J Fourier Anal Appl (2012) 18:592–608
DOI 10.1007/s00041-011-9212-3

Sets of Injectivity for Weighted Twisted Spherical Means and Support Theorems

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Received: 13 January 2011 / Revised: 31 October 2011 / Published online: 29 December 2011

Abstract We prove that the spheres centered at origin are sets of injectivity for certain weighted twisted spherical means on $\mathbb{C}^n$. We also prove an analogue of Helgason's support theorem for weighted Euclidean and twisted spherical means.

Keywords Hecke-Bochner identity · Heisenberg group · Laguerre polynomials · Spherical harmonics · Support theorems · Twisted convolution

Mathematics Subject Classification (2000) Primary 43A85 · Secondary 44A35

1 Introduction

In this article, we show that the spheres $S_R(o) = \{z \in \mathbb{C}^n : |z| = R\}$ are sets of injectivity for the weighted twisted spherical means (WTSM) for a suitable class of functions on $\mathbb{C}^n$. The weights here are spherical harmonics on S^{2n-1}. In general, the question of sets of injectivity for the twisted spherical means (TSM) with real analytic weight is still open. We would like to refer to [7], for some results on the sets of injectivity for the spherical means with real analytic weights in the Euclidean setup.

Our main result, Theorem 1.3 is a natural generalization of a result by Thangavelu et al. [9], where it has been proved that the spheres $S_R(o)$'s are sets of injectivity for the TSM on $\mathbb{C}^n$. The twisted spherical mean arises in the study of spherical mean on the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$. These result can also be interpreted for the weighted spherical means on the Heisenberg group. The set $S = \{(z, t) : \ |z| = R, \ t \in \mathbb{R}\} \subset \mathbb{H}^n$ is a set of injectivity for the weighted spherical means on $\mathbb{H}^n$ defined by (1.3).

Communicated by Hans G. Feichtinger.

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In a fundamental result, Helgason proved a support theorem for continuous function having vanishing spherical means over a family of spheres, sitting in the exterior of a ball. That is, if f is a continuous function on $\mathbb{R}^n$, $(n \geq 2)$ such that $|x|^k f(x)$ is bounded for each non-negative integer k, then f is supported in ball $B_r(o)$ if and only if $f * \mu_s(x) = 0, \forall x \in \mathbb{R}^n$ and $\forall s > |x| + r$, (see [6]). In a recent work [8], Thangavelu and Narayanan prove a support theorem, for the TSM for certain subspace of Schwartz class functions on $\mathbb{C}^n$. In our previous work [13], we have given an exact analogue of Helgason's support theorem for the TSM on $\mathbb{C}^n$ $(n \geq 2)$. For $n = 1$, we have proved a surprisingly stronger result where we do not need any decay condition. This result has no analogue in the Euclidean set up. In Theorem 1.4, we generalize our idea of support theorem for the TSM to the WTSM. At the end, we revisit Euclidean spherical means and prove Theorem 1.5, which is an analogue of Helgason's support theorem for the weighted spherical means. For some results on support theorem with real analytic weight, in non-Euclidean set up, we refer to Quinto's works [10–12].

Let μ_r be the normalized surface measure on sphere $S_r(x)$. Let $\mathcal{F} \subseteq L^1_{\text{loc}}(\mathbb{R}^n)$. We say that $S \subseteq \mathbb{R}^n$ is a set of injectivity for the spherical means for $\mathcal{F}$ if for $f \in \mathcal{F}$ with $f * \mu_r(x) = 0, \forall r > 0$ and $\forall x \in S$, implies $f = 0$ a.e.

The results on sets of injectivity differ in the choice of sets and the class of functions considered. The following result by Agranovsky et al. [1] partially describe the sets of injectivity in $\mathbb{R}^n$. The boundary of bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ is set of injectivity for the spherical means on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \frac{2n}{n-1}$. For $p > \frac{2n}{n-1}$, unit sphere S^{n-1} is an example of non-injectivity set in $\mathbb{R}^n$.

The range for p in the above result is optimal. That can be seen as follows. For $\lambda > 0$, define the radial function φ_λ on $\mathbb{R}^n$ by

$$\varphi_\lambda(x) = c_n(\lambda|x|)^{-\frac{n}{2}+1} J_{\frac{n}{2}-1}(\lambda|x|),$$

where $J_{\frac{n}{2}-1}$ is the Bessel function of order $\frac{n}{2} - 1$ and c_n is the constant such that $\varphi_\lambda(o) = 1$. Then the spherical means of φ_λ satisfy the relation

$$\varphi_\lambda * \mu_r(x) = \varphi_\lambda(r)\varphi_\lambda(x).$$

This shows that if λR is zero of Bessel function $J_{\frac{n}{2}-1}$ then $\varphi_\lambda * \mu_r(x) = 0$ on sphere $S_R(o)$ and for all $r > 0$. Since $\varphi_\lambda \in L^p(\mathbb{R}^n)$ if and only if $p > 2n/(n-1)$, it follows that spheres are not sets of injectivity for spherical means for L^p for $p > 2n/(n-1)$. In a recent result of Narayanan et al. [7], it has been shown that the boundary of a bounded domain in $\mathbb{R}^n$ is a set of injectivity for the weighted spherical means for $L^p(\mathbb{R}^n)$, with $1 \leq p \leq \frac{2n}{n-1}$.

Next, we come up with twisted spherical means which arises in the study of spherical means on Heisenberg group. The group $\mathbb{H}^n$ as a manifold, is $\mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z,t)(w,s) = \left(z + w, t + s + \frac{1}{2}\text{Im}(z.\bar{w})\right), \quad z, w \in \mathbb{C}^n \text{ and } t,\ s \in \mathbb{R}.$$

The spherical means of a function f in $L^1(\mathbb{H}^n)$ are defined by

$$f * \mu_s(z,t) = \int_{|w|=s} f((z,t)(-w,0))\, d\mu_s(w). \tag{1.1}$$

Thus the spherical means can be thought of as convolution operators. An important technique in many problems on $\mathbb{H}^n$ is to take partial Fourier transform in the t-variable to reduce matters to $\mathbb{C}^n$. This technique works very well with convolution operator on $\mathbb{H}^n$ and we will make use of it to analyze spherical means on $\mathbb{H}^n$. Let

$$f^\lambda(z) = \int_{\mathbb{R}} f(z,t) e^{i\lambda t}\, dt$$

be the inverse Fourier transform of f in the t-variable. Then a simple calculation shows that

$$\begin{aligned}(f * \mu_s)^\lambda(z) &= \int_{-\infty}^{\infty} f * \mu_s(z,t) e^{i\lambda t}\, dt \\ &= \int_{|w|=s} f^\lambda(z-w) e^{\frac{i\lambda}{2}\operatorname{Im}(z.\bar{w})}\, d\mu_s(w) \\ &= f^\lambda \times_\lambda \mu_s(z),\end{aligned}$$

where μ_s is now being thought of as normalized surface measure on the sphere $S_s(o) = \{z \in \mathbb{C}^n : |z| = s\}$ in $\mathbb{C}^n$. Thus the spherical mean $f * \mu_s$ on the Heisenberg group can be studied using the λ-twisted spherical mean $f^\lambda \times_\lambda \mu_s$ on $\mathbb{C}^n$. For $\lambda \neq 0$, a further scaling argument shows that it is enough to study these means for the case of $\lambda = 1$.

