J Fourier Anal Appl
DOI 10.1007/s00041-012-9222-9

Asymptotic Properties of Fourier Transforms of b-Decomposable Distributions

Toshiro Watanabe

Received: 28 January 2011

Abstract Erdös-Kahane numbers (EK numbers) are introduced in relation to the decay of the Fourier transforms of non-symmetric Bernoulli convolutions. The PV, PS, and EK numbers are characterized by using a certain trigonometric series $H_b(u)$. The relations between those numbers and the asymptotic properties of the Fourier transforms of full b-decomposable distributions are shown. A sufficient condition for the absolute continuity of one-dimensional b-decomposable distributions is given. As an application, an open problem on the uniform decay of the Fourier transforms of refinable distributions, raised by Dai et al. (J. Funct. Anal. 250(1):1–20, 2007), is solved. Finally, temporal evolution on continuity properties of distributions of some Lévy processes is discussed.

Keywords b-decomposable distribution · EK number · Bernoulli convolution · Lévy process

Mathematics Subject Classification 42B10 · 60G30

1 Introduction

In what follows, denote the Euclidean inner product of z and x and the norm of x in $\mathbb{R}^d$ by $\langle z, x\rangle$ and $|x|$, respectively. Let $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$, $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, $\mathbb{N} := \{1, 2, 3, \dots\}$, and $\mathbb{R}_+ := [0, \infty)$. The symbol $\delta_a(dx)$ stands for the delta measure at a in $\mathbb{R}^d$. Denote the convolution of probability distributions ρ and μ on $\mathbb{R}^d$ by

Communicated by Christian Houdré.

T. Watanabe (✉)
Center for Mathematical Sciences, University of Aizu, Aizu-Wakamatsu 965-8580, Japan
e-mail: t-watanb@u-aizu.ac.jp

Published online: 15 March 2012

$\rho * \mu$ and the characteristic function (Fourier transform) of a probability distribution μ on $\mathbb{R}^d$ by $\widehat{\mu}(z)$, namely,

$$\widehat{\mu}(z) := \int_{\mathbb{R}^d} e^{i\langle z,x\rangle}\mu(dx).$$

Denote by η^{n*} the nth convolution power of a finite measure η on $\mathbb{R}^d$ with the understanding that $\eta^{0*}(dx) = \delta_0(dx)$, and by $\bar{\eta}$ the reflection of η, that is, $\bar{\eta}(dx) = \eta(-dx)$. A probability distribution on $\mathbb{R}^d$ is said to be *full* if its support is not contained in any hyperplane in $\mathbb{R}^d$. Let $0 < b < 1$. A probability distribution μ on $\mathbb{R}^d$ is said to be *b-decomposable* if there exists a probability distribution ρ on $\mathbb{R}^d$ such that

$$\widehat{\mu}(z) = \widehat{\mu}(bz)\widehat{\rho}(z). \tag{1.1}$$

The decomposition (1.1) is equivalent to

$$\widehat{\mu}(z) = \prod_{n=0}^{\infty} \widehat{\rho}\big(b^n z\big).$$

The infinite product above converges if and only if ρ has a finite log-moment, namely,

$$\int_{\mathbb{R}^d} \log\big(2 + |x|\big)\rho(dx) < \infty. \tag{1.2}$$

See Lemma 1 of Bunge [2]. We see from Lemma 5 of Watanabe [28] that a b-decomposable distribution μ satisfying (1.1) with ρ is full if and only if ρ is full. A distribution μ on $\mathbb{R}^d$ is called *self-decomposable* if it is b-decomposable for every $b \in (0, 1)$. Self-decomposable distributions are infinitely divisible. Many statistically important distributions are known to be self-decomposable and all full self-decomposable distributions on $\mathbb{R}^d$ are absolutely continuous. See Sato [25]. A distribution μ on $\mathbb{R}^d$ is called a *homogeneous self-similar measure* with contraction ratio b if it is b-decomposable with some $b \in (0, 1)$ and the support of ρ is a finite set. The idea of b-decomposability was introduced by Loève [17] and then extended by Bunge [2] and Maejima et al. [18] in the theory of limit distributions for some sequence of normalized sums of independent random variables. Denote by $L(b)$ the totality of b-decomposable distributions on $\mathbb{R}^d$. The class $\bigcup_{b\in(0,1)} L(b)$ is a rich class which contains the semi-stable (including Gaussian) distributions and the semi-self-decomposable distributions. Wolfe [36] proved that every full distribution in $L(b)$ is either singular or absolutely continuous. The author [28, 29] studied continuity properties of some distributions in $L(b)$. Applications of the class $L(b)$ to limit theorems for shift self-similar additive random sequences, laws of iterated logarithm for Brownian motions on nested fractals, and exact Hausdorff and packing measures of random fractals on Galton-Watson trees are found in a series of papers [31–34]. Another application to generalized Ornstein-Uhlenbeck processes is found in Lindner and Sato [15, 16].

Let b and p be real numbers in $(0, 1)$. We define a Bernoulli convolution $\nu_{b,p}$ on $\mathbb{R}$ by

$$\widehat{\nu_{b,p}}(z) = \prod_{n=0}^{\infty}\big(p + (1-p)\exp\big(ib^n z\big)\big). \tag{1.3}$$

Note that $\nu_{b,p}$ is b-decomposable with $\rho(dx) = p\delta_0(dx) + (1-p)\delta_1(dx)$ in (1.1). Let $\nu_b := \nu_{b,p}$ with $p = 2^{-1}$. The probability distribution ν_b is called a symmetric Bernoulli convolution. An algebraic integer $\theta > 1$ is said to be a *Pisot-Vijayaraghavan number* (PV number, for short) if all its Galois conjugates θ_j satisfy $|\theta_j| < 1$. Kershner and Wintner [13] proved that, for $0 < b < 2^{-1}$, ν_b is a distribution on a Cantor set and hence it is singular. Wintner [35] noted that, for $b = 2^{-1}$, ν_b is the uniform distribution on $[0, 2]$. Jessen and Wintner [11] observed that, for $2^{-1} < b < 1$, ν_b is either singular or absolutely continuous. Erdös [6] showed that it is singular in case b^{-1} in $(1, 2)$ is a PV number. Garsia [9] gave concrete examples of b in $(2^{-1}, 1)$ such that ν_b is absolutely continuous. In 1995, Solomyak [26] proved that ν_b has an L^2-density for a.e. $b \in (2^{-1}, 1)$. Peres and Solomyak [21] showed that, for $1/3 \le p \le 2/3$, $\nu_{b,p}$ has an L^2-density for a.e. $b \in (p^2 + (1-p)^2, 1)$. We say that b in $(0, 1)$ is a *Peres-Solomyak number* (PS number, for short) if there exist p in $(0, 2^{-1})$ and a positive number q such that $\widehat{\nu_{b,p}}(z)$ belongs to $L^q(\mathbb{R})$, that is, $\int_{-\infty}^{\infty} |\widehat{\nu_{b,p}}(z)|^q dz < \infty$. We say that a nonnegative function $f(z)$ on $\mathbb{R}^d$ has *uniform decay at infinity* if there exists $\gamma > 0$ such that $f(z) = O(|z|^{-\gamma})$ as $|z| \to \infty$. Erdös [7] proved that $|\widehat{\nu_b}(z)|$ has uniform decay at infinity and ν_b is absolutely continuous for a.e. $b \in (a, 1)$ with a sufficiently close to 1. Kahane [12] pointed out that the Hausdorff dimension of the set of b in $(a, 1)$ such that ν_b is singular tends to 0 as $a \uparrow 1$. We say that b in $(0, 1)$ is an *Erdös-Kahane number* (EK number, for short) if there exists p in $(0, 2^{-1})$ such that $|\widehat{\nu_{b,p}}(z)|$ has uniform decay at infinity. EK numbers are always PS numbers, but its converse remains open. Applications of PS numbers to Lévy processes are found in Watanabe [29, 30] and Lindner and Sato [15]. They showed that the distribution of a certain Lévy process is singular for small time and absolutely continuous for large time. Using EK numbers, we see in Theorem 5.1 in Sect. 5 that more drastic temporal evolution can occur in continuities of the distribution of such a Lévy process.

In the following, we characterize the PV, PS, and EK numbers by using the function $H_b(u)$ defined by (1.4) below and discuss the relations between those numbers and the asymptotic properties of the characteristic functions of distributions in the class $L(b)$. Let $0 < b < 1$. Define functions $H_b(u)$, $I_b(t)$, and $J_b(t)$ on $\mathbb{R}_+$ as

$$\begin{aligned} H_b(u) &:= \sum_{k=0}^{\infty}\big(1-\cos\big(b^k u\big)\big), \\ I_b(t) &:= \int_0^{\infty} \exp\big(-tH_b(u)\big)du, \\ J_b(t) &:= \int_0^{\infty} u\exp\big(-tH_b(u)\big)du. \end{aligned} \tag{1.4}$$

Define the subsets $\mathbf{B}_j$ in $(0, 1)$ for $j = 1, 2, 3$ as

$$\mathbf{B}_1 := \left\{ b \in (0,1) : \lim_{u \to \infty} \exp\bigl(-H_b(u)\bigr) = 0 \right\},$$
$$\mathbf{B}_2 := \left\{ b \in (0,1) : I_b(t) < \infty \text{ for some } t > 0 \right\},$$
$$\mathbf{B}_3 := \left\{ b \in (0,1) : J_b(t) < \infty \text{ for some } t > 0 \right\}.$$

Our main results are as follows.

Theorem 1.1 *The following are equivalent*:

(1) $b \in \mathbf{B}_1$.
(2) b^{-1} *is not a PV number.*
(3) *For some* $p \in (0, 2^{-1})$, $\lim_{|z|\to\infty} \widehat{\nu_{b,p}}(z) = 0$.
(4) *For every full* $\mu \in L(b)$ *on* $\mathbb{R}^d$, $\lim_{|z|\to\infty} \widehat{\mu}(z) = 0$.

Remark 1.1

(i) In Theorem II on P. 40 of [24], Salem proved that, for $b \neq 1/2$, the condition (2) in Theorem 1.1 is equivalent to $\lim_{|z|\to\infty} \widehat{\nu}_b(z) = 0$. An operator version of a part of Theorem 1.1 is found in Theorem 2 of Watanabe [28].
(ii) Theorem 2.3 of Hu [10] gave a necessary and sufficient condition in order that the characteristic function $\widehat{\mu}(z)$ of a one-dimensional homogeneous self-similar measure μ tends to 0 as $|z| \to \infty$. Our Theorem 1.1 says that if the contraction ratio b is not the reciplocal of a PV number, then the characteristic function $\widehat{\mu}(z)$ of every full homogeneous self-similar measure μ on $\mathbb{R}^d$ tends to 0 as $|z| \to \infty$.

Theorem 1.2 *The following are equivalent*:

(1) $b \in \mathbf{B}_2$.
(2) b *is a PS number.*
(3) *For every full* $\mu \in L(b)$ *on* $\mathbb{R}^d$, $\widehat{\mu}(z) \in L^q(\mathbb{R}^d)$ *for some* $q > 0$.

Remark 1.2

(i) Let $n \in \mathbb{Z}_+$. For a.e. $b \in ((5/9)^{2^{-n}}, (5/9)^{2^{-n-1}})$, we have

$$I_b\bigl(2^{-n} \log 3\bigr) < \infty.$$

(ii) Let $c > 0$. For all $b \in (0, e^{-2c})$, we have $I_b(c) = \infty$.

