J Fourier Anal Appl (2012) 18:609–625
DOI 10.1007/s00041-011-9213-2

Coherent States Quantization for Generalized Bargmann Spaces with Formulae for Their Attached Berezin Transforms in Terms of the Laplacian on $\mathbb{C}^N$

Zouhaïr Mouayn

Received: 19 April 2011 / Revised: 27 September 2011 / Published online: 29 December 2011

Abstract While dealing with a class of generalized Bargmann spaces, we rederive their reproducing kernels from the knowledge of an orthonormal basis by using an addition formula for Laguerre polynomials involving the disk polynomials. We construct for each of these spaces a set of coherent states to apply a coherent states quantization method. This provides us with another way to recover the Berezin transforms attached to these spaces. Finally, two new formulae representing these transforms as functions of the Euclidean Laplacian are established and a possible physics direction for the application of such formulae is discussed.

Keywords Coherent states · Schrödinger operator with magnetic field · Generalized Bargmann spaces · Berezin transform · Euclidean Laplacian

Mathematics Subject Classification (2000) 47G10 · 47B35 · 81R30 · 44A35 · 42A38

1 Introduction

The Berezin transform introduced in [8] for certain classical symmetric domains in $\mathbb{C}^n$ is a transform linking the Berezin symbols and the symbols for Toeplitz operators. It is present in the study of the correspondence principle. The formula representing the Berezin transform as a function of the Laplace-Beltrami operator plays a key role in the Berezin quantization [7].

Communicated by Hans G. Feichtinger.

Z. Mouayn (✉)
Department of Mathematics, Faculty of Sciences and Technics (M'Ghila), Sultan Moulay Slimane University, PO. Box 523, Béni Mellal, Morocco
e-mail: mouayn@fstbm.ac.ma

In this paper we are concerned with the Berezin transforms associated with

$$A_m^2(\mathbb{C}^n) = \left\{ \psi \in L^2(\mathbb{C}^n; e^{-|z|^2} d\mu); \quad \widetilde{\Delta}\psi = m\psi \right\}, \tag{1.1}$$

the eigenspaces ([5]) of the second order differential operator

$$\widetilde{\Delta} = -\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_j} + \sum_{j=1}^{n} \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \tag{1.2}$$

corresponding to the eigenvalues $m = 0, 1, 2, \ldots$. Here, $d\mu$ denotes the Lebesgue measure on $\mathbb{C}^n$. The eigenspaces in (1.1) are called generalized Bargmann spaces and are reproducing kernel Hilbert spaces with reproducing kernels given by [5]:

$$K_m(z, w) := \pi^{-n} e^{\langle z, w \rangle} L_m^{(n-1)}\left(|z-w|^2\right), \quad z, w \in \mathbb{C}^n, \tag{1.3}$$

where $L_k^{(\alpha)}(\cdot)$ is the Laguerre polynomial [22]. The associated Berezin transform was obtained via the well known formalism of Toeplitz operators by the convolution product over $\mathbb{C}^n$ as ([6, 24]):

$$B_m[\phi](z) := (h_m * \phi)(z), \quad \phi \in L^2(\mathbb{C}^n, d\mu), \tag{1.4}$$

where

$$h_m(w) := \frac{\pi^{-n} m!}{(n)_m} e^{-|w|^2} \left(L_m^{(n-1)}\left(|w|^2\right) \right)^2, \quad w \in \mathbb{C}^n. \tag{1.5}$$

Here, our aim is first to present a direct proof for obtaining the reproducing kernel in (1.3) starting from the knowledge of an orthonormal basis of the eigenspace in (1.1) and by exploiting an addition formula for Laguerre polynomials involving the disk polynomials [20]. Next, we construct for each of the eigenspaces in (1.1) a set of coherent states by following a generalized formalism [16]. in order to apply a coherent states quantization method. This provides us with another way to recover the Berezin transforms in (1.4) attached to the generalized Bargmann spaces under consideration. Finally, direct calculations of the Fourier transform of the function in (1.5) enables us to present two other formulae expressing the transforms in (1.4) as functions of the Euclidean Laplacian. These two formulae together with a first one [6] could be of help in physics problems having a close analogy with the diamagnetism of spinless Bose systems [28].

This paper is organized as follows. In Sect. 2 we recall briefly the formalism of coherent states quantization we will be using. Section 3 deals with some needed facts on the generalized Bargmann spaces. In Sect. 4, we construct for each of these spaces a set of coherent states and we apply the corresponding quantization scheme in order to recover the Berezin transforms. In Sect. 5, we present two other formulae expressing these Berezin transforms as functions of the Laplacian in the Euclidean complex n-space. Section 6 is devoted to some concluding remarks.

2 Coherent States Quantization

Coherent states are mathematical tools which provide a close connection between classical and quantum formalisms. In general, they are a specific overcomplete set of vectors in a Hilbert space satisfying a certain resolution of the identity condition. Here, we adopt the prototypical model of coherent states presented in [16] and described as follows. Let $X = \{x|x \in X\}$ be a set equipped with a measure $d\nu$ and $L^2(X, d\nu)$ the Hilbert space of $d\nu$-square integrable functions $f(x)$ on X. Let $\mathcal{A}^2 \subset L^2(X, d\nu)$ be a subspace with an orthonormal basis $\{\Phi_k\}_{k=0}^{\infty}$ such that

$$\mathcal{N}(x) := \sum_{k=0}^{\infty} |\Phi_k(x)|^2 < +\infty, \quad x \in X. \tag{2.1}$$

Let $\mathcal{H}$ be another (functional) Hilbert space with $\dim \mathcal{H} = \dim \mathcal{A}^2$ and $\{\varphi_k\}_{k=0}^{\infty}$ is a given orthonormal basis of $\mathcal{H}$. Then consider the family of states $\{|x\rangle\}_{x\in X}$ in $\mathcal{H}$, through the following linear superpositions:

$$|x\rangle := (\mathcal{N}(x))^{-\frac{1}{2}} \sum_{k=0}^{\infty} \Phi_k(x)\varphi_k. \tag{2.2}$$

These *coherent states* obey the normalization condition

$$\langle x|x\rangle_{\mathcal{H}} = 1 \tag{2.3}$$

and the following resolution of the unity in $\mathcal{H}$

$$\mathbf{1}_{\mathcal{H}} = \int_X |x\rangle\langle x|\mathcal{N}(x)d\mu(x), \tag{2.4}$$

which is expressed in terms of Dirac's bra-ket notation $|x\rangle\langle x|$ meaning the rank-one-operator $\varphi \mapsto \langle \varphi|x\rangle_{\mathcal{H}}.|x\rangle$, $\varphi \in \mathcal{H}$.