Let $\mathcal{F} \subseteq L^1_{\text{loc}}(\mathbb{C}^n)$. We say $S \subseteq \mathbb{C}^n$ is a set of injectivity for twisted spherical means for $\mathcal{F}$ if for $f \in \mathcal{F}$ with $f \times \mu_r(z) = 0, \forall r > 0$ and $\forall z \in S$, implies $f = 0$ a.e. on $\mathbb{C}^n$.

As in the Euclidean case, it would be natural to ask if the boundaries of bounded domains in $\mathbb{C}^n$ continue to be sets of injectivity for L^p spaces for the twisted spherical means. However, this is no longer true as can be seen by considering the Laguerre functions $\varphi_k^{n-1}, k \in \mathbb{Z}_+$, given by $\varphi_k^{n-1}(z) = L_k^{n-1}(\frac{1}{2}|z|^2)e^{-\frac{1}{4}|z|^2}$, where L_k^{n-1}'s are the Laguerre polynomials of degree k and type $n-1$. These functions satisfy the functional relations

$$\varphi_k^{n-1} \times \mu_r(z) = \frac{k!(n-1)!}{(n+k-1)!}\varphi_k^{n-1}(r)\varphi_k^{n-1}(z), \quad k \in \mathbb{Z}_+. \tag{1.2}$$

For $k = 0, \varphi_0^{n-1}(z) = e^{-\frac{1}{4}|z|^2}$, which is never zero. Otherwise, if $\frac{1}{2}R^2$ is a zero of L_k^{n-1} for $k = 1, 2, \dots$, then $\varphi_k^{n-1} \times \mu_r(z) = 0$ on sphere $S_R(o)$ for all $r > 0$. Since φ_k^{n-1} are in Schwartz class, it follows that spheres, and hence boundaries of bounded domains are not sets of injectivity for $L^p(\mathbb{C}^n)$ for any p , $1 \leq p \leq \infty$. As $e^{\frac{1}{4}|z|^2}\varphi_k^{n-1}$, $k = 1, 2, \dots$, does not belong to $L^p(\mathbb{C}^n)$ for $1 \leq p \leq \infty$, it would be interesting to know if boundaries of bounded domains in $\mathbb{C}^n$ are sets of injectivity for the class of functions f such that $f(z)e^{\frac{1}{4}|z|^2} \in L^p(\mathbb{C}^n)$ for some p, $1 \le p \le \infty$. In [2] the authors answer this for a yet smaller function space. The boundary of a bounded domain in $\mathbb{C}^n$ is set of injectivity for function f with $f(z)e^{(\frac{1}{4}+\epsilon)|z|^2} \in L^p(\mathbb{C}^n)$ for some $\epsilon > 0$ and $1 \le p \le \infty$. In the light of the above discussion an optimal result would be proving this result for $\epsilon = 0$. This in general is an open problem, but in the special case of $\Gamma = S^{2n-1}$, the result has been established by Narayanan and Thangavelu [9].

Theorem 1.1 [9] *Let f be a function on $\mathbb{C}^n$ such that $e^{\frac{1}{4}|z|^2} f(z) \in L^p(\mathbb{C}^n)$, for $1 \le p \le \infty$. If $f \times \mu_r(z) = 0$ on sphere $S_R(o)$ and for all $r > 0$, then $f = 0$ a.e. on $\mathbb{C}^n$.*

Remark 1.2 For $\eta \in \mathbb{C}^n$, define the left twisted translate by

$$\tau_\eta f(\xi) = f(\xi - \eta)e^{\frac{i}{2}\mathrm{Im}(\eta.\bar{\xi})}.$$

Then $\tau_\eta(f \times \mu_r) = \tau_\eta f \times \mu_r$. Since the function space considered as in the above Theorem 1.1 is not twisted translation invariant, it follows that a sphere centered off origin is not set of injectivity for the TSM on $\mathbb{C}^n$.

Our aim is to consider some special weighted twisted spherical means and prove that Theorem 1.1 can be extended for those means. For this, let $\mathbb{Z}_+$ denote the set of all non negative-integers. For $s, t \in \mathbb{Z}_+$, let $P_{s,t}$ denote the space of all polynomials P in z and $\bar{z}$ of the form

$$P(z) = \sum_{|\alpha|=s} \sum_{|\beta|=t} c_{\alpha\beta} z^\alpha \bar{z}^\beta.$$

Let $H_{s,t} = \{P \in P_{s,t} : \Delta P = 0\}$, where Δ is the standard Laplacian on $\mathbb{C}^n$. Let $\{P \in P^j_{s,t} : 1 \le j \le d(s,t) = \dim H_{s,t}\}$ be an orthonormal basis of $H_{s,t}$ and $d\nu_{r,j} = P^j_{st} d\mu_r$. Then $d\nu_{r,j}$ is a signed measure on the sphere $S_r(o)$ in $\mathbb{C}^n$. As similar to (1.1), we can define the weighted spherical means of a function $f \in L^1(\mathbb{H}^n)$ by

$$f * \nu_{r,j}(z,t) = \int_{|w|=r} f((z,t)(-w,0))\ P^j_{st}(w)\, d\mu_r(w). \tag{1.3}$$

By taking the inverse Fourier transform in t variable at $\lambda = 1$, we can write

$$f \times \nu_{r,j}(z) = \int_{S_r(o)} f(z-w)e^{\frac{i}{2}\mathrm{Im}(z.\bar{w})} P^j_{st}(w)\, d\mu_r(w).$$

We call $f \times \nu_{r,j}$ the weighted twisted spherical mean (WTSM) of function $f \in L_{\mathrm{loc}}(\mathbb{C}^n)$. We prove the following result for the injectivity of the WTSM.

Theorem 1.3 *Let f be a function on $\mathbb{C}^n$ such that $e^{\frac{1}{4}|z|^2} f(z) \in L^p(\mathbb{C}^n)$, $1 \le p < \infty$. If $f \times \nu_{r,j}(z) = 0$ on sphere $S_R(o)$, $\forall r > 0$ and $\forall j$, $1 \le j \le d(s,t)$, then $f = 0$ a.e.*

For $p=\infty$, Theorem 1.3 does not hold as can be seen in Remark 2.4. Further, we prove a support theorem for the weighted twisted spherical means.