Theorem 1.3 *The following are equivalent*:

(1) $b \in \mathbf{B}_3$.
(2) b *is an EK number.*
(3) $\exp(-H_b(|z|))$ *has uniform decay at infinity.*
(4) *For every full* $\mu \in L(b)$ *on* $\mathbb{R}^d$, $|\widehat{\mu}(z)|$ *has uniform decay at infinity.*

Remark 1.3

(i) Let $n \in \mathbb{Z}_+$. For a.e. $b \in (2^{-2^{-n}}, 2^{-2^{-n-1}})$, we have

$$\exp\bigl(-H_b(u)\bigr) = O\bigl(u^{(-0.0027)2^n}\bigr).$$

(ii) Let $c > 0$. For all $b \in (0, e^{-c})$, we have $J_b(c) = \infty$ and

$$\exp\bigl(-H_b(u)\bigr) \neq O\bigl(u^{-1/c}\bigr).$$

Theorem 1.4 *Let μ be a full b-decomposable distribution on $\mathbb{R}$ satisfying (1.1). Let $q := \rho * \bar{\rho}(\{0\})$. Then we have the following.*

(i) *Let N be the smallest positive integer satisfying $2^{-N}\log 3 + q < 1$. Then, for a.e. $b \in ((5/9)^{2^{-N}}, 1)$, μ has an L^2-density. In particular, if ρ has no point mass, then μ has an L^2-density for a.e. $b \in (\sqrt{5}/3, 1)$.*

(ii) *Let $n \in \mathbb{Z}_+$. We have, for a.e. $b \in (2^{-2^{-n}}, 2^{-2^{-n-1}})$,*

$$\bigl|\widehat{\mu}(z)\bigr| = O\bigl(|z|^{(-0.0013)(1-q)2^n}\bigr).$$

Remark 1.4 Theorem 1.4 holds for $\mu = \nu_{b,p}$ with $1 - q = 2p(1 - p)$. We apply Theorem 1.4 to a one-dimensional homogeneous self-similar measure in Proposition 4.1 in Sect. 4. Peres and Solomyak [21] discussed the absolute continuity of a one-dimensional homogeneous self-similar measure under the assumption of the transversality condition. Ngai and Wang [19] extended the results of [21] to the inhomogeneous case. But, we do not know whether the transversality condition holds on an interval $(a, 1)$ with a close to 1.

We do not yet know an explicit example of EK numbers or PS numbers. However, modifying the Erdös-Kahane argument for symmetric Bernoulli convolutions, we have the following.

Theorem 1.5 *All real numbers in $(0, 1)$ outside a set of Hausdorff dimension 0 are EK numbers and hence PS numbers.*

Corollary 1.1 *If a full distribution μ on $\mathbb{R}^d$ is b-decomposable for all b in a set of positive Hausdorff dimension, then $|\widehat{\mu}(z)|$ has uniform decay at infinity. In particular, for every full self-decomposable distribution μ on $\mathbb{R}^d$, $|\widehat{\mu}(z)|$ has uniform decay at infinity.*

In Sect. 2, we prove our main results. In Sect. 3, we discuss the structures of the PS numbers and the EK numbers. In Sects. 4 and 5, we give applications of our main results to refinable distributions and to some Lévy processes, respectively.

2 Proofs of the Main Results

First we give several preliminary lemmas for the proofs of the main results mentioned in Sect. 1.

Lemma 2.1 *Let $0 < b < 1$. Then we have the following.*

(i) *For $p \in (0,1)$,*

$$\left|\widehat{\nu_{b,p}}(z)\right|^2 \le \exp\bigl(-2p(1-p)H_b(|z|)\bigr).$$

(ii) *For $p \in (0, 2^{-1}) \cup (2^{-1}, 1)$,*

$$\left|\widehat{\nu_{b,p}}(z)\right|^2 \ge \exp\bigl(-\bigl|\log|2p-1|\bigr|H_b(|z|)\bigr).$$

Proof Let $p \in (0,1)$. Note that $1 + 2p(1-p)(\cos(b^k z) - 1) \ge 0$. Thus, using the inequality $x \le e^{x-1}$ for $x \ge 0$, we have by (1.3)

$$\begin{aligned}\left|\widehat{\nu_{b,p}}(z)\right|^2 &= \prod_{k=0}^{\infty}\bigl(1 + 2p(1-p)\bigl(\cos\bigl(b^k z\bigr) - 1\bigr)\bigr)\\ &\le \exp\bigl(-2p(1-p)H_b(|z|)\bigr).\end{aligned}$$

Next let $p \in (0, 2^{-1}) \cup (2^{-1}, 1)$. Using the inequality $1 - x \ge \exp(a^{-1}x\log(1-a))$ for $0 \le x \le a < 1$, we see that

$$\begin{aligned}\left|\widehat{\nu_{b,p}}(z)\right|^2 &= \prod_{k=0}^{\infty}\bigl(1 + 2p(1-p)\bigl(\cos\bigl(b^k z\bigr) - 1\bigr)\bigr)\\ &\ge \exp\bigl(-\bigl|\log|2p-1|\bigr|H_b(|z|)\bigr).\end{aligned}$$

Thus we have proved the lemma. □

Lemma 2.2 *Let μ be a full b-decomposable distribution μ on $\mathbb{R}^d$ satisfying (1.1).*

(i) *There are positive constants K_1 and K_2 depending only on ρ such that*

$$\left|\widehat{\mu}(z)\right|^2 \le \sup_{u \ge K_2|z|} \exp\bigl(-K_1 H_b(u)\bigr). \tag{2.1}$$

(ii) *Let $d = 1$. Let $C_\delta := \rho * \bar{\rho}(|x| > \delta)$ for $\delta > 0$. Then we have*

$$\left|\widehat{\mu}(z)\right|^2 \le C_\delta^{-1} \int_{|x|>\delta} \exp\bigl(-C_\delta H_b(|zx|)\bigr)\rho * \bar{\rho}(dx). \tag{2.2}$$

Proof Note that $\int_{\mathbb{R}^d} \cos\langle z, x\rangle \rho * \bar{\rho}(dx) = |\widehat{\rho}(z)|^2 \ge 0$. Using the inequality $x \le e^{x-1}$ for $x \ge 0$, we have

$$\begin{aligned}\left|\widehat{\mu}(z)\right|^2 &= \prod_{k=0}^{\infty} \int_{\mathbb{R}^d} \cos\bigl\langle b^k z, x\bigr\rangle \rho * \bar{\rho}(dx)\\ &\le \exp\left(\sum_{k=0}^{\infty} \int_{\mathbb{R}^d} \bigl(\cos\bigl\langle b^k z, x\bigr\rangle - 1\bigr)\rho * \bar{\rho}(dx)\right)\\ &= \exp\left(-\int_{\mathbb{R}^d} H_b\bigl(\bigl|\langle z, x\rangle\bigr|\bigr)\rho * \bar{\rho}(dx)\right).\end{aligned} \tag{2.3}$$

We find from Lemma 5 of Watanabe [28] that ρ and also $\rho * \bar{\rho}$ are full on $\mathbb{R}^d$. Thus there exist disjoint closed balls D_j for $1 \le j \le d$ in $\mathbb{R}^d$ such that $C_j := \rho * \bar{\rho}(D_j) > 0$ and, for any choice of $x_j \in D_j$, $\{x_j\}_{j=1}^d$ is a basis of $\mathbb{R}^d$. Let $K_1 := \min_{1\le j\le d} C_j$. There exists $K_2 > 0$ independent of any choice of $\{x_j\}$ such that $\max_{1\le j\le d} |\langle z, x_j\rangle| \ge K_2|z|$ for $z \in \mathbb{R}^d$. Applying Jensen's inequality, we conclude from (2.3) that

$$\begin{aligned}
\left|\widehat{\mu}(z)\right|^2 &\le \exp\left\{-\sum_{j=1}^d \int_{D_j} H_b\big(\left|\langle z, x_j\rangle\right|\big)\rho * \bar{\rho}(dx_j)\right\} \\
&\le \prod_{j=1}^d C_j^{-1} \int_{D_j} \exp\big\{-C_j H_b\big(\left|\langle z, x_j\rangle\right|\big)\big\}\rho * \bar{\rho}(dx_j) \\
&= \left(\prod_{j=1}^d C_j^{-1}\right) \int_{D_1\times\cdots\times D_d} \exp\left\{-\sum_{j=1}^d C_j H_b\big(\left|\langle z, x_j\rangle\right|\big)\right\} \prod_{j=1}^d \rho * \bar{\rho}(dx_j) \\
&\le \sup_{u\ge K_2|z|} \exp\big(-K_1 H_b(u)\big). \qquad (2.4)
\end{aligned}$$

Thus we have proved (i). Let $d = 1$. Applying Jensen's inequality again, we have by (2.3)

$$\begin{aligned}
\left|\widehat{\mu}(z)\right|^2 &\le \exp\left\{-\int_{|x|>\delta} H_b\big(|zx|\big)\rho * \bar{\rho}(dx)\right\} \\
&\le C_\delta^{-1} \int_{|x|>\delta} \exp\big(-C_\delta H_b\big(|zx|\big)\big)\rho * \bar{\rho}(dx).
\end{aligned}$$

Thus we have proved (ii). □

Lemma 2.3 *Let* $0 < b < 1$. *Then we have the following.*

(i) *If* $u, v \ge 0$ *and* $|u - v| \le 1$, *then* $|H_b(u) - H_b(v)| \le (1 - b)^{-1}$.
(ii) *There are positive constants* C_1 *and* C_2 *such that, for* $u \ge 0$,

$$H_b(u) \le \frac{2}{|\log b|}\log(2 + u) + C_1 \le C_2 \log(2 + u).$$

Proof Let $u, v \ge 0$ and $|u - v| \le 1$. We have

$$\big|H_b(u) - H_b(v)\big| \le \sum_{k=0}^\infty \big|\cos\big(b^k u\big) - \cos\big(b^k v\big)\big| \le \sum_{k=0}^\infty b^k|u - v| \le (1 - b)^{-1}.$$

Thus (i) is true. Let $N := \log u/|\log b|$ for $u \ge 1$. Then we have, for $u \ge 1$,

$$H_b(u) \le \sum_{0\le k\le N} 2 + \sum_{k>N} 2^{-1}u^2 b^{2k} \le 2\left(\frac{\log u}{|\log b|} + 1\right) + \frac{1}{2(1 - b^2)}.$$

Thus we have proved (ii). □

The following lemma is due to Pisot [22].

Lemma 2.4 *A real number $\theta > 1$ is a PV number if and only if there exists $t \neq 0$ such that $\sum_{n=1}^{\infty} \sin^2(t\theta^n)$ converges.*

Let $\theta > 1$. For $1 \le t < \theta$ and $n \in \mathbb{N}$, we can uniquely represent $\theta^n t$ as

$$\theta^n t = C_n + \epsilon_n,$$

where $C_n \in \mathbb{N}$ and $\epsilon_n \in [-2^{-1}, 2^{-1})$. The symbol $[x]$ stands for the largest integer not exceeding $x \in \mathbb{R}$ and $\sharp(A)$ does for the cardinality of a finite set A. We denote by $\dim E$ the Hausdorff dimension of a Borel set E in $\mathbb{R}^d$. The following lemma is due to Lemma 6.3 of Peres et al. [20]. Its idea goes back to Erdös [7].