The choice of the Hilbert space $\mathcal{H}$ defines in fact a quantization of the space X by the coherent states in (2.2), via the inclusion map $X \ni x \mapsto |x\rangle \in \mathcal{H}$ and the property (2.4) is crucial in setting the bridge between the classical and the quantum worlds. It encodes the quality of being canonical quantizers along a guideline established by Klauder [19] and Berezin [8]. This Klauder-Berezin coherent state quantization, also named anti-Wick quantization or Toeplitz quantization [18] by many authors, consists in associating to a classical observable that is a function $f(x)$ on X having specific properties the operator-valued integral

$$A_f := \int_X |x\rangle\langle x|f(x)\mathcal{N}(x)d\mu(x). \tag{2.5}$$

The function $f(x) \equiv \widehat{A}_f(x)$ is called upper (or contravariant) symbol of the operator A_f and is nonunique in general. On the other hand, the expectation value $\langle x|A_f|x\rangle$ of A_f with respect to the set of coherent states $\{|x\rangle\}_{x\in X}$ is called lower (or covariant) symbol of A_f. Finally, associating to the classical observable $f(x)$ the obtained mean value $\langle x|A_f|x\rangle$, we get the Berezin transform of this observable. That is,

$$B[f](x) := \langle x|A_f|x\rangle, \quad x \in X. \tag{2.6}$$

For all aspect of the theory of coherent states and their genesis, we refer to the survey [11] by V.V. Dodonov or the recent book [16] of J.P. Gazeau.

3 The Generalized Bargmann Spaces $A_m^2(\mathbb{C}^n)$

In [5], we have introduced a class of generalized Bargmann spaces indexed by integer numbers $m = 0, 1, 2, \dots$, as eigenspaces of a single elliptic second-order differential operator as

$$A_m^2(\mathbb{C}^n) := \left\{\psi \in L^2(\mathbb{C}^n; e^{-|z|^2}d\mu); \quad \widetilde{\Delta}\psi = m\psi\right\}, \tag{3.1}$$

where

$$\widetilde{\Delta} = -\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_j} + \sum_{j=1}^{n} \overline{z}_j \frac{\partial}{\partial \overline{z}_j}. \tag{3.2}$$

The operator $\widetilde{\Delta}$ is considered with $C_0^\infty(\mathbb{C}^n)$ as its regular domain in the Hilbert space $L^2(\mathbb{C}^n; e^{-|z|^2}d\mu)$ of $e^{-|z|^2}d\mu$-square integrable complex-valued functions on $\mathbb{C}^n$. Its concrete L^2 spectral theory have been discussed in [4]. Actually, for $m = 0$, the eigenspace $A_0^2(\mathbb{C}^n)$ turns out to be the realization by harmonic functions with respect to $\widetilde{\Delta}$ of the classical Bargmann space whose elements are the entire functions in $L^2(\mathbb{C}^n; e^{-|z|^2}d\mu)$ see [5].

Now, for general $m = 0, 1, 2, \dots$, a complete description of the expansion of elements $f \in A_m^2(\mathbb{C}^n)$ in terms of the appropriate Fourier series in $\mathbb{C}^n$ have been given [5]. Precisely, a function $f : \mathbb{C}^n \to \mathbb{C}$ belongs to $A_m^2(\mathbb{C}^n)$ if and only if it can be expanded in the form

$$f(z) = \sum_{p=0}^{+\infty}\sum_{q=0}^{m} {}_1F_1\left(-m+q, n+p+q, \rho^2\right)\rho^{p+q}\gamma_{p,q} \cdot h_{p,q}(\omega) \tag{3.3}$$

in $C^\infty(\mathbb{C}^n)$, $z = \rho\omega$, $\rho > 0$, $|\omega| = 1$, ${}_1F_1$ is the confluent hypergeometric function [22], $\gamma_{p,q} = (\gamma_{p,q,j}) \in \mathbb{C}^{d(n;p,q)}$ are coefficients such that

$$\sum_{p=0}^{+\infty}\sum_{q=0}^{m} (m-q)!\,(p+q+n-1)!\Gamma(n+p+q)\,\frac{|\gamma_{p,q}|^2}{2\Gamma(n+m+p)} < +\infty \tag{3.4}$$

and

$$d(n; p, q) := \frac{(p+q+n-1)\,(p+n-2)!\,(q+n-2)!}{p!q!\,(n-1)!\,(n-2)!} \tag{3.5}$$

is the dimension of the space $H(p,q)$ of restrictions to the unit sphere $\mathbb{S}^{2n-1} = \{\omega \in \mathbb{C}^n, |\omega| = 1\}$ of harmonic polynomials $h(z)$ on $\mathbb{C}^n$, which are homogeneous of degree

p in z and degree q in $\overline{z}$ (see [26]). The notation "$\cdot$" in (3.3) means the following finite sum

$$\gamma_{p,q} \cdot h_{p,q}(\omega) := \sum_{j=0}^{d(n,p,q)} \gamma_{p,q,j} h_{p,q}^{j}(\omega), \tag{3.6}$$

where $\{h_{p,q}^{j}\}_{1\leq j\leq d(n;p,q)}$ is an orthonormal basis of $H(p,q)$.

Now, from (3.3) and using the relation ([22]):

$${}_1F_1(-k,\alpha;u) = \frac{k!\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)} L_k^{(\alpha)}(u) \tag{3.7}$$

for k being a positive integer, an orthonormal basis in the space $A_m^2(\mathbb{C}^n)$ can be written explicitly in terms of the Laguerre polynomials $L_k^{(\alpha)}(.)$ and the polynomials $h_{p,q}^{j}(z,\overline{z})$ as in [6]:

$$\Phi_{j,p,q}^{m}(z) := \left(\frac{2(m-q)!}{\Gamma(n+m+p)}\right)^{\frac{1}{2}} L_{m-q}^{(n+p+q-1)}\left(|z|^2\right) h_{p,q}^{j}(z,\overline{z}) \tag{3.8}$$

for varying $p = 0,1,2,\ldots; q = 0,1,\ldots,m$ and $j = 1,\ldots,d(n;p,q)$. These functions possess nice properties and represent a principal tool in the present work. For instance, we have to check by hand the following fact.

Lemma 3.1 *The functions in* (3.8) *satisfy*

$$\sum_{p=0}^{+\infty}\sum_{q=0}^{m}\sum_{j=1}^{d(n,p,q)} \Phi_{j,p,q}^{m}(z)\overline{\Phi_{j,p,q}^{m}(w)} = \pi^{-n} e^{\langle z,w\rangle} L_m^{(n-1)}\left(|z-w|^2\right) \tag{3.9}$$

for all $z, w \in \mathbb{C}^n$.