Theorem 1.4 *Let f be a smooth function on $\mathbb{C}^n$ such that for each non-negative integer k, $|z|^k|f(z)|\le C_k\, e^{-\frac{1}{4}|z|^2}$. Let $f\times \nu_{r,j}(z)=0$, for all $z\in\mathbb{C}^n$ and $r>|z|+B$ and for all j, $1\le j\le d(s,t)$. Then $f=0$, whenever $|z|>B$.*

In the end, we revisit Euclidean spherical means and prove the support Theorem 1.5 for the weighted spherical means. For $k\in\mathbb{Z}^+$, let P_k denote the space of all homogeneous polynomials P of degree k. Let $H_k=\{P\in P_k\ :\ \Delta P=0\}$. The elements of H_k are called the solid spherical harmonics of degree k. Let $\{P_{kj}:\ 1\le j\le d_k=\dim H_k\}$ be an orthonormal basis for H_k. Define the weighted spherical mean of function $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$f*\mu^k_{r,j}(x)=\int_{S_r(o)} f(x+y)P_{kj}(y)\,d\mu_r(y).$$

Theorem 1.5 *Let f be a smooth function on $\mathbb{R}^n$ such that $|x|^m f(x)$ is bounded for each $m\in\mathbb{Z}_+$. Let $f*\mu^k_{r,j}(x)=0$, for all $x\in\mathbb{R}^n$, $r>|x|+B$ and for all j, $1\le j\le d_k$. Then $f=0$ whenever $|x|>B$.*

2 Preliminaries

We need the following basic facts from the theory of bigraded spherical harmonics (see [15], p. 62 for details). We shall use the notation $K=U(n)$ and $M=U(n-1)$. Then $S^{2n-1}\cong K/M$ under the map $kM\to k.e_n$, $k\in U(n)$ and $e_n=(0,\dots,1)\in\mathbb{C}^n$. Let $\hat{K}_M$ denote the set of all equivalence classes of irreducible unitary representations of K, which have a nonzero M-fixed vector. It is known that for each representation in $\hat{K}_M$ has a unique nonzero M-fixed vector, up to a scalar multiple.

For a $\delta\in\hat{K}_M$, which is realized on V_δ, let $\{e_1,\dots,e_{d(\delta)}\}$ be an orthonormal basis of V_δ with e_1 as the M-fixed vector. Let $t^\delta_{ij}(k)=\langle e_i,\delta(k)e_j\rangle$, $k\in K$ and $\langle,\rangle$ stand for the innerproduct on V_δ. By Peter-Weyl theorem, it follows that $\{\sqrt{d(\delta)}t^\delta_{j1}:1\le j\le d(\delta),\delta\in\hat{K}_M\}$ is an orthonormal basis of $L^2(K/M)$ (see [15], p. 14 for details). Define $Y^\delta_j(\omega)=\sqrt{d(\delta)}t^\delta_{j1}(k)$, where $\omega=k.e_n\in S^{2n-1}$, $k\in K$. It then follows that $\{Y^\delta_j:1\le j\le d(\delta),\delta\in\hat{K}_M,\}$ forms an orthonormal basis for $L^2(S^{2n-1})$.

For our purpose, we need a concrete realization of the representations in $\hat{K}_M$, which can be done in the following way. See [14], p. 253, for details.

For $p,q\in\mathbb{Z}_+$, let $P_{p,q}$ denote the space of all polynomials P in z and $\bar{z}$ of the form

$$P(z)=\sum_{|\alpha|=p}\sum_{|\beta|=q} c_{\alpha\beta}z^\alpha\bar{z}^\beta.$$

Let $H_{p,q}=\{P\in P_{p,q}:\Delta P=0\}$. The elements of $H_{p,q}$ are called the bigraded solid harmonics on $\mathbb{C}^n$. The group K acts on $H_{p,q}$ in a natural way. It is easy to see that the space $H_{p,q}$ is K-invariant. Let $\pi_{p,q}$ denote the corresponding representation of K on $H_{p,q}$. Then representations in $\hat{K}_M$ can be identified, up to unitary equivalence, with the collection $\{\pi_{p,q} : p, q \in \mathbb{Z}_+\}$.

Define the bigraded spherical harmonic by $Y_j^{p,q}(\omega) = \sqrt{d(p,q)}t_{j1}^{p,q}(k)$. Then $\{Y_j^{p,q} : 1 \leq j \leq d(p,q) \text{ and } p, q \in \mathbb{Z}_+\}$ forms an orthonormal basis for $L^2(S^{2n-1})$. Therefore for a continuous function f on $\mathbb{C}^n$, writing $z = \rho\omega$, where $\rho > 0$ and $\omega \in S^{2n-1}$, we can expand the function f in terms of spherical harmonics as

$$f(\rho\omega) = \sum_{p,q\geq 0} \sum_{j=1}^{d(p,q)} a_j^{p,q}(\rho) Y_j^{p,q}(\omega). \tag{2.1}$$

The functions $a_j^{p,q}$ are called the spherical harmonic coefficients of the function f. The (p,q)th spherical harmonic projection, $\Pi_{p,q}(f)$ of the function f is then defined as

$$\Pi_{p,q}(f)(\rho,\omega) = \sum_{j=1}^{d(p,q)} a_j^{p,q}(\rho) Y_j^{p,q}(\omega). \tag{2.2}$$

We will replace the spherical harmonic $Y_j^{p,q}(\omega)$ on the sphere by the solid harmonic $P_j^{p,q}(z) = |z|^{p+q} Y_j^{p,q}(\frac{z}{|z|})$ on $\mathbb{C}^n$ and accordingly for a function f. Define $\tilde{a}_j^{p,q}(\rho) = \rho^{-(p+q)} a_j^{p,q}(\rho)$, where $a_j^{p,q}$ are defined by (2.1). We shall continue to call the functions $\tilde{a}_j^{p,q}$ the spherical harmonic coefficients of f.

In the proof of Theorem 1.3, we also need an expansion of functions on $\mathbb{C}^n$ in terms of Laguerre functions φ_k^{n-1}'s. Let $f \in L^2(\mathbb{C}^n)$. Then the special Hermite expansion for f is given by

$$f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k^{n-1}(z). \tag{2.3}$$

For radial functions, this expansion further simplifies as can be seen from the following lemma.

Lemma 2.1 [15] *Let f be a radial function in $L^2(\mathbb{C}^n)$. Then*

$$f = \sum_{k=0}^{\infty} B_k^n \langle f, \varphi_k^{n-1} \rangle \varphi_k^{n-1}, \quad \text{where } B_k^n = \frac{k!(n-1)!}{(n+k-1)!}.$$

We would also need the following Hecke-Bochner identities for the spectral projections $f \times \varphi_k^{n-1}$ (see [15], p. 70).

Lemma 2.2 [15] *Let $\tilde{a}P \in L^2(\mathbb{C}^n)$, where $\tilde{a}$ is radial and $P \in H_{p,q}$. Then*

$$(\tilde{a}P) \times \varphi_k^{n-1}(z) = (2\pi)^{-n} P(z)\, \tilde{a} \times \varphi_{k-p}^{n+p+q-1}(z),$$

if $k \geq p$ and 0 otherwise. The convolution in the right hand side is on the space $\mathbb{C}^{n+p+q}$.

Using the Hecke-Bochner identities, a weighted functional equation for spherical function φ_k^{n-1} has been proved in [15], p. 98.

Lemma 2.3 [15] *For $z\in\mathbb{C}^n$, let $P\in H_{p,q}$ and $d\nu_r=P\,d\mu_r$. Then*

$$\varphi_k^{n-1}\times\nu_r(z)=(2\pi)^{-n}C(n,p,q)r^{2(p+q)}\varphi_{k-q}^{n+p+q-1}(r)P(z)\varphi_{k-q}^{n+p+q-1}(z),$$

if $k\geq q$ and 0 otherwise.

