}\prod_{i=1}^m 2^{l_i - ju_i}\int |z|^s_{\mathbf{u}}\left|\prod_{i=1}^m \psi_{G_i}\left(2^{l_i - ju_i}z_i\right)\right| dz.
\end{aligned}$$

From (11), (12), (13) and the localization of the wavelets it follows that

$$\left|c^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}\right| \le C 2^{-js}.$$

Conversely, assume that F is expressed as in (16). Since

$$|C_{k,\mathcal{B}}| \le C \quad \forall k,$$

the function $\sum_{k\in\mathbb{Z}^m} C_{k,\mathcal{B}}\,\Phi_{k,\mathcal{B}}(x)$ has uniformly the same regularity as ψ_F. It suffices to examine the regularity of $\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^m}\sum_{(G,\mathbf{l})\in I_{j,\mathbf{u}}} c^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}\,\Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}(x)$. Set

$$F_j(x) = \sum_{k\in\mathbb{Z}^m}\sum_{(G,\mathbf{l})\in I_{j,\mathbf{u}}} c^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}\,\Psi^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}(x).$$

It follows from (11), (12), (13), relation (14) and the localization of the wavelets that

$$\|F_j\|_{L^\infty} \le C2^{-js} \tag{31}$$

and

$$\left\|\partial^I F_j\right\|_{L^\infty} \le C2^{-j(s-d_{\mathbf{u}}(I))}. \tag{32}$$

Let $y\in\mathbb{R}^m$. Set $\delta_{\mathbf{u}} = \max\{d_{\mathbf{u}}(I)\ :\ d_{\mathbf{u}}(I) < s\}$. For $j\ge 0$, denote by $P_j(x-y)$ the Taylor polynomial of F_j at y of $\mathbf{u}$-homogeneous degree $\delta_{\mathbf{u}}$ (which was defined in (10))

$$P_j(x-y) = \sum_{I\,:\,d_{\mathbf{u}}(I)\le\delta_{\mathbf{u}}} \frac{\partial^I F_j(y)}{I!}(x-y)^I.$$

Then the $\mathbf{u}$-Taylor polynomial of $F - \sum_{k\in\mathbb{Z}^m} C_{k,\mathcal{B}}\,\Phi_{k,\mathcal{B}}$ is $P(x-y) := \sum_{j=0}^{\infty} P_j(x-y)$. This series converges because of (32) and the fact that $s > d_{\mathbf{u}}(I)$.

Let j_0 be the unique integer such that $2^{-j_0}\le |x-y|_{\mathbf{u}} < 2.\,2^{-j_0}$. Therefore

$$\begin{aligned}\left|F(x) - \sum_{k\in\mathbb{Z}^m} C_{k,\mathcal{B}}\,\Phi_{k,\mathcal{B}}(x) - P(x-y)\right| &\le \sum_{j=0}^{j_0}\left|F_j(x) - P_j(x-y)\right|\\ &\quad + \sum_{j>j_0}\left|F_j(x)\right| + \sum_{j>j_0}\left|P_j(x-y)\right|.\end{aligned}$$

It follows from (31) and (32), that

$$\sum_{j>j_0}\left|F_j(x)\right| \le \sum_{j>j_0} C2^{-js} \le C2^{-j_0 s} \le C|x-y|^s_{\mathbf{u}}$$

and

$$\sum_{j>j_0}\left|P_j(x-x_0)\right| \le \sum_{j>j_0}\sum_{I\,:\,d_{\mathbf{u}}(I)\le\delta_{\mathbf{u}}} C2^{-j(s-d_{\mathbf{u}}(I))}\left|(x-y)^I\right|.$$

But from the definition of $|.|_{\mathbf{u}}$

$$\left|(x-y)^I\right| \le |x-y|^{d_{\mathbf{u}}(I)}_{\mathbf{u}}. \tag{33}$$

Furthermore, since $s > d_{\mathbf{u}}(I)$ then we obtain

$$\sum_{j>j_0} \left|P_j(x-y)\right| \le C2^{-j_0 s} \le C|x-y|_{\mathbf{u}}^s.$$