Proof Denoting the sum in the left hand side of (3.9) by $S_{n,m}^{z,w}$ and replacing the functions $\Phi_{j,p,q}^{m}(z)$ by their expressions in (3.8), we obtain that

$$S_{n,m}^{z,w} := \sum_{\substack{1\leq j\leq d(n,p,q)\\ 0\leq q\leq m, 0\leq p<+\infty}} \Phi_{j,p,q}^{m}(z)\overline{\Phi_{j,p,q}^{m}(w)} \tag{3.10}$$

$$= \sum_{\substack{1\leq j\leq d(n,p,q)\\ 0\leq q\leq m, 0\leq p<+\infty}} \frac{2(m-q)!}{\Gamma(n+m+p)} L_{m-q}^{(n+p+q-1)}\left(|z|^2\right) L_{m-q}^{(n+p+q-1)}\left(|w|^2\right)$$

$$\times\, h_{p,q}^{j}(z,\overline{z})\overline{h_{p,q}^{j}(w,\overline{w})} \tag{3.11}$$

$$= \sum_{\substack{0\leq q\leq m,\\ 0\leq p<+\infty}} \frac{2(m-q)!}{\Gamma(n+m+p)} L_{m-q}^{(n+p+q-1)}\left(|z|^2\right) L_{m-q}^{(n+p+q-1)}\left(|w|^2\right) \mathfrak{G}_{n;p,q}^{z,w}, \tag{3.12}$$

where

$$\mathfrak{G}^{z,w}_{n;p,q} := \sum_{1\le j\le d(n,p,q)} h^{j}_{p,q}(z,\overline{z})\,\overline{h^{j}_{p,q}(w,\overline{w})} \tag{3.13}$$

$$= (|z|\,|w|)^{p+q} \sum_{1\le j\le d(n,p,q)} h^{j}_{p,q}\left(\frac{z}{|z|},\frac{\overline{z}}{|z|}\right)\overline{h^{j}_{p,q}\left(\frac{w}{|w|},\frac{\overline{w}}{|w|}\right)}. \tag{3.14}$$

To calculate the finite sum in (3.13), we make use of the formula ([15]):

$$\begin{aligned}
&\sum_{1\le j\le d(n,p,q)} h^{j}_{p,q}\left(\frac{z}{|z|},\frac{\overline{z}}{|z|}\right)\overline{h^{j}_{p,q}\left(\frac{w}{|w|},\frac{\overline{w}}{|w|}\right)}\\
&\quad=\frac{\Gamma(n)}{2\pi^n}d(n,p,q)\left(P^{(n-2,|p-q|)}_{\min(p,q)}(1)\right)^{-1}\\
&\quad\times\left|\left\langle\frac{z}{|z|},\frac{w}{|w|}\right\rangle\right|^{|p-q|} e^{i(p-q)\arg\langle\frac{z}{|z|},\frac{w}{|w|}\rangle}.\\
&\quad\times P^{(n-2,|p-q|)}_{\min(p,q)}\left(2\left|\left\langle\frac{z}{|z|},\frac{w}{|w|}\right\rangle\right|^2-1\right),
\end{aligned}\tag{3.15}$$

where $P^{(\alpha,\beta)}_k(.)$ is the Jacobi polynomial [22]. Now, making appeal to the normalized Jacobi polynomial

$$R^{(\alpha,\beta)}_k(u) := \frac{P^{(\alpha,\beta)}_k(u)}{P^{(\alpha,\beta)}_k(1)} \tag{3.16}$$

together with the disk polynomials that were first studied by Zernike and Brinkman [30], [20] and are given by

$$R^{\gamma}_{p,q}(\xi) := |\xi|^{|p-q|}e^{i(p-q)\arg\xi}R^{(\gamma,|p-q|)}_{\min(p,q)}\left(2|\xi|^2-1\right), \tag{3.17}$$

where $R^{(\gamma,|p-q|)}_{\min(p,q)}(.)$ is defined according to (3.16) for the parameter $\gamma := n-2$, the finite sum takes the form

$$\mathfrak{G}^{z,w}_{n;p,q} = \left(2\pi^n\right)^{-1}\Gamma(n)\,d(n,p,q)\,(|z|\,|w|)^{p+q}R^{n-2}_{p,q}\left(\left\langle\frac{z}{|z|},\frac{w}{|w|}\right\rangle\right). \tag{3.18}$$

Returning back to (3.12) and inserting (3.18), then the sum $S^{z,w}_{n,m}$ takes the form

$$\begin{aligned}
S^{z,w}_{n,m} = &\sum_{\substack{0\le q\le m,\\ 0\le p<+\infty}} \frac{(m-q)!\Gamma(n)\,d(n,p,q)}{\Gamma(n+m+p)\,\pi^n}(|z|\,|w|)^{p+q}\\
&\times L^{(n+p+q-1)}_{m-q}\left(|z|^2\right)L^{(n+p+q-1)}_{m-q}\left(|w|^2\right)R^{n-2}_{p,q}\left(\left\langle\frac{z}{|z|},\frac{w}{|w|}\right\rangle\right).
\end{aligned}\tag{3.19}$$

Now, we use the slightly different function $\mathcal{L}_k^{(\alpha)}(u)$ for the Laguerre polynomial, which is such that

$$L_k^{(\alpha)}(u) = \frac{\Gamma(k+\alpha+1)}{k!\Gamma(\alpha+1)} e^{\frac{1}{2}u} \mathcal{L}_k^{(\alpha)}(u) \tag{3.20}$$

for $\alpha = n-1+p+q$, $u = |z|^2$, $u = |w|^2$ and $k = m-q$ to rewrite (3.19) as

$$S_{n,m}^{z,w} = \pi^{-n} e^{\frac{1}{2}|z|^2+\frac{1}{2}|w|^2} \sum_{\substack{0\le q\le m,\\ 0\le p<+\infty}} \frac{\Gamma(n)\, d(n,p,q)\, \Gamma(m+n+p)}{(m-q)!\Gamma^2(n+p+q)} \times (|z|\,|w|)^{p+q} \mathcal{L}_{m-q}^{(n-1+p+q)}\big(|z|^2\big) \mathcal{L}_{m-q}^{(n-1+p+q)}\big(|w|^2\big) R_{p,q}^{n-2}\left(\left\langle \frac{z}{|z|}, \frac{w}{|w|}\right\rangle\right). \tag{3.21}$$

Next, we use the explicit expression of the dimension $d(n,p,q)$ in (3.5) to write the coefficient

$$C_{n,p,q} := \frac{\Gamma(n)\, d(n,p,q)\, \Gamma(m+n+p)}{(m-q)!\Gamma^2(n+p+q)} \tag{3.22}$$

in the following form

$$C_{n,p,q} = \frac{1}{(n-2)!}\left[\frac{\Gamma(n+m)}{m!\,(n-1)}\right] \frac{\sigma}{\sigma+p+q} \binom{m}{q} \frac{(\sigma+m+1)_p}{p!\,(\sigma+q)_p\,(\sigma+p)_q}, \tag{3.23}$$

where $\sigma = n-1$ and $(a)_k$ denotes the Pochammer symbol. Therefore, the sum in (3.21) can be presented as

$$\begin{aligned} S_{n,m}^{z,w} = \frac{\Gamma(n+m)\, e^{\frac{1}{2}(|z|^2+|w|^2)}}{\pi^n m!\,(n-1)!} \sum_{p=0}^{+\infty}\sum_{q=0}^{m} \frac{\sigma}{\sigma+p+q}\binom{m}{q} \\ \times \frac{(\sigma+m+1)_p}{p!\,(\sigma+q)_p\,(\sigma+p)_q} |z|^{p+q} \mathcal{L}_{m-q}^{(\sigma+p+q)}\big(|z|^2\big) \\ \times |w|^{p+q} \mathcal{L}_{m-q}^{(\sigma+p+q)}\big(|w|^2\big) R_{p,q}^{\sigma-1}\big(\rho e^{i\theta}\big), \end{aligned} \tag{3.24}$$

where $\langle \frac{z}{|z|}, \frac{w}{|w|}\rangle = \rho e^{i\theta}$.