Remark 2.4 From Lemma 2.3, it can be seen that Theorem 1.3 does not hold for $p=\infty$. For instance, take $P\in H_{0,1}$ and let $d\nu=P\,d\mu_r$. Then $\varphi_0^{n-1}\times\nu(z)=0$, where $\varphi_0^{n-1}(z)=e^{-\frac{1}{4}|z|^2}$.

3 Injectivity of the Weighted Twisted Spherical Means

In this section, we prove that the spheres are sets of injectivity for the weighted twisted spherical means on $\mathbb{C}^n$. Let

$$f_{lm}(z)=d(s,t)\int_{U(n)}f(\sigma^{-1}z)t_{lm}^{s,t}(\sigma)\,d\sigma \tag{3.1}$$

for $1\leq l,m\leq d(s,t)$.

Lemma 3.1 *Let f be a continuous function on $\mathbb{C}^n$. Suppose $f\times\nu_{r,j}(z)=0$ on sphere $S_R(o)$, for all j, $1\leq j\leq d(p,q)$ and for all $r>0$. Then $f_{lm}\times\nu_{r,j}(z)=0$, on $S_R(o)$, whenever $1\leq l,m\leq d(s,t)$, $1\leq j\leq d(p,q)$ and $r>0$.*

Proof We have

$$f_{lm}\times\nu_{r,j}(z)=d(s,t)\int_{S_r(o)}\int_{U(n)}f(\sigma^{-1}(z-w))e^{\frac{i}{2}\mathrm{Im}(z.\bar{w})}t_{lm}^{s,t}(\sigma)P_{s,t}^{j}(w)\,d\sigma\,d\mu_r(w).$$

Since the space $H_{p,q}$ is $U(n)$-invariant, the function $P_{s,t}^j(\sigma^{-1}w)$ is linear combination of polynomials in $H_{p,q}$. By hypothesis, it follows that

$$\int_{U(n)}t_{lm}^{s,t}(\sigma)\int_{S_r(o)}f(\sigma^{-1}z-w)e^{\frac{i}{2}\mathrm{Im}(\sigma^{-1}z.\bar{w})}P_1^j(\sigma w)\,d\mu_r(w)\,d\sigma=0.$$

□

Remark 3.2 In view of Lemma 3.1, it is enough to work with the function of type $f(z)=\tilde{a}(|z|)P_{s,t}(z)$ and measure $d\nu_r=z_1^p\bar{z}_2^q\,d\mu_r$ for the proof of Theorem 1.3. We therefore drop the index j and write $P_1(z)=z_1^p\bar{z}_2^q$ and $d\nu_r=P_1\,d\mu_r$.

We need the following result of Filaseta and Lam [4], about the irreducibility of Laguerre polynomials. Define the Laguerre polynomials by

$$L_k^\alpha(x) = \sum_{i=0}^{k}(-1)^i \binom{\alpha+k}{k-i}\frac{x^i}{i!},$$

where $k \in \mathbb{Z}_+$ and $\alpha \in \mathbb{C}$.

Theorem 3.3 ([4]) *Let α be a rational number, which is not a negative integer. Then for all but finitely many $k \in \mathbb{Z}_+$, the polynomial $L_k^\alpha(x)$ is irreducible over the rationals.*

Using Theorem 3.3, we obtain the following corollary about the zeros of Laguerre polynomials.

Corollary 3.4 *Let $k \in \mathbb{Z}_+$. Then for all but finitely many k, the Laguerre polynomials $L_k^{n-1}(x)$'s have distinct zeros over the reals.*

Proof By Theorem 3.3, there exists $k_o \in \mathbb{Z}_+$ such that L_k^{n-1}'s are irreducible over $\mathbb{Q}$ whenever $k \geq k_o$. Therefore, we can find polynomials $P_1, P_2 \in \mathbb{Q}[x]$ such that $P_1 L_{k_1}^{n-1} + P_2 L_{k_2}^{n-1} = 1$, over $\mathbb{Q}$ with $k_1, k_2 \geq k_o$. Since this identity continues to hold on $\mathbb{R}$, it follows that $L_{k_1}^{n-1}$ and $L_{k_2}^{n-1}$ have no common zero over $\mathbb{R}$. □

In the proof of Theorem 1.3, we use the following right invariant differential operators for twisted convolution:

$$\tilde{A}_j = \frac{\partial}{\partial z_j} + \frac{1}{4}\bar{z}_j \quad \text{and} \quad \tilde{A}_j^* = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{4}z_j; \quad j = 1, 2, \ldots, n.$$

In addition, we have the left invariant differential operators

$$\tilde{Z}_j = \frac{\partial}{\partial z_j} - \frac{1}{4}\bar{z}_j \quad \text{and} \quad \tilde{Z}_j^* = \frac{\partial}{\partial \bar{z}_j} + \frac{1}{4}z_j; \quad j = 1, 2, \ldots, n$$

for twisted convolution. Let P be a non-commutative homogeneous harmonic polynomial on $\mathbb{C}^n$ with expression

$$P(z) = \sum_{|\alpha|=p}\sum_{|\beta|=q} c_{\alpha\beta} z^\alpha \bar{z}^\beta.$$

Using the result of Geller ([5], Proposition 2.7) about Weyl correspondence of the spherical harmonics, the operator analogue of $P(z)$, accordingly the left and right invariant vector fields can be expressed as

$$P(\widetilde{Z}) = \sum_{|\alpha|=p}\sum_{|\beta|=q} c_{\alpha\beta} \widetilde{Z^*}^\alpha \widetilde{Z}^\beta \quad \text{and} \quad P(\tilde{A}) = \sum_{|\alpha|=p}\sum_{|\beta|=q} c_{\alpha\beta} \tilde{A^*}^\alpha \tilde{A}^\beta.$$

In order to prove Theorem 1.3, We need to prove the following lemma.

Lemma 3.5 *For* $P_1(z) = z_1^p \bar{z}_2^q \in H_{p,q}$ *we have*

$$P_1\big(\tilde{A}\big)\varphi_k^{n-1}(z) = \bar{P}_1\big(\widetilde{Z}\big)\varphi_k^{n-1}(z) = (-2)^{-p-q} P_1(z)\varphi_{k-q}^{n+p+q-1}(z), \tag{3.2}$$

if $k \geq q$ *and* 0 *otherwise.*

Proof We have

$$\tilde{A}_1^*\varphi_k^{n-1}(z) = \left(\frac{\partial}{\partial \bar{z}_1} - \frac{1}{4}z_1\right)\varphi_k^{n-1}(z).$$

For $z \in \mathbb{C}^n$, let $z.\bar{z} = 2t$. By chain rule $\frac{\partial}{\partial \bar{z}_1} = \frac{1}{2}z_1\frac{\partial}{\partial t}$. Therefore,

$$\begin{aligned}\tilde{A}_1^*\varphi_k^{n-1}(z) &= \left(\frac{1}{2}z_1\frac{\partial}{\partial t} - \frac{1}{4}z_1\right)\left(L_k^{n-1}(t)e^{-\frac{1}{2}t}\right)\\ &= \frac{1}{2}z_1\left(\frac{\partial}{\partial t}L_k^{n-1}(t) - \frac{1}{2}L_k^{n-1}(t) - \frac{1}{2}L_k^{n-1}(t)\right)e^{-\frac{1}{2}t}.\end{aligned}$$

