J Fourier Anal Appl (2012) 18:27–44
DOI 10.1007/s00041-011-9183-4

Fourier Series with the Continuous Primitive Integral

Erik Talvila

Received: 20 May 2010 / Revised: 23 May 2011 / Published online: 1 July 2011

Abstract Fourier series are considered on the one-dimensional torus for the space of periodic distributions that are the distributional derivative of a continuous function. This space of distributions is denoted $\mathcal{A}_c(\mathbb{T})$ and is a Banach space under the Alexiewicz norm, $\|f\|_{\mathbb{T}} = \sup_{|I|\leq 2\pi} |\int_I f|$, the supremum being taken over intervals of length not exceeding 2π. It contains the periodic functions integrable in the sense of Lebesgue and Henstock–Kurzweil. Many of the properties of L^1 Fourier series continue to hold for this larger space, with the L^1 norm replaced by the Alexiewicz norm. The Riemann–Lebesgue lemma takes the form $\hat{f}(n) = o(n)$ as $|n| \to \infty$. The convolution is defined for $f \in \mathcal{A}_c(\mathbb{T})$ and g a periodic function of bounded variation. The convolution commutes with translations and is commutative and associative. There is the estimate $\|f * g\|_\infty \leq \|f\|_{\mathbb{T}}\|g\|_{\mathcal{BV}}$. For $g \in L^1(\mathbb{T})$, $\|f * g\|_{\mathbb{T}} \leq \|f\|_{\mathbb{T}}\|g\|_1$. As well, $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$. There are versions of the Salem–Zygmund–Rudin–Cohen factorization theorem, Fejér's lemma and the Parseval equality. The trigonometric polynomials are dense in $\mathcal{A}_c(\mathbb{T})$. The convolution of f with a sequence of summability kernels converges to f in the Alexiewicz norm. Let D_n be the Dirichlet kernel and let $f \in L^1(\mathbb{T})$. Then $\|D_n * f - f\|_{\mathbb{T}} \to 0$ as $n \to \infty$. Fourier coefficients of functions of bounded variation are characterized. The Appendix contains a type of Fubini theorem.

Keywords Fourier series · Convolution · Distributional integral · Continuous primitive integral · Henstock–Kurzweil integral · Schwartz distribution · Generalized function

Communicated by Hans G. Feichtinger.

Supported by the Natural Sciences and Engineering Research Council of Canada.

E. Talvila (✉)
Department of Mathematics and Statistics, University of the Fraser Valley, Abbotsford, BC, V2S 7M8, Canada
e-mail: Erik.Talvila@ufv.ca

Birkhäuser

Mathematics Subject Classification (2000) 26A39 · 42A16 · 46F10

1 Introduction and Notation

In this paper we consider Fourier series on the one dimensional torus. Progress in Fourier analysis has gone hand in hand with progress in theories of integration. This is perhaps best exemplified by the work of Riemann and Lebesgue using the integrals named after them. We describe below the *continuous primitive integral*. This is an integral that includes the Lebesgue, Henstock–Kurzweil and wide Denjoy integrals. It has a simple definition in terms of distributions. The space of distributions integrable in this sense is a Banach space under the Alexiewicz norm. Many properties of Fourier series that hold for L^1 functions continue to hold in this larger space with the L^1 norm replaced by the Alexiewicz norm.