Lemma 2.5 *Fix $a > 1$ and $\delta > 0$. For any $\epsilon > 0$, the following holds for all sufficiently large integers $n \ge n_0(a, \delta, \epsilon)$.*

(i) *Given C_n and C_{n+1}, there are at most $[1 + (a + \delta + 1)^2 + \epsilon]$ possibilities for C_{n+2}, independent of $\theta \in [a, a + \delta]$ and $t \in [1, \theta)$.*
(ii) *If*

$$\max\{|\epsilon_n|, |\epsilon_{n+1}|, |\epsilon_{n+2}|\} < \frac{1}{2(a + \delta + 1)^2 + \epsilon},$$

then C_{n+2} is uniquely determined by C_n and C_{n+1}, independent of $\theta \in [a, a+\delta]$ and $t \in [1, \theta)$.

Lemma 2.6 *Fix $a > 1$ and $\delta > 0$. Let $K \ge 3$. Suppose that*

$$0 < B < \frac{1 - \cos(\pi/(a + \delta + 1)^2)}{\log(a + \delta)}. \tag{2.5}$$

Then we have

$$\dim\left\{b \in \left[\frac{1}{a + \delta}, \frac{1}{a}\right] : \exp\bigl(-H_b(u)\bigr) \neq O\bigl(u^{-B/K}\bigr)\right\} \le \frac{\log K - (K - 1)\log(1 - K^{-1}) + 3\log[1 + (a + \delta + 1)^2]}{K \log a}. \tag{2.6}$$

Proof Let $N \in \mathbb{N}$ and $\epsilon > 0$. Define $\tau > 0$ and a set E_N as $\tau := (2(a+\delta+1)^2+\epsilon)^{-1}$ and

$$E_N := \bigl\{\theta \in [a, a + \delta] : \text{for some } t \in [1, \theta), \sharp\bigl\{n \in [1, N] : |\epsilon_n| \ge \tau\bigr\} \le N/K\bigr\}.$$

Let $E := \limsup_{N\to\infty} E_N$. Let $\theta \in [a, a + \delta]$. Choose $\epsilon > 0$ such that

$$B < \frac{1 - \cos(2\pi\tau)}{\log(a + \delta)}.$$

Note that there exists $M > 0$ independent of N such that, for $N \geq 2$,

$$\left|\frac{C_N}{C_{N-1}} - \theta\right| = \frac{|\theta\epsilon_{N-1} - \epsilon_N|}{C_{N-1}} \leq Ma^{-N}.$$

We see from Lemma 2.5 that, for $\theta \in E_N$, the possibilities of $\{C_j\}_{j=1}^N$ are at most $\text{Const} \cdot \binom{N}{[N/K]}[1 + (a + \delta + 1)^2 + \epsilon]^{3N/K}$. Thus E_N can be covered by at most $\text{Const} \cdot \binom{N}{[N/K]}[1 + (a + \delta + 1)^2 + \epsilon]^{3N/K}$ intervals with length $2Ma^{-N}$. Hence, using Stirling's formula, we have

$$\begin{aligned}\dim(E) &\leq \lim_{N\to\infty} \frac{\log(\binom{N}{[N/K]}[1 + (a+\delta+1)^2 + \epsilon]^{3N/K})}{N \log a} \\ &= \frac{\log K - (K-1)\log(1 - K^{-1}) + 3\log[1 + (a+\delta+1)^2 + \epsilon]}{K \log a}. \end{aligned} \tag{2.7}$$

Let $b = \theta^{-1}$ and represent $u \in [2\pi\theta, \infty)$ as $u = 2\pi t\theta^N$ with some $t \in [1, \theta)$ and $N \in \mathbb{N}$. Then we see that

$$\exp\bigl(-H_b(u)\bigr) \leq \exp\left(-\sum_{n=1}^{N}\bigl(1 - \cos(2\pi\epsilon_n)\bigr)\right).$$

Thus if $b \in [\frac{1}{a+\delta}, \frac{1}{a}]$ and

$$\exp\bigl(-H_b(u)\bigr) \neq O\bigl(u^{-B/K}\bigr),$$

then we have $b^{-1} = \theta \in E$. Hence, letting $\epsilon \to 0$, we obtain (2.6) from (2.7). □

Proof of Theorem 1.1 The equivalence (1) $\Longleftrightarrow$ (3) is clear from Lemma 2.1. It is obvious that (4) $\Longrightarrow$ (3). We see from (i) of Lemma 2.2 that (1) $\Longrightarrow$ (4). Suppose that (2) is not true, that is, b^{-1} is a PV number. Then we see from Lemma 2.4 that there exists $t > 0$ such that $\sum_{n=1}^{\infty} \sin^2(tb^{-n})$ converges. Thus we have

$$\lim_{n\to\infty} \exp\bigl(-H_b\bigl(2tb^{-n}\bigr)\bigr) = \exp\left(-H_b(2t) - 2\sum_{n=1}^{\infty}\sin^2\bigl(tb^{-n}\bigr)\right) > 0.$$

That is, (1) is not true, and hence (1) $\Longrightarrow$ (2). Finally, suppose that (1) is not true. Then there exist a sequence $\{2t_n b^{-m_n}\}_{n=1}^{\infty}$ such that $1 \leq t_n < b^{-1}$, $m_n \in \mathbb{N}$, $m_n \uparrow \infty$ as $n \to \infty$, and $\lim_{n\to\infty} t_n = t \in [1, b^{-1}]$ and that

$$\lim_{n\to\infty} \exp\bigl(-H_b\bigl(2t_n b^{-m_n}\bigr)\bigr) > 0.$$

We find that, for $n \geq N$ with $N \in \mathbb{N}$,

$$\begin{aligned}\exp\bigl(-H_b\bigl(2t_n b^{-m_n}\bigr)\bigr) &= \exp\left(-H_b(2t_n) - 2\sum_{k=1}^{m_n}\sin^2\bigl(t_n b^{-k}\bigr)\right) \\ &\leq \exp\left(-2\sum_{k=1}^{m_N}\sin^2\bigl(t_n b^{-k}\bigr)\right).\end{aligned}$$

Letting $n \to \infty$ and then $N \to \infty$, we have

$$\exp\left(-2\sum_{k=1}^{\infty}\sin^2\left(tb^{-k}\right)\right) > 0.$$

Thus, by Lemma 2.4, b^{-1} is a PV number, and hence (2) $\Longrightarrow$ (1). □

Proof of Theorem 1.2 The equivalence (1) $\Longleftrightarrow$ (2) is clear from Lemma 2.1. It is obvious that (3) $\Longrightarrow$ (2). Suppose that (1) is true. Then $I_b(t_0) < \infty$ for some $t_0 > 0$. Let $\{D_j\}$, $\{C_j\}$, and $\{x_j\}$ be the same as in the proof of (i) of Lemma 2.2. Let $x_j = {}^t(x_{1j}, x_{2j}, \dots, x_{dj})$ and define a real $d \times d$ matrix X as $X = (x_{ij})$. Clearly, if $x_j \in D_j$ for $1 \le j \le d$, then $\det X \neq 0$. Take any positive number q satisfying $2^{-1}qC_j \ge t_0$ for $1 \le j \le d$. Changing variables as $\langle z, x_j\rangle = u_j$ for $1 \le j \le d$, we obtain as in (2.4) that

$$\begin{aligned}
\int_{\mathbb{R}^d}\left|\widehat{\mu}(z)\right|^q dz &\le \int_{\mathbb{R}^d}\exp\left\{-2^{-1}q\sum_{j=1}^{d}\int_{D_j}H_b\left(\left|\langle z, x_j\rangle\right|\right)\rho*\bar{\rho}(dx_j)\right\}dz\\
&\le \int_{\mathbb{R}^d}dz\prod_{j=1}^{d}C_j^{-1}\int_{D_j}\exp\left\{-2^{-1}qC_jH_b\left(\left|\langle z, x_j\rangle\right|\right)\right\}\rho*\bar{\rho}(dx_j)\\
&= \left(\prod_{j=1}^{d}C_j^{-1}2I_b\left(2^{-1}qC_j\right)\right)\int_{D_1\times\cdots\times D_d}|\det X|^{-1}\prod_{j=1}^{d}\rho*\bar{\rho}(dx_j)\\
&< \infty.
\end{aligned}$$

Thus we see that (1) $\Longrightarrow$ (3). □

Proof of Remark 1.2 Let $n \in \mathbb{Z}_+$. Corollary 1.4 of Peres and Solomyak [21] says that, for a.e. $b \in (5/9, \sqrt{5/9})$ with $p = 3^{-1}$, $\widehat{\nu_{b,p}}(z) \in L^2(\mathbb{R})$. Thus, by (ii) of Lemma 2.1, the assertion is true for $n = 0$. Note that, for all $m \in \mathbb{N}$,

$$H_{b^{1/m}}(u) = \sum_{k=0}^{m-1}H_b\left(b^{k/m}u\right). \tag{2.8}$$

For general n, using (2.8) with $m = 2^n$ and applying the generalized Hölder's inequality, we have, for a.e. $b \in (5/9, \sqrt{5/9})$,

$$\begin{aligned}
I_{b^{1/m}}\left(m^{-1}\log 3\right) &= \int_0^{\infty}\prod_{k=0}^{m-1}\exp\left(-m^{-1}(\log 3)H_b\left(b^{k/m}u\right)\right)du\\
&\le \prod_{k=0}^{m-1}\left(\int_0^{\infty}\exp\left(-(\log 3)H_b\left(b^{k/m}u\right)\right)du\right)^{1/m}\\
&= b^{(-m+1)/(2m)}I_b(\log 3) < \infty.
\end{aligned}$$

Thus we proved (i). Assertion (ii) follows from (ii) of Lemma 2.3. □

Proof of Theorem 1.3 The equivalence (2) $\Longleftrightarrow$ (3) is clear from Lemma 2.1. Obviously, (3) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (2). We see from (i) of Lemma 2.2 that (3) $\Longrightarrow$ (4). Let $t > 0$ be arbitrary. We obtain from (i) of Lemma 2.3 that, for $v \geq 0$,

$$\begin{aligned} J_b(t) &\geq \int_v^{v+1} u \exp\bigl(-t H_b(u)\bigr) du \\ &\geq \exp\left(-\frac{t}{1-b}\right) v \exp\bigl(-t H_b(v)\bigr). \end{aligned}$$

Thus we have, for $v \geq 1$,

$$\exp\bigl(-H_b(v)\bigr) \leq \exp\left(\frac{1}{1-b}\right)\bigl(J_b(t)\bigr)^{1/t} v^{-1/t}. \tag{2.9}$$

Hence we have proved that (1) $\Longrightarrow$ (3). □

Proof of Remark 1.3 Let $n \in \mathbb{Z}_+$. For $n = 0$, take $K = 32$, $a = \sqrt{2}$, and $a + \delta = 2-$ in (2.6). Then we can take $B/K = 0.0027$ and make the dimension in (2.6) less than 1. Thus, by using (2.8) with $m = 2^n$, we find that (i) is true. By virtue of Theorem 4.5.2 of Bertin et al. [1], for a.e. $t > 0$, the sequence $\{\theta^n t\}_{n=1}^{\infty}$ is uniformly distributed mod 1. Fix such a t. We have

$$\lim_{n\to\infty} n^{-1} H_b\bigl(2\pi\theta^n t\bigr) = \int_0^1 \bigl(1 - \cos(2\pi u)\bigr) du = 1.$$