Let $\delta > \delta_{\mathbf{u}}/u_1$ and $l = [\delta]$, since F_j is of class $C^{(l+1)}$, then using the $\mathbf{u}$-Taylor inequality

$$\begin{aligned}
&\sum_{j=0}^{j_0} \left|F_j(x) - P_j(x-y)\right| \\
&\quad \le \sum_{j=0}^{j_0} C_\delta \sum_{|J|\le l+1,\, d(J)>\delta} |x-y|_{\mathbf{u}}^{d_{\mathbf{u}}(J)} \sup_{|h|_{\mathbf{u}} \le \nu^{l+1}|x-y|_{\mathbf{u}}} \left|\partial^J F_j(y+h)\right| \\
&\quad \le C_\delta \sum_{|J|\le l+1,\, d(J)>\delta} |x-y|_{\mathbf{u}}^{d_{\mathbf{u}}(J)} \sum_{j=0}^{j_0} 2^{-j(s-d_{\mathbf{u}}(J))}.
\end{aligned}$$

It follows from the definition of $\delta_{\mathbf{u}}$ and the fact that $\delta > \delta_{\mathbf{u}}/u_1$ that

$$\sum_{j=0}^{j_0} \left|F_j(x) - P_j(x-y)\right| \le C|x-y|_{\mathbf{u}}^s \quad \big(\text{because } s < d_{\mathbf{u}}(J)\big).$$

2. If $F \in C_{\mathbf{u}}^s(y, \mathcal{B})$, then

$$\begin{aligned}
\left|c_{j,k,\mathbf{u},\mathcal{B}}^{(G,\mathbf{l})}(F)\right| &= 2^{|\mathbf{l}|} \left| \int \big(F(x) - P(x-y)\big) \prod_{i=1}^m \psi_{G_i}\big(2^{l_i}x_i - k_i\big)\, dx \right| \\
&\le 2^{|\mathbf{l}|} \int C|x-y|_{\mathbf{u}}^s \left| \prod_{i=1}^m \psi_{G_i}\big(2^{l_i}x_i - k_i\big) \right| dx \\
&\le C2^{|\mathbf{l}|} \int \big(\left|x - 2^{-\mathbf{l}}k\right|_{\mathbf{u}}^s + \left|y - 2^{-\mathbf{l}}k\right|_{\mathbf{u}}^s\big) \left| \prod_{i=1}^m \psi_{G_i}\big(2^{l_i}x_i - k_i\big) \right| dx.
\end{aligned}$$

As previously, using the localization of the wavelets, we get

$$\begin{aligned}
\left|c_{j,k,\mathbf{u},\mathcal{B}}^{(G,\mathbf{l})}(F)\right| &\le C2^{-js} + C\left|y - 2^{-\mathbf{l}}k\right|_{\mathbf{u}}^s \\
&\le C2^{-js}\big(1 + 2^j\left|y - 2^{-\mathbf{l}}k\right|_{\mathbf{u}}\big)^s.
\end{aligned}$$

3. Conversely, if (28) holds then

$$\left|F_j(x)\right| \le \sum_{k\in\mathbb{Z}^m} \sum_{(G,\mathbf{l})\in I_{j,\mathbf{u}}} C\big(2^{-js} + \left|y - 2^{-\mathbf{l}}k\right|_{\mathbf{u}}^s\big) \left| \prod_{i=1}^m \psi_{G_i}\big(2^{l_i}x_i - k_i\big) \right|$$

$$\leq \sum_{k\in\mathbb{Z}^m} \sum_{(G,\mathbf{l})\in I_{j,\mathbf{u}}} C\big(2^{-js} + |x-y|_{\mathbf{u}}^s + \big|x - 2^{-\mathbf{l}}k\big|_{\mathbf{u}}^s\big) \left|\prod_{i=1}^m \psi_{G_i}\big(2^{l_i} x_i - k_i\big)\right|.$$

It follows from (11), (12), (13), relation (14) and the localization of the wavelets that

$$\big|F_j(x)\big| \leq C\big(2^{-js} + |x-y|_{\mathbf{u}}^s\big). \tag{34}$$

Similarly we obtain

$$\big|\partial^I F_j(x)\big| \leq C 2^{-j(s-d_{\mathbf{u}}(I))}\big(1 + 2^j |x-y|_{\mathbf{u}}\big)^s. \tag{35}$$