The Laguerre polynomials satisfy

$$\frac{d}{dx}L_k^n(x) = -L_{k-1}^{n+1}(x) \quad \text{and} \quad L_{k-1}^{n+1}(x) + L_k^n(x) = L_k^{n+1}(x). \tag{3.3}$$

Thus we have $\tilde{A}_1^*\varphi_k^{n-1}(z) = -\frac{1}{2}z_1\varphi_k^n(z)$. Similarly

$$\begin{aligned}\tilde{A}_2\varphi_k^{n-1}(z) &= \left(\frac{1}{2}\bar{z}_2\frac{\partial}{\partial t} + \frac{1}{4}\bar{z}_2\right)\left(L_k^{n-1}(t)e^{-\frac{1}{2}t}\right)\\ &= \frac{1}{2}\bar{z}_2\left(\frac{\partial}{\partial t}L_k^{n-1}(t) - \frac{1}{2}L_k^{n-1}(t) + \frac{1}{2}L_k^{n-1}(t)\right)e^{-\frac{1}{2}t}\\ &= -\frac{1}{2}\bar{z}_2\varphi_{k-1}^n(z).\end{aligned}$$

Therefore,

$$\tilde{A}_1^*\tilde{A}_2\varphi_k^{n-1}(z) = 2^{-2}z_1\bar{z}_2\varphi_{k-1}^{n+1}(z).$$

Since the operators $\tilde{A}_1^*$ and $\tilde{A}_2$ commute with each other, we can conclude that

$$\tilde{A}_1^{*p}\,\tilde{A}_2^q\varphi_k^{n-1}(z) = (-2)^{-p-q}\; z_1^p\bar{z}_2^q\;\varphi_{k-q}^{n+1}(z).$$

A similar computation shows that

$$\widetilde{Z}_1^{*p}\,\tilde{Z}_2^q\varphi_k^{n-1}(z) = (-2)^{-p-q}\; z_1^p\bar{z}_2^q\;\varphi_{k-p}^{n+1}(z).$$

□

Remark 3.6 Using the result of Geller ([5], Lemma 2.4), the identity (3.2) can be generalized for any $P \in H_{p,q}$. The complete proof of this identity requires some of the preliminaries about Weyl correspondence of spherical harmonic from the work of Geller [5] and will be presented elsewhere.

Lemma 3.7 *For $\rho > 0$, write $\widetilde{D} = \frac{\partial}{\partial \rho} - \frac{1}{2}\rho$ and $\widetilde{D}^* = \frac{\partial}{\partial \rho} + \frac{1}{2}\rho$. Then $\frac{1}{\rho}\widetilde{D}\varphi_k^{n-1}(\rho) = \varphi_k^n(\rho)$ and $\frac{1}{\rho}\widetilde{D}^*\varphi_k^{n-1}(\rho) = \varphi_{k-1}^n(\rho)$.*

Proof Let $\rho^2 = 2t$, then $\frac{\partial}{\partial \rho} = \rho \frac{\partial}{\partial t}$. Therefore,

$$\widetilde{D}\varphi_k^{n-1}(\rho) = \rho\left(\frac{\partial}{\partial t} - \frac{1}{2}\right)\left(L_k^{n-1}(t)e^{-\frac{1}{2}t}\right)$$
$$= \rho\left(\frac{\partial}{\partial t}L_k^{n-1}(t) - \frac{1}{2}L_k^{n-1}(t) - \frac{1}{2}L_k^{n-1}(t)\right)e^{-\frac{1}{2}t}.$$

Using (3.3), we have $\frac{1}{\rho}\widetilde{D}\varphi_k^{n-1}(\rho) = \varphi_k^n(\rho)$. Similarly,

$$\widetilde{D}^*\varphi_k^{n-1}(\rho) = \rho\left(\frac{\partial}{\partial t} + \frac{1}{2}\right)\left(L_k^{n-1}(t)e^{-\frac{1}{2}t}\right)$$
$$= \rho\left(\frac{\partial}{\partial t}L_k^{n-1}(t) - \frac{1}{2}L_k^{n-1}(t) + \frac{1}{2}L_k^{n-1}(t)\right)e^{-\frac{1}{2}t}.$$

Therefore $\frac{1}{\rho}\widetilde{D}^*\varphi_k^{n-1}(\rho) = \varphi_{k-1}^n(\rho)$. □

Suppose f be a function on $\mathbb{C}^n$ such that $e^{\frac{1}{4}|z|^2}f(z) \in L^p(\mathbb{C}^n)$, for $1 \leq p < \infty$. Let φ_ϵ be a smooth, radial compactly supported approximate identity on $\mathbb{C}^n$. Then $f \times \varphi_\epsilon \in L^1 \cap L^\infty(\mathbb{C}^n)$ and in particular $f \times \varphi_\epsilon \in L^2(\mathbb{C}^n)$. Let $d\nu_r = P\, d\mu_r$. Suppose $f \times \nu_r(z) = 0, \forall r > 0$. Then by polar decomposition $f \times P\varphi_{k-q}^{n+p+q-1}(z) = 0$, $\forall k \geq q$. Since φ_ϵ is radial, we can write

$$f \times \varphi_\epsilon \times \nu_r(z) = \sum_{k\geq 0} B_k^n \langle \varphi_\epsilon, \varphi_k^{n-1}\rangle f \times \varphi_k^{n-1} \times \nu_r(z).$$

By Lemma 2.3, it follows that $f \times \varphi_\epsilon \times \nu_r(z) = 0, \forall k \geq q$. Thus without loss of generality, we can assume $f \in L^2(\mathbb{C}^n)$. Hence to prove the Theorem 1.3, in view of Lemma 3.1, it is enough to prove the following result.

Proposition 3.8 *Let $P_{s,t} \in H_{s,t}$ and $f = \tilde{a}P_{s,t} \in L^2(\mathbb{C}^n)$ be a smooth function such that $e^{\frac{1}{4}|z|^2}f(z) \in L^p(\mathbb{C}^n)$, for $1 \leq p < \infty$. If $f \times \nu_r(z) = 0$ on $S_R(o)$ and for all $r > 0$, then $f = 0$ a.e.*

Proof We have

$$f = (2\pi)^{-n}\sum_{k\geq 0} f \times \varphi_k^{n-1}.$$

Therefore

$$\sum_{k\geq 0} f \times (\varphi_k^{n-1} \times P_1\mu_r)(z) = 0,$$

whenever $z \in S_R(o)$ and $r > 0$. By Lemma 2.3, we get

$$\sum_{k\geq q} C(n,p,q)\varphi_{k-q}^{n+p+q-1}(r) f \times P_1\varphi_{k-q}^{n+p+q-1}(z) = 0,$$

for $|z| = R$ and for all $r > 0$. As the functions $\{\varphi_{k-q}^{n+p+q-1}(r) : k \geq q\}$ form an orthonormal basis for $L^2(\mathbb{R}^+, r^{2(n+p+q)-1}\,dr)$, the above implies that

$$f \times P_1\varphi_{k-q}^{n+p+q-1}(z) = 0, \quad \forall k \geq q \text{ and } |z| = R.$$