Thus we see that, for all $b \in (0, 1)$ and any $\epsilon > 0$,

$$\limsup_{u\to\infty} \exp\bigl(-H_b(u)\bigr) u^{\frac{1}{|\log b|}+\epsilon} = \infty.$$

Thus the second assertion of (ii) is true. The first assertion follows from (2.9). □

Proof of Theorem 1.4 Let $C_\delta := \rho * \bar{\rho}(|x| > \delta)$ for $\delta > 0$. We can choose $\delta > 0$ such that $2^{-N} \log 3 < C_\delta$. We obtain from (ii) of Lemma 2.2 and (i) of Remark 1.2 that, for a.e. $b \in ((5/9)^{2^{-N}}, 1)$,

$$\int_{-\infty}^{\infty} \bigl|\widehat{\mu}(z)\bigr|^2 dz \leq 2 I_b(C_\delta) C_\delta^{-1} \int_{|x|>\delta} |x|^{-1} \rho * \bar{\rho}(dx) < \infty.$$

Thus μ has an L^2-density. If ρ has no point mass, we have $q = 0$ and hence $N = 1$. Thus (i) is true. Assertion (ii) follows from (ii) of Lemma 2.2 and (i) of Remark 1.3. □

Proof of Theorem 1.5 Since $\mathbf{B}_3 \subset \mathbf{B}_2$, it is enough to prove that

$$\dim\bigl((0, 1) \setminus \mathbf{B}_3\bigr) = 0. \tag{2.10}$$

Let $K \geq 3$ and take B satisfying (2.5). We see from Theorem 1.3 that

$$\left[\frac{1}{a+\delta}, \frac{1}{a}\right] \setminus \mathbf{B}_3 \subset \left\{b \in \left[\frac{1}{a+\delta}, \frac{1}{a}\right] : \exp\bigl(-H_b(u)\bigr) \neq O\bigl(u^{-B/K}\bigr)\right\}.$$

Thus we obtain from Lemma 2.6 that

$$\dim\left(\left[\frac{1}{a+\delta},\frac{1}{a}\right]\setminus \mathbf{B}_3\right)$$
$$\le \frac{\log K-(K-1)\log(1-K^{-1})+3\log[1+(a+\delta+1)^2]}{K\log a}.$$

Letting $K\uparrow\infty$, and then $a\downarrow 1$ and $\delta\uparrow\infty$, we have (2.10). □

Proof of Corollary 1.1 The corollary follows from Theorems 1.3 and 1.5. □

Finally, we add a theorem of the Erdös-Kahane type for b-decomposable distributions. For an integer $k\ge 0$, let $C^k(\mathbb{R}^d)$ be the class of real-valued functions on $\mathbb{R}^d$ all of whose partial derivatives of order up to and including k are continuous.

Theorem 2.1

(i) *Let μ be a full b-decomposable distribution on $\mathbb{R}^d$. Suppose that b is an EK number. Then $|\widehat{\mu}(z)|$ has uniform decay at infinity. Thus, for every integer $k\ge 0$, the convolution power μ^{n*} is absolutely continuous with a bounded density of class $C^k(\mathbb{R}^d)$ for all sufficiently large integers n.*

(ii) *Fix a full probability distribution ρ on $\mathbb{R}^d$ satisfying* (1.2). *Then there are a positive strictly increasing sequence $\{a_k\}_{k=0}^{\infty}$ and a positive strictly decreasing sequence $\{c_k\}_{k=0}^{\infty}$ with $\lim_{k\to\infty}a_k=1$ and $\lim_{k\to\infty}c_k=0$ such that the following statement is true for every integer $k\ge 0$: For $b\in(a_k,1)$ outside a set of Hausdorff dimension less than c_k, any b-decomposable distribution μ on $\mathbb{R}^d$ satisfying* (1.1) *has a density of class $C^k(\mathbb{R}^d)$.*

Proof The first assertion of (i) follows from Theorem 1.3. The second one is a direct consequence of the first one. Next we prove (ii). We see from (2.1), (2.8), and Lemma 2.6 that we can take the two sequences $\{a_k\}_{k=0}^{\infty}$ and $\{c_k\}_{k=0}^{\infty}$. □

3 Structures of the Sets $\mathbf{B}_j$

Denote by $\|x\|$ the distance of $x\in\mathbb{R}$ to the nearest integer. We define a function $D(\theta)$ for $\theta>1$ as

$$D(\theta):=\inf_{1\le t<\theta}\limsup_{n\to\infty}\|\theta^n t\|.$$

Let $\mathbf{S}$ be the totality of PV numbers. Salem [23] proved that the set $\mathbf{S}$ is a closed set. All positive integers bigger than 1 are PV numbers. The minimum PV number is the positive zero of x^3-x-1. An algebraic integer $\theta>1$ is said to be a *Salem number* if all its Galois conjugates θ_j satisfy $|\theta_j|\le 1$ and at least one of the conjugates has modulus equal to one. Let $\mathbf{T}$ be the totality of Salem numbers and $\mathbf{T}^{-1}$ be the set of their reciprocals. Salem numbers are not PV numbers. We say that a real number $\theta>1$ is a *generalized Salem number* if $D(\theta)=0$. Let $\mathbf{T}^*$ be the totality of generalized

Salem numbers and $(\mathbf{T}^*)^{-1}$ be the set of their reciprocals. Let $\mathbf{S}_\perp$ be the set of b in $(2^{-1}, 1)$ such that ν_b is singular, and let $\mathbf{S}_\perp^* (\supset \mathbf{S}_\perp)$ be the set of b in $(2^{-1}, 1)$ such that ν_b does not have an L^2-density. Let E be a set in $(0, 1)$. We say that E is of *type A* if E contains a left neighborhood of 1, that is, E contains an interval $(1-\epsilon, 1)$ for some $\epsilon \in (0, 1)$. We say that E is of *type B* if, for any nonempty open interval I in $(0, 1)$, $I \setminus E$ is an uncountable set. Clearly, if E is of type B, then it is totally disconnected. We do not yet know whether $\mathbf{S}_\perp$ is a countable set. However, we prove in Theorem 3.1 below that if 1 is a limit point of $\mathbf{S}_\perp$, then $\mathbf{S}_\perp$ is an uncountable dense set in $(2^{-1}, 1)$ and $\mathbf{B}_2$ and $\mathbf{B}_3$ are of type B.

Lemma 3.1 *Suppose that a set E in $(0, 1)$ is a F_σ set and that if $b \in E$, then $b^{1/n} \in E$ for all $n \in \mathbb{N}$. Then we have the following.*

(i) *E is either of type A or of type B.*
(ii) *If $(0, 1) \setminus E$ is a countable set, then E is of type A and $(0, 1) \setminus E$ is nowhere dense in $(0, 1)$.*

Proof Suppose that there exists a nonempty open interval I in $(0, 1)$ such that $I \setminus E$ is a countable set. Then, by virtue of the Baire-Hausdorff theorem, $I \cap E$ contains a nonempty open interval. We see that if 1 is a limit point of $(0, 1) \setminus E$, then $I \cap E$ cannot include a nonempty open interval. Thus E contains a left neighborhood of 1 and the proof of (i) is complete. Suppose that $(0, 1) \setminus E$ is a countable set. Then E is not of type B and hence it is of type A. Let I be any nonempty open interval in $(0, 1)$. Then, as in the proof of (i), $I \cap E$ includes a nonempty open interval and thereby $(0, 1) \setminus E$ is nowhere dense in $(0, 1)$. Thus (ii) is true. □

Proposition 3.1 *We have the following.*

(i) $\mathbf{B}_3 \subset \mathbf{B}_2 \subset \mathbf{B}_1$. $\mathbf{B}_1$ *is an open set.* $\mathbf{B}_2$ *and* $\mathbf{B}_3$ *are F_σ sets.*
(ii) $\mathbf{S} \cup \mathbf{T} \subset \mathbf{T}^*$, $(\mathbf{T}^*)^{-1} \cap \mathbf{B}_3 = \emptyset$, *and* $\mathbf{T}^{-1} \subset \mathbf{B}_1 \setminus \mathbf{B}_3$.
(iii) *Let $j = 1, 2, 3$. If $b \in \mathbf{B}_j$, then $b^{1/n} \in \mathbf{B}_j$ for all $n \in \mathbb{N}$.*
(iv) $\mathbf{B}_1$ *is of type A. Assertions* (i) *and* (ii) *of Lemma* 3.1 *are true for $E = \mathbf{B}_2$ and for $E = \mathbf{B}_3$.*

Proof First we prove (i). It is clear that $\mathbf{B}_3 \subset \mathbf{B}_2$. Suppose that $b \notin \mathbf{B}_1$. Then there is a sequence $\{u_n\}_{n=0}^\infty$ such that $u_0 = 1$, $u_{n+1} \geq u_n + 1$ for $n \geq 0$ and that

$$\lim_{n\to\infty} \exp\bigl(-H_b(u_n)\bigr) \in (0, 1].$$

Let $t > 0$ be arbitrary. Thus we see from (i) of Lemma 2.3 that

$$\begin{aligned} I_b(t) &\geq \sum_{n=0}^\infty \int_{u_n}^{u_n+1} \exp\bigl(-t H_b(u)\bigr) du \\ &\geq \exp\left(-\frac{t}{1-b}\right) \sum_{n=0}^\infty \exp\bigl(-t H_b(u_n)\bigr) = \infty. \end{aligned}$$

Thus $b \notin \mathbf{B}_2$ and hence $\mathbf{B}_2 \subset \mathbf{B}_1$. Since $\mathbf{S}$ is a closed set, we find from Theorem 1.1 that $\mathbf{B}_1$ is an open set. Note that

$$H_b(u) \le \frac{1}{2}\sum_{n=0}^{\infty} b^{2n}u^2 = \frac{u^2}{2(1-b^2)}.$$

By virtue of the dominated convergence theorem, we see that $H_b(u)$ is continuous in $b \in [1/(n+1), n/(n+1)]$ for $n \in \mathbb{N}$. We have

$$\begin{aligned}\mathbf{B}_2 &= \bigcup_{n\in\mathbb{N}}\bigcup_{N\in\mathbb{N}}\bigcup_{t\in\mathbb{N}}\left\{b \in \left[1/(n+1), n/(n+1)\right] : I_b(t) \le N\right\}, \quad \text{and}\\ \mathbf{B}_3 &= \bigcup_{n\in\mathbb{N}}\bigcup_{N\in\mathbb{N}}\bigcup_{t\in\mathbb{N}}\left\{b \in \left[1/(n+1), n/(n+1)\right] : J_b(t) \le N\right\}.\end{aligned} \tag{3.1}$$