Using the same notations as previously with besides $j_1 = s j_0/\beta$

$$\left|F(x) - \sum_{k\in\mathbb{Z}^m} C_{k,\mathcal{B}}\, \Phi_{k,\mathcal{B}}(x) - P(x-y)\right|$$
$$\leq \sum_{j=0}^{j_0} \big|F_j(x) - P_j(x-y)\big| + \sum_{j=j_0}^{j_1} \big|F_j(x)\big| + \sum_{j>j_1} \big|F_j(x)\big| + \sum_{j>j_0} \big|P_j(x-y)\big|.$$

It follows from (34) that

$$\begin{aligned}\sum_{j=j_0}^{j_1} \big|F_j(x)\big| &\leq \sum_{j=j_0}^{j_1} C\big(2^{-js} + |x-y|_{\mathbf{u}}^s\big)\\ &\leq C\big(2^{-j_0 s} + (j_1 - j_0)|x-y|_{\mathbf{u}}^s\big)\\ &\leq C\left(|x-y|_{\mathbf{u}}^s + |x-y|_{\mathbf{u}}^s \log \frac{1}{|x-y|_{\mathbf{u}}}\right)\\ &\leq C|x-y|_{\mathbf{u}}^s \log\left(\frac{1}{|x-y|_{\mathbf{u}}}\right).\end{aligned}$$

It follows from the fact that $F \in C_{\mathbf{u}}^{\beta}(\mathbb{R}^m, \mathcal{B})$ for a $\beta > 0$, that

$$\sum_{j>j_1} \big|F_j(x)\big| \leq \sum_{j>j_1} C 2^{-\beta j} \leq C 2^{-\beta j_1} \leq C|x-y|_{\mathbf{u}}^s.$$

It follows from (35) that

$$\sum_{j>j_0} \big|P_j(x-y)\big| \leq \sum_{j>j_0} \sum_{I:d_{\mathbf{u}}(I)\leq\delta_{\mathbf{u}}} C 2^{-j(s-d_{\mathbf{u}}(I))}\big(1 + 2^j |y-y|_{\mathbf{u}}\big)^s \big|(x-y)^I\big|.$$

Using (33) and the fact that $s > d_{\mathbf{u}}(I)$, we get

$$\sum_{j>j_0} \big|P_j(x-y)\big| \leq C \sum_{I:d_{\mathbf{u}}(I)\leq\delta_{\mathbf{u}}} |x-y|_{\mathbf{u}}^{d_{\mathbf{u}}(I)} \sum_{j>j_0} 2^{-j(s-d_{\mathbf{u}}(I))} \leq C|x-y|_{\mathbf{u}}^s.$$

Let $\delta = \delta_{\mathbf{u}}/u_1$ and $l = [\delta]$, since F_j is of class $C^{(l+1)}$, then using the $\mathbf{u}$-Taylor inequality

$$\begin{aligned}
&\sum_{j=0}^{j_0} \big|F_j(x) - P_j(x-y)\big| \\
&\quad \le \sum_{j=0}^{j_0} C_\delta \sum_{|J|\le l+1\,,\,d(J)>\delta} |x-y|_{\mathbf{u}}^{d_{\mathbf{u}}(J)} \sup_{|h|_{\mathbf{u}}\le \nu^{l+1}\rho_{\mathbf{u}}(x-y)} \big|\partial^J F_j(y+h)\big| \\
&\quad \le \sum_{j=0}^{j_0} C_\delta \sum_{|J|\le l+1\,,\,d(J)>\delta} |x-y|_{\mathbf{u}}^{d_{\mathbf{u}}(J)} \sup_{|h|_{\mathbf{u}}\le \nu^{l+1}\rho_{\mathbf{u}}(x-y)} 2^{-j(s-d_{\mathbf{u}}(J))}\big(1+2^j|h|_{\mathbf{u}}\big)^s \\
&\quad \le C_\delta \sum_{|J|\le l+1\,,\,d(J)>\delta} |x-y|_{\mathbf{u}}^{d_{\mathbf{u}}(J)} \sum_{j=0}^{j_0} \big(2^{-j(s-d_{\mathbf{u}}(J))} + 2^{jd_{\mathbf{u}}(J)}|x-y|_{\mathbf{u}}^s\big) \\
&\quad \le C|x-y|_{\mathbf{u}}^s \quad \big(\text{because } s < d_{\mathbf{u}}(J)\big).
\end{aligned}$$