We use the following notation for distributions. The space of *test functions* is $\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R}) = \{\phi : \mathbb{R} \to \mathbb{R} \mid \phi \in C^\infty(\mathbb{R})$ and $\mathrm{supp}(\phi)$ is compact$\}$. The *support* of function ϕ is the closure of the set on which ϕ does not vanish and is denoted $\mathrm{supp}(\phi)$. Under usual pointwise operations $\mathcal{D}(\mathbb{R})$ is a linear space over field $\mathbb{R}$. In $\mathcal{D}(\mathbb{R})$ we have a notion of convergence. If $\{\phi_n\} \subset \mathcal{D}(\mathbb{R})$ then $\phi_n \to 0$ as $n \to \infty$ if there is a compact set $K \subset \mathbb{R}$ such that for each n, $\mathrm{supp}(\phi_n) \subset K$, and for each $m \geq 0$ we have $\phi_n^{(m)} \to 0$ uniformly on K as $n \to \infty$. The *distributions* are denoted $\mathcal{D}'(\mathbb{R})$ and are the continuous linear functionals on $\mathcal{D}(\mathbb{R})$. For $T \in \mathcal{D}'(\mathbb{R})$ and $\phi \in \mathcal{D}(\mathbb{R})$ we write $\langle T, \phi\rangle \in \mathbb{R}$. For $\phi, \psi \in \mathcal{D}(\mathbb{R})$ and $a, b \in \mathbb{R}$ we have $\langle T, a\phi + b\psi\rangle = a\langle T, \phi\rangle + b\langle T, \psi\rangle$. And, if $\phi_n \to 0$ in $\mathcal{D}(\mathbb{R})$ then $\langle T, \phi_n\rangle \to 0$ in $\mathbb{R}$. Linear operations are defined in $\mathcal{D}'(\mathbb{R})$ by $\langle aS + bT, \phi\rangle = a\langle S, \phi\rangle + b\langle T, \phi\rangle$ for $S, T \in \mathcal{D}'(\mathbb{R})$; $a, b \in \mathbb{R}$ and $\phi \in \mathcal{D}(\mathbb{R})$. If $f \in L^1_{loc}$ then $\langle T_f, \phi\rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$ defines a distribution $T_f \in \mathcal{D}'(\mathbb{R})$. The integral exists as a Lebesgue integral. All distributions have derivatives of all orders that are themselves distributions. For $T \in \mathcal{D}'(\mathbb{R})$ and $\phi \in \mathcal{D}(\mathbb{R})$ the distributional derivative of T is T' where $\langle T', \phi\rangle = -\langle T, \phi'\rangle$. If $p : \mathbb{R} \to \mathbb{R}$ is a function that is differentiable in the pointwise sense at $x \in \mathbb{R}$ then we write its derivative as $p'(x)$. If p is a C^∞ bijection such that $p'(x) \neq 0$ for any $x \in \mathbb{R}$ then the composition with distribution T is defined by $\langle T \circ p, \phi\rangle = \langle T, \frac{\phi\circ p^{-1}}{p'\circ p^{-1}}\rangle$ for all $\phi \in \mathcal{D}(\mathbb{R})$. Translations are a special case. For $x \in \mathbb{R}$ define the *translation* τ_x on distribution $T \in \mathcal{D}'(\mathbb{R})$ by $\langle \tau_x T, \phi\rangle = \langle T, \tau_{-x}\phi\rangle$ for test function $\phi \in \mathcal{D}(\mathbb{R})$ where $\tau_x\phi(y) = \phi(y - x)$. A distribution $T \in \mathcal{D}'(\mathbb{R})$ is *periodic* if $\langle \tau_p T, \phi\rangle = \langle T, \phi\rangle$ for some $p > 0$ and all $\phi \in \mathcal{D}(\mathbb{R})$. The least such positive p is the *period*. In this paper, periodic will always mean periodic with period 2π. Periodic distributions are defined in an alternative manner in [11] and [19]. All of the results on distributions we use can be found in these works and [6].

We define the torus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\} = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$. The real interval $[-\pi, \pi)$ will be used as a model for $\mathbb{T}$.

The continuous primitive integral was discussed on the real line in [15]. As the name suggests, this integral is characterized by having a primitive that is a continuous function; the integrable distributions are those that are the distributional derivative of a continuous function. Take the space of primitives as $\mathcal{B}_c(\mathbb{T}) = \{F : \mathbb{R} \to \mathbb{R} | F \in C^0(\mathbb{R}), F(-\pi) = 0, F(x) = F(y) + nF(\pi)$ if $y \in [-\pi, \pi), x = y + 2n\pi$ for $n \in \mathbb{Z}\}$.

Note that $F \in \mathcal{B}_c(\mathbb{T})$ is periodic on $\mathbb{R}$ if and only if $F(\pi) = 0$. If $x \in \mathbb{R}$ and $n \in \mathbb{Z}$ then $F(x + 2n\pi) = F(x) + nF(\pi)$ and $F(x) = (x - x \bmod 2\pi)F(\pi)/(2\pi) + F(x \bmod 2\pi)$. It is easy to see that $\mathcal{B}_c(\mathbb{T})$ is a Banach space under the uniform norm $\|F\|_{\mathbb{T},\infty} = \sup_{|\alpha-\beta|\leq 2\pi} |F(\alpha) - F(\beta)|$. The integrable distributions on the torus are then given by $\mathcal{A}_c(\mathbb{T}) = \{f \in \mathcal{D}'(\mathbb{R}) \mid f = F' \text{ for some } F \in \mathcal{B}_c(\mathbb{T})\}$. For $a, b \in \mathbb{R}$ the integral of $f \in \mathcal{A}_c(\mathbb{T})$ is $\int_a^b f = F(b) - F(a)$ where $F \in \mathcal{B}_c(\mathbb{T})$ and $F' = f$. Note that for all $a, b \in \mathbb{R}$ and all $m, n \in \mathbb{Z}$ we have $\int_{a+2m\pi}^{b+2n\pi} f = \int_a^b f + (n - m)\int_{-\pi}^{\pi} f$. If f is complex-valued, the real and imaginary parts are integrated separately. The distributional differential equation $T' = 0$ has only constant solutions and we have made our primitives in $\mathcal{B}_c(\mathbb{T})$ vanish at $-\pi$ so the primitive of a distribution in $\mathcal{A}_c(\mathbb{T})$ is unique.