Thus $\mathbf{B}_2$ and $\mathbf{B}_3$ are F_σ sets. Next we prove (ii). It is well known that if $\theta \in \mathbf{S}$, then $\lim_{n\to\infty}\|\theta^n\| = 0$. Hence $\mathbf{S} \subset \mathbf{T}^*$. We see from Theorem 5.5.1 of Bertin et al. [1] that $\mathbf{T} \subset \mathbf{T}^*$. Thus, $\mathbf{S} \cup \mathbf{T} \subset \mathbf{T}^*$. Suppose that $b := \theta^{-1} \in (\mathbf{T}^*)^{-1}$. Then, for any $\delta > 0$, there is $t \in [1, \theta)$ such that $\|\theta^n t\| \le \delta$ for all sufficiently large $n \in \mathbb{N}$. Let $\epsilon := 4\pi^2\delta^2/\log\theta$. Since $\delta > 0$ is arbitrary, so is $\epsilon > 0$. We obtain that

$$\limsup_{u\to\infty}\exp\bigl(-H_b(u)\bigr)u^\epsilon \ge \limsup_{n\to\infty}\exp\bigl(-H_b\bigl(2\pi t\theta^n\bigr)\bigr)\bigl(2\pi t\theta^n\bigr)^\epsilon = \infty.$$

Namely, by Theorem 1.3, $b \notin \mathbf{B}_3$ and thereby $(\mathbf{T}^*)^{-1} \cap \mathbf{B}_3 = \emptyset$. Since Salem numbers are not PV numbers, the inclusion $\mathbf{T}^{-1} \subset \mathbf{B}_1$ is clear. Thus $\mathbf{T}^{-1} \subset \mathbf{B}_1 \setminus \mathbf{B}_3$ because $\mathbf{T} \subset \mathbf{T}^*$. To prove (iii), note from (2.8) that, for all $n \in \mathbb{N}$, $H_{b^{1/n}}(u) \ge H_b(u)$. Thus, if $b \in \mathbf{B}_j$, then $b^{1/n} \in \mathbf{B}_j$ for all $n \in \mathbb{N}$ and $j = 1, 2, 3$. Next, we prove (iv). Since there exists the minimum PV number greater than 1, we find from Theorem 1.1 that $\mathbf{B}_1$ is of type A. Remaining assertions follow from Lemma 3.1. □

Remark 3.1

(i) Theorem 5.6.1 of Bertin et al. [1] says that the set $\{\theta > 1 : D(\theta) < (2(\theta+1)^2)^{-1}\}$ is a countable set and hence $\mathbf{T}^*$ is a countable set.

(ii) Let $M(\theta) := \sum_{k=0}^n |a_k|$ for an algebraic number $\theta > 1$ where $P(x) := \sum_{k=0}^n a_k x^k$ is the minimal polynomial of θ. Theorem 1 of Dubickas [4] says that if $\theta > 1$ is an algebraic number satisfying $\theta \notin \mathbf{S} \cup \mathbf{T}$, then we have

$$D(\theta) \ge \bigl(M(\theta)\bigr)^{-1}.$$

Thus if $\theta \in \mathbf{T}^*$ is an algebraic number, then $\theta \in \mathbf{S} \cup \mathbf{T}$.

(iii) We do not yet know whether the following equations are true: $\mathbf{S}_\perp = \mathbf{S}_\perp^*$; $\mathbf{S}_\perp \cap \mathbf{B}_1 = \emptyset$; $\mathbf{B}_1 = \mathbf{B}_2$; $\mathbf{B}_2 = \mathbf{B}_3$; $\mathbf{S} \cup \mathbf{T} = \mathbf{T}^*$; and $(\mathbf{T}^*)^{-1} \cup \mathbf{B}_3 = (0, 1)$. We see from (ii) of Proposition 3.1 that $\mathbf{B}_3 \subsetneq \mathbf{B}_2$ or $\mathbf{B}_2 \subsetneq \mathbf{B}_1$ holds. Feng and Wang [8] showed that $\mathbf{S}_\perp \subsetneq \mathbf{S}_\perp^*$ or $\mathbf{S}_\perp \cap \mathbf{B}_1 \ne \emptyset$ holds.

At this point, it is not easy to answer the fascinating question whether $\mathbf{B}_2$ and $\mathbf{B}_3$ are of type A. However, we can find that some strong assertions are true under the assumption that $\mathbf{B}_2$ and $\mathbf{B}_3$ are of type A.

Proposition 3.2 *Suppose that* $\mathbf{B}_2$ *is of type A. Then we have the following.*

(i) *There are* $N_1 \in \mathbb{N}$ *and* $t_1 \in \mathbb{N}$ *such that* $I_b(t_1) \le N_1$ *for all* b *in a left neighborhood of* 1.
(ii) *Fix a full probability distribution* ρ *on* $\mathbb{R}^d$ *satisfying* (1.2). *Then, for all* b *in a left neighborhood of* 1, *any* b*-decomposable distribution* μ *on* $\mathbb{R}^d$ *satisfying* (1.1) *has a bounded continuous density.*
(iii) 1 *is not a limit point of* $\mathbf{S}^*_\perp$.

Proof Suppose that $\mathbf{B}_2$ is of type A, that is, $\mathbf{B}_2$ contains a left neighborhood I_0 of 1. Then, by virtue of the Baire-Hausdorff theorem, we see from (3.1) that $I_0 \cap \{b \in (0,1) : I_b(t_1) \le N_1\}$ includes a non-empty open interval for some $N_1 \in \mathbb{N}$ and $t_1 \in \mathbb{N}$. Let $B := \{b \in (0,1) : I_b(t_1) \le N_1\}$. Like in (iii) of Proposition 3.1, if $b \in B$, then $b^{1/n} \in B$ for all $n \in \mathbb{N}$. In case 1 is a limit point of $(0,1) \setminus B$, B cannot include a non-empty open interval. Thus we see that B contains a left neighborhood I_1 of 1. Hence (i) is true. Next to prove (ii), let $\{D_j\}$, $\{C_j\}$, $\{x_j\}$, and the matrix X be the same as in the proof of Theorem 1.2. Choose $n \in \mathbb{N}$ satisfying $2^{-1} n C_j \ge t_1$ for $1 \le j \le d$ and let $b \in I_1$. Fix a full probability distribution ρ on $\mathbb{R}^d$ satisfying (1.2). Define a $b^{1/n}$-decomposable distribution $\mu_{(n)}$ on $\mathbb{R}^d$ by

$$\widehat{\mu_{(n)}}(z) = \prod_{k=0}^{\infty} \widehat{\rho}\big(b^{k/n} z\big). \tag{3.2}$$

Define a constant K as

$$K := \int_{D_1 \times \cdots \times D_d} |\det X|^{-1} \prod_{j=1}^{d} \rho * \bar{\rho}(dx_j).$$

In the following calculation, we use Jensen's inequality together with (2.8), change variables as $\langle z, x_j\rangle = u_j$ for $1 \le j \le d$, and apply the generalized Hölder's inequality. Thus, replacing b by $b^{1/n}$, we obtain from (2.3) that

$$\begin{aligned}
\int_{\mathbb{R}^d} \big|\widehat{\mu_{(n)}}(z)\big| dz &\le \int_{\mathbb{R}^d} \exp\left\{ -2^{-1} \sum_{j=1}^{d} \int_{D_j} H_{b^{1/n}}\big(|\langle z, x_j\rangle|\big) \rho * \bar{\rho}(dx_j) \right\} dz \\
&\le \int_{\mathbb{R}^d} dz \prod_{j=1}^{d} C_j^{-1} \int_{D_j} \prod_{\ell=0}^{n-1} \exp\big\{-2^{-1} C_j H_b\big(\big|\big\langle b^{\ell/n} z, x_j\big\rangle\big|\big)\big\} \rho * \bar{\rho}(dx_j) \\
&= K \prod_{j=1}^{d} C_j^{-1} \int_{-\infty}^{\infty} \prod_{\ell=0}^{n-1} \exp\big(-2^{-1} C_j H_b\big(b^{\ell/n} |u_j|\big)\big) du_j \\
&\le K \prod_{j=1}^{d} C_j^{-1} \prod_{\ell=0}^{n-1} \left(\int_{-\infty}^{\infty} \exp\big(-2^{-1} n C_j H_b\big(b^{\ell/n} |u_j|\big)\big) du_j \right)^{1/n} \\
&\le K \left(\prod_{j=1}^{d} C_j^{-1} \right) b^{(-n+1)d/(2n)} \left(\int_{-\infty}^{\infty} \exp\big(-t_1 H_b(|u|)\big) du \right)^{d} < \infty.
\end{aligned}$$

Therefore, the b-decomposable distribution μ satisfying (1.1) has a bounded continuous density for all b in a left neighborhood of 1. Hence, setting $\mu = \nu_b$, we see that 1 is not a limit point of $\mathbf{S}_\perp^*$. Thus (ii) and (iii) are true. □

Remark 3.2 Peres et al. [20] showed that $\mathbf{S}_\perp$ is a G_δ set. Since $|\widehat{\nu_b}(z)|$ is continuous in $b \in [1/(n+1), n/(n+1)]$ for $n \in \mathbb{N}$, $\mathbf{S}_\perp^*$ is clearly a G_δ set. Note that $\widehat{\nu_{b^{1/n}}}(z) = \prod_{k=0}^{n-1} \widehat{\nu_b}(b^{k/n}z)$ for all $n \in \mathbb{N}$. Hence it is clear that $b^{1/n} \in (2^{-1}, 1) \setminus \mathbf{S}_\perp$ (resp. $b^{1/n} \in (2^{-1}, 1) \setminus \mathbf{S}_\perp^*$) for all $n \in \mathbb{N}$ provided that $b \in (2^{-1}, 1) \setminus \mathbf{S}_\perp$ (resp. $b \in (2^{-1}, 1) \setminus \mathbf{S}_\perp^*$). Thus we see that the assertions of Lemma 3.1 are true for the sets $(2^{-1}, 1) \setminus \mathbf{S}_\perp$ and $(2^{-1}, 1) \setminus \mathbf{S}_\perp^*$ instead of E by replacing the interval $(0, 1)$ with the interval $(2^{-1}, 1)$.