We are now in position to apply the addition formula for Laguerre polynomials due to Koornwinder [21]:

$$\begin{aligned} &\exp(ixyr\sin\psi)\, \mathcal{L}_s^{(\sigma)}\Big(x^2+y^2-2xy\cos\psi\Big) \\ &= \sum_{k=0}^{+\infty}\sum_{l=0}^{s} \frac{\sigma}{\sigma+k+l}\binom{s}{l} \frac{(\sigma+s+1)_k}{k!\,(\sigma+l)_k\,(\sigma+k)_l} \\ &\quad \times x^{k+l}\mathcal{L}_{s-l}^{(\sigma+k+l)}\big(x^2\big) y^{k+l} \mathcal{L}_{s-l}^{(\sigma+k+l)}\big(y^2\big) R_{k,l}^{\sigma-1}\big(re^{i\psi}\big), \end{aligned} \tag{3.25}$$

where $x \geq 0$, $y \geq 0$, $0 \leq r \leq 1$, $0 \leq \psi < 2\pi$, $\sigma > 0$, $s = 0, 1, 2, \ldots$, for $s = m$, $\sigma = n-1$, $re^{i\psi} = \langle \frac{z}{|z|}, \frac{w}{|w|} \rangle$, $x = |z|$ and $y = |w|$. After computations, we arrive at

$$S_{n,m}^{z,w} = \pi^{-n} e^{\langle z,w \rangle} L_m^{(n-1)}\left(|z-w|^2\right). \tag{3.26}$$

This ends the proof. □

By a general fact on reproducing kernels [3], Lemma 3.1 says that the knowledge of the explicit orthonormal basis in (3.8) leads directly to the expression of the reproducing kernel in (1.3) via calculations using properties of the spherical harmonics together with the addition formula (3.25).

Remark 3.2 The motion of charged particle in a constant uniform magnetic field in $\mathbb{R}^{2n}$ is described (in suitable units and up to an additive constant) by the Schrödinger operator

$$H_\beta := -\frac{1}{4} \sum_{j=1}^{n} \left(\partial_{x_j} + \beta y_j\right)^2 + \left(\partial_{y_j} - i\beta x_j\right)^2 - \frac{n}{2} \tag{3.27}$$

acting on $L^2(\mathbb{R}^{2n}, d\mu)$, $\beta > 0$ is a constant proportional to the magnetic field strength. We identify the Euclidean space $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the usual way. The operator H_β in (3.27) can be represented by the operator

$$\widetilde{H}_\beta = e^{\frac{1}{2}\beta|z|^2} H_\beta e^{-\frac{1}{2}\beta|z|^2}. \tag{3.28}$$

According to (3.28), an arbitrary state ϕ of $L^2(\mathbb{R}^{2n}, d\mu)$ is represented by the function $Q[\phi]$ of $L^2(\mathbb{C}^n, e^{-|z|^2} d\mu)$ defined by

$$Q[\phi](z) := e^{\frac{1}{2}\beta|z|^2} \phi(z), \quad z \in \mathbb{C}^n. \tag{3.29}$$

The unitary map Q in (3.29) is called a ground state transformation. For $\beta = 1$, the explicit expression for the operator in (3.28) turns out to be given by the operator $\widetilde{\Delta}$ introduced in (1.2). That is, $\widetilde{H}_1 = \widetilde{\Delta}$ with eigenvalues corresponding to well known *Landau levels* of the operator in (3.27).

4 A Coherent States Quantization for $A_m^2(\mathbb{C}^n)$

Now, to adapt the definition (2.2) of coherent states for the context of the generalized Bargmann spaces $A_m^2(\mathbb{C}^n)$, we first list the following notations.

- $(X, d\nu) := (\mathbb{C}^n, e^{-|z|^2} d\mu)$, $d\nu := e^{-|z|^2} d\mu$.
- $x \equiv z \in \mathbb{C}^n$.
- $A^2 := A_m^2(\mathbb{C}^n) \subset L^2(\mathbb{C}^n, e^{-|z|^2} d\mu)$.
- $\{\Phi_k(x)\} \equiv \{\Psi_{j,p,q}^m(z)\}$ is the orthonormal basis of $A_m^2(\mathbb{C}^n)$ in (3.8).
- $\mathcal{N}(x) \equiv \mathcal{N}(z)$ is a normalization factor.

- $\{\varphi_k\} \equiv \{\varphi_{j,p,q}\}$ is an orthonormal basis of another (functional) Hilbert space $\mathcal{H}$ having the same dimension (∞) of $A_m^2(\mathbb{C}^n)$.

Definition 4.1 For each fixed integer $m = 0, 1, 2, \ldots$, a class of the generalized coherent states associated with the space $A_m^2(\mathbb{C}^n)$ is defined according to (2.2) by the form

$$\phi_{z,m} \equiv |z, m\rangle := (\mathcal{N}(z))^{-\frac{1}{2}} \sum_{\substack{1 \le j \le d(n,p,q) \\ 0 \le q \le m, 0 \le p < +\infty}} \Phi_{j,p,q}^m(z)\varphi_{j,p,q}, \tag{4.1}$$

where $\mathcal{N}(z)$ is a factor such that $\langle \phi_{z,m}, \phi_{z,m}\rangle_{\mathcal{H}} = 1$.

Proposition 4.2 *The normalization factor in* (4.1) *is given by*

$$\mathcal{N}(z) = \frac{\pi^{-n}\Gamma(n+m)}{\Gamma(m+1)\Gamma(n)} e^{\langle z,z\rangle} \tag{4.2}$$

for every $z \in \mathbb{C}^n$.