From Lemma 3.5, $P_1(\tilde{A})\varphi_k^{n-1}(z) = (-2)^{-p-q} P_1(z)\varphi_{k-q}^{n+p+q-1}(z)$, moreover $P_1(\tilde{A})$ is right invariant, therefore it follows that

$$P_1(\tilde{A})(\tilde{a} P_{s,t} \times \varphi_k^{n-1})(z) = 0, \quad \forall k \geq q \text{ and } |z| = R.$$

Using Hecke-Bochner identity (Lemma 2.2), we get

$$\langle \tilde{a}, \varphi_{k-s}^{n+s+t-1}\rangle P_1(\tilde{A}) P_{s,t}\varphi_{k-s}^{n+s+t-1}(z) = 0, \quad \forall k \geq \max(q,s) \text{ and } |z| = R.$$

If $\tilde{A_1^*}^p \tilde{A_2}^q (P_{s,t}\varphi_{k-s}^{n+s+t-1})(R) = 0$ for some $k \geq \max(q,s)$, then by a computation similar as done for $Z_j^* f$ in [13], pp. 2516–2517, we have

$$\tilde{A_1^*}^p \tilde{A_2}^{q-1}\left[\frac{1}{2\rho}\widetilde{D}^*\varphi_{k-s}^{\gamma-1} P_{s+1,t} + \left\{\left(\frac{1}{2(\gamma-1)}\rho\widetilde{D}^* + 1\right)\varphi_{k-s}^{\gamma-1}\right\}\frac{\partial P_{s,t}}{\partial \bar{z}_2}\right] = 0,$$

for $|z| = R$ and $\gamma = n + s + t$. Since $\{P_{s,t}|_{S^{2n-1}} : s,t \geq 0\}$ form an orthonormal basis for $L^2(S^{2n-1})$. An inductive process then gives the coefficient of highest degree polynomial $P_{p+s,q+t}$ as

$$\left(\frac{1}{\rho}\widetilde{D}\right)^p\left(\frac{1}{\rho}\widetilde{D}^*\right)^q \varphi_{k-s}^{\gamma-1}(R) = 0.$$

Using Lemma 3.7, the above equation implies that $\varphi_{k-s-q}^{\gamma+p+q-1}(R) = 0$. In view of Corollary 3.4, without loss of generality, we can assume, the Laguerre polynomials $L_{k-s-q}^{\gamma+p+q-1}$ have distinct zeros. Hence $L_{k-s-q}^{\gamma+p+q-1}(\frac{1}{2}R^2)$ can vanish for at most one value say $k_0 \geq s+q$ of $k \geq \max(q,s)$. Therefore $\langle \tilde{a}, \varphi_{k-s}^{\gamma-1}\rangle = 0$, for $k \geq \max(q,s)$, except for $k \neq k_0$. Hence $\tilde{a}(\rho)$ is finite linear combination of $\varphi_{k-s}^{\gamma-1}$'s. As $\tilde{a}$ satisfies the same decay condition as f, it follows that $\tilde{a} = 0$. This completes the proof. □

Remark 3.9 In the proof of Theorem 1.3, we have used the fact that the WTSM $f \times \nu_{r,j}$ vanishes for each $j : 1 \leq j \leq d(s,t)$. It would be an interesting question to consider a single weight or, in general, a real analytic weight, which we leave open for the time being.

4 Support Theorems for the Weighted Spherical Means

In this section, we prove Theorem 1.4, which is an analogue of the author's support theorem ([13], Theorem 1.2) for the TSM to the WTSM on $\mathbb{C}^n$. Our previous result ([13], Theorem 1.2) is a special case of Theorem 1.4, for $p=q=0$. We would like to quote support theorem for the case $n=1$. In the end, we would revisit Euclidean spherical means and indicate a corresponding support theorem for weighted spherical means.

We need the following result from [13]. Let $Z_{B,\infty}$ be a class of continuous functions on $\mathrm{Ann}(B,\infty)=\{z\in\mathbb{C}^n : B<|z|<\infty\}$ such that $f\times\mu_r(z)=0$ for all $z\in\mathbb{C}^n$ and $r>|z|+B$.

Theorem 4.1 [13] *A necessary and sufficient condition for $f\in Z_{B,\infty}(\mathbb{C}^n)$ is that for all $p,q\in\mathbb{Z}_+$, $1\le j\le d(p,q)$, the spherical harmonic coefficients $\tilde{a}_j^{p,q}$ of f satisfy the following conditions*:

(1) *For $p=0, q=0$ and $r<\rho<R$, $\tilde{a}^0(\rho)=0$.*
(2) *For $p,q\ge 1$ and $r<\rho<R$, there exists $c_i, d_k\in\mathbb{C}$ such that*

$$\tilde{a}_j^{p,q}(\rho)=\sum_{i=1}^{p} c_i e^{\frac{1}{4}\rho^2}\rho^{-2(n+p+q-i)}+\sum_{k=1}^{q} d_k e^{-\frac{1}{4}\rho^2}\rho^{-2(n+p+q-k)}.$$

(3) *For $q=0$ and $p\ge 1$ or $p=0$ and $q\ge 1$ and $r<\rho<R$, there exists $c_i, d_k\in\mathbb{C}$ such that*

$$\tilde{a}_j^{p,0}(\rho)=\sum_{i=1}^{p} c_i e^{\frac{1}{4}\rho^2}\rho^{-2(n+p-i)},\qquad \tilde{a}_j^{0,q}(\rho)=\sum_{k=1}^{q} d_k e^{-\frac{1}{4}\rho^2}\rho^{-2(n+q-k)}.$$

Since the Heisenberg group H^n is non-commutative, the twisted spherical means $f\times\mu_r$ and $\mu_r\times f$ are not equal, in general. Using this fact, we have proved the following support theorem which do not require any decay condition.

Theorem 4.2 ([13]) *Let f be a continuous function on $\mathbb{C}$. Then f is supported in $|z|\le B$ if and only if $f\times\mu_r=\mu_r\times f=0$ for $s>B+|z|$ and $\forall z\in\mathbb{C}$.*

We shall need the following lemmas in the proof of Theorem 1.4.

Lemma 4.3 *Let $d\nu_\rho^{p,q}=P_1\,d\mu_\rho$. Let f be a smooth function on $\mathbb{C}^n$ such that $f\times\nu_\rho^{p,q}(z)=0$, for all $z\in\mathbb{C}^n$ and for all $\rho>|z|+B$. Then $P_1(\widetilde{Z})f\times\mu_\rho(z)=0$, for all $z\in\mathbb{C}^n$ and for all $\rho>|z|+B$. Equivalently, $P_1(\widetilde{Z})f\in Z_{B,\infty}(\mathbb{C}^n)$.*

Proof We first prove

$$\widetilde{Z}_1^* f\times\nu_\rho^{p-1,q}(z)=0,\quad z\in\mathbb{C}^n \text{ for } \rho>|z|+B. \tag{4.1}$$

