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Uniform Estimates for the Fourier Transform of Surface Carried Measures in $\mathbb{R}^3$ and an Application to Fourier Restriction

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Abstract Let S be a hypersurface in $\mathbb{R}^3$ which is the graph of a smooth, finite type function ϕ, and let $\mu = \rho\, d\sigma$ be a surface carried measure on S, where $d\sigma$ denotes the surface element on S and ρ a smooth density with sufficiently small support. We derive uniform estimates for the Fourier transform $\hat{\mu}$ of μ, which are sharp except for the case where the principal face of the Newton polyhedron of ϕ, when expressed in adapted coordinates, is unbounded. As an application, we prove a sharp L^p-L^2 Fourier restriction theorem for S in the case where the original coordinates are adapted to ϕ. This improves on earlier joint work with M. Kempe.

Keywords Oscillatory integral · Newton diagram · Fourier restriction

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1 Introduction

The goal of this article is to improve on two results from our previous article [11] concerning uniform estimates for two-dimensional oscillatory integrals with smooth,

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finite type phase functions, and L^p-L^2 Fourier restriction for smooth, finite type hypersurfaces S in $\mathbb{R}^3$ which are locally the graph of a function ϕ in adapted coordinates. Note that we may and shall assume that $\phi(0,0)=0$, $\nabla\phi(0,0)=0$.

More precisely, the estimate in Theorem 1.9 of [11] for the Fourier transform of a surface carried measure of S can be re-written as

$$\left|\int_{\mathbb{R}^2} e^{i(\xi_3\phi(x_1,x_2)+\xi_1x_1+\xi_2x_2)}\eta(x)\,dx\right| \le C\,\|\eta\|_{C^3(\mathbb{R}^2)}\,(\log(2+|\xi|))(1+|\xi|)^{-1/h},$$

where h is the so-called height of ϕ in the sense of Varchenko (we shall recall some basic notions like adaptedness of coordinates, height, etc., subsequently).

In Theorems 1.1 and 1.3, we shall identify exactly when the logarithmic factor in this estimate will be present, with the exception of the case where the principal face of the Newton polyhedron of ϕ, when expressed in adapted coordinates, is unbounded and ϕ is non-analytic. Examples by A. Iosevich and E. Sawyer show that a different behavior can indeed occur in the latter case.

Secondly, we shall improve on the following Fourier restriction estimate from Corollary 1.10 in [11]: Assume that the given coordinates are adapted to ϕ. Then

$$\left(\int_S |\widehat{f}|^2\,\rho\,d\sigma\right)^{1/2} \le C_p\|f\|_{L^p(\mathbb{R}^3)}, \quad f\in\mathcal{S}(\mathbb{R}^3), \tag{1.1}$$

holds true for every p such that $1\le p<(2h+2)/(2h+1)$, provided that the support of the smooth density ρ lies in a sufficiently small neighborhood of the origin.

In Theorem 1.7 we shall prove that this restriction estimate holds true also at the endpoint $p=(2h+2)/(2h+1)$, provided that the coordinates (x_1,x_2) are adapted to ϕ, possibly after applying a linear change of coordinates.

If the coordinates (x_1,x_2) are not adapted to ϕ, then we will show in a sequel to this article that the restriction estimate can be extended to an even wider range of p's.

We shall build in this article on the results and techniques developed in [12] and [11], which will be our main references, also for cross-references to earlier and related work. Let us first recall some basic notions from [12], which essentially go back to A.N. Varchenko [21].

Let ϕ be a smooth real-valued function defined on a neighborhood of the origin in $\mathbb{R}^2$ with $\phi(0,0)=0$, $\nabla\phi(0,0)=0$, and consider the associated Taylor series

$$\phi(x_1,x_2)\sim\sum_{j,k=0}^{\infty} c_{jk}x_1^jx_2^k$$

of ϕ centered at the origin. The set

$$\mathcal{T}(\phi):=\left\{(j,k)\in\mathbb{N}^2 : c_{jk}=\frac{1}{j!k!}\partial_1^j\partial_2^k\phi(0,0)\neq 0\right\}$$

will be called the *Taylor support* of ϕ at $(0,0)$. We shall always assume that

$$\mathcal{T}(\phi)\neq\emptyset,$$

i.e., that the function ϕ is of finite type at the origin. The *Newton polyhedron* $\mathcal{N}(\phi)$ of ϕ at the origin is defined to be the convex hull of the union of all the quadrants $(j,k)+\mathbb{R}^2_+$ in $\mathbb{R}^2$, with $(j,k)\in\mathcal{T}(\phi)$. The associated *Newton diagram* $\mathcal{N}_d(\phi)$ in the sense of Varchenko [21] is the union of all compact faces of the Newton polyhedron; here, by a *face*, we shall mean an edge or a vertex.

We shall use coordinates (t_1,t_2) for points in the plane containing the Newton polyhedron, in order to distinguish this plane from the (x_1,x_2)-plane.

The *Newton distance*, or shorter *distance* $d=d(\phi)$ between the Newton polyhedron and the origin in the sense of Varchenko is given by the coordinate d of the point (d,d) at which the bi-sectrix $t_1=t_2$ intersects the boundary of the Newton polyhedron.

The *principal face* $\pi(\phi)$ of the Newton polyhedron of ϕ is the face of minimal dimension containing the point (d,d). Deviating from the notation in [21], we shall call the series

$$\phi_{\mathrm{pr}}(x_1,x_2):=\sum_{(j,k)\in\pi(\phi)} c_{jk}x_1^j x_2^k$$

the *principal part* of ϕ. In case that $\pi(\phi)$ is compact, ϕ_{pr} is a mixed homogeneous polynomial; otherwise, we shall consider ϕ_{pr} as a formal power series.

Note that the distance between the Newton polyhedron and the origin depends on the chosen local coordinate system in which ϕ is expressed. By a *local coordinate system at the origin* we shall mean a smooth coordinate system defined near the origin which preserves 0. The *height* of the smooth function ϕ is defined by

$$h(\phi):=\sup\{d_x\},$$

where the supremum is taken over all local coordinate systems $x=(x_1,x_2)$ at the origin, and where d_x is the distance between the Newton polyhedron and the origin in the coordinates x.

A given coordinate system x is said to be *adapted* to ϕ if $h(\phi)=d_x$. In [12] we proved that one can always find an adapted local coordinate system in two dimensions, thus generalizing the fundamental work by Varchenko [21] who worked in the setting of real-analytic functions ϕ. For real analytic functions ϕ, an alternative proof of Varchenko's result, based on Puiseux series expansions of roots and a clustering of roots has been given by D.H. Phong, E.M. Stein and J. Sturm in [17]. Our proof in the smooth case in [12] makes use of ideas from both approaches.

Following [21] (with as slight modification), we next define what we like to call *Varchenko's exponent* $\nu(\phi)\in\{0,1\}$ as follows (this number had been identified by Varchenko in [21] as what Karpushkin calls the "multiplicity of the oscillation of ϕ at $(0,0)$" in [15]):

If there exists an adapted local coordinate system y near the origin such that the principal face $\pi(\phi^a)$ of ϕ, when expressed by the function ϕ^a in the new coordinates (i.e. $\phi(x)=\phi^a(y)$), is a vertex, and if $h(\phi)\ge 2$, then we put $\nu(\phi):=1$; otherwise, we put $\nu(\phi):=0$.

As has been shown by Varchenko in [21], the number $\nu(\phi)$ arises as the exponent of a logarithmic factor in the principal part of the asymptotic expansion of two-dimensional oscillatory integrals with real analytic phase functions ϕ.

Analogously, we can prove the following uniform estimate for two-dimensional oscillatory integrals with smooth, finite type phase functions ϕ, which improves on Theorem 11.1 in [11].

Theorem 1.1 *Let ϕ be a smooth, real-valued phase function of finite type, defined near the origin, as before, and let $h := h(\phi)$, $\nu := \nu(\phi)$. Then there exist a neighborhood $\Omega \subset \mathbb{R}^2$ of the origin and a constant C such that for every $\eta \in C_0^\infty(\Omega)$ the following estimate holds true for every $\xi \in \mathbb{R}^3$:*

$$\left| \int_{\mathbb{R}^2} e^{i(\xi_3\phi(x_1,x_2)+\xi_1 x_1+\xi_2 x_2)} \eta(x)\, dx \right| \le C\, \|\eta\|_{C^3(\mathbb{R}^2)} \, (\log(2+|\xi|))^\nu (1+|\xi|)^{-1/h}. \tag{1.2}$$

Remarks 1.2

(a) For some special classes of hypersurfaces, related results have been derived by L. Erdös and M. Salmhofer in [8], which, however, are not necessarily uniform in all directions. For estimates with $\xi_1 = \xi_2 = 0$, we refer to the recent work of M. Greenblatt [9].

(b) For real analytic phase functions ϕ, if we restrict ourselves to the direction where $\xi_1 = \xi_2 = 0$, then the asymptotic expansion of the corresponding oscillatory integrals in [21] shows that the estimate (1.2) is essentially sharp as an estimate in terms of $|\xi|$.

(c) For real analytic phase functions, our result is covered by Karpushkin's work [15], who proved the following:

If ϕ is a real analytic function defined near the origin with $\phi(0,0) = 0$, $\nabla\phi(0,0) = 0$, and if r is a real analytic function with sufficiently small norm (in the space of real analytic functions) then

$$\left| \int_{\mathbb{R}^2} e^{i\lambda(\phi(x)+r(x))} \eta(x)\, dx \right| \le C \|\eta\|_{C^3} \frac{(\log(2+|\lambda|))^\nu}{(2+|\lambda|)^{1/h}}, \quad \lambda \in \mathbb{R},$$

provided the amplitude η is supported in a sufficiently small neighborhood of the origin. Moreover, the constant C then does not depend on the function r. These estimates also imply closely related stability results for integrals

$$\int_B |\phi(x)|^{-\delta}\, dx$$

over compact balls B, under small analytic perturbations of ϕ. A simpler, alternative proof to such stability estimates has been devised in the work of Phong, Stein and Sturm [17].

(d) If $h(\phi) < 2$, results analogous to Karpushkin's have been obtained by J.J. Duistermaat [6] in the smooth setting. In this case one always has $\nu(\phi) = 0$.

(e) If $h(\phi) = 2$, and if the principal part of ϕ, when expressed in an adapted coordinate system, has a critical point of finite multiplicity at the origin (so that it is isolated), then an analogue to Karpushkin's estimate has been established by Colin de Verdière [3] in the smooth setting. Notice that if the principal part of ϕ

has an isolated critical point at the origin, then the coordinate system is adapted to ϕ and $\nu(\phi)=0$.

The next result, which improves on corresponding results by M. Greenblatt, shows in particular that, in most cases, the uniform estimates from Theorem 1.1 are sharp if $(\xi_1,\xi_2)=(0,0)$.

Theorem 1.3 *Let us put*

$$J_\pm(\lambda) := \int_{\mathbb{R}^2} e^{\pm i\lambda\phi(x_1,x_2)}\eta(x)\,dx, \qquad \lambda>0,$$

with ϕ and η as in Theorem 1.1. If the principal face $\pi(\phi^a)$ of ϕ, when given in adapted coordinates, is a compact set (i.e., a compact edge or a vertex), then there exists a neighborhood Ω of the origin such that for every η supported in Ω the following limits

$$\lim_{\lambda\to+\infty}\frac{\lambda^{1/h}}{(\log\lambda)^\nu}J_\pm(\lambda)=c_\pm\,\eta(0) \tag{1.3}$$

exist, where the constants $c_\pm$ are non-zero and depend on the phase function ϕ only.

Remarks 1.4

(a) The proof of Theorem 1.3 reveals the following additional facts:
If $\nu(\phi)=0$ in the theorem, then the principal face $\pi(\phi^a)$ is a compact edge, and the constants $c_\pm$ are completely determined by the principal part ϕ^a_{pr} of ϕ^a. And, if $\nu(\phi)=1$, and if we work in super-adapted coordinates in the sense of Greenblatt (as explained in Lemma 3.4), so that in particular $\pi(\phi^a)$ consists of the vertex (h,h), then the constants $c_\pm$ are completely determined by the principal part ϕ^a_{pr} of ϕ^a and the slopes of those compact edges of $\mathcal{N}(\phi^a)$ which contain this vertex.

(b) An analogous result for real analytic phase functions ϕ has been proven by M. Greenblatt (Theorem 1.2 in [9]). For non-analytic, but smooth and finite type ϕ, the following weaker result had been obtained in Theorem 1.6b of the same article:

$$\limsup_{\lambda\to+\infty}\left|\frac{\lambda^{1/h}}{(\log\lambda)^\nu}J_\pm(\lambda)\right|>0.$$

(c) If the principal face $\pi(\phi^a)$ is unbounded, then the estimate in Theorem 1.1 may fail to be sharp, if ϕ is non-analytic, as the following class of examples by A. Iosevich and E. Sawyer [13] shows: If

$$\phi(x_1,x_2):=x_2^2+e^{-1/|x_1|^\alpha},$$

with $\alpha>0$, then

$$|J_\pm(\lambda)|\asymp\frac{1}{\lambda^{1/2}\log\lambda^{1/\alpha}}\quad \text{as } \lambda\to+\infty,$$

whereas $\nu(\phi)=0$. These examples also indicate that a precise determination of the asymptotic behavior of $J_\pm(\lambda)$ may be difficult when the principal face is non-compact.

(d) For real-analytic phase functions depending on more than two variables and satisfying an appropriate non-degeneracy condition, the explicit form of the principal part of the asymptotic expansion of the corresponding oscillatory integrals has been obtained by J. Denef, J. Nicaise and P. Sargos [4].

The existence of an adapted coordinate system in which the principal face is a vertex is a priori not so easily verified, but there exists an equivalent, more accessible condition. In order to describe this, we first recall that if the principal face of the Newton polyhedron $\mathcal{N}(\phi)$ is a compact edge, then it lies on a unique line $\kappa_1 t_1 + \kappa_2 t_2 = 1$, with $\kappa_1, \kappa_2 > 0$. By permuting the coordinates x_1 and x_2, if necessary, we shall always assume that $\kappa_1 \le \kappa_2$. We shall call this weight $\kappa = (\kappa_1, \kappa_2)$ the *principal weight* associated to ϕ, and denote it also by κ^{pr}. It induces dilations $\delta_r(x_1, x_2) := (r^{\kappa_1} x_1, r^{\kappa_2} x_2),\ r > 0$, on $\mathbb{R}^2$, so that the principal part ϕ_{pr} of ϕ is κ- homogeneous of degree one with respect to these dilations, i.e., $\phi_{\text{pr}}(\delta_r(x_1, x_2)) = r\phi_{\text{pr}}(x_1, x_2)$ for every $r > 0$, and

$$d = \frac{1}{\kappa_1^{\text{pr}} + \kappa_2^{\text{pr}}} = \frac{1}{|\kappa^{\text{pr}}|}. \tag{1.4}$$

Denote by

$$m(\phi_{\text{pr}}) := \operatorname{ord}_{S^1} \phi_{\text{pr}}$$

the maximal order of vanishing of ϕ_{pr} along the unit circle S^1 centered at the origin.

We also recall from [12] that the *homogeneous distance* of a κ-homogeneous polynomial P (such as $P = \phi_{\text{pr}}$) is given by $d_h(P) := 1/(\kappa_1 + \kappa_2) = 1/|\kappa|$, and that

$$h(P) = \max\{m(P), d_h(P)\}. \tag{1.5}$$

According to [12], Corollary 4.3 and Corollary 2.3, *the coordinates x are adapted to ϕ if and only if one of the following conditions is satisfied:*

(a) *The principal face $\pi(\phi)$ of the Newton polyhedron is a compact edge, and $m(\phi_{\text{pr}}) \le d(\phi)$.*

(b) *$\pi(\phi)$ is a vertex.*

(c) *$\pi(\phi)$ is an unbounded edge.*

We like to mention that in case (a) we have $h(\phi) = h(\phi_{\text{pr}}) = d_h(\phi_{\text{pr}})$. Notice also that (a) applies whenever $\pi(\phi)$ is a compact edge and $\kappa_2/\kappa_1 \notin \mathbb{N}$; in this case we even have $m(\phi_{\text{pr}}) < d(\phi)$ (cf. [12], Corollary 2.3).