Proposition 3.3 *Suppose that* $\mathbf{B}_3$ *is of type A. Then the following hold.*

(i) 1 *is not a limit point of* $\mathbf{S}_\perp^* \cup \mathbf{T}^*$.
(ii) *There are* $N_2 \in \mathbb{N}$ *and* $t_2 \in \mathbb{N}$ *such that* $J_b(t_2) \le N_2$ *for all* b *in a left neighborhood of* 1.
(iii) *Fix a full probability distribution* ρ *on* $\mathbb{R}^d$ *satisfying* (1.2). *Fix an integer* $k \ge 0$. *Then, for all* b *in a left neighborhood of* 1, *any* b*-decomposable distribution* μ *on* $\mathbb{R}^d$ *satisfying* (1.1) *has a bounded density of class* $C^k(\mathbb{R}^d)$.
(iv) *We have*

$$\liminf_{\theta \downarrow 1} D(\theta) > 0. \tag{3.3}$$

Proof Suppose that $\mathbf{B}_3$ is of type A. Assertion (i) is obvious from (i) and (ii) of Proposition 3.1 and (iii) of Proposition 3.2. The proof of assertion (ii) is similar to that of (i) of Proposition 3.2 by using the Baire-Hausdorff theorem again and is omitted. Thus there are $N_2 \in \mathbb{N}$ and $t_2 \in \mathbb{N}$ such that $J_b(t_2) \le N_2$ for all b in a left neighborhood I_2 of 1. Fix a full probability distribution ρ on $\mathbb{R}^d$ satisfying (1.2). Let $\mu_{(n)}$ be the $b^{1/n}$-decomposable distribution defined by (3.2) with $b \in I_2$. Then we see from (2.1), (2.8), and (2.9) that, for every integer $k \ge 0$, there is a sufficiently large $n \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^d} |z|^k \left|\widehat{\mu_{(n)}}(z)\right| dz < \infty.$$

Hence the b-decomposable distribution μ satisfying (1.1) has a bounded density of class $C^k(\mathbb{R}^d)$ for all b in a left neighborhood of 1. Thus (iii) is true. Finally, we prove (iv). Let $\gamma := t_2^{-1}$ and $I_2 := (\delta, 1)$ with some $\delta \in (0, 1)$. We see from (ii) and (2.9) that, for all $b \in (\delta, 1)$,

$$\exp\bigl(-H_b(u)\bigr) = O\bigl(u^{-\gamma}\bigr).$$

Hence we obtain from (2.8) that, for all $b \in (\delta^{2^{-k}}, \delta^{2^{-k-1}}]$ with $k \in \mathbb{Z}_+$,

$$\exp\bigl(-H_b(u)\bigr) = O\bigl(u^{-\gamma 2^k}\bigr). \tag{3.4}$$

Let $\epsilon := \sqrt{2\gamma|\log\delta|}/(4\pi)$. Suppose that (3.3) fails. Then there exists a sufficiently large $k \in \mathbb{N}$ such that, for some $\theta \in [\delta^{-2^{-k-1}}, \delta^{-2^{-k}})$ and $t \in [1, \theta)$,

$$\limsup_{n\to\infty} \|\theta^n t\| < \epsilon.$$

Therefore we have

$$\lim_{n\to\infty} \exp\bigl(-H_{\theta^{-1}}\bigl(2\pi t\theta^n\bigr)\bigr)\bigl(2\pi t\theta^n\bigr)^{4\pi^2\epsilon^2/\log\theta} = \infty.$$

Since $\gamma 2^k \ge 4\pi^2\epsilon^2/\log\theta$, we see from (3.4) that this is a contradiction. Thus we have proved (iv). □

Consequently, we can establish the following facts on the types of $\mathbf{B}_2$ and $\mathbf{B}_3$.

Theorem 3.1

(i) *If* 1 *is a limit point of* $\mathbf{S}_\perp$ (*resp.* $\mathbf{S}_\perp^*$), *then* $\mathbf{S}_\perp$ (*resp.* $\mathbf{S}_\perp^*$) *is an uncountable dense set in* $(2^{-1}, 1)$ *and* $\mathbf{B}_2$ *and* $\mathbf{B}_3$ *are of type B.*

(ii) $\mathbf{B}_2$ *is of type A if and only if, for some fixed* $p \neq 2^{-1}$, $\nu_{b,p}$ *has a bounded continuous density for all b in a left neighborhood of* 1.

(iii) $\mathbf{B}_3$ *is of type A if and only if, for some fixed* $p \neq 2^{-1}$, $\nu_{b,p}$ *has a density of class* $C^1(\mathbb{R})$ *for all b in a left neighborhood of* 1.

Proof The proof of (i) is clear from Remark 3.2 and Propositions 3.1, 3.2, and 3.3. Fix $p \in (0, 2^{-1}) \cup (2^{-1}, 1)$. Suppose that $\mathbf{B}_2$ is of type A. Then we see from Proposition 3.2 that $\nu_{b,p}$ has a bounded continuous density for all b in a left neighborhood of 1. Conversely, suppose that $\mathbf{B}_2$ is of type B. Then we find from (ii) of Lemma 2.1 that $\nu_{b,p}$ does not have an L^2-density for some b arbitrarily close to 1. Thus (ii) is true. Next suppose that $\mathbf{B}_3$ is of type A. Then we see from Proposition 3.3 that $\nu_{b,p}$ has a density of class $C^1(\mathbb{R})$ for all b in a left neighborhood of 1. Conversely, suppose that $\mathbf{B}_3$ is of type B. Note that $\nu_{b,p}$ has a compact support. Thus we find from (ii) of Lemma 2.1 that, for some b arbitrarily close to 1, $\widehat{\nu_{b,p}}(z)$ does not have uniform decay at infinity and hence $\nu_{b,p}$ does not have a density of class $C^1(\mathbb{R})$. Thus the proof of (iii) is complete. □

We finish this section by posing three open problems which are of interest from the view-point of number theory. It is known that if the answer to Lehmer's problem in [14] on the minimum Mahler measure for integer polynomials is "no" (that is, so-called Lehmer's conjecture is true), then 1 is not a limit point of $\mathbf{T}$. It should be noted that if the answer to one of those open problems is "yes", then 1 is not a limit point of $\mathbf{T}^*(\supset \mathbf{T})$.

Problem 1 Is it true that $\liminf_{\theta\downarrow 1} D(\theta) > 0$?

Problem 2 Is it true that all real numbers in a left neighborhood of 1 are EK numbers?

Problem 3 Is it true that if $b^{-1} \in (1,\infty)$ is not a PV nor Salem number, then b is an EK number?

4 Applications to Refinable Distributions

Let $\lambda \in \mathbb{R}$ with $|\lambda| > 1$. A Schwartz distribution function f_λ on $\mathbb{R}$ is called λ-*refinable function* if there exist $m \geq 2$ and $c_j, d_j \in \mathbb{R}$ for $1 \leq j \leq m$ with $\sum_{j=1}^m c_j = |\lambda|$ such that, in the sense of Schwartz distribution theory,

$$f_\lambda(x) = \sum_{j=1}^m c_j f_\lambda(\lambda x - d_j). \tag{4.1}$$

Note that f_λ always has a compact support. A probability distribution μ_λ on $\mathbb{R}$ is called λ-*refinable distribution* if there exist $m \geq 2$ and $c_j > 0$, $d_j \in \mathbb{R}$ for $1 \leq j \leq m$ with $\sum_{j=1}^m c_j = |\lambda|$ such that

$$\mu_\lambda(dx) = \sum_{j=1}^m \frac{c_j}{|\lambda|}\mu_\lambda(\lambda dx - d_j) \tag{4.2}$$

with the understanding that $\int_B \mu_\lambda(\lambda dx - d_j) = \mu_\lambda(\lambda B - d_j)$ for Borel sets B in $\mathbb{R}$. We assume that μ_λ is not a delta measure. If the λ-refinable distribution μ_λ is absolutely continuous, then its density $f_\lambda(x)$ satisfies (4.1) in the usual sense. The Bernoulli convolution $\nu_{b,p}$ is a typical example of b^{-1}-refinable distributions with $m = 2, c_1 = b^{-1}p, c_2 = b^{-1}(1-p), d_1 = 0$, and $d_2 = b^{-1}$. Note that λ-refinable distributions μ_λ are λ^{-2}-decomposable because they satisfy (1.1) with $b = \lambda^{-2}$ and

$$\rho(dx) = \left(\sum_{j=1}^m c_j|\lambda|^{-1}\delta_{d_j}(\lambda dx)\right) * \left(\sum_{j=1}^m c_j|\lambda|^{-1}\delta_{d_j}\left(\lambda^2 dx\right)\right).$$

Thus they are homogeneous self-similar measures. In particular, if $\lambda > 1$, then μ_λ is λ^{-1}-decomposable with $\rho(dx) = \sum_{j=1}^m c_j\lambda^{-1}\delta_{d_j}(\lambda dx)$.

Dai et al. [3] discussed several cases where λ-refinable distributions have uniform decay at infinity. In the first assertion of the following theorem, we solve the third open problem raised by Dai et al. [3] in their appendix.

Theorem 4.1

(i) *Every real number λ in $(-\infty,-1)\cup(1,\infty)$ outside a set of Hausdorff dimension 0 has the property that, for any λ-refinable distribution μ_λ on $\mathbb{R}$, $|\widehat{\mu_\lambda}(z)|$ has uniform decay at infinity.*

(ii) *Fix $c'_j > 0$ and $d'_j \in \mathbb{R}$ for $1 \leq j \leq m$. Then there are positive strictly decreasing sequences $\{a_k\}_{k=0}^\infty$ and $\{e_k\}_{k=0}^\infty$ with $\lim_{k\to\infty} a_k = 1$ and $\lim_{k\to\infty} e_k = 0$ such that the following statement is true for each integer $k \geq 0$: For $\lambda \in (1, a_k)$ outside a set of Hausdorff dimension less than e_k, any λ-refinable distribution μ_λ on $\mathbb{R}$ satisfying (4.2) with $c_j = c'_j\lambda$ and $d_j = d'_j\lambda$ has a density of class $C^k(\mathbb{R})$.*

Proof Theorem 4.1 is a direct consequence of Theorems 1.5 and 2.1. □

We can show the absolute continuity of refinable distributions μ_λ for a.e. $\lambda \in (1, a)$ with some $a > 1$ without assuming the transversality condition used in Theorem 1.3 of Peres and Solomyak [21]. Moreover, we can give the decay order of the Fourier transforms of μ_λ for a.e. $\lambda \in (1, 2)$.

Proposition 4.1 *Let $\lambda > 1$ and $q := \sum_{j=1}^m c_j^2 \lambda^{-2}$. Then we have the following.*

(i) *Let N be the smallest positive integer satisfying $2^{-N} \log 3 + q < 1$. Then, for a.e. $\lambda \in (1, (9/5)^{2^{-N}})$, μ_λ has an L^2-density. In particular, let $m \geq 3$ and $c_j = \lambda/m$ for $1 \leq j \leq m$. Then $N = 1$ and μ_λ has an L^2-density for a.e. $\lambda \in (1, 3/\sqrt{5})$.*

(ii) *Let $n \in \mathbb{Z}_+$. We have, for a.e. $\lambda \in (2^{2^{-n-1}}, 2^{2^{-n}})$,*

$$\left|\widehat{\mu_\lambda}(z)\right| = O\big(|z|^{(-0.0013)(1-q)2^n}\big).$$

Proof The distribution μ_λ is λ^{-1}-decomposable with $\rho(dx) = \sum_{j=1}^m c_j \lambda^{-1} \delta_{d_j}(\lambda dx)$. Note that $\rho * \bar{\rho}(\{0\}) = q$. Thus the proof is clear from Theorem 1.4. □

We add a result on the non-smoothness of the distributions in $L(b)$. We say that a probability distribution μ on $\mathbb{R}^d$ belongs to the class $W^\infty(\mathbb{R}^d)$ if, for every $\gamma > 0$, $|\widehat{\mu}(z)| = O(|z|^{-\gamma})$ as $|z| \to \infty$. If $\mu \in W^\infty(\mathbb{R}^d)$, then μ has a bounded density of class $C^\infty(\mathbb{R}^d)$. The converse is also true provided that the support of μ is compact. Dubickas and Xu [5] proved in their Theorem 1.1 that a λ-refinable distribution μ_λ does not have a density of class $C^\infty(\mathbb{R})$, provided that $\lambda > 1$ and that all d_j/d_{j_0} for $1 \leq j \leq m$ are rational with some $d_{j_0} \neq 0$. Thus $\nu_{b,p}$ does not have a density of class $C^\infty(\mathbb{R})$ for every b and p in $(0, 1)$. Recently, Wang and Xu [27] proved that all λ-refinable distributions μ_λ do not have densities of class $C^\infty(\mathbb{R})$. We also give another proof. Define the Laplace transform $L_\rho(u)$ on $[0, \infty)$ for a probability distribution ρ on $[0, \infty)$ as

$$L_\rho(u) := \int_{0-}^{\infty} e^{-ux} \rho(dx).$$

Under the assumption that $0 < \rho(\{0\}) < 1$, we define a regularly varying function $K_{\lambda_0}(x)$ on $(0, \infty)$ with the index $-\lambda_0 := \log \rho(\{0\})/|\log b|$ for a probability distribution ρ on $[0, \infty)$ as

$$K_{\lambda_0}(x) := x^{-\lambda_0} \exp\left(\int_1^x \frac{\log \rho(\{0\}) - \log L_\rho(u)}{u \log b} du\right).$$

For two positive functions $f_1(x)$ and $f_2(x)$ on $(0, \infty)$, we define the relation $f_1(x) \asymp f_2(x)$ by $0 < \liminf_{x\to\infty} f_1(x)/f_2(x)$ and $\limsup_{x\to\infty} f_1(x)/f_2(x) < \infty$. The following lemma is due to Proposition 4.1 of Watanabe [31].