Let $\partial_{\bar{w}_1}=2\frac{\partial}{\partial\bar{w_1}}=\frac{\partial}{\partial\xi_1}+i\frac{\partial}{\partial\eta_1}$, $w_1=\xi_1+i\eta_1$. Then

$$\int_{\mathrm{Ann}(r,\rho)}\partial_{\bar{w}_1}\left(f(z-w)e^{\frac{i}{2}\mathrm{Im}(z.\bar{w})}w_1^{p-1}\bar{w_2}^q\right)dw$$

$$= \int_{|w|=\rho} f(z-w)e^{\frac{i}{2}\mathrm{Im}(z.\bar{w})} w_1^{p-1} \bar{w_2}^q \frac{w_1}{\rho}\, d\mu_\rho(w)$$
$$- \int_{|w|=r} f(z-w)e^{-\frac{i}{2}\mathrm{Im}(z.\bar{w})} w_1^{p-1} \bar{w_2}^q \frac{w_1}{r}\, d\mu_r(w) = 0.$$

Thus we have the following equation

$$\int_{\mathrm{Ann}(r,\rho)} \partial_{\bar{w}_1} \left(f(z-w)e^{\frac{i}{2}\mathrm{Im}(z.\bar{w})} w_1^{p-1} \bar{w_2}^q \right) dw = 0.$$

Rewriting this equation in the polar form, we get

$$\int_{s=r}^{\rho} \int_{|w|=s} \partial_{\bar{w}_1} \left(f(z-w)e^{\frac{i}{2}\mathrm{Im}(z.\bar{w})} w_1^{p-1} \bar{w_2}^q \right) d\mu_s(w)\, s^{2n-1} ds = 0.$$

Differentiating the above equation with respect to ρ, we have

$$\int_{|w|=\rho} \partial_{\bar{w}_1} \left(f(z-w)e^{\frac{i}{2}\mathrm{Im}(z.\bar{w})} w_1^{p-1} \bar{w_2}^q \right) d\mu_\rho(w) = 0,$$

whenever $z \in \mathbb{C}^n$ and $\rho > |z| + B$. Computing the differential inside the integral and rearranging the terms, we get

$$\int_{|w|=\rho} \left(-\frac{\partial}{\partial \bar{w}_1} f(z-w) + \frac{1}{4} z_1 f(z-w) \right) e^{\frac{i}{2}\mathrm{Im}(z.\bar{w})} w_1^{p-1} \bar{w_2}^q \, d\mu_\rho(w) = 0.$$

That is

$$\int_{|w|=\rho} \left(\frac{\partial}{\partial \bar{z}_1} f(z-w) + \frac{1}{4}(z_1 - w_1) f(z-w) \right) e^{\frac{i}{2}\mathrm{Im}(z.\bar{w})} w_1^{p-1} \bar{w_2}^q \, d\mu_\rho(w) = 0,$$

which is (4.1). Proceeding in a similar way, it can be shown that $P_1(\widetilde{Z}) f \times \mu_\rho(z) = 0$, whenever $z \in \mathbb{C}^n$ and $\rho > |z| + B$. □

As before, it is enough to prove Theorem 1.4 for the function of type $\tilde{a}(\rho) P_{s,t}(z)$. We can see this in the following lemma.

Lemma 4.4 *Fix $p, q \in \mathbb{Z}^+$ and let $f \times \nu_{r,j}(z) = 0$, for all $z \in \mathbb{C}^n$ and $r > |z| + B$ and for all $j, 1 \le j \le d(p,q)$. Then $f_{lm} \times \nu_{r,j}(z) = 0$, for all $z \in \mathbb{C}^n$ and for all $\rho > |z| + B$.*

Proof The proof of this lemma is similar to the proof of Lemma 3.1 and hence omitted. □

To prove Theorem 1.4, in view of Lemma 4.4, it is enough to prove the following result.

Proposition 4.5 *Let $f(z)=\tilde{a}P_{s,t}$ be a smooth function on $\mathbb{C}^n$ such that $|f(z)||z|^k \leq C_k\, e^{-\frac{1}{4}|z|^2}, k\in\mathbb{Z}_+$. Let $f\times\nu_r(z)=0$, for all $z\in\mathbb{C}^n$ and $r>|z|+B$ and for all j, $1\leq j\leq d(p,q)$. Then $f=0$ whenever $|z|>B$.*

Proof We first prove the result in case when $p=1, q=0$. The argument for general p,q is very similar. In this case, by Lemma 4.3, we have $\widetilde{Z}_1^* f\in Z_{B,\infty}(\mathbb{C}^n)$. Since $f=\tilde{a}P_{s,t}$, a similar calculation as in [13], pp. 2516–2517, gives that

$$\widetilde{Z}_1^* f=\frac{1}{2\rho}\widetilde{D}^*\tilde{a}P_{s+1,t}+\left\{\left(\frac{1}{2(\gamma-1)}\rho\widetilde{D}^*+1\right)\tilde{a}\right\}\frac{\partial P_{s,t}}{\partial\bar{z_1}},$$

where $\gamma=n+s+t$. Since $\tilde{Z}_1 f\in Z_{B,\infty}(\mathbb{C}^n)$, by Lemma 4.4 and Theorem 4.1, it follows that

$$\left(\frac{1}{2(\gamma-1)}\rho\widetilde{D}^*+1\right)\tilde{a}=\sum_{i=1}^{s} c_i'\, e^{\frac{1}{4}\rho^2}\rho^{-2(\gamma-1-i)}+\sum_{k=1}^{t-1} d_k'\, e^{-\frac{1}{4}\rho^2}\rho^{-2(\gamma-1-k)}$$

and

$$\frac{1}{2\rho}\widetilde{D}^*\tilde{a}=\sum_{i=1}^{s+1} c_i e^{\frac{1}{4}\rho^2}\rho^{-2(\gamma+1-i)}+\sum_{k=1}^{t} d_k e^{-\frac{1}{4}\rho^2}\rho^{-2(\gamma+1-k)}.$$

Solving these equations for $\tilde{a}$ we get

$$\tilde{a}(\rho)=\sum_{i=1}^{s+1} C_i\, e^{\frac{1}{4}\rho^2}\rho^{-2(\gamma-i)}+\sum_{k=1}^{t} D_k\, e^{-\frac{1}{4}\rho^2}\rho^{-2(\gamma-k)},\quad C_i, D_k\in\mathbb{C}.$$

But the given decay condition on the function f then implies that $\tilde{a}(\rho)=0$, whenever $\rho>B$. Hence $f=0$ for $\rho>B$. For the weight $z_1^p\bar{z}_2^q$, the computations are similar and therefore omitted. □

Next we take up the case of Euclidean weighed spherical means. We prove the following lemma which is key to the proof of Theorem 1.5. As in [3], let

$$f_{lm}(x)=d_s\int_{SO(n)} f(\tau^{-1}x)t_{\pi_s}^{lm}(\tau)\,d\tau,$$

for any l,m with $1\leq l,m\leq d_s$.