Proof To calculate this factor, we start by writing the condition

$$\langle \phi_{z,m}, \phi_{z,m}\rangle_{\mathcal{H}} = 1. \tag{4.3}$$

Equation (4.3) is equivalent to

$$(\mathcal{N}(z))^{-1} \sum_{p=0}^{+\infty} \sum_{q=0}^{m} \sum_{j=1}^{d(n,p,q)} \Phi_{j,p,q}^m(z) \overline{\Phi_{j,p,q}^m(z)} = 1. \tag{4.4}$$

Making use of Lemma 3.1 for the particular case $z = w$, we get that

$$\mathcal{N}(z) = \pi^{-n} e^{\langle z,z\rangle} L_m^{(n-1)}(0). \tag{4.5}$$

Next, by the fact that ([17]):

$$L_m^{(n-1)}(0) = \frac{\Gamma(n+m)}{\Gamma(m+1)\Gamma(n)}, \tag{4.6}$$

we arrive at the announced result. □

Now, the states $\phi_{z,m} \equiv |z, m\rangle$ satisfy the resolution of the identity

$$1_{\mathcal{H}} = \int_{\mathbb{C}^n} |z, m\rangle\langle z, m| \mathcal{N}(z) d\nu(z) \tag{4.7}$$

and with the help of them we can achieve the coherent states quantization scheme described in Sect. 2 to rederive the Berezin transform B_m in (1.4) which was defined by the Toeplitz operators formalism in previous works. For this let us first associate to any arbitrary function $\varphi \in L^2(\mathbb{C}^n, d\mu)$ the operator-valued integral

$$A_\varphi := \int_{\mathbb{C}^n} |z, m\rangle\langle m, z|\varphi(z)\mathcal{N}(z)d\nu(z). \tag{4.8}$$

The function $\varphi(z)$ is a upper symbol of the operator A_φ. On the other hand, we need to calculate the expectation value

$$\mathbb{E}_{\{|z,m\rangle\}}\left(A_\varphi\right) := \langle z, m|A_\varphi|z, m\rangle \tag{4.9}$$

of A_φ with respect to the set of coherent states $\{|z, m\rangle\}_{z\in\mathbb{C}^n}$ defined in (4.1). This will constitute a lower symbol of the operator A_φ.

Proposition 4.3 *Let $\varphi \in L^2(\mathbb{C}^n, d\mu)$. Then, the expectation value in (4.9) has the following expression*

$$\mathbb{E}_{\{|z,m\rangle\}}\left(A_\varphi\right) = \frac{m!}{(n)_m\pi^n}\int_{\mathbb{C}^n} e^{-|z-w|^2}\left(L_m^{(n-1)}\left(|z-w|^2\right)\right)^2\varphi(w)d\mu(w) \tag{4.10}$$

for every $z \in \mathbb{C}^n$.

Proof We first write the action of the operator A_φ in (4.8) on an arbitrary coherent state $|z, m\rangle$ in terms of Dirac's bra-ket notation as

$$A_\varphi|z, m\rangle = \int_{\mathbb{C}^n} |w, m\rangle\langle w, m|z, m\rangle\varphi(w)\mathcal{N}(w)e^{-|w|^2}d\mu(w). \tag{4.11}$$

Therefore, the expectation value reads

$$\langle z, m|A_\varphi|z, m\rangle = \int_{\mathbb{C}^n} \langle z, m|w, m\rangle\langle w, m|z, m\rangle\varphi(w)\mathcal{N}(w)e^{-|w|^2}d\mu(w) \tag{4.12}$$

$$= \int_{\mathbb{C}^n} \langle z, m|w, m\rangle\overline{\langle z, m|w, m\rangle}\varphi(w)\mathcal{N}(w)e^{-|w|^2}d\mu(w) \tag{4.13}$$

$$= \int_{\mathbb{C}^n} |\langle z, m|w, m\rangle|^2\varphi(w)\mathcal{N}(w)e^{-|w|^2}d\mu(w). \tag{4.14}$$

Note that we have used the fact that $d\nu(w) := e^{-|w|^2}d\mu(w)$. Now, we need to evaluate the quantity $|\langle z, m|w, m\rangle|^2$ in (4.14). For this, we write the scalar product as

$$\langle z, m|w, m\rangle = \sum_{p=0}^{+\infty}\sum_{q=0}^{m}\sum_{j=1}^{d(n,p,q)}\sum_{r=0}^{+\infty}\sum_{s=0}^{m}\sum_{l=1}^{d(n,r,s)} \frac{\Phi^m_{j,p,q}(z)\overline{\Phi^m_{l,r,s}(w)}}{\sqrt{\mathcal{N}(z)\mathcal{N}(w)}}\left\langle\varphi_{j,p,q}, \varphi_{l,r,s}\right\rangle_{\mathcal{H}}. \tag{4.15}$$

Recalling that

$$\left\langle\varphi_{j,p,q}, \varphi_{l,r,s}\right\rangle_{\mathcal{H}} = \delta_{j,l}\delta_{p,r}\delta_{q,s} \tag{4.16}$$

since $\{\varphi_{j,p,q}\}$ is an orthonormal basis of $\mathcal{H}$, the above sum in (4.15) reduces to

$$\langle z,m|w,m\rangle = (\mathcal{N}(z)\mathcal{N}(w))^{-\frac{1}{2}} \sum_{p=0}^{+\infty}\sum_{q=0}^{m}\sum_{j=1}^{d(n,p,q)} \Phi^m_{j,p,q}(z)\overline{\Phi^m_{j,p,q}(w)}. \tag{4.17}$$

Now, by Lemma 3.1, Equation (4.17) takes the form

$$\langle z,m|w,m\rangle = (\mathcal{N}(z)\mathcal{N}(w))^{-\frac{1}{2}}\pi^{-n}e^{\langle z,w\rangle}L_m^{(n-1)}\big(|z-w|^2\big). \tag{4.18}$$

So that the squared modulus of the scalar product in (4.18) reads

$$|\langle z,m|w,m\rangle|^2 = (\mathcal{N}(z)\mathcal{N}(w))^{-1}\pi^{-2n}e^{2\Re\langle z,w\rangle}\left(L_m^{(n-1)}\big(|z-w|^2\big)\right)^2. \tag{4.19}$$

Returning back to (4.14) and inserting (4.19), we obtain that

$$\mathbb{E}_{\{|z,m\rangle\}}\left(A_\varphi\right) = \int_{\mathbb{C}^n}\mathcal{N}(z)^{-1}\pi^{-2n}e^{2\Re\langle z,w\rangle-|w|^2}\left(L_m^{(n-1)}\big(|z-w|^2\big)\right)^2\varphi(w)\,d\mu(w). \tag{4.20}$$

Replacing $\mathcal{N}(z)$ by its expression in Proposition 4.2, we arrive at the result in (4.10). This ends the proof. □

Finally, we summarize the above discussion by considering the following definition.

Definition 4.4 The Berezin transform of the classical observable $\varphi \in L^2(\mathbb{C}^n, d\mu)$ constructed according to the quantization by the coherent states $\{|z,m\rangle\}$ in (4.1) is obtained by associating to φ the obtained mean value in (4.10). That is,

$$B_m[\varphi](z) := \mathbb{E}_{\{|z,m\rangle\}}\left(A_\varphi\right) \tag{4.21}$$

for every $z \in \mathbb{C}^n$.