Lemma 4.1 *Let μ be a b-decomposable distribution on $[0, \infty)$ with ρ in (1.1). If $0 < \rho(\{0\}) < 1$, then*

$$\mu\big([0, 1/t]\big) \asymp K_{\lambda_0}(t) \quad \text{as } t \to \infty.$$

Proposition 4.2 *Let $\mu \in L(b)$ on $\mathbb{R}^d$ satisfying* (1.1).

(i) *Suppose that $\rho * \bar{\rho}(\{0\}) > 2^{-1}$. Then $\mu \notin W^{\infty}(\mathbb{R}^d)$. Moreover, if the support of ρ is compact, then μ does not have a density of class $C^{\infty}(\mathbb{R}^d)$.*

(ii) *Suppose that $d = 1$ and there exists $a \in \mathbb{R}$ such that $\rho((-\infty, a)) = 0$ and $0 < \rho(\{a\}) < 1$. Let $k_0 := [\log \rho(\{a\})/\log b]$. Then μ does not have a density of class $C^{k_0}(\mathbb{R})$. In particular, all λ-refinable distributions μ_λ with $\lambda > 1$ do not have densities of class $C^{\infty}(\mathbb{R})$.*

Proof First we prove (i). If $\rho * \bar{\rho}(\{0\}) = 1$, then μ is a delta measure. Thus, let $q := \rho * \bar{\rho}(\{0\}) \in (2^{-1}, 1)$ and set $\eta(dx) := (\rho * \bar{\rho}(dx) - q\delta_0(dx))/(1-q)$. Note from (1.2) that η has a finite log-moment. Using the inequality $1 - x \geq \exp(a^{-1}x \log(1 - a))$ for $0 \leq x \leq a < 1$, we see from (ii) of Lemma 2.3 that

$$\begin{aligned}
\left|\widehat{\mu}(z)\right|^2 &= \prod_{k=0}^{\infty}\left(1 - (1-q)\int_{\mathbb{R}^d}\left(1 - \cos\left\langle b^k z, x\right\rangle\right)\eta(dx)\right) \\
&\geq \exp\left(-2^{-1}\left|\log(2q-1)\right|\int_{\mathbb{R}^d} H_b\left(\left|\langle z, x\rangle\right|\right)\eta(dx)\right) \\
&\geq \exp\left(-2^{-1}\left|\log(2q-1)\right|C_2\int_{\mathbb{R}^d}\log\left(2 + \left|\langle z, x\rangle\right|\right)\eta(dx)\right) \\
&\geq \left(2 + |z|\right)^{-\gamma}\exp\left(-2^{-1}C_2\left|\log(2q-1)\right|\int_{\mathbb{R}^d}\log\left(2 + |x|\right)\eta(dx)\right),
\end{aligned}$$

where $\gamma := 2^{-1}C_2|\log(2q-1)| > 0$. Thus $\mu \notin W^{\infty}(\mathbb{R}^d)$. Note that the support of μ is compact if and only if so is that of ρ. Thus the second assertion is clear. Next we prove (ii). Suppose that $d = 1$ and there exists $a \in \mathbb{R}$ such that $\rho((-\infty, a)) = 0$ and $0 < \rho(\{a\}) < 1$ and assume that μ has a density of class $C^k(\mathbb{R})$ with $k \in \mathbb{Z}_+$. By translating the support of ρ, we can assume that $a = 0$. Noting that $\mu((-\infty, 0)) = 0$, we see that

$$\mu\left([0, 1/t]\right) = o\left(t^{-k-1}\right) \quad \text{as } t \to \infty.$$

Thus we find from Lemma 4.1 that $k + 1 \leq \lambda_0$, but see from the definitions of k_0 and λ_0 that $\lambda_0 < k_0 + 1$. Hence μ does not have a density of class $C^{k_0}(\mathbb{R})$. The second assertion is obvious. □

5 Applications to Lévy Processes

Let $0 < b < 1$ and $t > 0$. Let ζ_t be an infinitely divisible distribution on $\mathbb{R}^d$ defined by

$$\widehat{\zeta_t}(z) = \exp\left(t\sum_{n=0}^{\infty}\int_{\mathbb{R}^d}\left(\exp\left(i\left\langle b^n z, x\right\rangle\right) - 1 - \frac{i\langle b^n z, x\rangle}{1 + |x|^2}\right)\nu(dx) + it\langle z, x_0\rangle\right), \quad (5.1)$$

where $x_0 \in \mathbb{R}^d$ and ν is a measure on $\mathbb{R}^d$ with $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \big(|x|^2 \wedge \log\big(2 + |x|\big)\big)\nu(dx) < \infty. \tag{5.2}$$

Then $\mu = \zeta_t$ satisfies (1.1) with $\rho = \rho_t$ defined by

$$\widehat{\rho_t}(z) = \exp\left(t \int_{\mathbb{R}^d} \left(\exp\big(i\langle z, x\rangle\big) - 1 - \frac{i\langle z, x\rangle}{1 + |x|^2}\right)\nu(dx) + it(1-b)\langle z, x_0\rangle\right).$$

Moreover (5.2) is equivalent to (1.2). Hence $\zeta_t \in L(b)$ and it is the distribution of X_t for some Lévy process $\{X_t\}$ on $\mathbb{R}^d$ without Gaussian part. See Lemma 2.1 of Watanabe [29] and Sato [25]. Thus $\{\zeta_t\}$ satisfies the convolution semigroup property, that is, $\zeta_s * \zeta_t = \zeta_{s+t}$ for all $s, t > 0$. We assume that the linear span of the support of ν is the whole space $\mathbb{R}^d$. By virtue of Wolfe's theorem in [36], ζ_t is either singular or absolutely continuous for $t > 0$. Thus there are logically three cases:

Case A. ζ_t is absolutely continuous for all $t > 0$;
Case B. ζ_t is singular for all $t > 0$;
Case C. There is $T \in (0, \infty)$ such that ζ_t is singular for $0 < t < T$ and absolutely continuous for $t > T$.

We show the existence of Cases A, B, and C, respectively, in Propositions 5.1, 5.3, and Theorem 5.1 below.

Proposition 5.1 *Let $d = 1$. Suppose that ν in (5.1) satisfies $\nu(\mathbb{R}) = \infty$. Then we have the following.*

(i) *If b is a PS number, then ζ_t is absolutely continuous with a bounded continuous density for all $t > 0$.*
(ii) *If b is an EK number, then $\zeta_t \in W^\infty(\mathbb{R})$ for all $t > 0$.*

Proof Let $\delta > 0$ and set $C_\delta := \nu(|x| > \delta)$. Applying Jensen's inequality, we have

$$\big|\widehat{\zeta_t}(z)\big| = \exp\left(-t \int_{\mathbb{R}} H_b\big(|zx|\big)\nu(dx)\right) \le C_\delta^{-1} \int_{|x|>\delta} \exp\big(-tC_\delta H_b\big(|zx|\big)\big)\nu(dx). \tag{5.3}$$

Note that as $\delta \to 0$, $C_\delta \to \infty$. Thus, if b is a PS number, then we see from Theorem 1.2 that $\int_{\mathbb{R}} |\widehat{\zeta_t}(z)|dz < \infty$ for all $t > 0$. Moreover, if b is an EK number, then we find from Theorem 1.3 that $\zeta_t \in W^\infty(\mathbb{R})$ for all $t > 0$. □

Proposition 5.2 *Let $d = 1$. Suppose that ν in (5.1) satisfies $\nu(\mathbb{R}) < \infty$. Then we have the following.*

(i) *Let $n \in \mathbb{Z}_+$. For a.e. $b \in ((5/9)^{2^{-n}}, (5/9)^{2^{-n-1}})$, ζ_t is absolutely continuous with a bounded continuous density for all $t > 2^{-n}(\nu(\mathbb{R}))^{-1} \log 3$.*
(ii) *Let $n, k \in \mathbb{Z}_+$. For a.e. $b \in (2^{-2^{-n}}, 2^{-2^{-n-1}})$, ζ_t is absolutely continuous with a bounded density of class $C^k(\mathbb{R})$ for all $t > 371(k+1)2^{-n}(\nu(\mathbb{R}))^{-1}$.*

Proof Note that (5.3) holds. Thus assertions (i) and (ii) follow from (i) of Remark 1.2 and (i) of Remark 1.3, respectively. □

Proposition 5.3 *Let $d = 1$. Suppose that ν in (5.1) is a finite discrete measure on $\mathbb{Z}$. If b^{-1} is a PV number, then ζ_t is singular for all $t > 0$.*

Proof Let $\{\theta_j\}_{j=1}^m$ be the Galois conjugates of a PV number b^{-1}. Note that $|\theta_j| < 1$ for $1 \le j \le m$. Thus, as in the proof of (ii) of Lemma 2.3, there is a constant $C > 0$ such that, for $x \in \mathbb{Z}$,

$$\sum_{k=1}^{\infty}\left(1 - \cos\left(2\pi \sum_{j=1}^{m} \theta_j^k x\right)\right) \le C \log\bigl(2 + |x|\bigr). \tag{5.4}$$

Since b^{-1} is an algebraic integer, we find that, for $n \in \mathbb{N}$, there is $N_n \in \mathbb{Z}$ such that $b^{-n} = N_n - \sum_{j=1}^m \theta_j^n$. Thus we see from (5.4) and (ii) of Lemma 2.3 that, for $x \in \mathbb{Z}$,

$$\begin{aligned} H_b\bigl(b^{-n}2\pi|x|\bigr) &= \sum_{k=1}^{n}\left(1 - \cos\left(2\pi \sum_{j=1}^{m} \theta_j^k |x|\right)\right) + H_b\bigl(2\pi|x|\bigr) \\ &\le (C + C_2)\log\bigl(2 + 2\pi|x|\bigr). \end{aligned}$$

It follows from (5.2) that

$$\begin{aligned} \liminf_{n\to\infty}\bigl|\widehat{\zeta_t}\bigl(b^{-n}2\pi\bigr)\bigr| &= \liminf_{n\to\infty}\exp\left(-t\int_{\mathbb{Z}} H_b\bigl(b^{-n}2\pi|x|\bigr)\nu(dx)\right) \\ &\ge \exp\left(-t\int_{\mathbb{Z}} (C + C_2)\log\bigl(2 + 2\pi|x|\bigr)\nu(dx)\right) > 0. \end{aligned}$$