Lemma 4.6 *Let $f*\mu_{\rho,j}^k(x)=0$, for all $x\in\mathbb{R}^n$, $\rho>|x|+B$ and for all $j, 1\leq j\leq d_k$. Then $f_{lm}*\mu_{\rho,j}^k(x)=0$, for all $x\in\mathbb{R}^n$ and for all $\rho>|x|+B$.*

Proof Since space H_k is $SO(n)$-invariant by change of variables, it follows that

$$f_{lm}*\mu_{\rho,j}^k(x)=d_s\int_{SO(n)} t_{\pi_s}^{lm}(\tau)\int_{S_\rho(o)} f(\tau^{-1}x+y)P_{kj}(\tau y)\,d\mu_\rho(w)\,d\tau=0,$$

whenever $x \in \mathbb{R}^n$ and $\rho > |x| + B$. □

For $x = (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n$, we realize the function $f(x_1, x_2, x_3, \ldots, x_n)$ as $f(x_1 + ix_2, x_3, \ldots, x_n)$. Let $z_1 = x_1 + ix_2$. Then we can write

$$\partial_{\bar{z}_1} = 2\frac{\partial}{\partial \bar{z}_1} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}.$$

We need the following result from [3]. Let $Z_{B,\infty}$ be a class of continuous functions on $\mathrm{Ann}(B, \infty) = \{x \in \mathbb{R}^n : B < |x| < \infty\}$ such that $f * \mu_r(x) = 0$ for all $x \in \mathbb{R}^n$ and $r > |x| + B$.

Theorem 4.7 [3] *A necessary and sufficient condition for $f \in Z_{B,\infty}(\mathbb{R}^n)$ is that for all $k \in \mathbb{Z}_+$, the spherical harmonic coefficients a_{kj} of f satisfy the following conditions.*

$$a_{kj}(\rho) = \sum_{i=0}^{k-1} \alpha_{kj}^i \rho^{k-d-2i}, \quad \alpha_{kj}^i \in \mathbb{C},$$

for all $k > 0$, $1 \le j \le d_k$, and $a_0(\rho) = 0$ whenever $r < \rho < R$.

Lemma 4.8 *Let $P_k(x) = (x_1 + ix_2)^k$. Suppose $f * \mu_\rho^k(x) = 0$, for all $x \in \mathbb{R}^n$ and for all $\rho > |x| + B$. Then $\partial_{\bar{z}_1}^k f * \mu_\rho(x) = 0$ for all $x \in \mathbb{R}^n$ and for all $\rho > |x| + B$. Equivalently, $\partial_{\bar{z}_1}^k f \in Z_{B,\infty}(\mathbb{R}^n)$.*

Proof We first prove

$$\partial_{\bar{z}_1} f * \mu_\rho^{k-1}(x) = 0, \quad x \in \mathbb{R}^n \text{ for } \rho > |x| + B. \tag{4.2}$$

Let $\partial_{\bar{w}_1} = 2\frac{\partial}{\partial \bar{w}_1} = \frac{\partial}{\partial y_1} + i\frac{\partial}{\partial y_2}$, $w_1 = y_1 + iy_2$. Then

$$\begin{aligned}
&\int_{\mathrm{Ann}(r,\rho)} \partial_{\bar{w}_1}\Big(f(z_1 + w_1, x_3 + y_3, \ldots, x_n + y_n)w_1^{k-1}\Big)\,dy \\
&\quad = \int_{|y|=\rho} f(z_1 + w_1, x_3 + y_3, \ldots, x_n + y_n)w_1^{k-1}\frac{w_1}{\rho}\,d\mu_\rho(y) \\
&\qquad - \int_{|y|=r} f(z_1 + w_1, x_3 + y_3, \ldots, x_n + y_n)w_1^{k-1}\frac{w_1}{r}\,d\mu_r(w) = 0.
\end{aligned}$$

Thus we have the following equation

$$\int_{\mathrm{Ann}(r,\rho)} \partial_{\bar{w}_1}\big(f(z_1 + w_1, x_3 + y_3, \ldots, x_n + y_n)w_1^{k-1}\big)\,dy = 0.$$

Rewriting this equation into polar form, we get

$$\int_{s=r}^{\rho}\int_{|y|=s} \partial_{\bar{w}_1}\big(f(z_1 + w_1, x_3 + y_3, \ldots, x_n + y_n)w_1^{k-1}\big)d\mu_s(y)\, s^{n-1}\,ds = 0.$$

Differentiating the above equation with respect to ρ, we have

$$\int_{|y|=\rho} \partial_{\bar{w}_1} f\big(f(z_1+w_1, x_3+y_3, \ldots, x_n+y_n) w_1^{k-1}\big)\, d\mu_\rho(w) = 0,$$

whenever $x \in \mathbb{R}^n$ and $\rho > |x| + B$. Computing the differential inside integral, we obtain (4.2). Proceeding in a similar way, it can be shown that $\partial_{\bar{z}_1}^k f * \mu_\rho^k(x) = 0$, whenever $x \in \mathbb{R}^n$ and $\rho > |x| + B$. □

To prove Theorem 1.5, in view of Lemma 4.6, it is enough to prove the following result.

Proposition 4.9 *Let $f(x) = \tilde{a}(|x|)P_s(x) \in C^\infty(\mathbb{R}^n)$ such that $|x|^m f(x)$ is bounded for each $m \in \mathbb{Z}_+$. Let $f * \mu_\rho^k(x) = 0$, for all $x \in \mathbb{R}^n$, $\rho > |x| + B$. Then $f = 0$ whenever $|x| > B$.*

Proof First we find $\tilde{a}(\rho)$ for $k = 1$. For this, by Lemma 4.8 we have $\bar{\partial}_1 f \in Z_{B,\infty}(\mathbb{R}^n)$. A computation similar to that in [3], p. 445–446, we can write

$$\frac{\partial f}{\partial x_j} = \frac{1}{\rho}\frac{\partial \tilde{a}}{\partial \rho} P_{s+1}^j + \left\{\left(\frac{1}{n+2(s-1)}\rho\frac{\partial}{\partial \rho} + 1\right)\tilde{a}\right\}\frac{\partial P_s}{\partial x_j},$$

where $P_{s+1}^j \in H_{s+1}$. Therefore,

$$\bar{\partial}_1 f = \frac{1}{\rho}\frac{\partial \tilde{a}}{\partial \rho} P_{s+1} + \left\{\left(\frac{1}{n+2(s-1)}\rho\frac{\partial}{\partial \rho} + 1\right)\tilde{a}\right\}\bar{\partial}_1 P_s$$

for some $P_{s+1} \in H_{s+1}$. By Lemmas [4.6, 4.8] and Theorem 4.7, it follows that

$$\frac{1}{\rho}\frac{\partial \tilde{a}}{\partial \rho} = \sum_{i=0}^{s} c_i \rho^{-n-2i}$$

and

$$\left(\frac{1}{n+2(s-1)}\rho\frac{\partial}{\partial \rho} + 1\right)\tilde{a} = \sum_{i=0}^{s-2} d_i \rho^{-n-2i},$$

where $c_i, d_i \in \mathbb{C}$. Solving these equations for $\tilde{a}$ we get

$$\tilde{a}(\rho) = \sum_{i=-1}^{s-1} c_i' \rho^{-n-2i}, \quad c_i' \in \mathbb{C}.$$

The given decay condition on the function f then implies that $\tilde{a}(\rho) = 0$, whenever $\rho > B$. Hence $f = 0$ for $\rho > B$. The case of general weight $(x_1 + ix_2)^k$ follows from induction. This completes the proof. □

Acknowledgements The author wishes to thank Rama Rawat for several fruitful discussions during preparation of this article. The author would also like to gratefully acknowledge the support provided by the Department of Atomic Energy, government of India.

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