The proofs of the last two results in Proposition 2 are similar. □

6 Wavelet Leaders Characterization of u-Regularity

We will now deduce an equivalent characterization of $\mathbf{u}$-Hölder regularity by decay conditions of anisotropic wavelet leaders. For that we start by introducing some definitions and notations. By $\lambda_{\mathbf{u}}(\mathcal{B}) = \lambda^{\mathbf{l}}_{j,k,\mathbf{u},\mathcal{B}}$ we denote a $\mathbf{u}$-dyadic rectangle in $\mathbb{R}^m$ of scale j *oriented with respect to the basis* $\mathcal{B}$, which has the form

$$\lambda_{\mathbf{u}}(\mathcal{B}) = \lambda^{\mathbf{l}}_{j,k,\mathbf{u},\mathcal{B}} = 2^{-\mathbf{l}}k + \prod_{i=1}^{m} [0, 2^{-l_i}),$$

in the coordinates of the basis $\mathcal{B}$, where $2^{-\mathbf{l}}k$ was defined in (27). Set

$$|c_{\lambda_{\mathbf{u}}(\mathcal{B})}| = \max \big|c^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}\big|, \tag{36}$$

where the maximum is taken over all indices G giving the same $\mathbf{l}$ at scale j.

Definition 6 The $\mathbf{u}$-wavelet leaders (oriented with respect to the basis $\mathcal{B}$) are defined by

$$d_{\lambda_{\mathbf{u}}(\mathcal{B})} = \sup_{\lambda'_{\mathbf{u}}(\mathcal{B})\subset\lambda_{\mathbf{u}}(\mathcal{B})} |c_{\lambda'_{\mathbf{u}}(\mathcal{B})}|.$$

Note that since we are interested to pointwise regularity we can assume that $f \in L^\infty_{loc}$ then the $\mathbf{u}$-wavelet leaders are finite because

$$|c_{\lambda_{\mathbf{u}}(\mathcal{B})}| \le C\|f\|_{L^\infty_{loc}}.$$

Definition 7 We say that two **u**-dyadic rectangles are adjacent if they are at the same scale and if the distance between them equals 0 (note that a **u**-dyadic rectangle is adjacent to itself). We denote by $\lambda_{j,\mathbf{u}}(y,\mathcal{B})$ the **u**-dyadic rectangle at scale j containing y and by $Adj(\lambda_{\mathbf{u}}(\mathcal{B}))$ the set of **u**-dyadic rectangles adjacent to $\lambda_{\mathbf{u}}(\mathcal{B})$. Then

$$d_{j,\mathbf{u}}(y,\mathcal{B}) = \max_{\lambda'_{\mathbf{u}}(\mathcal{B})\subset Adj(\lambda_{j,\mathbf{u}}(y,\mathcal{B}))} d_{\lambda'_{\mathbf{u}}(\mathcal{B})}.$$

The following proposition is reminiscent of [14] where Jaffard proved similar results for the isotropic Hölder regularity in the canonical basis. The following proposition characterizes the **u**-uniform (resp. **u**-pointwise) regularity by a decay condition of the $d_{\lambda_{\mathbf{u}}(\mathcal{B})}$ (resp. $d_{j,\mathbf{u}}(y,\mathcal{B})$) when $j\to\infty$.

Proposition 3

1. $F\in C^s_{\mathbf{u}}(\mathbb{R}^m,\mathcal{B})$ *if and only if there exists a constant* $C>0$ *such that*

$$d_{\lambda_{\mathbf{u}}(\mathcal{B})} \le C2^{-js} \quad \forall\lambda_{\mathbf{u}}.$$

2. *If* $F\in C^s_{\mathbf{u}}(y,\mathcal{B})$ *then there exists a constant* $C>0$ *such that*

$$d_{j,\mathbf{u}}(y,\mathcal{B}) \le C2^{-js} \quad \forall j\in\mathbb{N}. \tag{37}$$

3. *If* (37) *holds and if* $F\in C^\beta_{\mathbf{u}}(\mathbb{R}^m,\mathcal{B})$ *for* $\beta>0$, *there exist a constant* $C>0$ *and a polynomial* P *of homogeneous degree smaller than* s *such that if* $|x-y|_{\mathbf{u}}\le 1/2$, *then* (29) *holds.*

Proof of Proposition 3 The first result is immediate because of Proposition 2 and the fact that $|c_{\lambda_{\mathbf{u}}(\mathcal{B})}|\le d_{\lambda_{\mathbf{u}}(\mathcal{B})}$.