Remark 4.5 As mentioned above, for $m = 0$, the eigenspace $A_0^2(\mathbb{C}^n)$ coincides with the Bargmann space of analytic functions on $\mathbb{C}^n$ that are $e^{-|z|^2}d\mu$-square integrable with the reproducing kernel $K_0(z,w) := \pi^{-n}e^{\langle z,w\rangle}$ and the associated Berezin transform B_0 is given by a convolution product over the group $\mathbb{C}^n$ as

$$B_0[\phi](z) := \big(\pi^{-n}e^{-|w|^2} * \phi\big)(z); \quad \phi \in L^2(\mathbb{C}^n; d\mu). \tag{4.22}$$

Furthermore, it can be expressed in terms of the Euclidean Laplacian on $\mathbb{C}^n$ as $B_0 = e^{\frac{1}{4}\Delta_{\mathbb{C}^n}}$ [7, 25, 29].

5 The Transform B_m and the Euclidean Laplacian

From (4.10) and (4.21) it is easy to see from that the transform

$$B_m[\varphi](z) = \frac{m!}{(n)_m\pi^n}\int_{\mathbb{C}^n} e^{-|z-w|^2}\left(L_m^{(n-1)}\big(|z-w|^2\big)\right)^2\varphi(w)d\mu(w) \tag{5.1}$$

can written as a convolution operator as

$$B_m[\varphi] = h_m * \varphi, \quad \varphi \in L^2\left(\mathbb{C}^n, d\mu\right), \tag{5.2}$$

where

$$h_m(z) = \frac{m!}{(n)_m \pi^n} e^{-|z|^2} \left(L_m^{(n-1)}\left(|z|^2\right)\right)^2, \quad z \in \mathbb{C}^n. \tag{5.3}$$

In view of (5.2) a general principle [9, p. 200] guaranties that B_m is a function of the Euclidean Laplacian $\Delta_{\mathbb{C}^n}$. As in [6] we start from the fact that B_m should be the Fourier transform of the function $h_m(z)$ evaluated at $\frac{1}{i}$ times the gradient operator ∇. i.e.,

$$B_m = \mathfrak{F}[h_m]\left(\frac{1}{i}\nabla\right). \tag{5.4}$$

Here, our method is based on straightforward calculations.

Theorem 5.1 *Let $m = 0, 1, 2, \ldots$. Then, the Berezin transform B_m can be expressed in terms of the Laplacian $\Delta_{\mathbb{C}^n}$ as*

$$B_m = e^{\frac{1}{4}\Delta_{\mathbb{C}^n}} \sum_{j=0}^{2m} \sigma_j^{(n,m)} L_j^{(n-1)}\left(\frac{-1}{4}\Delta_{\mathbb{C}^n}\right) \tag{5.5}$$

with

$$\sigma_j^{(n,m)} = \frac{m!}{(n)_m}(-1)^j \sum_{s=0}^{j} \binom{j}{s} \binom{m+n-1}{m-j+s} \binom{m+n-1}{m-s}. \tag{5.6}$$

Proof We start by looking at the integral

$$\widehat{h_m}(\xi) = \int_{\mathbb{C}^n} e^{-i\langle \xi, z\rangle} h_m(z) d\mu(z). \tag{5.7}$$

Inserting (5.3) in (5.7) and using polar coordinates $z = \rho\omega$, $\rho > 0$ and $\omega \in \mathbb{S}^{2n-1}$, then (5.7) takes the form

$$\begin{aligned} \widehat{h}_m(\xi) &= \frac{m!}{\pi^n (n)_m} \int_0^{+\infty} \int_{S^{2n-1}} e^{-i\langle \xi, z\rangle} e^{-\rho^2} \\ &\quad \times \left(L_m^{(n-1)}\left(\rho^2\right)\right)^2 \rho^{2n-1} d\rho d\sigma(\omega) \qquad (5.8) \\ &= \frac{m!}{\pi^n (n)_m} \int_0^{+\infty} e^{-\rho^2} \left(L_m^{(n-1)}\left(\rho^2\right)\right)^2 \rho^{2n-1} \\ &\quad \times \left(\int_{S^{2n-1}} e^{-i\langle \xi, \rho\omega\rangle} d\sigma(\omega)\right) d\rho. \qquad (5.9) \end{aligned}$$

The last integral in (5.9) can be identified as a Bochner integral ([2, p. 646]), as:

$$\int_{S^{2n-1}} e^{-i2\pi\langle\rho\zeta,\omega\rangle} d\sigma(\omega) = 2\pi\rho^{-n+1} |\zeta|^{-n+1} J_{n-1}(2\pi\rho|\zeta|), \tag{5.10}$$

$J_\nu(\cdot)$ being the Bessel function. Therefore, we set $\zeta = (2\pi)^{-1}\xi$ and we insert (5.10) into (5.9) to get that

$$\widehat{h}_m(\xi) = \frac{2^n m!}{(n)_m}|\xi|^{-n+1}\int_0^{+\infty} \rho^n e^{-\rho^2}\left(L_m^{(n-1)}(\rho^2)\right)^2 J_{n-1}(\rho|\xi|)\, d\rho. \tag{5.11}$$

Now, the Feldheim formula [14], which expresses the product of Laguerre polynomials as a sum of Laguerre polynomials, is given by

$$L_k^{(\alpha)}(x)L_l^{(\beta)}(x) = \sum_{j=0}^{k+l} A_j(k,l,\alpha,\beta)\, L_j^{(\alpha+\beta)}(x) \tag{5.12}$$

$$= (-1)^{k+l}\sum_{j=0}^{k+l} A_j(k,l,\beta-k+l,\alpha+k-l)\,\frac{x^j}{j!} \tag{5.13}$$

with

$$A_j(k,l,\alpha,\beta) = (-1)^{k+l+j}\sum_{s=0}^{j}\binom{j}{s}\binom{k+\alpha}{l-j+s}\binom{l+\beta}{k-s}, \tag{5.14}$$

$\Re\alpha > -1$, $\Re\beta > -1$, $\Re(\alpha+\beta) > -1$. We make use of this formula for the particular values of $k = l = m$, $\alpha = \beta = n-1$ and $x = \rho^2$, we obtain

$$\left(L_m^{(n-1)}(\rho^2)\right)^2 = \sum_{j=0}^{2m}\gamma_j^{(n,m)}\frac{\rho^{2j}}{j!} \tag{5.15}$$

with

$$\gamma_j^{(n,m)} := A_j(m,m,n-1,n-1). \tag{5.16}$$

Returning back to (5.11) and replacing (5.15), we get

$$\widehat{h}_m(\xi) = \frac{2^n m!}{(n)_m}|\xi|^{-n+1}\int_0^{+\infty}\rho^n e^{-\rho^2}\left(\sum_{j=0}^{2m}\gamma_j^{(n,m)}\frac{\rho^{2j}}{j!}\right) J_{n-1}(\rho|\xi|)\, d\rho \tag{5.17}$$

$$= \frac{2^n m!}{(n)_m}|\xi|^{-n+1}\sum_{j=0}^{2m}\gamma_j^{(n,m)}\frac{1}{j!}\int_0^{+\infty} e^{-\rho^2}\rho^{2j+n}J_{n-1}(\rho|\xi|)d\rho. \tag{5.18}$$