Lemma 1.5 *The following conditions on ϕ are equivalent*:

(a) *There exists an adapted local coordinate system y at the origin such that the principal face $\pi(\phi^a)$ is a vertex.*

(b) *If y is any adapted local coordinate system at the origin, then either $\pi(\phi^a)$ is a vertex, or a compact edge and $m(\phi^a_{\text{pr}}) = d(\phi^a)$.*

Consider for example the function $\phi(x_1, x_2) := (x_2 - 2x_1^2)^2(x_2 - x_1^2)$. Then $\phi = \phi_{\mathrm{pr}}$, $\pi(\phi)$ is a compact edge and $m(\phi_{\mathrm{pr}}) = 2 = d(\phi)$, so that case (b) above applies and the coordinates x are adapted to ϕ. Moreover, $\nu(\phi) = 1$. If we introduce new coordinates y given by $y_1 := x_1$, $y_2 := x_2 - 2x_1^2$, then $\phi(x) = \tilde{\phi}(y)$, where $\tilde{\phi}(y) = y_2^2(y_2 + y_1^2)$. The principal face of $\mathcal{N}(\tilde{\phi})$ is the vertex $(2, 2)$, so that also the coordinates y are adapted.

In the case where the coordinates are not adapted to ϕ, we see that the principal face $\pi(\phi)$ is a compact edge such that

$$m_1 := \kappa_2/\kappa_1 \in \mathbb{N}. \tag{1.6}$$

Then, by Theorem 5.1 in [12], there exists a smooth real-valued function ψ of the form

$$\psi(x_1) = b_1 x_1^{m_1} + O(x_1^{m_1+1}), \tag{1.7}$$

with $b_1 \neq 0$, defined on a neighborhood of the origin such that an adapted coordinate system (y_1, y_2) for ϕ is given locally near the origin by means of the (in general non-linear) shear

$$y_1 := x_1, \qquad y_2 := x_2 - \psi(x_1).$$

In these coordinates, ϕ is given by

$$\phi^a(y) := \phi(y_1, y_2 + \psi(y_1)). \tag{1.8}$$

As an immediate consequence of Theorem 1.1 we obtain uniform estimates for the Fourier transform

$$\widehat{\rho d\sigma}(\xi) = \int_S e^{-i\xi\cdot x} \rho(x)\, d\sigma(x), \quad \xi \in \mathbb{R}^3,$$

of surface carried measures on smooth, finite type hypersurfaces S in $\mathbb{R}^3$. Here, $d\sigma$ denotes the Riemannian volume element on S.

If a point x^0 on such a hypersurface S is given, which we may assume to be the origin after a translation of coordinates, and if we represent S locally near $x^0 = (0, 0)$ as the graph $x_3 = \phi(x_1, x_2)$ of smooth, finite type function ϕ with $\phi(0, 0) = 0$, $\nabla\phi(0, 0) = 0$ as before, then we define the *height* of S at x^0 by $h(x^0, S) := h(\phi)$. This notion is invariant under affine linear coordinate changes of the ambient space, as has been shown in [11]. Similarly, we define $\nu(x^0, S) := \nu(\phi)$. Denote by $d\sigma$ the surface element of S. Then we have the following improvement of Theorem 1.9 in [11]:

Corollary 1.6 *Let S be a smooth hypersurface of finite type in $\mathbb{R}^3$ and let x^0 be a fixed point on S. Then there exists a neighborhood $U \subset S$ of the point x^0 such that for every $\rho \in C_0^\infty(U)$ the following estimate holds true*:

$$|\widehat{\rho d\sigma}(\xi)| \le C\|\rho\|_{C^3(S)} (\log(2 + |\xi|))^{\nu(x^0,S)} (1 + |\xi|)^{-1/h(x^0,S)} \quad \textit{for every } \xi \in \mathbb{R}^3.$$

Our second result concerns Fourier restriction to S. We shall prove that the L^p-L^2 Fourier restriction theorem of Corollary 1.10 in [11] also holds true at the endpoint, if S is locally given as the graph of a function ϕ which is given in adapted coordinates:

Theorem 1.7 *Let S be a smooth hypersurface of finite type in $\mathbb{R}^3$, and let x^0 be a fixed point on S. Assume that, possibly after a translation of coordinates, $x^0 = 0$, and that S is locally near x^0 given as the graph $x_3 = \phi(x_1, x_2)$ of a smooth, finite type function ϕ with $\phi(0,0) = 0$, $\nabla\phi(0,0) = 0$ as before.*

We also assume that, after applying a suitable linear change of coordinates, the coordinates (x_1, x_2) are adapted to ϕ, so that $d = h$, where $d = d(\phi)$ denotes the Newton distance of ϕ and $h = h = h(x^0, S) = h(\phi)$ its height. We then define the critical exponent p_c by

$$p_c' := 2h + 2, \tag{1.9}$$

where p' denotes the exponent conjugate to p, i.e., $1/p + 1/p' = 1$.

Then there exists a neighborhood $U \subset S$ of the point x^0 such that for every non-negative density $\rho \in C_0^\infty(U)$ the Fourier restriction estimate

$$\left(\int_S |\widehat{f}|^2 \rho \, d\sigma\right)^{1/2} \le C_p \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in \mathcal{S}(\mathbb{R}^3), \tag{1.10}$$

holds true for every p such that

$$1 \le p \le p_c. \tag{1.11}$$

Moreover, if $\rho(x^0) \neq 0$, then the condition (1.11) on p is also necessary for the validity of (1.10).

The second statement about sharpness had already been proven in [11], Sect. 12, and is based, as usually, on Knapp type examples. The idea is very simple: if we assume that the coordinates are adapted to ϕ and if, for instance, the principal face of the Newton polyhedron of ϕ is a compact edge, then we can use the dilations δ_r associated to the principal weight κ^{pr} to construct Knapp boxes by dilating a fixed cube. Since the Jacobian of δ_r is given by $r^{|\kappa^{\mathrm{pr}}|} = r^{1/d} = r^{1/h}$ (cf. (1.4)) and since ϕ can be well-approximated by the homogeneous polynomial ϕ_{pr}, the necessity of the condition (1.11) follows easily.

Remarks 1.8

(a) The case where the coordinates (x_1, x_2) are not adapted to ϕ will be treated in a subsequent article. It has turned out that in this case the restriction estimate (1.10) is valid in a wider range of p's, with a critical exponent which is strictly bigger than p_c and which can be determined explicitly by means of Varchenko's algorithm (cf. [12]) for the construction of adapted coordinates.
(b) If the surface S is of finite line type and convex, and if the restriction property (1.10) holds true also in the endpoint $p_c = (2h+2)/(2h+1)$, then it has been shown by A. Iosevich in [14] that necessarily the Fourier transform of $\rho \, d\sigma$ must

decay of order $O(|\xi|^{-1/h})$ as $|\xi| \to +\infty$ (it can easily be shown by means of Schulz' [18] decomposition of convex smooth functions of finite line type), i.e., $\nu(x^0, S) = 0$. Conversely, the decay rate $O(|\xi|^{-1/h})$ immediately implies the restriction estimate (1.10) also for the endpoint $p = p_c$; this is an immediate consequence of A. Greenleaf's work in [10].

However, if $\nu(x^0, S) = 1$, which can only happen in the non-convex case, the logarithmic factor in (1.2) is necessary, so that one cannot apply Greenleaf's result directly.

(c) Of course, ultimately one would be interested in more general L^p-L^q Fourier restriction estimates for S. Since the full range of such estimates is not known yet even for manifolds with non-vanishing Gaussian curvature, such as spheres or paraboloids, at present one cannot expect to find the full range for more general surfaces. A first step in this direction would be bilinear restriction estimates for more general hypersurfaces, which is the subject of recent ongoing joint work of the second author with A. Vargas.

Restriction theorems for the Fourier transform go back to E.M. Stein and have a long history by now. Recall for instance the seminal work by E.M. Stein, and P. Tomas, for the case of the Euclidean sphere (see, e.g., [19]). Some restriction estimates for analytic hypersurfaces in $\mathbb{R}^3$ have been obtained by A. Magyar [16], whose results were sharp for particular classes of hypersurfaces given as graphs of functions in adapted coordinates, with the exception of the endpoint.

2 Uniform Estimates for Oscillatory Integrals with Finite Type Phase Functions of Two Variables

In this section we shall give a proof of Theorem 1.1. We shall closely follow the proof of Theorem 11.1 in [11], which did already provide the uniform estimates in Theorem 1.1, except for a logarithmic factor which is not really needed in many cases, as we shall see.

The reader is strongly recommended to have [11] at hand when reading this article, since we shall make use of the notation and many results from [11] without repeating all of them here.

By decomposing $\mathbb{R}^2$ into its four quadrants, we may reduce ourselves to the estimation of oscillatory integrals of the form

$$J(\xi) := \int_{(\mathbb{R}_+)^2} e^{i(\xi_3\phi(x_1,x_2)+\xi_1x_1+\xi_2x_2)}\eta(x_1,x_2)\,dx.$$

Notice also that we may assume in the sequel that

$$|\xi_1| + |\xi_2| \le \delta|\xi_3|, \quad \text{hence } |\xi| \sim |\xi_3|, \tag{2.1}$$

where $0 < \delta \ll 1$ is a sufficiently small constant to be chosen later, since for $|\xi_1| + |\xi_2| > \delta|\xi_3|$ the estimate (1.2) follows by an integration by parts, if Ω is chosen small enough. Of course, we may in addition always assume that $|\xi| \ge 2$.

If χ is any integrable function defined on Ω, we shall put

$$J^\chi(\xi) := \int_{(\mathbb{R}_+)^2} e^{i(\xi_3\phi(x_1,x_2)+\xi_1x_1+\xi_2x_2)}\eta(x_1,x_2)\chi(x)\,dx.$$

The case where $h(\phi) < 2$ is contained in Duistermaat's work [6] (notice that Duistermaat proves estimates of the form (1.2) without the presence of a logarithmic factor $\log(2+|\xi|)$, even for a wider class of phase functions), so let us assume from now on that

$$h := h(\phi) \geq 2.$$

Moreover, if $h = 2$, then we shall make use of the following special property:

Lemma 2.1 *If $h(\phi) = 2$, then, after applying a suitable linear change of coordinates, we may assume that one of the following conditions are satisfied*:

(i) *The coordinates are adapted to ϕ.*
(ii) *The coordinates are not adapted to ϕ, but $h(\phi_{\mathrm{pr}}) = h(\phi)$. In this case, we have $\nu(\phi) = 0$ and $m(\phi_{\mathrm{pr}}) = 2$.*

Note that in general we only have $h(\phi_{\mathrm{pr}}) \geq h(\phi)$, and the inequality may be strict.

Proof Let us assume that the coordinates x are not adapted to ϕ. Then the principal face $\pi(\phi)$ is a compact edge and $m(\phi_{\mathrm{pr}}) > d(\phi) = d_x$. In particular, the principal part ϕ_{pr} of ϕ is a polynomial which is κ-homogeneous of degree 1, where we may assume that $0 < \kappa_1 \leq \kappa_2$, so that $m := m_1 = \kappa_2/\kappa_1 \geq 1$ is an integer. According to [12], ϕ_{pr} can be written as

$$\phi_{\mathrm{pr}}(x_1,x_2) = c x_1^\alpha x_2^\beta \prod_l (x_2 - c_l x_1^m)^{n_l},$$

where the c_l's are the non-trivial distinct complex roots of the polynomial $t \mapsto \phi_{\mathrm{pr}}(1,t)$ and the n_l's are their multiplicities. Moreover, there exists an l_0 such that $m(\phi_{\mathrm{pr}}) = n_{l_0}$ and such that c_{l_0} is real. Notice also that $\alpha \leq 1, \beta \leq 1$, since otherwise the coordinates were adapted.

Assume first that $\kappa_1 = \kappa_2$. Then $m = 1$, and applying the first step in Varchenko's algorithm (see [12], or Sect. 2.5 in [11]), we see that we can transform ϕ into $\tilde\phi$ by means of the linear change of variables $y_1 = x_1, y_2 = x_2 - c_{l_0}x_1$ such that either the coordinates y are adapted to $\tilde\phi$, hence $\tilde\phi = \phi^a$, or they are not, but then $\tilde\kappa_1 < \tilde\kappa_2$ (where $\tilde\phi_{\mathrm{pr}}$ is assumed to be $\tilde\kappa$-homogeneous of degree 1).

After applying a suitable linear change of coordinates, we are thus reduced to the situation where $\kappa_1 < \kappa_2$, hence $m \geq 2$. Let us denote by (A_0, B_0) and (A_1, B_1) the two vertices of $\pi(\phi)$, and assume that $A_0 < A_1$. Recall from [12], displays (3.2) and (3.3), that

$$A_0 = \alpha, \qquad B_0 = \beta + N, \qquad A_1 = \alpha + mN, \qquad B_1 = \beta,$$

and that

$$d_x = \frac{\alpha + m(\beta+N)}{1+m}, \tag{2.2}$$

with $N := \sum_l n_l$. Recall also that the point (A_0, B_0), with $A_0 < B_0$, will be a vertex of all the Newton diagrams that arise when running Varchenko's algorithm on ϕ, so that we must have $\beta + N = B_0 \geq 2$, since $h(\phi) = 2$. Then (2.2) implies that $d_x \geq \frac{2m}{1+m} > 1$, so that $n_{l_0} = m(\phi) \geq 2$.

Since $d_x \leq h(\phi) = 2$, (2.2) implies that $\beta + N \leq 2 + \frac{2}{m} \leq 3$. But, if we had $\beta + N = 3$, then the conditions $d_x \leq 2$ and $m \geq 2$ would imply $\alpha = 0, m = 2$, hence $d_x = 2$, and so the coordinates x would be adapted, contradicting our assumption.

Therefore, we must have $\beta + N = 2$. Then $\beta = 0$, $N = n_{l_0} = 2$ and $\alpha < 2$, and thus the change of coordinates

$$y_1 := x_1, \qquad y_2 := x_2 - c_{l_0} x_1^m$$

transforms the principal part ϕ_{pr} into $\widetilde{\phi_{\text{pr}}}(y) = c y_1^\alpha y_2^2$. This implies $h(\phi_{\text{pr}}) = 2 = h(\phi)$. Notice also that in this case the principal face of the Newton polyhedron of ϕ, when expressed in adapted coordinates, must be the unbounded half-line with left endpoint $(\alpha, 2)$, so that $\nu(\phi) = 0$. □

We recall the following lemma, which is a (not quite straight-forward) consequence of van der Corput's lemma and whose formulation goes back to J.E. Björk (see [5]) and G.I. Arhipov [1].

Lemma 2.2 *Assume that f is a smooth real valued function defined on an interval $I \subset \mathbb{R}$ which is of polynomial type $m \geq 2$ ($m \in \mathbb{N}$), i.e., there are positive constants $c_1, c_2 > 0$ such that*

$$c_1 \leq \sum_{j=1}^{m} |f^{(j)}(s)| \leq c_2 \quad \textit{for every } s \in I.$$

Then for $\lambda \in \mathbb{R}$,

$$\left| \int_I e^{i\lambda f(s)} g(s)\, ds \right| \leq C \|g\|_{C^1(I)} (1 + |\lambda|)^{-1/m},$$

where the constant C depends only on the constants c_1 and c_2.

2.1 The Case where the Coordinates Are Adapted to ϕ, or where $h = 2$

We shall begin with the easiest case where either the coordinates x are adapted to ϕ, or $h = 2$ and condition (ii) in Lemma 2.1 is satisfied.

Recall from [12] that if $\kappa = (\kappa_1, \kappa_2)$ is any weight with $0 < \kappa_1 \leq \kappa_2$ such that the line $L_\kappa := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\}$ is a supporting line to the Newton polyhedron $\mathcal{N}(\phi)$ of ϕ, then the *κ-principal part* of ϕ

$$\phi_\kappa(x_1, x_2) := \sum_{(j,k) \in L_\kappa} c_{jk} x_1^j x_2^k$$

is a non-trivial polynomial which is κ-homogeneous of degree 1. By definition, we then have

$$\phi(x_1, x_2) = \phi_\kappa(x_1, x_2) + \text{terms of higher } \kappa\text{-degree}$$

(see [12] for the precise meaning of this notion).