For the second result, thanks to Proposition 2 we have

$$|c_{\lambda'_{\mathbf{u}}(\mathcal{B})}| \le C2^{-j's}\left(1+2^{j'}\left|y-2^{-\mathbf{l}'}k'\right|_{\mathbf{u}}\right)^s \quad \forall j'\in\mathbb{N},\ \forall k',\ \left(G,\mathbf{l}'\right). \tag{38}$$

Let $j\in\mathbb{N}$, if $\lambda'_{\mathbf{u}}(\mathcal{B})\subset Adj(\lambda_{\mathbf{u}}(\mathcal{B}))$ then $j'\ge j-1$ and $|y-2^{-\mathbf{l}'}k'|_{\mathbf{u}}\le C2^{-j}$, hence (38) implies that $|c_{\lambda'_{\mathbf{u}}(\mathcal{B})}|\le C2^{-js}$, so that $d_{j,\mathbf{u}}(y,\mathcal{B})\le C2^{-js}$.

For the third result, let j' be given. We will first estimate the size of $F_{j'}$ and of its partial derivatives. If $\lambda'_{\mathbf{u}}(\mathcal{B})$ is a **u**-dyadic rectangle at scale j', denote by $\lambda_{\mathbf{u}}(\mathcal{B})$ the **u**-dyadic rectangle defined by

- If $\lambda'_{\mathbf{u}}(\mathcal{B})\subset Adj(\lambda_{j',\mathbf{u}}(y,\mathcal{B}))$, then $\lambda_{\mathbf{u}}(\mathcal{B})=\lambda_{j',\mathbf{u}}(y,\mathcal{B})$,
- else, if $j=\sup\{n:\lambda'_{\mathbf{u}}(\mathcal{B})\subset Adj(\lambda_{n,\mathbf{u}}(y,\mathcal{B}))\}$, then $\lambda_{\mathbf{u}}(\mathcal{B})=\lambda_{j,\mathbf{u}}(y,\mathcal{B})$ and it follows that $C_1 2^{-j}\le|y-2^{-\mathbf{l}'}k'|_{\mathbf{u}}\le C2^{-j}$.

In the first case, by hypothesis, $|c_{\lambda'_{\mathbf{u}}(\mathcal{B})}|\le d_{j',\mathbf{u}}(y,\mathcal{B})\le C2^{-j's}$, and as in (31), the sum $F_{1,j'}$ on the corresponding $\lambda'_{\mathbf{u}}(\mathcal{B})$ satisfies $\|F_{1,j'}\|_{L^\infty}\le C2^{-sj'}$.

In the second case, $|c_{\lambda'_{\mathbf{u}}(\mathcal{B})}|\le d_{j,\mathbf{u}}(y,\mathcal{B})\le C2^{-js}\le C|y-2^{-\mathbf{l}'}k'|^s_{\mathbf{u}}$, and as in (34) and (35), the sum $F_{2,j'}$ on the corresponding $\lambda'_{\mathbf{u}}(\mathcal{B})$ satisfies $\|F_{2,j'}\|_{L^\infty}\le C(2^{-j's}+$

$|x-y|_{\mathbf{u}}^{s}$) and $|\partial^I F_{2,j'}(x)| \le C2^{-j'(s-d_{\mathbf{u}}(I))}(1+2^{j'}|x-y|_{\mathbf{u}})^s$. And the conclusion is the same. □

As a consequence of the second and third results of Proposition 3, we have the following theorem.

Theorem 4 *If $f \in C_{\mathbf{u}}^{\beta}(\mathbb{R}^m, \mathcal{B})$ for $\beta > 0$, the $\mathbf{u}$-Hölder exponent of f can be expressed at every point by the formula*

$$h_{\mathbf{u},f}(x,\mathcal{B}) = \liminf_{j\to\infty} \frac{\log(d_{j,\mathbf{u}}(x,\mathcal{B}))}{\log(2^{-j})}.$$

Theorems 3 and 4 yield the following corollary.