Next, we use the identity ([17, p. 704]):

$$\int_0^{+\infty} x^{2s+\nu+1}e^{-x^2}J_\nu\left(2x\sqrt{z}\right)dx = \frac{s!}{2}e^{-z}z^{\frac{1}{2}\nu}L_s^{(\nu)}(z); \tag{5.19}$$

$s=0,1,2,\dots,s+\Re\nu>-1$, for $x=\rho$, $\nu=n-1$, $s=j$ and $2\sqrt{z}=|\xi|$. This gives

$$\widehat{h}_m(\xi)=\frac{m!}{(n)_m}e^{-\frac{1}{4}|\xi|^2}\sum_{j=0}^{2m}\gamma_j^{(n,m)}L_j^{(n-1)}\left(\frac{1}{4}|\xi|^2\right). \tag{5.20}$$

Finally, we replace ξ by $\frac{1}{i}\nabla$ and we state the first result as follows. □

Another way to write the Berezin transform B_m as function of Laplacian $\triangle_{\mathbb{C}^n}$ is as follows.

Theorem 5.2 *Let $m=0,1,2,\dots$. Then, the Berezin transform B_m can be expressed in terms of the Laplacian $\triangle_{\mathbb{C}^n}$ as*

$$B_m=e^{\frac{1}{4}\triangle_{\mathbb{C}^n}}\sum_{j=0}^{2m}\kappa_j^{(n,m)}\left(\triangle_{\mathbb{C}^n}\right)^j \tag{5.21}$$

with

$$\kappa_j^{(n,m)}=\frac{2^{2m}(m!)^3(-1)^j\,{}_3F_2(\frac{j}{2}-m,\frac{j+1}{2}-m,j+n,j-m+1,j-m+1;1)}{(n)_m\,j!2^{3j}(2m-j)!\,(\Gamma(j-m+1))^2}. \tag{5.22}$$

Proof We return back to (5.11) and we make us of the following linearization of the product of Laguerre polynomials ([27, p. 7361]):

$$L_k^{(\alpha)}(x)L_l^{(\alpha)}(x)=\sum_{j=|k-l|}^{k+l}C_j\,(k,l,\alpha)\,L_j^{(\alpha)}(x), \tag{5.23}$$

where the coefficients are given in terms of ${}_3F_2$ hypergeometric function [2] as

$$\begin{aligned}C_j\,(k,l,\alpha)=&\frac{2^{k+l-j}k!l!}{(k+l-j)!\Gamma\,(j-k+1)\,\Gamma(j-l+1)}\\&\times{}_3F_2\left(\frac{j-k-l}{2},\frac{j-k-l+1}{2},j+\alpha+1,j-k+1,j-l+1;1\right)\end{aligned} \tag{5.24}$$

for the particular case $\alpha=n-1$, $k=l=m$ and $x=\rho^2$. We obtain

$$\left(L_m^{(n-1)}\left(\rho^2\right)\right)^2=\sum_{j=0}^{2m}c_j^{(n,m)}L_j^{(n-1)}\left(\rho^2\right), \tag{5.25}$$

where

$$c_j^{(n,m)}=\frac{2^{2m-j}(m!)^2.{}_3F_2\big(\frac{j}{2}-m,\frac{j+1}{2}-m,j+n,j-m+1,j-m+1;1\big)}{(2m-j)!(\Gamma(j-m+1))^2}. \tag{5.26}$$

Therefore, (5.11) takes the form

$$\widehat{h}_m(\xi) = \frac{2^n m!}{(n)_m}|\xi|^{-n+1}\int_0^{+\infty}\rho^n e^{-\rho^2}\left(\sum_{j=0}^{2m} c_j^{(n,m)} L_j^{(n-1)}(\rho^2)\right) J_{n-1}(\rho|\xi|)\,d\rho \tag{5.27}$$

$$= \frac{2^n m!}{(n)_m}|\xi|^{-n+1}\sum_{j=0}^{2m} c_j^{(n,m)}\int_0^{+\infty} e^{-\rho^2}\rho^n L_j^{(n-1)}(\rho^2) J_{n-1}(\rho|\xi|)\,d\rho. \tag{5.28}$$

Next, making use of the identity [17, p. 812]:

$$\int_0^{+\infty} e^{-x^2}x^{\nu+1}L_s^{(\nu)}(x^2)J_\nu(xu)\,dx = \frac{1}{2s!}\left(\frac{u}{2}\right)^{2s+\nu} e^{-\frac{1}{4}u^2} \tag{5.29}$$

for $\nu = n-1, x=\rho, s=j$ and $u=|\xi|$, the integral in (5.29) takes the form

$$\int_0^{+\infty} e^{-\rho^2}\rho^n L_j^{(n-1)}(\rho^2) J_{n-1}(\rho|\xi|)\,d\rho = \frac{1}{2j!}\left(\frac{|\xi|}{2}\right)^{2j+n-1} e^{-\frac{1}{4}|\xi|^2} \tag{5.30}$$

and (5.28) becomes

$$\widehat{h}_m(\xi) = \frac{m!}{(n)_m} e^{-\frac{1}{4}|\xi|^2}\sum_{j=0}^{2m} c_j^{(n,m)}\frac{(-1)^j}{j!2^{2j}}\left(-|\xi|^2\right)^j. \tag{5.31}$$

Finally, we replace ξ by $\frac{1}{i}\nabla$ and we state the result. □

Remark 5.3 In [6], we have proved that

$$B_m = e^{\frac{1}{4}\Delta_{\mathbb{C}^n}}\sum_{k=0}^{m}\frac{(n-1)_k\,(m-k)!}{(n)_m k!}\left(\frac{\Delta_{\mathbb{C}^n}}{4}\right)^k L_{m-k}^{(k)}\left(\frac{-\Delta_{\mathbb{C}^n}}{4}\right) L_{m-k}^{(n-1+k)}\left(\frac{-\Delta_{\mathbb{C}^n}}{4}\right) \tag{5.32}$$

so that formulae in (5.5) and (5.21) represent other ways to write the transform B_m as function of $\Delta_{\mathbb{C}^n}$.