We claim that we can choose a weight κ with $0 < \kappa_1 \le \kappa_2 < 1$ such that L_κ is a supporting line to the Newton polyhedron of ϕ and

$$\frac{1}{|\kappa|} = d_h(\phi_\kappa) \le h(\phi_\kappa) = h.$$

Indeed, in case that the coordinates are adapted to ϕ, this has been shown in [11], Lemma 2.4. And, if the coordinates are not adapted to ϕ but $h(\phi_{\mathrm{pr}}) = h(\phi)$, then the principal face is a compact edge, and we can choose for κ the principal weight, so that $\phi_\kappa = \phi_{\mathrm{pr}}$. Notice that we have $\kappa_2 < 1$, since $\nabla\phi(0,0) = 0$.

Let us denote by δ_r the dilation by the factor $r > 0$ associated to the weight κ, i.e., $\delta_r(x_1, x_2) = (r^{\kappa_1} x_1, r^{\kappa_2} x_2)$.

In analogy with the proof of Theorem 11.1 in [11] we fix a suitable smooth cut-off function χ on $\mathbb{R}^2$ supported in an annulus D such that the functions $\chi_k := \chi \circ \delta_{2^k}$ form a partition of unity, and then decompose

$$J(\xi) = \sum_{k=k_0}^{\infty} J_k(\xi),$$

where

$$\begin{aligned} J_k(\xi) &:= \int_{(\mathbb{R}_+)^2} e^{i(\xi_3\phi(x) + \xi_1 x_1 + \xi_2 x_2)} \eta(x) \chi_k(x)\, dx \\ &= 2^{-k|\kappa|} \int_{(\mathbb{R}_+)^2} e^{i(2^{-k}\xi_3 \phi^k(x) + 2^{-k\kappa_1}\xi_1 x_1 + 2^{-k\kappa_2}\xi_2 x_2)} \eta(\delta_{2^{-k}}(x)) \chi(x)\, dx, \end{aligned}$$

with $\phi^k(x) := 2^k \phi(\delta_{2^{-k}} x) = \phi_\kappa(x) + \text{error term}$.

We claim that given any point $x^0 \in D$, we can find a unit vector $e \in \mathbb{R}^2$ and some $j \in \mathbb{N}$ with $2 \le j \le h(\phi_\kappa) = h$ such that $\partial_e^j \phi_\kappa(x^0) \ne 0$.

Indeed, if the coordinates are adapted to ϕ, then this has been shown in Sect. 7 of [11], and if they are not adapted to ϕ, then the same is true whenever x^0 does not lie on the principal root of ϕ_κ, as shown in Sect. 8 of [11]. However, if x^0 does lie an the principal root of ϕ_κ, then according to Lemma 2.1 we may choose $j = 2$.

For $k \ge k_0$ sufficiently large we can thus apply Lemma 2.2 to the integration along lines parallel to the direction e in the integral defining $J_k(\xi)$ near the point x^0. Applying Fubini's theorem and a partition of unity argument, we thus obtain

$$\begin{aligned} |J_k(\xi)| &\le C \|\eta\|_{C^3(\mathbb{R}^2)} 2^{-k|\kappa|} (1 + 2^{-k}|\xi_3|)^{-1/j} \\ &\le C \|\eta\|_{C^3(\mathbb{R}^2)} 2^{-k|\kappa|} (1 + 2^{-k}|\xi|)^{-1/m}, \end{aligned} \tag{2.3}$$

where m denotes the maximal j that arises in this context.

Summation in k then yields the following estimates:

$$|J(\xi)| \le C\|\eta\|_{C^3(\mathbb{R}^2)} \begin{cases} (1+|\xi|)^{-1/m}, & \text{if } m|\kappa| > 1, \\ \log(2+|\xi|)(1+|\xi|)^{-1/m}, & \text{if } m|\kappa| = 1, \\ (1+|\xi|)^{-|\kappa|}, & \text{if } m|\kappa| < 1. \end{cases} \tag{2.4}$$

Now, if $h(\phi) = 2$ and if the coordinates are not adapted to ϕ, then $m = m(\phi_{\text{pr}}) > d(\phi) = 1/|\kappa|$, so the first case in (2.4) applies. This implies (1.2), in view of Lemma 2.1(ii).

Next, assume that the coordinates are adapted. If the principal face $\pi(\phi)$ is a compact edge, then $\phi_\kappa = \phi_{\text{pr}}$, hence $1/|\kappa| = d(\phi) = h$, and moreover $m \le h$. This implies $|\kappa|m \le 1$. Since in this case, by Lemma 1.5, $\nu(\phi) = 1$ if and only if $m = m(\phi_{\text{pr}}) = h(\phi)$, i.e., if and only if $m|\kappa| = 1$, we again obtain estimate (1.2).

If $\pi(\phi)$ is unbounded, then $m = h$ and $1/|\kappa| < h$, so that the first case in (2.4) applies and we again verify (1.2).

Finally, if $\pi(\phi)$ is a vertex, then $1/|\kappa| = h = m$, so that the second case in (2.4) applies and we obtain (1.2) also in this case.

2.2 The Case of Non-adapted Coordinates: The Contribution of Regions Away from the Principal Root Jet

Assume next that the coordinates x are not adapted to ϕ and that $h > 2$.

As we already explained in Sect. 1, based on Varchenko's algorithm we can then locally find a smooth real-valued function ψ which defines an adapted coordinate system

$$y_1 := x_1, \qquad y_2 := x_2 - \psi(x_1) \tag{2.5}$$

for the function ϕ near the origin. In these coordinates, ϕ is given by

$$\phi^a(y) := \phi(y_1, y_2 + \psi(y_1)).$$

In the case where Varchenko's algorithm stops after a finite number of steps because the principal face $\pi(\phi^a)$ is a compact edge and $m(\phi^a_{\text{pr}}) = d(\phi^a)$, we meet the following *convention*:

We assume that we have then run the algorithm one further step (as in the proof of the implication (b) $\Rightarrow$ (a) in Lemma 1.5), so that we may assume that $\pi(\phi^a)$ is a vertex, i.e., the point (h,h). *Under this convention,* $\pi(\phi^a)$ *will be a vertex whenever* $\nu(\phi) = 1$.

Consider the Taylor series

$$\psi(x_1) \approx \sum_{l\ge1} b_l x_1^{m_l} \tag{2.6}$$

of ψ, where the b_l are assumed to be non-zero. After applying a linear change of coordinates, if necessary, we may and shall assume that $b_1 \ne 0$ and that the $m_l \in \mathbb{N}$ form a strictly increasing sequence

$$2 \le m_1 < m_2 < \cdots.$$

Suppose that the vertices of the Newton diagram $\mathcal{N}_d(\phi^a)$ of ϕ^a are the points (A_l, B_l), $l = 0, \ldots, n$, so that the Newton polyhedron $\mathcal{N}(\phi^a)$ is the convex hull of the set $\bigcup_l ((A_l, B_l) + \mathbb{R}^2_+)$, where $A_l < A_{l+1}$ for every $l \geq 0$.

Let $L_l := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1^l t_1 + \kappa_2^l t_2 = 1\}$ denote the line passing through the points (A_{l-1}, B_{l-1}) and (A_l, B_l), and let $a_l := \kappa_2^l / \kappa_1^l$. The a_l can be identified as the distinct leading exponents of all the roots of ϕ^a in case that ϕ^a is analytic (see Sect. 3 of [11]), and the cluster of roots whose leading exponent in their Puiseux series expansion is given by a_l is associated to the edge $\gamma_l := [(A_{l-1}, B_{l-1}), (A_l, B_l)]$ of $\mathcal{N}(\phi^a)$.

As in Sect. 8.2 of [11], we choose the integer $l_0 \geq 1$ such that

$$a_0 < \cdots < a_{l_0-1} \leq m_1 < a_{l_0} < \cdots < a_l < a_{l+1} < \cdots < a_n.$$

As has been shown in Sect. 3 of [11], the vertex (A_{l_0-1}, B_{l_0-1}) lies strictly above the bisectrix, i.e., $A_{l_0-1} < B_{l_0-1}$, since the original coordinates x were assumed to be non-adapted.

Distinguishing the cases listed below, we single out a particular edge by fixing the corresponding index $\lambda \geq l_0$ as in Sect. 3 of [11]:

Cases:

(a) In case (a), where the principal face of ϕ^a is a compact edge, we choose λ so that the edge $\gamma_\lambda = [(A_{\lambda-1}, B_{\lambda-1}), (A_\lambda, B_\lambda)]$ is the principal face $\pi(\phi^a)$ of the Newton polyhedron of ϕ^a.

(b) In case (b), where $\pi(\phi^a)$ is the vertex (h, h), we choose λ so that $(h, h) = (A_\lambda, B_\lambda)$. Then $\lambda \geq 1$, and (h, h) is the right endpoint of the compact edge γ_λ.

(c) Consider finally case (c), in which the principal face $\pi(\phi^a)$ is unbounded, namely a half-line given by $t_1 \geq \nu_1$ and $t_2 = h$, where $\nu_1 < h$. Here, we distinguish two sub-cases:
   - (c1) If the point (ν_1, h) is the right endpoint of a compact edge of $\mathcal{N}(\phi^a)$, then we choose again $\lambda \geq 1$ so that this edge is given by γ_λ.
   - (c2) Otherwise, $(\nu_1, h) = (A_0, B_0)$ is the only vertex of $\mathcal{N}(\phi^a)$, i.e., $\mathcal{N}(\phi^a) = (\nu_1, h) + \mathbb{R}^2_+$. In this case, there is no non-trivial root r, hence $n = 0$.

In the cases (a), (b) and (c1), let us put

$$a := a_\lambda = \frac{\kappa_2^\lambda}{\kappa_1^\lambda}. \tag{2.7}$$

We shall assume in the sequel that ϕ is analytic, since the general case can be reduced to this case as in [11].

In a first step, we now decompose $J(\xi) = J^{1-\rho_1}(\xi) + J^{\rho_1}(\xi)$, where ρ_1 is the cut-off function introduced in Sect. 8.1 of [11] which localizes to a narrow κ-homogeneous neighborhood of the form

$$|x_2 - b_1 x_1^{m_1}| \leq \varepsilon_1 x_1^{m_1} \tag{2.8}$$

of the curve $x_2 = b_1 x_1^{m_1}$.

Lemma 2.3 *Let* $\varepsilon_1 > 0$. *If the neighborhood* Ω *of the point* $(0,0)$ *is chosen sufficiently small, then* $J^{1-\rho_1}(\xi)$ *satisfies estimate* (1.2).

Moreover, if $\mathcal{N}(\phi^a)$ *is of the form* $(\nu_1, h) + \mathbb{R}^2_+$, *with* $\nu_1 < h$, (*case* (c2) *above*), *then the same statement holds true for* $J(\xi)$ *in place of* $J^{1-\rho_1}(\xi)$.

Proof The oscillatory integral $J^{1-\rho_1}(\xi)$ can be estimated in a similar way as in the case of adapted coordinates by means of Lemma 2.2, and no logarithmic factor is needed. The reason for this is that any root of $\phi_{\rm pr}$ which does not agree with the principal root $x_2 = b_1 x_1^{m_1}$ has multiplicity strictly less then $d(\phi)$, as can be seen from Corollary 2.3 in [12], so that the third case of (2.4) applies.

Moreover, if $\mathcal{N}(\phi^a) = (\nu_1, h) + \mathbb{R}^2_+$, with $\nu_1 < h$, then we recall from the proof of Lemma 8.1 in [11] that $\phi_\kappa(x) = c x_1^{\nu_1}(x_2 - b_1 x_1^{m_1})^h$, which implies $h(\phi_\kappa) = h(\phi^a_\kappa) = h$, and we see that in this case we can again apply Lemma 2.2 to the x_2-integration in order to see that also $J^{\rho_1}(\xi)$ satisfies (1.2), without logarithmic factor, since here $1/|\kappa| = d_h(\phi) < h$. This proves also the second statement in the lemma. □

We may and shall therefore from now on assume that the Newton polyhedron of ϕ^a has at least one compact edge "lying above" the principal face, i.e., that one of the cases (a), (b) or (c1) applies. There remains $J^{\rho_1}(\xi)$ to be considered.

In a next step we shall narrow down the domain (2.8) to a neighborhood of the principal root jet of the form

$$|x_2 - \psi(x_1)| \le N_\lambda x_1^{a_\lambda}, \tag{2.9}$$

where N_λ is a constant to be chosen later. This domain is κ^λ-homogeneous in the adapted coordinates y. More precisely, we fix a cut-off function $\rho \in C_0^\infty(\mathbb{R})$ supported in a neighborhood of the origin such that $\rho = 1$ near the origin, and put

$$\rho_\lambda(x_1, x_2) := \rho\left(\frac{x_2 - \psi(x_1)}{N_\lambda x_1^a}\right).$$

Proposition 2.4 *Let* $N_\lambda > 0$. *If the neighborhood* Ω *of the point* $(0,0)$ *is chosen sufficiently small, then the oscillatory integral* $J^{1-\rho_\lambda}(\xi)$ *satisfies estimate* (1.2).

Moreover, if the principal face $\pi(\phi^a)$ *is a vertex or unbounded, then the same holds true for* $J(\xi)$ *in place of* $J^{1-\rho_\lambda}(\xi)$.

Proof To prove the first statement in the proposition, we decompose the difference set of the domains (2.8) and (2.9) in a similar way as in Sect. 8.2 of [11] into domains

$$D_l := \{(x_1, x_2) : \varepsilon_l x_1^{a_l} < |x_2 - \psi(x_1)| \le N_l x_1^{a_l}\}, \quad l = l_0, \dots, \lambda - 1,$$

which are κ^l-homogeneous in the adapted coordinates y given by (2.5), and intermediate domains

$$E_l := \{(x_1, x_2) : N_{l+1} x_1^{a_{l+1}} < |x_2 - \psi(x_1)| \le \varepsilon_l x_1^{a_l}\}, \quad l = l_0, \dots, \lambda - 1,$$

and

$$E_{l_0 - 1} := \{(x_1, x_2) : N_{l_0} x_1^{a_{l_0}} < |x_2 - \psi(x_1)| \le \varepsilon_1 x_1^{m_1}\}.$$

Here, the $\varepsilon_l > 0$ are small and the $N_l > 0$ are large parameters to be determined later.

Notice that what will remain is the domain in (2.9). Deviating somewhat from our previous notation for $l < \lambda$ (and the one in [11]), we shall denote this domain by D_λ, i.e.,

$$D_\lambda := \{(x_1, x_2) : |x_2 - \psi(x_1)| \leq N_\lambda x_1^{a_\lambda}\}.$$

The localizations to these domains will be performed by means of cut-off functions

$$\rho_l(x_1, x_2) := \rho\left(\frac{x_2 - \psi(x_1)}{N_l x_1^{a_l}}\right) - \rho\left(\frac{x_2 - \psi(x_1)}{\varepsilon_l x_1^{a_l}}\right), \quad l = l_0, \dots, \lambda - 1,$$

$$\tau_l(x_1, x_2) := \rho\left(\frac{x_2 - \psi(x_1)}{\varepsilon_l x_1^{a_l}}\right)(1 - \rho)\left(\frac{x_2 - \psi(x_1)}{N_{l+1} x_1^{a_{l+1}}}\right), \quad l = l_0, \dots, \lambda - 1,$$

and

$$\tau_{l_0-1}(x_1, x_2) := \rho\left(\frac{x_2 - \psi(x_1)}{\varepsilon_1 x_1^{m_1}}\right)(1 - \rho)\left(\frac{x_2 - \psi(x_1)}{N_{l_0} x_1^{a_{l_0}}}\right),$$

respectively by ρ_λ for the domain D_λ.

Here, in each instance $\rho \in C_0^\infty(\mathbb{R})$ is a suitable cut-off function supported in the interval $[-1, 1]$ such that $\rho = 1$ near the origin. Accordingly, we decompose

$$J^{\rho_1}(\xi) - J^{\rho_\lambda}(\xi) = \sum_{l=l_0}^{\lambda-1} J^{\rho_l}(\xi) + \sum_{l=l_0-1}^{\lambda-1} J^{\tau_l}(\xi).$$

The first part of Proposition 2.4 will be verified if we show that each of the oscillatory integrals $J^{\rho_l}(\xi)$ and $J^{\tau_l}(\xi)$ arising in this sum satisfies estimate (1.2).