Corollary 2 *Let $f \in C^{\varepsilon}(\mathbb{R}^m)$ for $\varepsilon > 0$. Let $e \in \mathbb{R}^m$ with $|e| = 1$. Let $\mathcal{B}$ be any orthonormal basis starting with the vector e. Let E be the set of all anisotropies $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$ satisfying $0 < u_1 \le 1$ and $u_2 = \dots = u_m = \frac{m-u_1}{m-1}$. The Hölder exponent of f in the direction e at y is given by*

$$\alpha_f(y,e) = \sup_{\mathbf{u}\in E}\left(\liminf_{j\to\infty} \frac{\log(d_{j,\mathbf{u}}(y,\mathcal{B}))}{\log(2^{-ju_1})}\right).$$

Remark 3 In Definition 5, if the signal $f(x)$ behaves like a $\mathbf{u}$-cusp-like singularity $f(y) + |x-y|_{\mathbf{u}}^{h}$ in a neighborhood of y then the $\mathbf{u}$-Hölder exponent of f at y will be given by the formula

$$h_{\mathbf{u},f}(y,\mathcal{B}) = \liminf_{j\to\infty} \inf_{k\in\mathbb{Z}^m,(G,\mathbf{l})\in I_{j,\mathbf{u}}} \frac{\log(|c_{j,k,\mathbf{u},\mathcal{B}}^{(G,\mathbf{l})}|)}{\log(2^{-j})}. \tag{39}$$

This is the case for self-affine functions F on bounded domains Ω (see [2])

$$F(x) = \sum_{i=1}^{L} \lambda_i F\big(S_i^{-1}(x)\big) + g(x), \tag{40}$$

where $L \ge 2$, g is a smooth and well localized function, the λ_i are scalars with $|\lambda_i| < 1$, $S_i(x) = \mu_i^{\frac{1}{u_1}\mathbf{u}} x + V_i$, $0 < \mu_i < 1$, , V_i are vectors in $\mathbb{R}^m$ and $\sum_{i=1}^{L} |\lambda_i| \mu_i^m < 1$.

Assume that

$$S_i(\Omega) \subset \Omega \quad \forall i \tag{41}$$

and

$$S_i(\Omega) \cap S_j(\Omega) = \emptyset \quad \forall i \ne j. \tag{42}$$

Let K be the non-empty compact set K that satisfies $K = \bigcup_{i=1}^{L} S_i(K)$. In [2], the following results were found:

- if $y \notin K$ then F is $C_{\mathbf{u}}^{k}$ in a neighborhood of y,

– if $y \in K$ and if $B_{j,\mathbf{u}}(y)$ is the set of $i = (i_1, \ldots, i_n)$ such that

$$\left|S_{i_1} \cdots S_{i_n}(0) - y\right|_{\mathbf{u}}^{u_1} \leq \mu_{i_1} \cdots \mu_{i_n}$$

and

$$2^{-j} \leq \mu_{i_1} \cdots \mu_{i_n} < 2^{-(j-1)},$$

then

$$h_{\mathbf{u},F}(y, \mathcal{B}) = u_1 \liminf_{j\to\infty} \inf_{i\in B_{j,\mathbf{u}}(y)} \frac{\log|\lambda_{i_1}|\cdots|\lambda_{i_n}|}{\log \mu_{i_1}\cdots\mu_{i_n}}.$$

So if a function f exhibits only $\mathbf{u}$-cusp-like pointwise singularities for all $\mathbf{u}$'s in the set E given in Theorem 1 of Sect. 2, then the Hölder exponent of f in the direction e at y is given by

$$\alpha_f(y, e) = \sup_{\mathbf{u}\in E}\left(\liminf_{j\to\infty} \inf_{k\in\mathbb{Z}^m,(G,\mathbf{l})\in I_{j,\mathbf{u}}} \frac{\log(|c^{(G,\mathbf{l})}_{j,k,\mathbf{u},\mathcal{B}}|)}{\log(2^{-ju_1})}\right).$$

Hence this exponent can be numerically obtained from Sect. 4 by discretizing E.

On the opposite from $\mathbf{u}$-cusp singularities are the $\mathbf{u}$-chirp-like singularities which display very strong oscillations in the neighborhood of y, such as

$$f(y) + |x - y|_{\mathbf{u}}^h \sin\left(\frac{1}{|x-y|_{\mathbf{u}}^{\beta}}\right)$$

where $\beta > 0$. In this case, the $\mathbf{u}$-Hölder exponent of f at y cannot be deduced from formula (39), but from Theorem 4. And numerically, the Hölder exponent of f in the direction e at y should be obtained from Sect. 4 by discretizing E.

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