Thus, by virtue of the Riemann-Lebesgue theorem, ζ_t is singular for all $t > 0$. □

Proposition 5.4

(i) *If ν in (5.1) satisfies $\nu(\mathbb{R}^d) < \infty$, then $\zeta_t \notin W^\infty(\mathbb{R}^d)$ for all $t > 0$.*
(ii) *If $d = 1$, $\nu((-\infty, 0)) = 0$, and $0 < \nu((0, \infty)) < \infty$, then ζ_t does not have a density of class $C^\infty(\mathbb{R})$ for all $t > 0$.*

Proof Define a finite measure σ on $\mathbb{R}^d$ as

$$\sigma(dx) := \nu(dx) + \nu(-dx).$$

Then it holds that, for sufficiently small $t > 0$,

$$\begin{aligned} \rho_t * \bar{\rho}_t\bigl(\{0\}\bigr) &= \exp\bigl(-t\sigma\bigl(\mathbb{R}^d\bigr)\bigr)\sum_{n=0}^{\infty}(n!)^{-1}t^n\sigma^{n*}\bigl(\{0\}\bigr) \\ &\ge \exp\bigl(-2t\nu\bigl(\mathbb{R}^d\bigr)\bigr) > 1/2. \end{aligned}$$

Thus we see from Proposition 4.2 that $\zeta_t \notin W^\infty(\mathbb{R}^d)$ for sufficiently small $t > 0$, and hence so is for all $t > 0$. We have proved (i). Suppose that $d = 1$, $\nu((-\infty, 0)) = 0$, and $0 < \nu((0,\infty)) < \infty$. Since $\rho = \rho_t$ satisfies the assumption of (ii) of Proposition 4.2 with $a = -t\int_0^\infty x(1+x^2)^{-1}\nu(dx) + t(1-b)x_0$, the assertion (ii) is clear from Proposition 4.2. □

The *upper Hausdorff dimension* of a probability distribution μ on $\mathbb{R}^d$ is denoted by $\dim^* \mu$, that is,

$$\dim^* \mu := \inf\bigl\{\dim E : \mu(E) = 1\bigr\}.$$

The *entropy* of a discrete probability measure μ on $\mathbb{R}^d$ is denoted by $H(\mu)$, that is,

$$H(\mu) := -\sum_{a\in A} \mu\bigl(\{a\}\bigr) \log \mu\bigl(\{a\}\bigr),$$

where the set A is given by $A = \{a \in \mathbb{R}^d : \mu(\{a\}) > 0\}$. For a finite discrete measure μ on $\mathbb{R}^d$, define $H(\mu)$ by $H(\mu) := H((\mu(\mathbb{R}^d))^{-1}\mu)$. Let η be a finite discrete measure on $\mathbb{R}^d$ with $\eta(\{0\}) = 0$. Define a compound Poisson distribution η_t for $t \ge 0$ as

$$\eta_t := \exp\bigl(-t\eta\bigl(\mathbb{R}^d\bigr)\bigr) \sum_{n=0}^{\infty} (n!)^{-1} t^n \eta^{n*}.$$

Note that η_t is a discrete distribution for all $t \ge 0$. Define a function $h_\eta(t)$ on $\mathbb{R}_+$ as $h_\eta(t) := H(\eta_t)$. The following is due to Proposition 5.1 of Watanabe [29].

Lemma 5.1 *Let η be a finite discrete measure on $\mathbb{R}^d$ with $\eta(\{0\}) = 0$. Then we have the following.*

(i) *If $H(\eta) = \infty$, then $h_\eta(t) = \infty$ for all $t > 0$.*
(ii) *If $H(\eta) < \infty$, then $h_\eta(t) < \infty$ for all $t \ge 0$ and it is positive, continuous, and strictly increasing for $t > 0$ with $h_\eta(0+) = h_\eta(0) = 0$ and $\lim_{t\to\infty} h_\eta(t) = \infty$.*

In the last theorem, we discover that, as time increases, the distribution of a certain Lévy process on $\mathbb{R}^d$ can change from singular with arbitrarily small dimension to absolutely continuous with a density of class $C^k(\mathbb{R}^d)$ of any order k.

Theorem 5.1 *Suppose that ν in (5.1) is a finite discrete measure on $\mathbb{R}^d$ with $H(\nu) < \infty$. Then the following hold.*

(i) *We have*

$$\dim^* \zeta_t \le \frac{h_\nu(t)}{|\log b|}$$

and ζ_t is singular for $0 < t < h_\nu^{-1}(d|\log b|) < \infty$.
(ii) *If b is a PS number, then ζ_t is absolutely continuous with a bounded continuous density for all sufficiently large $t > 0$.*
(iii) *If b is an EK number, then, for every $k \in \mathbb{Z}_+$, ζ_t is absolutely continuous with a bounded density of class $C^k(\mathbb{R}^d)$ for all sufficiently large $t > 0$.*

Proof Assertion (i) is due to Theorem 5.1 of Watanabe [29]. Since ζ_t is full and $\zeta_t \in L(b)$, assertions (ii) and (iii) follow from Theorems 1.2 and 1.3, respectively. □

Finally we raise two open problems on the temporal evolution of continuities of ζ_t.

Problem 4 Is it true that if ζ_t is singular for all $t > 0$, then b^{-1} is a PV number?

Problem 5 Is it true that if ζ_t is absolutely continuous for some $t > 0$, then it has a bounded continuous density for all sufficiently large $t > 0$?

Acknowledgements The author is sincerely grateful to Ken-iti Sato for his useful comments on the early draft.

References

1. Bertin, M.J., Decomps-Guilloux, A., Grandet-Hugot, M., Pathiaux-Delefosse, M., Schreiber, J.P.: Pisot and Salem Numbers. Birkhäuser, Boston (1992)
2. Bunge, J.: Nested classes of C-decomposable laws. Ann. Probab. **25**(1), 215–229 (1997)
3. Dai, X., Feng, D., Wang, Y.: Refinable functions with non-integer dilations. J. Funct. Anal. **250**(1), 1–20 (2007)
4. Dubickas, A.: On the distance from a rational power to the nearest integer. J. Number Theory **117**(1), 222–239 (2006)
5. Dubickas, A., Xu, Z.: Refinement equations and spline functions. Adv. Comput. Math. **32**(1), 1–23 (2010)
6. Erdös, P.: On a family of symmetric Bernoulli convolutions. Am. J. Math. **61**(4), 974–976 (1939)
7. Erdös, P.: On the smoothness properties of Bernoulli convolutions. Am. J. Math. **62**(1), 180–186 (1940)
8. Feng, D., Wang, Y.: Bernoulli convolutions associated with certain non-Pisot numbers. Adv. Math. **187**(1), 173–194 (2004)
9. Garsia, A.M.: Arithmetic properties of Bernoulli convolutions. Trans. Am. Math. Soc. **102**(3), 409–432 (1962)
10. Hu, T.Y.: Asymptotic behavior of Fourier transforms of self-similar measures. Proc. Am. Math. Soc. **129**(6), 1713–1720 (2001)
11. Jessen, B., Wintner, A.: Distribution functions and the Riemann zeta function. Trans. Am. Math. Soc. **38**(1), 48–88 (1935)
12. Kahane, J.P.: Sur la distribution de certaines séries aléatoires. In: Colloque de Théorie des Nombres. Univ. Bordeaux, Bordeaux, 1969. Bull. Soc. Math. France., vol. 25, pp. 119–122 (1971)
13. Kershner, R., Wintner, A.: On symmetric Bernoulli convolutions. Am. J. Math. **57**(3), 541–548 (1935)
14. Lehmer, D.H.: Factorization of certain cyclotomic functions. Ann. Math. **34**, 461–469 (1933)
15. Lindner, A., Sato, K.: Continuity properties and infinite divisibility of stationary distributions of some generalized Ornstein-Uhlenbeck processes. Ann. Probab. **37**(1), 250–274 (2009)
16. Lindner, A., Sato, K.: Properties of stationary distributions of a sequence of generalized Ornstein-Uhlenbeck processes. Math. Nachr. **284**(17–18), 2225–2248 (2011)
17. Loève, M.: Nouvelles classes de loi limites. Bull. Soc. Math. Fr. **73**, 107–126 (1945)
18. Maejima, M., Sato, K., Watanabe, T.: Operator semi-selfdecomposability, (C, Q)-decomposability and related nested classes. Tokyo J. Math. **22**(2), 473–509 (1999)
19. Ngai, S., Wang, Y.: Self-similar measures associated to IFS with non-uniform contraction ratios. Asian J. Math. **9**(2), 227–244 (2005)
20. Peres, Y., Schlag, W., Solomyak, B.: Sixty years of Bernoulli convolutions. In: Fractal geometry and stochastics, II. Progr. Probab., vol. 46, pp. 39–65. Birkhäuser, Basel (2000)
21. Peres, Y., Solomyak, B.: Self-similar measures and intersections of Cantor sets. Trans. Am. Math. Soc. **350**(10), 4065–4087 (1998)
22. Pisot, C.: La répartition modulo un et les nombres algebriques. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **2**, 205–248 (1938)

23. Salem, R.: A remarkable class of algebraic integers: proof of a conjecture of Vijayaraghavan. Duke Math. J. **11**(1), 103–108 (1944)
24. Salem, R.: Algebraic Numbers and Fourier Analysis. D.C. Heath and Co., Boston (1963)
25. Sato, K.: Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics, vol. 68. Cambridge University Press, Cambridge (1999)
26. Solomyak, B.: On the random series $\sum \pm\lambda^n$ (an Erdös problem). Ann. Math. **142**(3), 611–625 (1995)
27. Wang, Y., Xu, Z.: The regularity of refinable functions (2011). arXiv:1109.1100v1
28. Watanabe, T.: Continuity properties of distributions with some decomposability. J. Theor. Probab. **13**(1), 169–192 (2000)
29. Watanabe, T.: Absolute continuity of some semi-selfdecomposable distributions and self-similar measures. Probab. Theory Relat. Fields **117**(3), 386–405 (2000)
30. Watanabe, T.: Temporal Change in Distributional Properties of Lévy Processes, pp. 89–107. Birkhäuser, Boston (2001)
31. Watanabe, T.: Limit theorems for shift selfsimilar additive random sequences. Osaka J. Math. **39**(3), 561–603 (2002)
32. Watanabe, T.: Shift self-similar additive random sequences associated with supercritical branching processes. J. Theor. Probab. **15**(3), 631–665 (2002)
33. Watanabe, T.: Exact packing measure on the boundary of a Galton-Watson tree. J. Lond. Math. Soc. **69**(3), 801–816 (2004)
34. Watanabe, T.: Exact Hausdorff measure on the boundary of a Galton-Watson tree. Ann. Probab. **35**(3), 1007–1038 (2007)
35. Wintner, A.: On convergent Poisson convolutions. Am. J. Math. **57**(4), 827–838 (1935)
36. Wolfe, S.J.: Continuity properties of decomposable probability measures on Euclidean spaces. J. Multivar. Anal. **13**(4), 534–538 (1983)