Now, the estimates of Sect. 11 in [11] show that the $J^{\tau_l}(\xi)$ satisfy estimate (1.2), even without logarithmic factor, so we only need to consider the $J^{\rho_l}(\xi)$.

Estimation of $J^{\rho_l}(\xi)$ for $l_0 \leq l \leq \lambda - 1$ Applying the change of coordinates (2.5), performing a dyadic decomposition and re-scaling similarly as in the case of adapted coordinates, only with the weight κ replaced by the weight κ^l, we find that

$$J^{\rho_l}(\xi) = \sum_{k=k_0}^{\infty} J_k(\xi),$$

where

$$J_k(\xi) = 2^{-k|\kappa^l|} \int_{(\mathbb{R}_+)^2} e^{i(2^{-k}\xi_3 \phi^k(y) + 2^{-k\kappa_1^l}\xi_1 y_1 + 2^{-k\kappa_2^l}\xi_2 y_2 + 2^{-k\kappa_2^l}\xi_2 \psi^k(y_1))}$$
$$\times \rho_l^a(y)\, \eta^a(\delta_{2^{-k}}^l y) \chi(y)\, dy,$$

with $\psi^k(y_1) := 2^{k\kappa_2^l}\psi(2^{-k\kappa_1^l} y_1)$, $\phi^k(y) := \phi_{\kappa^l}^a(y) + 2^k \phi_r^a(\delta_{2^{-k}}^l y)$, and

$$\rho_l^a(y) := \rho\left(\frac{y_2}{N_l y_1^{a_l}}\right) - \rho\left(\frac{y_2}{\varepsilon_l y_1^{a_l}}\right).$$

Here, δ_r^l denotes the dilation by $r>0$ associated to the weight κ^l, and we have again decomposed

$$\phi^a = \phi^a_{\kappa^l} + \phi^a_r,$$

where ϕ^a_r depends in fact also on l and consists of terms of κ^l-degree higher than 1.

Since $l \le \lambda - 1$ we can then again estimate $J_k(\xi)$ by means of Lemma 2.2, applied to the y_2-integration, by using Corollary 3.2(i) in [11], and obtain

$$\begin{aligned} |J_k(\xi)| &\le C\|\eta\|_{C^3(\mathbb{R}^2)}\, 2^{-k|\kappa^l|}(1+2^{-k}|\xi_3|)^{-1/d_h(\phi^a_{\kappa^l})} \\ &\le C\|\eta\|_{C^3(\mathbb{R}^2)}\, 2^{-k|\kappa^l|}(1+2^{-k}|\xi_3|)^{-1/h} \end{aligned} \tag{2.10}$$

since $d_h(\phi^a_{\kappa^l}) < h$. This also implies $1 = |\kappa^l| d_h(\phi^a_{\kappa^l})| < |\kappa^l| h$, so that a comparison with (2.4) shows that summation over k yields

$$|J^{\rho_l}(\xi)| \le C\|\eta\|_{C^3(\mathbb{R}^2)}(1+|\xi|)^{-1/h}.$$

We next turn to the second statement in Proposition 2.4. We have to show that also $J^{\rho_\lambda}(\xi)$ satisfies estimate (1.2). However, if the principal face $\pi(\phi^a)$ is a vertex (case (b)) or unbounded (case (c1)) then Corollary 3.2 (ii) in [11] allows us to argue exactly as before in order to see that (2.10) also holds for $l=\lambda$. And, in case (b) we have $|\kappa_\lambda|h = 1$, whereas in case (c1) we have $|\kappa_\lambda|h > 1$, so that a comparison with (2.4) shows that estimate (1.2) is indeed valid for $J^{\rho_\lambda}(\xi)$. □

2.3 The Contribution of the Homogenous Domain D_λ Containing the Principal Root Jet

In view of Proposition 2.4, we may and shall from now on assume that the principal face of $\mathcal{N}(\phi^a)$ is a compact edge (case (a)). What remains to be estimated is the contribution of the domain (2.9) to $J(\xi)$, i.e., we are left with the oscillatory integral $J^{\rho_\lambda}(\xi)$. This will require different arguments then those used in [11]. We are also assuming that $x_1 > 0$. Recall also that according to our convention

$$m(\phi^a_{\mathrm{pr}}) < d(\phi^a) = h, \tag{2.11}$$

so that $\nu(\phi) = 0$. This means that we have to prove that $J^{\rho_\lambda}(\xi)$ satisfies (1.2), without the presence of a logarithmic factor.

2.3.1 Preliminary Reductions

Following [11], Sect. 9, it will be convenient at this point to defray our notation by writing ϕ in place of ϕ^a and η in place of η^a, κ in place of κ^λ, δ_r in place of δ_r^λ, etc. With some slight abuse of notation, we shall denote $J^{\rho_\lambda}(\xi)$ by $J(\xi)$.

After applying the change of coordinates (2.5), this means that from now on we shall have to consider oscillatory integrals

$$J(\xi) := \int_{\mathbb{R}^2_+} e^{i(\xi_3\phi(x)+\xi_1x_1+\xi_2(x_2+\psi(x_1)))}\rho\left(\frac{x_2}{N_0x_1^a}\right)\eta(x)\,dx,$$

where $a = \kappa_2/\kappa_1 > m_1$, and where N_0 is a given, possibly large positive number. Notice that the integration takes place only over the domain

$$|x_2| \le N_0 x_1^a. \tag{2.12}$$

We shall write $m := m_1$, so that ψ can be factored as $\psi(x_1) = x_1^m \sigma(x_1)$, with a smooth function σ satisfying $\sigma(0) \neq 0$. $J(\xi)$ can thus be written as an oscillatory integral

$$J(\xi) = \int_{\mathbb{R}^2_+} e^{iF(x,\xi)} \rho\left(\frac{x_2}{N_0 x_1^a}\right) \eta(x)\, dx, \tag{2.13}$$

with a phase function

$$F(x,\xi) := \xi_3 \phi(x) + \xi_1 x_1 + \xi_2 x_1^m \sigma(x_1) + \xi_2 x_2$$

depending on $\xi \in \mathbb{R}^3$. The coordinates x are now adapted to ϕ. We shall again decompose

$$\phi(x) = \phi_\kappa + \phi_r,$$

where ϕ_r consists of terms of κ-degree strictly bigger then 1, the κ-degree of ϕ_κ.

In order to estimate $J(\xi)$, in a first step we shall decompose the domain (2.12) into smaller, κ-homogeneous sub-domains. To this end, given any point $c \in [0, N_0]$, we define

$$J^c(\xi) := \int_{\mathbb{R}^2_+} e^{iF(x,\xi)} \rho\left(\frac{x_2 - c x_1^a}{\varepsilon_0 x_1^a}\right) \eta(x)\, dx,$$

where $\varepsilon_0 > 0$ will be a sufficiently small constant (the cut-off function ρ here is possibly different from the one in (2.13)).

In order to prove that $J(\xi)$ satisfies estimate (1.2), it will be sufficient to show that for every $c \in [0, N_0]$ there exists an $\varepsilon_0 > 0$ such that $J^c(\xi)$ satisfies (1.2), as can be seen easily be means of a partition of unity argument.

We therefore assume that c is fixed. Then we take again a smooth cut-off function χ which is supported in an annulus D such that

$$\sum_{k=k_0}^{\infty} \chi(\delta_{2^k}(x)) = 1 \quad \text{for every } x \in \operatorname{supp} \eta \setminus \{0\}.$$

Notice that we can assume that k_0 is a sufficiently large positive integer by choosing the support of η sufficiently small. Then we have

$$J^c(\xi) = \sum_{k=k_0}^{\infty} J_k(\xi),$$

where

$$J_k(\xi) := \int_{\mathbb{R}^2_+} e^{iF(x,\xi)} \rho\left(\frac{x_2 - c x_1^a}{\varepsilon_0 x_1^a}\right) \eta(x)\, \chi(\delta_{2^k}(x))\, dx.$$

After the change of variables $x \mapsto \delta_{2^{-k}}(x)$ we obtain

$$J_k(\xi) = 2^{-|\kappa|k} \int e^{i2^{-k}\xi_3 F_k(x,s)} \rho\left(\frac{x_2 - cx_1^a}{\varepsilon_0 x_1^a}\right) \eta(\delta_{2^{-k}}(x))\, \chi(x)\, dx, \tag{2.14}$$

where

$$\begin{aligned} F_k(x,s) &:= \phi_\kappa(x) + 2^k \phi_r(\delta_{2^{-k}}(x)) + s_1 x_1 + S_2 x_1^m \sigma(2^{-\kappa_1 k} x_1) + s_2 x_2, \\ s_1 &:= 2^{(1-\kappa_1)k} \frac{\xi_1}{\xi_3}, \qquad s_2 := 2^{(1-\kappa_2)k} \frac{\xi_2}{\xi_3}, \qquad S_2 := 2^{(\kappa_2 - m\kappa_1)k} s_2, \\ s &:= (s_1, s_2, S_2). \end{aligned}$$

Note that $2 \le m < a = \kappa_2/\kappa_1$ and $k \gg 1$, so that $|S_2| \gg |s_2|$. Observe also that there exists a compact interval I such that $x_1 \sim 1$ on I, so that for every (x_1, x_2) in the support of the integrand of $J_k(\xi)$ as given by (2.14), we have

$$x_1 \in I \quad \text{and} \quad |x_2 - cx_1^a| \lesssim \varepsilon_0.$$

Recall also from (2.1) that we are assuming that $|\xi| \sim |\xi_3|$.

2.3.2 Estimation of the Oscillatory Integrals $J_k(\xi)$

In order to estimate $J_k(\xi)$, we shall distinguish several cases depending on the size of the parameters s_1, s_2 and S_2. Recall here that ξ is a function of ξ_3, s_1, s_2 and S_2.

Case 1. $|S_2| \ge M$ for some sufficiently large constant $M \gg 1$. In this case we can apply Lemma 2.2 to the x_1-integration and obtain

$$|J_k(\xi)| \le C \frac{2^{-k|\kappa|} \|\eta\|_{C^1}}{(1 + 2^{-k}|\xi|)^{1/2}}. \tag{2.15}$$

Case 2. $|S_2| < M$, where M is chosen as in Case 1. Then $|s_2| \ll 1$, provided we have chosen k_0 sufficiently large.

If we assume that there is some $j \ge 1$ such that

$$\partial_2^j \phi_\kappa(1, c) \neq 0, \tag{2.16}$$

then we claim that

$$|J_k(\xi)| \le C \frac{2^{-k|\kappa|} \|\eta\|_{C^1}}{(1 + 2^{-k}|\xi|)^{1/j}}. \tag{2.17}$$

Indeed, by the homogeneity of ϕ_κ, if we choose ε_0 sufficiently small, then $\partial_2^j \phi_\kappa(x) \neq 0$ at every point x in the support of the integrand of $J_k(\xi)$, so that the estimate follows for $j \ge 2$ from Lemma 2.2 again, this time applied to the x_2-integration. Notice that the term $2^k \phi_r(\delta_{2^{-k}}(x))$ can be viewed as a perturbation term. Similarly, if $j = 1$, the estimate follows by an integration by parts with respect to x_2.

We observe that if (2.16) *holds for some* $1 \le j < h$, *then by* (2.15), (2.17) *and* (2.4) *we obtain the desired estimate* (1.2), *even without a logarithmic factor, since* $h > 2$.

We may and shall therefore henceforth assume that

$$\partial_2^j \phi_\kappa(1, c) = 0 \quad \text{for } 1 \le j < h. \tag{2.18}$$

Recall that we are assuming that the principal face of $\mathcal{N}(\phi)$ is a compact edge, so that $\phi_\kappa = \phi_{\rm pr}$ and $h = 1/|\kappa|$.

Assume first that $c = 0$ Then necessarily $\phi_{\rm pr}(1,0) \neq 0$, for otherwise $\phi_{\rm pr}$ would have a root of multiplicity at least h at $(1,0)$, which would contradict (2.11).

Assuming without loss of generality that $\phi_{\rm pr}(1,0) = 1$, we can then write (compare [11], Sect. 9.1)

$$\phi_{\rm pr}(x_1, x_2) = x_2^B Q(x_1, x_2) + x_1^n,$$

where Q is a κ-homogeneous polynomial such that $Q(1,0) \neq 0$, and where $B \ge h > 2$.

Recall that $|S_2| < M$, so that $|s_2| \ll 1$. We now distinguish two subcases:

Case 2.a. $|S_2| < M$, and $|s_1| \ge N$ for some sufficiently large constant $N \gg 1$. Then an integration by parts in x_1 leads to the estimate $|J_k(\xi)| \le C \frac{2^{-k|\kappa|}\|\eta\|_{C^1}}{1+2^{-k}|\xi|}$, which in return implies (1.2), even without logarithmic factor.

Case 2.b. $|S_2| < M$, hence $|s_2| \ll 1$, and $|s_1| < N$, where N is chosen as in Case 2.a.

We shall show that, given any point $(s_1^0, S_2^0) \in [-M, M] \times [-N, N]$ and any point $x_1^0 \in I$, there exist a neighborhood U of (s_1^0, S_2^0), a neighborhood V of x_1^0 and some $\omega > 1/h$ such that we have an estimate of the form

$$|J_k(\xi)| \le C \frac{2^{-k|\kappa|}\|\eta\|_{C^1}}{(1 + 2^{-k}|\xi|)^\omega} \tag{2.19}$$

for every $(s_1, S_2) \in U$, provided the function χ in the definition of $J_k(\xi)$ is supported in V and ε_0 and k are chosen sufficiently small, respectively large. The same type of estimate will then hold also for every $(s_1, S_2) \in [-M, M] \times [-N, N]$ and for the original function χ in the definition of $J_k(\xi)$, as can be seen by means of a partition of unity argument. Summing over all k, this will clearly imply the estimate (1.2), even without logarithmic factor.

To this end, first notice that for $(s_1, S_2) \in U$ and k sufficiently large, the function $F_k(x, s)$ can be viewed as a small C^∞-perturbation of the function

$$F_{\rm pr}(x) := x_2^B Q(x_1, x_2) + s_1^0 x_1 + S_2^0 \sigma(0) x_1^m + x_1^n.$$

Thus, if $\nabla F_{\rm pr}(x_1^0, 0) \neq 0$, then we obtain (2.19), with $\omega = 1$, simply by integration by parts.

Assume therefore that $(x_1^0, 0)$ is a critical point of $F_{\rm pr}$. Then x_1^0 is a critical point of the polynomial function

$$g(x_1) := s_1^0 x_1 + S_2^0 \sigma(0) x_1^m + x_1^n,$$

which comprises all terms of $F_{\rm pr}$ depending on the variable x_1 only. Note that $2 \le m < n$, since $n = 1/\kappa_1 > \kappa_2/\kappa_1 > m$. It is then easy to see that g'' and g''' cannot vanish simultaneously at the given point $x_1^0 \in I$, so that there are positive constants $c_1, c_2 > 0$ and a compact neighborhood V of x_1^0 such that

$$c_1 \le \sum_{j=1}^{3} |g^{(j)}(x_1)| \le c_2 \quad \text{for every } x_1 \in V.$$

This implies an analogous estimate for the partial derivatives $\partial_{x_1}^j F_k(x_1, x_2, s)$ of F_k, uniformly for $(s_1, S_2) \in U$ and x_2 satisfying (2.12), provided we choose U and ε_0 sufficiently small. Applying the van der Corput type estimate in Lemma 2.2, we thus obtain the estimate (2.19) with $\omega = 1/3$, so that we are done provided $h > 3$. Notice also that if $g''(x_1^0) \neq 0$, then by the same type of argument we see that (2.19) holds true with $\omega = 1/2 > 1/h$.

We may thus finally assume that $2 < h \le 3$, and that $g'(x_1^0) = g''(x_1^0) = 0$. In this case we have

$$\frac{1}{\kappa_1 + \kappa_2} = h \le 3 \quad \text{and} \quad \frac{\kappa_2}{\kappa_1} > m \ge 2,$$

so that $1/\kappa_2 < 9/2$.