6 Conclusions

While dealing with a class of generalized Bargmann spaces [5], we first have been concerned with a direct proof for obtaining the reproducing kernel of these spaces starting from the knowledge of an explicit orthonormal basis and by exploiting an addition formula for Laguerre polynomials involving the disk polynomials due to Koornwinder [21]. With the help of this basis, we have constructed for each of these spaces a set of coherent states by following a generalized formalism in order to apply a coherent states quantization method [16]. This has provided us with another way to recover the Berezin transforms attached to the generalized Bargmann spaces under consideration, which was constructed in [6, 24] by means of Toeplitz operators. For related recent works in the literature, we should mention the references [1, 10, 12, 13, 23]. We have also established two other formulae expressing these Berezin transforms as functions of Euclidean Laplacian. These two formulae together with a first one [6] could be of help in physics problems. Why? First, we should note that the Euclidean Laplacian represents (in suitable units) the Hamiltonian of a *free* particle in quantum mechanics. On the other hand, in view of Remark 3.1, the transform B_m encodes the *effect* of the magnetic field at the mth eigenenergy (*Landau level*) so that it could be useful to prepare for physicists formulae expressing B_m as function of this Laplacian in all possible different forms. In some sense, these formulae express, at some energy level, a relation linking a transform arising from a magnetic Schrödinger operator with a quantity involving the non-magnetic Schrödinger operator. This link is well defined through the exponential prefactor (which reflects the free particle case) times a polynomial function of the Laplacian. The degree and coefficients of this polynomial function are given explicitly as in Theorem 5.2 with the effort to describe the polynomial part in a precise way. The diamagnetism of spinless Bose systems [28] or diamagnetic inequalities illustrate very well this kind of picture.

References

1. Abreu, L.D.: Sampling and interpolation in Bargmann-Fock spaces of polyanalytic functions. Appl. Comput. Harmon. Anal. **29**(3), 287–302 (2010)
2. Andrews, G.E., Askey, R., Roy, R.: Special Functions. Cambridge University Press, Cambridge (1999)
3. Aronszajn, N.: Theory of reproducing kernels. Trans. Am. Math. Soc. **68**, 337–404 (1950)
4. Askour, N., Mouayn, Z.: Spectral decomposition and resolvent kernel for a magnetic Laplacian in $\mathbb{C}^n$. J. Math. Phys. **41**(10), 6937–6943 (2000)
5. Askour, N., Intissar, A., Mouayn, Z.: Espaces de Bargmann généralisés et formules explicites pour leurs noyaux reproduisants. C. R. Acad. Sci. Paris Sér. I Math. **325**(7), 707–712 (1997)
6. Askour, N., Intissar, A., Mouayn, Z.: A formula representing Berezin transforms as functions of the Laplacian on $\mathbb{C}^n$. Integral Transforms Spec. Funct. Available online: 24 Jun 2011
7. Berezin, F.A.: Quantization. Math. USSR, Izv. **8**, 1109–1165 (1974)
8. Berezin, F.A.: General concept of quantization. Commun. Math. Phys. **40**, 153–174 (1975)
9. Birman, M.S., Solomjak, Z.: Spectral theory of selfadjoint operators in Hilbert space. Translated from the 1980 Russian original by S. Khrushchëv and V. Peller. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht (1987)
10. de Gosson, M., Luef, F.: Spectral and regularity properties of a pseudo-differential calculus related to Landau quantization. J. Pseudo-Differential Oper. Appl. **1**(1), 3–34 (2010)
11. Dodonov, V.V.: 'Nonclassical' states in quantum optics: a squeezed review of the first 75 years. J. Opt. B, Quantum Semiclass. Opt. **4**, R1–R33 (2002)
12. Englis, M.: Toeplitz operators and Localization operators. Trans. Am. Math. Soc. **361**(2), 1039–1052 (2009)
13. Faustino, N.: Localization and Toeplitz operators on polyanalytic Fock spaces (2011). arXiv:1107.4680v1
14. Feldheim, E.: Expansions and integral transforms for products of Laguerre and Hermite polynomials. Quart. J. Math. Oxford Ser. **11**, 18–29 (1940)
15. Folland, G.B.: Spherical harmonic expansion of the Poisson-Szegö kernel for the ball. Proc. Am. Math. Soc. **47**(2), 401–407 (1975)
16. Gazeau, J.P.: Coherent States in Quantum Physics. Wiley, Weinheim (2009)

17. Gradshteyn, I.S., Ryzhik, I.M.: Table of integrals, series, and products. Translated from the Russian. Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger., Seventh edn. Elsevier/Academic Press, Amsterdam (2007)
18. Hall, B.C.: Holomorphic methods in analysis and mathematical physics. First Summer School in Analysis and Mathematical Physics (Cuernavaca, Morelos, 1998), Contemp. Math. 260 1, Am. Math. Soc. Providence RI, 2000
19. Klauder, J.R.: Beyond Conventional Quantization. Cambridge University Press, Cambridge (2000)
20. Koornwinder, T.H.: The addition formula for Jacobi polynomials II. The Laplace type integral representation and the product formula. Math. Centrum Amsterdam, Report TW 133 (1976)
21. Koornwinder, T.H.: The addition formula for Laguerre polynomials. SIAM J. Math. Anal. **8**, 535–540 (1977)
22. Magnus, W., Oberhettinger, F., Soni, R.P.: Formulas and Theorems for the Special Functions of Mathematical Physics, 3rd edn. Die Grundlehren der mathematischen Wissenschaften, vol. 52. Springer, New York (1966)
23. Molahajloo, S., Wong, M.W.: The Schrödinger kernel of the twisted Laplacian and cyclic models. Arkiv Math. **95**(6), 593–599 (2010)
24. Mouayn, Z.: Decomposition of magnetic Berezin transforms on the Euclidean complex space $\mathbb{C}^n$. Integral Transforms Spec. Funct. **19**(11–12), 903–912 (2008)
25. Peetre, J.: The Berezin transform and Ha-plitz operators. J. Oper. Theory **24**(1), 165–186 (1990)
26. Rudin, W.: Function Theory in the Unit Ball of $\mathbb{C}^n$. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 241. Springer, New York (1980)
27. Sanchez-Ruiz, J., Artès, P.L., Martinez-Finkelshtein, A., Dehesa, J.S.: General linearisation formula for product of continuous hypergeometric-type polynomials. J. Phys. A, Math. Gen. **32**, 7345–7366 (1999)
28. Simon, B.: Universal diamagnetism of spinless Bose systems. Phys. Rev. Lett. **36**, 1083–1084 (1976)
29. Unterberger, A., Upmeir, H.: The Berezin transform and invariant differential operators. Commun. Math. Phys. **164**, 563–597 (1994)
30. Zernike, F., Brinkman, H.C.: Hyperspharische Funktionen und die in spharischen Bereichen orthogonalen Polynome. Proc. Kon. Akad. v. wet., Amterdam **38**, 161–170 (1935)