Note that $B \le 1/\kappa_2$ is a positive integer, and $h \le B < 9/2$, so that either $B = 4$ or $B = 3$. We translate the critical point $(x_1^0, 0)$ of F_{pr} to the origin by considering the function

$$F_{\mathrm{pr}}^\sharp(y) := F_{\mathrm{pr}}(x_1^0 + y_1, y_2) - g(x_1^0) = y_2^B Q(x_1^0 + y_1, y_2) + \frac{1}{6} g^{(3)}(x_1^0)\, y_1^3 + \cdots.$$

It is easy to see that this function has height $h^\sharp := h(F_{\mathrm{pr}}^\sharp)$ given by $h^\sharp = \frac{1}{1/3+1/4} = 12/7$, if $B = 4$, and $h^\sharp = \frac{1}{1/3+1/3} = 3/2$, if $B = 3$.

In both cases, $F_{\mathrm{pr}}^\sharp$ has height $h^\sharp < 2$ (indeed, according to Arnol'd's classification of singularities, $F_{\mathrm{pr}}^\sharp$ is of type E_6 and D_4, respectively). We can therefore again apply Duistermaat's results in [6] to the oscillatory integral $J_k(\xi)$ and obtain the estimate (2.19), with $\omega = 1/h^\sharp > 1/h$. Note here that the estimates in [6] are stable under small perturbations.

Assume finally that $c > 0$ Then, by Corollary 3.2(iii) in [11], our assumption (2.18) implies that necessarily $a = \kappa_2/\kappa_1 \in \mathbb{N}$.

We can then reduce this case to the previous case $c = 0$ by performing another change of variables $x_2 \mapsto x_2 + c x_1^a$ in the integral defining $J_k(\xi)$.

Indeed, this is equivalent to replacing the function ψ in our previous argument by $\psi^\sharp(x_1) := \psi(x_1) + c x_1^a$, and assuming that $c = 0$. Denote by $\phi^\sharp(x_1, x_2) := \phi(x_1, x_2 + c x_1^a)$ the corresponding phase function. Then the coordinates (x_1, x_2) are adapted to $\phi^\sharp$ too, as can be seen as follows:

Lemma 3.1 in [12] shows that $(\phi_{\mathrm{pr}})^\sharp(x_1, x_2) := \phi_{\mathrm{pr}}(x_1, x_2 + c x_1^a)$ is again a κ-homogeneous polynomial whose principal face intersects the bi-sectrix, and $m(\phi_{\mathrm{pr}}) = m((\phi_{\mathrm{pr}})^\sharp)$. Therefore $(\phi_{\mathrm{pr}})^\sharp$ must be the principal part of $\phi^\sharp$.

This completes the proof of Theorem 1.1.

3 Sharpness of the Uniform Estimates

In this section, we shall give a proof of Theorem 1.3. Observe that we may assume for this purpose that the coordinates (x_1, x_2) are adapted to ϕ, so that $d := d(\phi) = h$.

We shall only consider the asymptotic behavior of $J_+(\lambda)$, since the result for $J_-(\lambda)$ follows from the one for $J_+(\lambda)$ by means of complex conjugation.

Remarks 3.1 If $h < 2$, then the phase function ϕ has a critical point at the origin with finite Milnor number, and can thus be reduced to a polynomial phase function by means of a smooth local change of coordinates at the origin (see [2]). Therefore, in this case we could apply the classical results for analytic phase functions by A.N. Varchenko [21]. However, we will give a more elementary argument which does not rely on this classification of singularities.

Notice also that if $h = 1$, then the phase function has a non-degenerate critical point at the origin in our adapted coordinates, and we could apply the method of stationary phase in order to prove the existence of the limits in Theorem 1.3 (see [19]).We shall, however, proceed somewhat differently also in this case.

3.1 The Case where the Principal Face is a Compact Edge

We begin with the simplest case where the principal face $\pi(\phi)$ is a compact edge. Arguing as in Sect. 2.2, we may then assume in addition that

$$m(\phi_{\mathrm{pr}}) < d,$$

since otherwise a suitable local change of coordinates would reduce us to the situation where the principal face is a vertex.

Then there is a unique weight κ such that $\pi(\phi)$ is lying on the line given by the equation $\kappa_1 t_1 + \kappa_2 t_2 = 1$. Without loss of generality we may assume that $0 < \kappa_1 \le \kappa_2$. Recall also that then $\phi_{\mathrm{pr}} = \phi_\kappa$ and $d = d_h(\phi_\kappa) = 1/|\kappa|$, and that if we decompose

$$\phi(x) = \phi_\kappa(x) + \phi_r(x),$$

then ϕ_r is an error term whose Newton polyhedron is contained in the set $\{(k_1, k_2) \in \mathbb{Z}^2 : \kappa_1 k_1 + \kappa_2 k_2 > 1\}$.

In a first step, we shall reduce ourselves to the situation where the amplitude a is constant on a neighborhood of the origin. To this end, if Ω is an open neighborhood of the origin in $\mathbb{R}^2$, let us introduce the subspace of amplitude functions

$$\dot{C}_0^3(\Omega) := \{a \in C_0^3(\Omega) : a(0,0) = 0\}.$$

If $a \in \dot{C}_0^3(\Omega)$ and if F is a smooth, real-valued phase function on Ω, we consider the oscillatory integral

$$J(\lambda, F, a) := \int e^{i\lambda(\phi_\kappa(x) + F(x))} a(x)\, dx, \quad \lambda > 0.$$

Proposition 3.2 *There exists a positive number ε such that for any smooth function $F \in C^\infty(\mathbb{R}^2)$ with $\mathcal{N}(F) \subset \{(k_1, k_2) \in \mathbb{Z}^2 : \kappa_1 k_1 + \kappa_2 k_2 > 1\}$ there exists a neighborhood Ω of the origin so that for any $a \in \dot{C}_0^3(\Omega)$ the following estimate*

$$|J(\lambda, F, a)| \leq \frac{C(F)\|a\|_{C^3(\Omega)}}{\lambda^{1/d+\varepsilon}}$$

holds true, with a constant $C(F)$ depending only on the $C^N(\Omega)$ norm of F, for some sufficiently large number N.

Proof If $a \in \dot{C}_0^2(\Omega)$, then a can be written as $a(x_1, x_2) = x_1 a_1(x_1, x_2) + x_2 a_2(x_1, x_2)$, with functions $a_1, a_2 \in C^1(\Omega)$ whose C^1-norms can be controlled by the C^2-norm of a. Consequently for the oscillatory integral we have

$$J(\lambda, F, a) = J(\lambda, F, x_1 a_1) + J(\lambda, F, x_2 a_2).$$

We shall therefore estimate $J(\lambda, F, x_1 a_1)$ ($J(\lambda, F, x_2 a_2)$ can be treated in a similar way). As before, we choose a suitable smooth cut-off function χ on $\mathbb{R}^2$ supported in an annulus D such that the functions $\chi_k := \chi \circ \delta_{2^k}$ form a partition of unity, and then decompose

$$J(\lambda, F, x_1 a_1) = \sum_{k=k_0}^{\infty} J(\lambda, F, x_1 a_1 \chi_k).$$

Here, δ_r denotes again the dilation by the factor $r > 0$ associated to the weight κ. Recall that by choosing Ω sufficiently small we may assume that k_0 is a sufficiently large number. After re-scaling, we may re-write

$$J_k(\lambda) := J(\lambda, F, x_1 a_1 \chi_k)$$

as

$$J_k(\lambda) = 2^{-(|\kappa|+\kappa_1)k} \int e^{i\lambda 2^{-k}(\phi_\kappa + 2^k F(\delta_{2^k}(x)))} x_1 a_1(\delta_{2^{-k}}(x)) \chi(x)\, dx.$$

If $\lambda 2^{-k} \leq M$ (with a fixed positive number M), a trivial estimate for the integral $J_k(\lambda)$ gives

$$|J_k(\lambda)| \leq C\|a_1\|_{C^0(\Omega)} 2^{-(|\kappa|+\kappa_1)k},$$

hence

$$\sum_{\lambda 2^{-k} \leq M} |J_k(\lambda)| \leq C_M \frac{\|a\|_{C^1(\Omega)}}{\lambda^{|\kappa|+\kappa_1}}, \tag{3.1}$$

if we assume without loss of generality that Ω is a ball.

Assume next that $\lambda 2^{-k} > M$. Since $\|2^k F \circ \delta_{2^{-k}}\|_{C^m(\Omega)} \to 0$ as $k \to +\infty$, by choosing Ω sufficiently small we may assume that $\|2^k F \circ \delta_{2^{-k}}\|_{C^m(\Omega)}$ is sufficiently small.

Now, if $m(\phi_\kappa) \geq 1$, we put $m := m(\phi_\kappa)$. Then $1 \leq m < d$. Notice that if $x^0 \in D$ is such that $\nabla\phi_\kappa(x^0) = 0$, then, by Euler's homogeneity relation, also $\phi(x^0) = 0$. Therefore, by applying Lemma 2.2, respectively an integration by parts, and assuming that M is sufficiently big, we see that

$$|J_k(\lambda)| \leq C(F)\|a_1\|_{C^1(\Omega)} 2^{-(|\kappa|+\kappa_1)k}(1 + 2^{-k}\lambda)^{-1/m}.$$

Summing in k, this implies

$$\sum_{\lambda 2^{-k} > M} |J_k(\lambda)| \leq C(F)\|a\|_{C^2(\Omega)} \begin{cases} (1+\lambda)^{-1/m}, & \text{if } m(|\kappa|+\kappa_1) > 1, \\ \log(2+\lambda)\,(1+\lambda)^{-1/m}, & \text{if } m(|\kappa|+\kappa_1) = 1, \\ (1+\lambda)^{-(|\kappa|+\kappa_1)}, & \text{if } m(|\kappa|+\kappa_1) < 1. \end{cases} \tag{3.2}$$

If we put $\varepsilon_0 := \min\{\kappa_1, 1/m - 1/d\}$, we see that (3.1) and (3.2) imply that

$$|J(\lambda,\ F, x_1 a_1)| \leq \frac{C(F)\|a\|_{C^2(\Omega)}}{\lambda^{1/d+\varepsilon}},$$

for every positive number $\varepsilon < \varepsilon_0$. Similar estimates hold true for $J(\lambda,\ F, x_2 a_2)$, only with κ_1 replaced by κ_2. Since $\kappa_1 \leq \kappa_2$, we see that we can use the same range of ε's also in this case and obtain the desired estimate in Proposition 3.2.

There remains the case where $m(\phi_\kappa) = 0$. Here, ϕ_κ does not vanish away from the origin, and thus $\nabla\phi_\kappa(x^0) \neq 0$ for every $x^0 \in D$. Thus, choosing $m = 1$ here and applying one integration by parts, we again obtain estimate (3.2), and can conclude as before, if $d > 1$.

Finally, if $d = 1$ (notice that necessarily $d \geq 1$, since $\nabla\phi(0,0) = 0$), applying two integrations by parts to $J_k(\lambda)$, we obtain

$$\sum_{\lambda 2^{-k} > M} |J_k(\lambda)| \leq C(F)\|a\|_{C^3(\Omega)}\, \lambda^{-(|\kappa|+\kappa_1)},$$

where $|\kappa| = 1$. Thus, we can choose $\varepsilon := \kappa_1$ in this case. □

Let us now consider the oscillatory integral

$$J_+(\lambda) := \int_{\mathbb{R}^2} e^{i\lambda\phi(x)}\eta(x)\,dx,$$

where $\eta \in C_0^\infty(\Omega)$. Choose a smooth bump function χ_0 supported in Ω which is identically 1 on a neighborhood of the origin. Then, if we choose Ω sufficiently small, Proposition 3.2 implies that the oscillatory integrals $J_+(\lambda)$ and $\eta(0,0)J(\lambda)$, with

$$J(\lambda) := \int_{\mathbb{R}^2} e^{i\lambda\phi(x)}\chi_0(x)\,dx,$$

differ by a term of decay rate $O(\lambda^{-1/d-\varepsilon})$. This shows that, in order to prove Theorem 1.3 in this case, it suffices to prove that the limit

$$\lim_{\lambda\to+\infty} \lambda^{1/d} J(\lambda) = c \tag{3.3}$$

exists and that $c \neq 0$.

To this end, put $\delta := \varepsilon/4$, with $\varepsilon > 0$ as in Proposition 3.2, and define the polynomial functions P and Q by

$$Q(x) := \sum_{|\alpha|\le 1/\delta+3} \frac{\partial^\alpha \phi(0)}{\alpha!} x^\alpha =: \phi_\kappa(x) + P(x).$$

Notice that all the derivatives of the function $e^{i\lambda(\phi(x)-Q(x))}$ up to order 3 are uniformly bounded with respect to λ on the set where $\lambda^\delta|x| < 1$. We therefore decompose

$$J(\lambda) = \int e^{i\lambda\phi(x)} \chi_0(x)\chi_0(\lambda^\delta x)\, dx + \int e^{i\lambda\phi(x)} \chi_0(x)(1-\chi_0(\lambda^\delta x))\, dx.$$

Due to Proposition 3.2 (with $F := P$), the second summand has decay rate of order $O(\lambda^{-1/d-\varepsilon+3\delta}) = O(\lambda^{-1/d-\delta})$ as $\lambda \to +\infty$, if Ω is supposed to be chosen sufficiently small. In order to prove (3.3), we may therefore replace $J(\lambda)$ by the first summand, $J_0(\lambda)$, which we again decompose as

$$\begin{aligned} J_0(\lambda) &= \int e^{i\lambda\phi(x)} \chi_0(x)\chi_0(\lambda^\delta x)\, dx \\ &= \int e^{i\lambda Q(x)} \chi_0(x)\, dx \\ &\quad + \int e^{i\lambda(\phi_\kappa(x)+P(x))} \chi_0(x)\big(\chi_0(\lambda^\delta x) e^{i\lambda(\phi(x)-Q(x))} - 1\big)\, dx. \end{aligned}$$

Again, by applying Proposition 3.2, we see that the second summand has decay rate $O(\lambda^{-1/d-\delta})$ as $\lambda \to +\infty$, and thus we are reduced to proving that the limit

$$\lim_{\lambda\to+\infty} \lambda^{1/d} \int e^{i\lambda Q(x)} \chi_0(x)\, dx = c$$

exists and that $c \ne 0$. But, $Q(x) = \phi_\kappa(x) + P(x)$ is a polynomial, with principal part ϕ_κ, and therefore this statement follows from the classical results for analytic phase functions due to Varchenko [21] (see also [9]).

3.2 The Case where the Principal Face Is a Vertex

Assume now that $\pi(\phi) = \{(d,d)\}$ is a vertex, so that in particular d is a positive integer. After multiplying the phase function with a suitable real constant (this can be implemented by means of a suitable scaling in λ and, possibly, complex conjugation of $J_+(\lambda)$), we may assume without loss of generality that the principal part of ϕ is given by

$$\phi_{\mathrm{pr}}(x) = x_1^d x_2^d.$$

We may also assume that the coordinates (x_1, x_2) are super-adapted, in the sense of Greenblatt [9]. Then, if $d = h = 1$, according to Lemma 1.0 in [9], the critical point of ϕ at the origin is non-degenerate, and thus the statement of Theorem 1.3 is a well-known consequence of the method of stationary phase.

Let us therefore henceforth assume that $d = h \ge 2$.

3.2.1 Two Compact Edges

First, assume that the Newton polyhedron $\mathcal{N}(\phi)$ has two compact edges containing the vertex (d,d) as one of their endpoints, say γ_a, lying "above" the bi-sectrix and on the line given by $\kappa_1^a t_1 + \kappa_2^a t_2 = 1$, and γ_b, lying "below" the principal face and on the line given by $\kappa_1^b t_1 + \kappa_2^b t_2 = 1$. Notice that then

$$a := \frac{\kappa_2^a}{\kappa_1^a} < \frac{\kappa_2^b}{\kappa_1^b} =: b. \tag{3.4}$$

Lemma 3.3 *The function ϕ can be written as*

$$\phi(x_1,x_2) = x_1^d x_2^d + \phi_a(x_1,x_2) + \phi_b(x_1,x_2),$$

where $\phi_a(x_1,x_2) = x_2^d \tilde{\phi}_a(x_1,x_2)$ and $\phi_b(x_1,x_2) = x_1^d \tilde{\phi}_b(x_1,x_2)$, with smooth functions $\tilde{\phi}_a$ and $\tilde{\phi}_b$.

The proof of Lemma 3.3 is straightforward. Notice also that we have

$$x_1^d x_2^d + \phi_a(x_1,x_2) = \phi_{\kappa^a}(x_1,x_2) + \phi_{a,r} \quad \text{and}$$
$$x_1^d x_2^d + \phi_b(x_1,x_2) = \phi_{\kappa^b}(x_1,x_2) + \phi_{b,r},$$

where ϕ_{κ^a} is $\kappa^a := (\kappa_1^a, \kappa_2^a)$-homogeneous of degree 1, and $\phi_{a,r}$ consists of terms of κ^a-degree higher than 1, and where the analogous statements holds true for ϕ_{κ^b} and $\phi_{b,r}$.

Lemma 3.4 *After applying a suitable smooth local change of coordinates at the origin, we may assume that the functions $x_1 \mapsto \phi_{\kappa^a}(x_1,\pm 1)$ and $x_2 \mapsto \phi_{\kappa^b}(\pm 1, x_2)$ have no root of multiplicity greater or equal to d, respectively.*

Proof We may assume that $b \geq 1$, for otherwise, after interchanging the coordinates x_1 and x_2, we will have $b \geq a \geq 1$.

Then, the proof of Theorem 7.1 in [9] for the existence of "super-adapted coordinates" shows that, after applying a suitable local change of coordinates at the origin, we may assume that $\phi_{\kappa^b}(\pm 1, x_2)$ has no non-zero root of order greater or equal to d (of course, the edge γ_b may have changed and even have become unbounded, but this would be a case to be considered later). We also remark that the change of coordinates in [9] is such that the edge γ_a remains to be an edge of the Newton diagram in the new coordinates.

According to Proposition 2.2 in [12], we can then write, say for $x_1 > 0$,

$$\phi_{\kappa^b}(x_1,x_2) = x_1^\alpha x_2^\beta \prod_l (x_2^q - c_l x_1^p)^{n_l},$$

for suitable integers $\alpha, \beta \geq 0$ and $p, q \geq 1$ such that $p/q = b$, where $c_l \in \mathbb{C} \setminus \{0\}$ and $n_l \in \mathbb{N} \setminus \{0\}$. Since we are assuming that (d,d) is the upper vertex of the edge γ_b, we see that $\alpha = d$ and $\beta + (\sum_l n_l)q = d$. Therefore, necessarily $\beta < d$, which shows that $\phi_{\kappa^b}(\pm 1, x_2)$ that also $x_2 = 0$ is no root of order greater or equal to d.

We now turn to ϕ_{κ^a}. If $a \leq 1$, after interchanging again the coordinates x_1 and x_2, hence also the edges γ_a and γ_b, we may assume that $x_1 \mapsto \phi_{\kappa^a}(x_1, \pm 1)$ has no root of multiplicity greater or equal to d, and that $a \geq 1$. Applying then the previous argument again to γ_b, we see that in addition we may assume that $\phi_{\kappa^b}(\pm 1, x_2)$ has no root of multiplicity greater or equal to d, and are done.

Assume finally that $a > 1$. Then we can accordingly write

$$\phi_{\kappa^a}(x_1, x_2) = x_1^\alpha x_2^\beta \prod_l (x_2^q - c_l x_1^p)^{n_l},$$

where now $p/q = a$. Since (d, d) is the lower vertex of γ_a, we see that $\beta = d$ and $\alpha + (\sum_l n_l)p = d$, hence $\alpha < d$. Moreover, if $a \notin \mathbb{N}$, then Corollary 2.3 in [12] shows that $n_l < d$ for every l, which shows that $\phi_{\kappa^a}(x_1, \pm 1)$ has no root of multiplicity greater or equal to d.

And, if $a \in \mathbb{N}$, then $q = 1$ and $p = a > 1$, hence $n_l < n_l p \leq d$, so that again $n_l < d$, and we can conclude as before. □

Let us assume in the sequel that the adapted coordinates are chosen so that the conclusions in Lemma 3.4 do apply, and consider again the oscillatory integral

$$J_+(\lambda) := \int_{\mathbb{R}^2} e^{i\lambda\phi(x)} \eta(x)\, dx.$$

Note that in this case we have to prove that

$$\lim_{\lambda \to +\infty} \frac{\lambda^{1/d}}{\log \lambda} J_+(\lambda) = c\, \eta(0),$$

where $c \neq 0$.

With χ_0 as before, let us consider the oscillatory integrals

$$J_1(\lambda) := \int e^{i\lambda\phi(x)} (\eta(x) - \eta(0)\chi_0(x))\, dx$$

and

$$J(\lambda) := \int_{\mathbb{R}^2} e^{i\lambda\phi(x)} \chi_0(x)\, dx.$$

We then have the following substitute for Proposition 3.2, which allows to reduce to proving that the following limit

$$\lim_{\lambda \to +\infty} \frac{\lambda^{1/d}}{\log \lambda} J(\lambda) = c \tag{3.5}$$

exists and is non-zero.

Lemma 3.5 *If Ω is chosen sufficiently small, then the following estimate*

$$|J_1(\lambda)| \le \frac{C\|\eta\|_{C^2(\Omega)}}{\lambda^{1/d}}$$

holds true.

Proof Permuting the coordinates x_1, x_2, if necessary, we may choose a weight $\kappa = (\kappa_1, \kappa_2)$ with $0 < \kappa_1 \le \kappa_2$, such that the line given by $\kappa_1 t_1 + \kappa_2 t_2 = 1$ is a supporting line to $\mathcal{N}(\phi)$ which contains only the point (d, d) of $\mathcal{N}(\phi)$. Arguing now in the same way is in the proof of Proposition 3.2, with $m := d$, we obtain the desired estimate. □

Choose a smooth cut-off function $\chi^0 \in C_0^\infty(\mathbb{R})$ supported in a sufficiently small neighborhood of the origin. In order to prove (3.5), let us decompose

$$J(\lambda) = J_0(\lambda) + J_\infty(\lambda),$$

where

$$J_0(\lambda) := \int e^{i\lambda\phi(x_1,x_2)} \chi_0(x_1,x_2)\chi^0\left(\frac{x_2}{\varepsilon|x_1|^a}\right)\chi^0\left(\frac{x_1}{\varepsilon|x_2|^{1/b}}\right) dx, \tag{3.6}$$

$$J_\infty(\lambda) := \int e^{i\lambda\phi(x_1,x_2)} \chi_0(x_1,x_2)\left(1-\chi^0\left(\frac{x_2}{\varepsilon|x_1|^a}\right)\chi^0\left(\frac{x_1}{\varepsilon|x_2|^{1/b}}\right)\right) dx, \tag{3.7}$$

where $\varepsilon > 0$ will be chosen later.

Lemma 3.6 *Let $\varepsilon > 0$. Then, if Ω is chosen sufficiently small, the following estimate*

$$|J_\infty(\lambda)| \le \frac{C\|\eta\|_{C^2(\Omega)}}{\lambda^{1/d}}$$

holds true.

Proof We decompose $J_\infty(\lambda) = J_a(\lambda) + J_b(\lambda)$, where

$$J_a(\lambda) := \int e^{i\lambda\phi(x_1,x_2)} \chi_0(x_1,x_2)\left(1-\chi^0\left(\frac{x_2}{\varepsilon|x_1|^a}\right)\right) dx,$$

$$J_b(\lambda) := \int e^{i\lambda\phi(x_1,x_2)} \chi_0(x_1,x_2)\left(1-\chi^0\left(\frac{x_1}{\varepsilon|x_2|^{1/b}}\right)\right)\chi^0\left(\frac{x_2}{\varepsilon|x_1|^a}\right) dx,$$

and show that both terms separately satisfy the estimate in Lemma 3.6.

We begin with $J_a(\lambda)$. Using the dilations δ_r associated to the weight κ^a, we dyadically decompose $J_a(\lambda) = \sum_{k=k_0}^\infty J_k(\lambda)$ in a similar way as in the proof of Lemma 3.2. Here, after re-scaling, $J_k(\lambda)$ is given by

$$J_k(\lambda) = 2^{-|\kappa^a|k} \int e^{i\lambda 2^{-k}(\phi_{\kappa^a} + 2^k\phi_r(\delta_{2^k}(x)))} \chi_0(\delta_{2^{-k}}(x)) \times \left(1-\chi^0\left(\frac{x_2}{\varepsilon|x_1|^a}\right)\right)\chi(x_1,x_2)\, dx,$$

where $|\kappa^a| = 1/d$. Notice that

$$|x_1| \lesssim 1 \quad \text{and} \quad \varepsilon \lesssim |x_2| \lesssim 1$$

for every (x_1, x_2) in the support of the integrand. Let m denote the maximal order of vanishing of ϕ_{κ^a} transversal to its roots on this domain. Then $m < d$, since, according to Lemma 3.4, we are assuming that $\phi_{\kappa^a}(x_1, \pm 1)$ has no root of order greater or equal to d. Consequently, we have $m|\kappa^a| < 1$. Arguing as in the proof of Lemma 3.2 in order to estimate the $J_k(\lambda)$, and summing in k, we then find that $|J_a(\lambda)| \le C\lambda^{-|\kappa^a|} = C\lambda^{-1/d}$.

Finally, $J_b(\lambda)$ can be estimated in a very similar way, making use of the dilations associated to the weight κ^b in place of κ^a. Note that the additional factor $\chi^0(x_2/(\varepsilon|x_1|^a))$ appearing in the integral defining $J_b(\lambda)$ is under control because of (3.4).

The proof of (3.5) is thus reduced to proving the next

Lemma 3.7 *The following limit*

$$\lim_{\lambda \to +\infty} \frac{\lambda^{1/d}}{\log \lambda} J_0(\lambda)$$

exists and is non-zero. Moreover, it does not depend on the choice of ε.

Proof Let us first assume that the integer $d \ge 2$ even.

We may also assume without loss of generality that χ^0 is an even function and that χ_0 is radial, so that in particular

$$\chi_0(x_1, x_2) = \chi_0(\pm x_1, \pm x_2).$$

This implies that, if we decompose the integral defining $J_0(\lambda)$ into the four integrals over each of the quadrants of $\mathbb{R}^2$, then, after an obvious change of coordinates, all four of them will have the same amplitude, as well as the same principal part $x_1^d x_2^d$ for their phases. Since we shall see that the leading term in the asymptotic expansion of $J_0(\lambda)$ will only depend on the principal part of the phase function, we may thus reduce ourselves to considering the integral $J_0^+(\lambda)$ over the positive quadrant only.

Notice that by (3.4) $b - a > 0$. In the integral for $J_0^+(\lambda)$ we apply the change of variables

$$x_2 = x_1^a y_2, \qquad x_1 = y_2^{\frac{1}{b-a}} y_1,$$

and denote by $\tilde{\phi}$ the phase function when expressed in the coordinates y, i.e., $\tilde{\phi}(y) = \phi(x)$.

Observe that this change of coordinates is of class C^∞ away from the coordinate axes, and that it leads to the following form of the phase function $\tilde{\phi}$:

$$\tilde{\phi}(y_1, y_2) = y_1^{d(1+a)} y_2^{\frac{d(1+b)}{b-a}} (1 + \rho(y_1^\delta, y_2^\delta)),$$

where $\rho(z_1, z_2)$ is a smooth function with $\rho(0,0) = 0$, and where $\delta = 1/p > 0$ is some rational number.

Indeed, the Newton polyhedron is transformed into $\mathcal{N}(\tilde{\phi}) = (d,d) + \mathbb{R}^2_+$ under this change of variables, and since

$$x_1 = y_2^{\frac{1}{b-a}} y_1, \qquad x_2 = y_1^a y_2^{1+\frac{a}{b-a}},$$

it is clear that if f is any smooth function of x which is flat at the origin, i.e., which vanishes to infinity oder at the origin, then $\tilde{f}$, defined by $\tilde{f}(y) = f(x)$, can be factored as $\tilde{f}(y) = y_1^{d(1+a)} y_2^{\frac{d(1+b)}{b-a}} g(y)$, where also g is smooth and flat at the origin.

The oscillatory integral $J_0^+(\lambda)$ then transforms into

$$J_0^+(\lambda) = \int e^{i\lambda\tilde{\phi}(y_1,y_2)} y_1^a y_2^{\frac{1+a}{b-a}} \chi_0\left(\frac{y_1}{\varepsilon}\right)\chi_0\left(\frac{y_2}{\varepsilon}\right)\tilde{\chi}_0(y_1,y_2)\,dy,$$

where $\tilde{\chi}_0$ is of class C^1 on the closed positive quadrant, and of class C^∞ away from the coordinate axes, and $\tilde{\chi}_0(0,0) = 1$.

Observe next that if M is any positive constant, then the contribution to the integral $J_0^+(\lambda)$ by the sub-domain where $\lambda y_1^{d(1+a)} \le M$ is trivially of order $O(\lambda^{-1/d})$ as $\lambda \to +\infty$.

We may therefore consider the oscillatory integral

$$\begin{aligned} I(\lambda) &:= \int\limits_{\lambda y_1^{d(1+a)} > M} \int e^{i\lambda\tilde{\phi}(y_1,y_2)} y_1^a y_2^{\frac{1+a}{b-a}} \chi_0\left(\frac{y_1}{\varepsilon}\right)\chi_0\left(\frac{y_2}{\varepsilon}\right)\tilde{\chi}_0(y_1,y_2)\,dy_2dy_1 \\ &= \int\limits_{\lambda y_1^{d(1+a)} > M} y_1^a \chi_0\left(\frac{y_1}{\varepsilon}\right) I_{\text{int}}(\lambda, y_1)\,dy_1 \end{aligned}$$

in place of $J_0^+(\lambda)$, where M is a fixed, sufficiently large positive number.

Assuming that $\varepsilon > 0$ is chosen sufficiently small, we may apply the change of variables

$$z_2 := y_2(1 + \rho(y_1^\delta, y_2^\delta))^{\frac{b-a}{d(1+b)}}$$

to the inner integral

$$I_{\text{int}}(\lambda, y_1) := \int e^{i\lambda\tilde{\phi}(y_1,y_2)} \chi_0\left(\frac{y_2}{\varepsilon}\right)\tilde{\chi}_0(y_1,y_2)\, y_2^{\frac{1+a}{b-a}}\,dy_2,$$

which leads to

$$I_{\text{int}}(\lambda, y_1) = \int e^{i\lambda y_1^{d(1+a)} z_2^{\frac{d(1+b)}{b-a}}} \chi_0\left(\frac{z_2(1+\tilde{\rho}(y_1,z_2))}{\varepsilon}\right)\tilde{\chi}_{0,0}(y_1,z_2)\, z_2^{\frac{1+a}{b-a}}\,dz_2,$$

where $\tilde{\rho}$ and $\tilde{\chi}_{0,0}$ have similar properties as ρ and $\tilde{\chi}_0$, respectively. Changing variables in this integral to $t := z_2^{\frac{1+b}{b-a}}$, and applying some classical results on one-dimensional oscillatory integrals with critical points of order d (see A. Erde'lyi [7], Sect. 2.9), we thus obtain

$$I_{\text{int}}(\lambda, y_1) = \frac{b-a}{1+b}\left(\frac{C_d}{(\lambda y_1^{d(1+a)})^{1/d}} + R(\lambda, y_1)\right),$$

where $C_d \neq 0$ is given explicitly by

$$C_d := \frac{\Gamma(1/d)}{d}\, e^{\frac{\pi i}{2d}}, \tag{3.8}$$

and where the remainder term satisfies an estimate

$$|R(\lambda, y_1)| \le \frac{C_d'}{(\lambda y_1^{d(1+a)})^{1/d+\delta_1}},$$

where $\delta_1 > 0$ is a positive number and where the constant C_d' can be chosen independently of a and b (we mention this here for later use). The latter estimate implies that

$$\left|\int\limits_{\lambda y_1^{d(1+a)} > M} y_1^a \chi_0\left(\frac{y_1}{\varepsilon}\right) R(\lambda, y_1)\, dy_1\right| \le \frac{C_2}{\lambda^{1/d}},$$

whereas the corresponding integral over the principal part of $I_{\text{int}}(\lambda, y_1)$ behaves asymptotically like $c \log \lambda/\lambda^{1/d}$, as required. Explicitly, our argument shows that

$$\lim_{\lambda\to+\infty} \frac{\lambda^{1/d}}{\log \lambda} J_0(\lambda) = 4\frac{b-a}{1+b} C_d, \tag{3.9}$$

if d is even.

Finally, if d is odd, a very similar reasoning shows that

$$\lim_{\lambda\to+\infty} \frac{\lambda^{1/d}}{\log \lambda} J_0(\lambda) = 2\frac{b-a}{1+b}(C_d + \overline{C_d}). \tag{3.10}$$

□

We have thus proved the theorem in the case where the Newton polyhedron $\mathcal{N}(\phi)$ has two compact edges containing the vertex (d, d) as one of their endpoints.

Assume therefore next that at least one of the two edges containing the vertex (d, d) is unbounded. We shall then argue in a similar way as in the previous case, however, by approximating the unbounded faces by compact line segments which have (d, d) as one of their vertices and which lie on supporting lines to $\mathcal{N}(\phi)$ whose angle with the unbounded face tend to zero.

3.2.2 *Two Unbounded Edges*

Assume next that both edges containing (d, d) are unbounded, i.e., that $\mathcal{N}(\phi) = (d, d) + \mathbb{R}_+^2$. Let us then choose numbers a, b such that $0 < a < 1 < b$, where later we shall let a tend to 0 and b to ∞. We associate to a and b weights

$$\kappa^a := \left(\frac{1}{(1+a)d}, \frac{a}{(1+a)d}\right), \qquad \kappa^b := \left(\frac{1}{(1+b)d}, \frac{b}{(1+b)d}\right).$$

Then the supporting lines mentioned before will be given by $\kappa_1^a t_1 + \kappa_2^a = 1$ and $\kappa_1^b t_1 + \kappa_2^b = 1$, respectively, and the identities (3.4) remain valid.

We can then proceed as in the previous case, reducing to the asymptotic analysis of $J(\lambda)$, which in return is decomposed into $J_0(\lambda)$ and $J_\infty(\lambda)$, given by (3.6) and (3.7), respectively. We further decompose $J_\infty(\lambda) = J_a(\lambda) + J_b(\lambda)$ as in the proof of Lemma 3.6.

In place of this lemma, we here have

Lemma 3.8 *Let $\varepsilon > 0$. Then, if Ω is chosen sufficiently small, the following estimates*

$$|J_a(\lambda)| \le A_d\left(1 + \frac{\log\lambda}{1+1/a}\right)\lambda^{-1/d}, \tag{3.11}$$

$$|J_b(\lambda)| \le A_d\left(1 + \frac{\log\lambda}{1+b}\right)\lambda^{-1/d} \tag{3.12}$$

hold true, with a constant A_d which does not depend on a and b, but only on d.

Proof We shall prove the estimate for $J_a(\lambda)$; the proof of the corresponding estimate for $J_b(\lambda)$ is obtained by the same kind of reasoning, essentially just by interchanging the roles of the variables x_1, x_2 in the argument. Assuming that ε is chosen sufficiently small, we may decompose $J_a(\lambda) = J_a^0(\lambda) + J_a^\infty(\lambda)$, where

$$J_a^0(\lambda) := \int e^{i\lambda\phi(x_1,x_2)}\chi_0(x_1,x_2)\chi^0\left(\frac{\varepsilon x_2}{|x_1|^a}\right)\left(1-\chi^0\left(\frac{x_2}{\varepsilon|x_1|^a}\right)\right)dx$$

and

$$J_a^\infty(\lambda) := \int e^{i\lambda\phi(x_1,x_2)}\chi_0(x_1,x_2)\left(1-\chi^0\left(\frac{\varepsilon x_2}{|x_1|^a}\right)\right)dx.$$

Notice that the integrand of $J_a^0(\lambda)$ is supported where

$$\varepsilon|x_1|^a \lesssim |x_2| \lesssim \frac{1}{\varepsilon}|x_1|^a,$$

and the integrand of $J_a^\infty(\lambda)$ is supported where

$$\frac{1}{\varepsilon}|x_1|^a \lesssim |x_2|.$$

Using a dyadic decomposition of $J_a^0(\lambda)$ by means of the dilations δ_r associated to the weight κ^a, we can estimate $J_a^0(\lambda)$ in the same way as we did estimate $J_a(\lambda)$

in the proof of Lemma 3.6. Notice to this end that the corresponding integrals $J_k(\lambda)$ will be performed here over a domain where

$$\varepsilon^{1/a} \lesssim |x_1| \lesssim 1 \quad \text{and} \quad \varepsilon \lesssim |x_2| \lesssim 1.$$

And, since now we have $\phi_{\kappa^a}(x_1, x_2) = x_1^d x_2^d$, there is no root of multiplicity d or greater of ϕ_{κ^a} on this domain, hence we obtain the estimate

$$|J_a^0(\lambda)| \le C_d \lambda^{-1/d}.$$

As for $J_a^\infty(\lambda)$, observe first that there is another smooth, even bump function $\tilde{\chi}^0$ which is identically 1 near the origin such that $1 - \chi^0(\varepsilon x_2/|x_1|^a) = \tilde{\chi}^0(x_1/(\varepsilon^{1/a}|x_2|^{1/a}))$. Moreover, even though this function will depend on a, we may assume that its derivatives are uniformly bounded for $0 < a < 1$. We accordingly re-write

$$J_a^\infty(\lambda) := \int e^{i\lambda\phi(x_1,x_2)} \chi_0(x_1, x_2) \tilde{\chi}^0\left(\frac{x_1}{\varepsilon^{1/a}|x_2|^{1/a}}\right) dx.$$

Decomposing the integral into the contributions by the four quadrants, we reduce our considerations to estimating the integral

$$I(\lambda) := \int_0^\infty \int_0^\infty e^{i\lambda\phi(x_1,x_2)}\, \chi_0(x_1, x_2)\, \tilde{\chi}^0\left(\frac{x_1}{\varepsilon^{1/a} x_2^{1/a}}\right) dx_1\, dx_2.$$

Observe next that the phase function can be written as

$$\phi(x_1, x_2) = x_1^d x_2^d a(x_1, x_2) + \sum_{\nu=0}^{d-1} (x_1^\nu \varphi_\nu(x_2) + x_2^\nu \psi_\nu(x_1)),$$

where the functions φ_ν, ψ_ν are smooth and flat at the origin and where a is a smooth function such that $a(0, 0) = 1$. This shows that the change of variables

$$x_1 := x_2^{1/a} y_1, \qquad x_2 := y_2$$

will transform the phase function ϕ into a phase function $\tilde{\phi}$ of the form

$$\tilde{\phi}(y_1, y_2) = y_2^{d(1+1/a)} \left(y_1^d\, \tilde{a}(y_1, y_2) + \sum_{\nu=0}^{d-1} y_1^\nu \tilde{\varphi}_\nu(y_2) \right) =: y_2^{d(1+1/a)} \psi(y_1, y_2),$$

where the functions $\tilde{\varphi}_\nu$ are again smooth and flat at the origin and where $\tilde{a}$ is smooth with $\tilde{a}(0, 0) = 1$.

Accordingly, we re-write

$$I(\lambda) = \int_0^\infty \int_0^\infty e^{i\lambda y_2^{d(1+1/a)} \psi(y_1,y_2)}\, \tilde{\chi}_0(y_1, y_2)\, \tilde{\chi}^0\left(\frac{y_1}{\varepsilon^{1/a}}\right)\, dy_1\, y_2^{1/a}\, dy_2.$$

Notice that if M is any fixed positive constant, than the contribution to $I(\lambda)$ by the region where $\lambda y_2^{d(1+1/a)} \le M$ is trivially bounded by $C_M \lambda^{-1/d}$, with a constant C_M

which does not depend on a, so that we may assume that $\lambda y_2^{d(1+1/a)} > M$ in the inner integral, where M is a sufficiently large constant.

In order to estimate the inner integral, observe that the C^M-norm of ψ as a function of y_1 and y_2 may be very large as $a \to 0$, due the type of change of coordinates that we applied. However, for y_2 fixed, the d'th derivative of ψ with respect to y_1 is bounded from below by a fixed constant not depending on y_2 and a. Indeed, by choosing Ω sufficiently small, it is easy to see that we may assume that $\partial_1^d \psi(y_1, y_2) \geq a(0,0)d!/2 = d!/2$.

We may thus apply van der Corput's estimate in order to estimate the inner integral with respect to y_1 by $C(\lambda y_2^{d(1+1/a)})^{-1/d}$, with a constant C which does not depend on a, and then perform the integration in y_2, to find that

$$|I(\lambda)| \leq A_d\left(1 + \frac{\log \lambda}{1 + 1/a}\right)\lambda^{-1/d},$$

as required. □

We are thus left with the main term $J_0(\lambda)$, which can be treated exactly as in the proof of Lemma 3.7, so that the conclusion of this lemma holds true. In particular, the limit relations (3.9) and (3.10) hold true. Letting $a \to 0$ and $b \to \infty$, we finally derive from those limit relations in combination with Lemma 3.8 that indeed

$$\lim_{\lambda \to +\infty} \frac{\lambda^{1/d}}{\log \lambda} J(\lambda) = c,$$

where c is given by $4C_d$, if d is even, and by $2(C_d + \overline{C_d})$, if d is odd. This proves Theorem 1.3 also in this case.

3.2.3 A Compact and an Unbounded Edge

Finally, if one of the edges containing (d, d) is compact and the other one is unbounded, then let us assume without loss of generality that the edge lying above the bi-sectrix is compact, and the one below is unbounded. Then we define $a := \kappa_2^a/\kappa_1^a$ associated to the upper, compact edge as in Sect. 3.2.1, and approximate the lower, horizontal edge by a compact line segment of slope $1/b$ as in Sect. 3.2.2, and consider what will happen to the integrals $J_a(\lambda)$, $J_b(\lambda)$ and $J_0(\lambda)$, defined in the same way as before, when $b \to +\infty$.

Applying the same kind of reasoning as before, one then finds that $J_a(\lambda) = O(\lambda^{-1/d})$ as $\lambda \to +\infty$, that $J_b(\lambda)$ satisfies estimate (3.12) from Lemma 3.8, and that the main contribution is again given by $J_0(\lambda)$, which can be treated as before by Lemma 3.6. We can then conclude as in the previous case by letting $b \to +\infty$.

This completes the proof of Theorem 1.3. □

4 Fourier Restriction in the Case of Adapted Coordinates

Let us turn to proving the restriction estimate (1.10) in Theorem 1.7. We may assume that $x^0 = (0, 0)$, and that the hypersurface S is given as the graph $x_3 = \phi(x_1, x_2)$ of a smooth, finite type function ϕ in adapted coordinates (x_1, x_2), which is defined in an open neighborhood Ω of the origin such that $\phi(0,0) = 0$, $\nabla\phi(0,0) = 0$. Recall that then $\nu(x^0, S) = \nu(\phi)$ and $h(x^0, S) = h = d(\phi)$.

If $\nu(\phi) = 0$, then by A. Greenleaf's work [10] (compare also [19], Chap. VIII, 5.15(b)), the $L^p(\mathbb{R}^3)$-$L^2(S)$ restriction theorem for the Fourier transform in Theorem 1.7 is an immediate consequence of the uniform estimate in Corollary 1.6 for the Fourier transform of the surface carried measure $\rho d\sigma$ of the hypersurface S.

We shall therefore assume in the sequel that $\nu(\phi) = 1$. This implies in particular that $h = h(\phi) \geq 2$. Note that in this case Greenleaf's theorem misses the endpoint $p = p_c = (2h+2)/(2h+1)$, on which we shall concentrate in the sequel. As we shall see, this endpoint can nevertheless be obtained if we invoke tools from Littlewood-Paley theory. Our approach has some resemblance to Stein's proof in [19], Chap. VIII, 5.16, of Strichartz' estimates for the Fourier restriction to quadratic surfaces from [20].

We shall denote by μ the surface carried measure $\rho d\sigma$ from Theorem 1.7. By decomposing $\mathbb{R}^2$ again into its four quadrants, we may assume without loss of generality that μ is of the form

$$\langle \mu, f\rangle = \int_{(\mathbb{R}+)^2} f(x', \phi(x'))\, \eta(x')\, dx', \qquad f \in C_0(\mathbb{R}^3),$$

where $\eta(x') := \rho(x', \phi(x'))\sqrt{1 + |\nabla\phi(x')|^2}$ is smooth and has its support in a sufficiently small neighborhood Ω of the origin.

In the sequel, we shall split the coordinates in $\mathbb{R}^3$ as $x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$. If χ is an integrable function defined on Ω, we put

$$\mu^\chi := (\chi \otimes 1)\mu, \quad \text{i.e.,}\ \langle \mu^\chi, f\rangle = \int_{(\mathbb{R}+)^2} f(x', \phi(x'))\, \eta(x')\chi(x')\, dx'.$$

Observe that then

$$\widehat{\mu^\chi}(-\xi) = J^\chi(\xi), \quad \xi \in \mathbb{R}^3, \tag{4.1}$$

with $J^\chi(\xi)$ defined as in Sect. 2.

We next choose a weight κ with $0 < \kappa_1 \leq \kappa_2$ such that the line L_κ is a supporting line to the Newton polyhedron $\mathcal{N}(\phi)$ and so that

$$\frac{1}{|\kappa|} = d_h(\phi_\kappa) = h(\phi_\kappa) = h.$$

This is possible, since according to Lemma 1.5 the principal face $\pi(\phi)$ of $\mathcal{N}(\phi)$ is either a vertex, or a compact edge such that $m(\phi_{\text{pr}}) = d(\phi)$. In the first case, we have $\phi_\kappa(x_1, x_2) = c x_1^h x_2^h$, and in the second $\phi_\kappa = \phi_{\text{pr}}$, so that in both cases

$$m(\phi_\kappa) = h. \tag{4.2}$$

The corresponding dilations will be denoted by δ_r. Fixing a suitable smooth cut-off function $\chi \geq 0$ on $\mathbb{R}^2$ supported in an annulus D such that the functions $\chi_k := \chi \circ \delta_{2^k}$

form a partition of unity, we then decompose the measure μ as

$$\mu = \sum_{k\geq k_0} \mu_k, \tag{4.3}$$

where $\mu_k := \mu^{\chi_k}$. Let us extend the dilations δ_r to $\mathbb{R}^3$ by putting

$$\delta^e_r(x', x_3) := (r^{\kappa_1}x_1, r^{\kappa_2}x_2, rx_3).$$

We re-scale the measure μ_k by defining $\mu_{0,(k)} := 2^{-k}\mu_k \circ \delta^e{}_{2^{-k}}$, i.e.,

$$\langle \mu_{0,(k)}, f\rangle = 2^{|\kappa|k}\langle \mu_k, f\circ \delta^e{}_{2^k}\rangle = \int_{(\mathbb{R}+)^2} f(x', \phi^k(x'))\,\eta(\delta_{2^{-k}}x')\chi(x')\,dx', \tag{4.4}$$

with $\phi^k(x) := 2^k\phi(\delta_{2^{-k}}x) = \phi_\kappa(x) +$ error terms. This shows that the measures $\mu_{0,(k)}$ are supported on the smooth hypersurfaces S^k defined as the graph of ϕ^k, their total variations are uniformly bounded, i.e., $\sup_k \|\mu_{0,(k)}\|_1 < \infty$, and that they are approaching the surface carried measure $\mu_{0,(\infty)}$ on S defined by

$$\langle \mu_{0,(\infty)}, f\rangle := \int_{(\mathbb{R}+)^2} f(x', \phi(x'))\,\eta(0)\chi(x')\,dx'$$

as $k \to \infty$.

We claim that there is a constant C such that

$$|\widehat{\mu_{0,(k)}}(\xi)| \leq C(1 + |\xi|)^{-1/h} \quad \text{for every } \xi \in \mathbb{R}^3, k \geq k_0. \tag{4.5}$$

Indeed, we may again assume that $|\xi_1| + |\xi_2| \leq \delta|\xi_3|$, where $0 < \delta \ll 1$ is a sufficiently small constant, since for $|\xi_1| + |\xi_2| > \delta|\xi_3|$ the estimate (4.5) follows by an integration by parts, if Ω is chosen small enough, i.e., k_0 sufficiently large.

We may thus in particular assume that $|\xi| \sim |\xi_3|$. Note that (4.1) and (4.4) show that

$$\widehat{\mu_{0,(k)}}(-\xi) = 2^{|\kappa|k} J^{\chi_k}(\delta^e{}_{2^k}\xi).$$

Therefore, in view of (4.2) the estimate (2.3) for $J_k(\xi) = J^{\chi_k}(\xi)$ in Sect. 2.1 implies in our case that

$$|\widehat{\mu_{0,(k)}}(\xi)| \leq C(1 + 2^{-k}|\delta^e{}_{2^k}\xi|)^{-1/h},$$

which yields (4.5) if $|\xi| \sim |\xi_3|$.

According to Theorem 1 in [10], the estimates in (4.5) imply the restriction estimates

$$\left(\int |\hat{f}(x)|^2\,d\mu_{0,(k)}(x)\right)^{1/2} \leq C\|f\|_p, \quad f \in \mathcal{S}(\mathbb{R}^3), \tag{4.6}$$

with $p = (2h+2)/(2h+1)$, and the proof in [10] reveals that the constant C can be chosen independently of k.

Let us re-scale these estimates, by putting

$$f_{(r)}(x) := r^{|\kappa|/2} f({\delta^e}_r x), \quad r > 0,$$

for any function f on $\mathbb{R}^3$. Then $\widehat{f_{(r)}} = r^{-|\kappa|/2-1} \widehat{f \circ {\delta^e}_{r^{-1}}}$, and (4.6) implies

$$\int |\hat{f}(x)|^2 \, d\mu_k(x) = \int |\widehat{f_{(2^{-k})}}(x)|^2 \, d\mu_{0,(k)}(x) \leq C^2 2^{(|\kappa|/2+1)k} \| f \circ {\delta^e}_{2^k} \|_p^2,$$

hence

$$\int |\hat{f}(x)|^2 \, d\mu_k(x) \leq C^2 \| f \|_p^2, \tag{4.7}$$

with a constant C which does not depend in k.

Fix a cut-off function $\tilde{\chi} \in C_0^\infty(\mathbb{R}^2)$ supported in an annulus centered at the origin such that $\tilde{\chi} = 1$ on the support of χ, and define dyadic decomposition operators Δ'_k by

$$\widehat{\Delta'_k f}(x) := \tilde{\chi}(\delta_{2^k} x') \, \hat{f}(x', x_3).$$

Then $\int |\hat{f}(x)|^2 d\mu_k(x) = \int |\widehat{\Delta'_k f}(x)|^2 d\mu_k(x)$, so that (4.7) yields in fact that

$$\int |\hat{f}(x)|^2 d\mu_k(x) \leq C^2 \, \| \widehat{\Delta'_k f} \|_p^2,$$

for any $k \geq k_0$. In combination with Minkowski's inequality, this implies

$$\begin{aligned}\left(\int |\hat{f}(x)|^2 d\mu(x) \right)^{1/2} &= \left(\sum_{k \geq k_0} \int |\hat{f}(x)|^2 d\mu_k(x) \right)^{1/2} \leq \left(\sum_{k \geq k_0} \| \Delta'_k f \|_p^2 \right)^{1/2} \\ &= C \left(\left(\sum_{k \geq k_0} \left(\int |\Delta'_k f(x)|^p dx \right)^{2/p} \right)^{p/2} \right)^{1/p} \\ &\leq C \left\| \left(\sum_{k \geq k_0} |\Delta'_k f(x)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^3)},\end{aligned}$$

since $p < 2$.

Thus, by Littlewood-Paley Theory [19], we obtain estimate (1.10). This completes the proof of Theorem 1.7.

Corrigenda We would like to take the opportunity to make some statements about uniqueness of certain roots of a quasi-homogeneous polynomial in [12] (e.g. in Corollary 2.3) and [11] (e.g. in Proposition 2.3) more precise, and correct statement (c) in the latter proposition: (1) When $P(x_1, x_2) = c x_1^{\nu_1} x_2^{\nu_2} \prod_{l=1}^M (x_2^q - \lambda_l x_1^p)^{n_l}$ is a quasi-homogenous polynomial, where $(p, q) = 1$, then its roots on the unit circle can be partitioned into classes consisting of pairs of roots $S^1_R := \{(x_1, x_2) \in S^1 : R(x_1, x_2) = 0\}$, where R is any of the factors x_1, x_2 or $x_2^q - \lambda_l x_1^p$ (with λ_l real) appearing in its factorization. We shall not distinguish between the roots within a given class. I.e., when we say that there is a *unique* root of maximal multiplicity, then what we mean precisely is that there is a unique class of roots of maximal multiplicity. The uniqueness statements in [12] and [11] have to be interpreted in this way. (2) The proof of Proposition 2.3(c) in [11] contains an error on p. 170: the estimate $qm_1(x^0) \le n_1$ has to be replaced by $m_1(x^0) \le n_1$. Tracing the corresponding arguments of the proof one finds that the correct statement should read as follows:

(c) *Assume that $\kappa_2/\kappa_1 \notin \mathbb{N}$, and that $\partial_2 P$ does not vanish identically. If $x^0 \in S^1$, then denote by $m_1(x^0)$ the order of vanishing of $\partial_2 P$ along S^1 in the point x^0. Then $m_1(x_0) < d_h(P) - 1$ for every root x^0 of $\partial_2 P$ with $x_1^0 \neq 0 \neq x_2^0$, unless $\kappa_2/\kappa_1 = 3/2$ and $\partial_2 P$ is of the form $\partial_2 P(x_1, x_2) = c(x_2^2 - \lambda x_1^3)^k$, $k = 1$ or $k = 2$ and $\lambda \neq 0$ real.*

In particular, if $\kappa_2/\kappa_1 \neq 3/2$, then for every point $x \in S^1$ which does not lie on a coordinate axis, there exists some j with $1 \le j < d_h(P)$ such that $\partial_2^j P(x) \neq 0$. Note that this error has no further consequences, since Proposition 2.3 is applied only in the proof of Corollary 3.2, which is not effected by it, since we assume that $\kappa_2/\kappa_1 > 2$ in that corollary.

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Appendix: Proof of Lemma 1.5

To prove Lemma 1.5, we shall apply the techniques and results from [12], in particular the reasoning in the proof of Lemma 3.2 of that article.

In order to show that (a) implies (b), we may assume without loss of generality that the coordinates x are adapted to ϕ, and that the principal face $\pi(\phi)$ is a vertex, say $\pi(\phi) = \{(\ell, \ell)\}$, i.e.,

$$\phi_{\text{pr}}(x_1, x_2) = c x_1^\ell x_2^\ell.$$

Assume that y is another adapted coordinate system, say $x = F(y)$, where F is a local, smooth diffeomorphism at the origin, and write $\tilde{\phi}(y) := \phi(F(y))$.

Possibly after permuting the coordinates x_1 and x_2, we may choose a weight $\kappa = (\kappa_1, \kappa_2)$ with $0 < \kappa_1 \le \kappa_2$ in the following way:

Case I. If (ℓ, ℓ) is the right endpoint of a compact edge γ of the Newton diagram of ϕ, then we choose the unique weight κ so that γ lies on the line $L_\kappa := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\}$ (which is then a supporting line to $\mathcal{N}(\phi)$).

Case II. Otherwise, i.e., if $\mathcal{N}(\phi)$ is contained in the half-plane $t_1 \ge \ell$, then we choose κ so that the vertex (ℓ, ℓ) is the unique point of the Newton polyhedron $\mathcal{N}(\phi)$ contained in the supporting line L_κ.

Permuting the coordinates y_1 and y_2, if necessary, we may assume without loss of generality that $(x_1, x_2) = (F_1(y_1, y_2), F_2(y_1, y_2))$ satisfies $\frac{\partial F_j(0,0)}{\partial y_j} \neq 0$ for $j = 1, 2$.

Therefore, we can write the functions F_1, F_2 in the form

$$F_1(y_1, y_2) = y_1\psi_1(y_1, y_2) + \eta_1(y_2), \qquad F_2(y_1, y_2) = y_2\psi_2(y_1, y_2) + \eta_2(y_1), \tag{5.1}$$

where $\psi_1, \psi_2, \eta_1, \eta_2$ are smooth functions satisfying

$$\psi_1(0,0) \neq 0, \qquad \psi_2(0,0) \neq 0, \qquad \eta_1(0) = \eta_2(0) = 0.$$

We may further assume that $\psi_1(0,0) = \psi_2(0,0) = 1$. Denote by k_j the order of vanishing of η_j at 0, $j = 1, 2$. Then clearly $k_j \geq 1$.

Notice that in Case II, we may and shall assume that $\kappa_2/\kappa_1 > k_2$.

We first recall some observation from [12]. If F_κ denotes the κ-principal part of F, then

$$\tilde{\phi}(y_1, y_2) = \phi_\kappa \circ F_\kappa(y_1, y_2) + \text{terms of higher } \kappa\text{-degree},$$

so that

$$\tilde{\phi}_\kappa = \phi_\kappa \circ F_\kappa.$$

Moreover, $\phi_\kappa \circ F_\kappa$ is a κ-homogeneous polynomial, so that its Newton diagram $\mathcal{N}_d(\tilde{\phi}_\kappa)$ is again a compact interval (possibly a single point). In case that this interval intersects the bi-sectrix too, then it contains the principal face of $\mathcal{N}(\tilde{\phi})$.

(a) *The case where $k_2 > \frac{\kappa_2}{\kappa_1}$, and either $\kappa_2 > \kappa_1$, or $\kappa_1 = \kappa_2$ and $k_1 > 1$.*

In this case, one finds that $F_\kappa(y_1, y_2) = (y_1, y_2)$ (see [12]), hence $\tilde{\phi}_\kappa = \phi_\kappa$, so that $\pi(\tilde{\phi}) = \pi(\phi)$ is a vertex.

(b) *The case where $k_2 > \frac{\kappa_2}{\kappa_1}$, $\kappa_1 = \kappa_2$ and $k_1 = 1$.*

Then $k_2 > 1, k_1 = 1$, so that $F_\kappa(y_1, y_2) = (y_1 + ay_2, y_2)$ for some constant $a \in \mathbb{R}$, hence $\tilde{\phi}_\kappa(y_1, y_2) = c(y_1 + ay_2)^\ell y_2^\ell$. From a view at the Newton diagram of this polynomial, we see that $\pi(\tilde{\phi}) = \pi(\phi)$ is a vertex.

(c) *The case where $k_2 < \frac{\kappa_2}{\kappa_1}$.*

As in the proof of Lemma 3.2 in [12], we then introduce a second weight $\mu := (1, k_2)$, and choose $d > 0$ so that the line $L_\mu := \{(t_1, t_2) \in \mathbb{R}^2 : t_1 + k_2 t_2 = d\}$ is the supporting line to the Newton polyhedron $\mathcal{N}(\phi)$. It has been shown in the proof of Lemma 3.2 in [12] (Case (c)) that the principal face of $\mathcal{N}(\tilde{\phi})$ then lies on the line L_μ. Noticing that the line L_μ is steeper than the line L_κ, we see that Case I cannot arise here, since otherwise we would have $d_y < d_x$, contradicting our assumption that also the coordinates y are adapted. And, in Case II, we see that (ℓ, ℓ) will be the only point of $\mathcal{N}(\phi)$ contained in L_μ, so that $\phi_\mu = \phi_{\text{pr}}$. This shows that $\tilde{\phi}_\mu = \phi_\mu \circ F_\mu = \phi_{\text{pr}} \circ F_\mu$.

Moreover, the μ-principal part of F is given by $F_\mu(y_1, y_2) = (y_1, y_2 + a_2 y_1^{k_2})$, if $k_2 > 1$, and by $F_\mu(y_1, y_2) = (y_1 + a_1 y_2, y_2 + a_2 y_1)$, if $k_2 = 1$, with $a_1 \neq 0$ if and only if $k_1 = 1$.

In the first case, we obtain $\tilde{\phi}_\mu = c y_1^\ell (y_2 + a_2 y_1^{k_2})^\ell$, so that $\pi(\tilde{\phi}) = \pi(\phi)$ is again a vertex. A similar reasoning applies in the second case, if $a_1 = 0$ or $a_2 = 0$. And, if $a_1 \neq 0 \neq a_2$, we find that $\tilde{\phi}_\mu = c(y_1 + a_1 y_2)^\ell (y_2 + a_2 y_1)^\ell$. This means that the principal face of $\mathcal{N}(\tilde{\phi})$ is a compact edge passing through the point (ℓ, ℓ), and clearly $m(\tilde{\phi}_{\rm pr}) = \ell$, so that $m(\tilde{\phi}_{\rm pr}) = \ell = d(\tilde{\phi})$.

(e) *The case where* $k_2 = \frac{\kappa_2}{\kappa_1}$.

Observe that $k_1\kappa_2 > \kappa_1$, unless $\kappa_1 = \kappa_2$ and $k_1 = 1$, since $\kappa_1/\kappa_2 \le 1$ (the latter will only arise in Case I).

Assuming first that $k_1\kappa_2 > \kappa_1$, we then see that $\phi_\kappa(y_1, y_2) = (y_1, y_2 + a_2 y_1^{k_2})$, hence $\tilde{\phi}_\kappa(y_1, y_2) = c y_1^\ell (y_2 + a_2 y_1^{k_2})^\ell$. This shows that again $\pi(\tilde{\phi}) = \pi(\phi)$ is a vertex.

Finally, assume that $\kappa_1 = \kappa_2$ and $k_1 = 1$, so that also $k_2 = 1$. Then ϕ_κ is of the form $\phi_\kappa(y_1, y_2) = (y_1 + a_1 y_2, y_2 + a_2 y_1)$, hence $\tilde{\phi}_\kappa(y_1, y_2) = c(y_1 + a_1 y_2)^\ell (y_2 + a_2 y_1)^\ell$. As before, this means that the principal face of $\mathcal{N}(\tilde{\phi})$ is a compact edge passing through the point (ℓ, ℓ), and we have $m(\tilde{\phi}_{\rm pr}) = \ell = d(\tilde{\phi})$.

There remains to show that (b) implies (a). To this end, we may assume without loss of generality that $y = x$, i.e., that x is an adapted coordinate system, and that $\pi(\phi)$ is a compact edge and $m(\phi_{\rm pr}) = d(\phi)$. We shall denote the latter by d. Let us denote by (A_0, B_0) and (A_1, B_1) the two vertices of $\pi(\phi)$, and assume that $A_0 < A_1$. According to [12], displays (3.2) and (3.3), we can then write the principal part of ϕ as

$$\phi_{\rm pr}(x_1, x_2) = c x_1^\alpha x_2^\beta \prod_l (x_2 - c_l x_1^m)^{n_l},$$

where the c_l's are the non-trivial distinct complex roots of the polynomial $\phi_{\rm pr}(1, x_2)$ and the n_l's are their multiplicities. Moreover, there exists an l_0 such that $d = n_{l_0}$ and such that c_{l_0} is real.

We then apply the change of coordinates $y_1 := x_1$, $y_2 := x_2 - c_{l_0} x_1^m$, which preserves the mixed homogeneity of $\phi_{\rm pr}$ and transforms this polynomial into a polynomial of the same form, $c x_1^{\tilde{\alpha}} x_2^{\tilde{\beta}} \prod_l (x_2 - \tilde{c}_l x_1^m)^{n_l}$, but now with $\tilde{\beta} = d$. The vertices of the corresponding Newton diagram are given by (A_0, B_0) and $(\tilde{A}_1, \tilde{B}_1)$ and lie on the same line as (A_0, B_0) and (A_1, B_1) (see [12]), where obviously $\tilde{B}_1 = \tilde{\beta} = d$. This shows that $(\tilde{A}_1, \tilde{B}_1) = (d, d)$, and consequently the principal face of the Newton polyhedron of $\tilde{\phi}$ is given by the vertex (d, d).

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