If $f : \mathbb{R} \to \mathbb{R}$ is a periodic function that is locally integrable in the Lebesgue, Henstock–Kurzweil or wide Denjoy sense then $T_f \in \mathcal{A}_c(\mathbb{T})$. Thus, if $f(t) = t^{-2}\cos(t^{-2})$ for $t \in (0, \pi)$ and $f(t) = 0$ for $t \in [-\pi, 0]$ with f extended periodically, then $T_f \in \mathcal{A}_c(\mathbb{T})$ but f is not Lebesgue integrable. In this case, f has an improper Riemann integral and 0 is the only point of nonabsolute summability. There are examples of functions integrable in the Henstock–Kurzweil sense but not in the Lebesgue sense for which the set of points of nonabsolute summability has positive measure. See [10]. Such functions correspond to distributions integrable in the continuous primitive sense. We will usually drop the distinction between f and T_f. As well, if $F \in \mathcal{B}_c(\mathbb{T})$ is a function of Weierstrass type that is continuous but has a pointwise derivative nowhere then the distributional derivative of F exists and $F' \in \mathcal{A}_c(\mathbb{T})$. If F is a continuous singular function, so that $F'(x) = 0$ a.e., then $F' \in \mathcal{A}_c(\mathbb{T})$ and the continuous primitive integral is $\int_a^b F' = F(b) - F(a)$. In this case, $F' \in L^1(\mathbb{T})$ but the Lebesgue integral gives $\int_a^b F'(x)\,dx = 0$.

If $f \in \mathcal{A}_c(\mathbb{T})$ and $F \in \mathcal{B}_c(\mathbb{T})$ is its primitive then the action of f on test function $\phi \in \mathcal{D}(\mathbb{R})$ is given by $\langle f, \phi\rangle = \langle F', \phi\rangle = -\langle F, \phi'\rangle = -\int_{-\infty}^{\infty} F(x)\phi'(x)\,dx$. This last integral exists as a Riemann integral. And, for $f \in \mathcal{A}_c(\mathbb{T})$ with primitive $F \in \mathcal{B}_c(\mathbb{T})$,

$$\begin{aligned}\langle \tau_{2\pi} f, \phi\rangle = \langle f, \tau_{-2\pi}\phi\rangle &= -\int_{-\infty}^{\infty} F(x)\phi'(x + 2\pi)\,dx \\ &= -\int_{-\infty}^{\infty} F(x - 2\pi)\phi'(x)\,dx \\ &= -\int_{-\infty}^{\infty} F(x)\phi'(x)\,dx + F(\pi)\int_{-\infty}^{\infty} \phi'(x)\,dx \\ &= -\langle F, \phi'\rangle = \langle f, \phi\rangle,\end{aligned}$$

so f is periodic. If $F \in C^0(\mathbb{T})$ then $F' \in \mathcal{A}_c(\mathbb{T})$. Note that distributions in $\mathcal{A}_c(\mathbb{T})$ are tempered and of order one. See [6] for the definitions.

Distributions in $\mathcal{A}_c(\mathbb{T})$ can be composed with continuous functions and this leads to a very powerful change of variables formula. See [15, Theorem 11].

The *Alexiewicz norm* of $f \in \mathcal{A}_c(\mathbb{T})$ is $\|f\|_{\mathbb{T}} = \sup_{|I|\leq 2\pi} |\int_I f|$, the supremum being taken over intervals of length not exceeding 2π. We have $\|f\|_{\mathbb{T}} = \|F\|_{\mathbb{T},\infty} =$

$\max_{|\beta-\alpha|\leq 2\pi}|F(\beta)-F(\alpha)|$ where $F\in\mathcal{B}_c(\mathbb{T})$ is the primitive of f. The integral provides a linear isometry and isomorphism between $\mathcal{A}_c(\mathbb{T})$ and $\mathcal{B}_c(\mathbb{T})$. Define $\Phi:\mathcal{A}_c(\mathbb{T})\to\mathcal{B}_c(\mathbb{T})$ by $\Phi[f](x)=\int_{-\pi}^x f$. Then Φ is a linear bijection and $\|f\|_{\mathbb{T}}=\|\Phi[f]\|_{\mathbb{T},\infty}$. Hence, $\mathcal{A}_c(\mathbb{T})$ is a Banach space. The spaces of periodic Lebesgue, Henstock–Kurzweil and wide Denjoy integrable functions are not complete under the Alexiewicz norm. The space $\mathcal{A}_c(\mathbb{T})$ furnishes their completion. An equivalent norm is $\|f\|'_{\mathbb{T}}=\sup_{-\pi\leq x\leq\pi}|\int_{-\pi}^x f|$.

The multipliers and dual space of $\mathcal{A}_c(\mathbb{T})$ are given by the functions of bounded variation. If $g:\mathbb{R}\to\mathbb{R}$ is periodic then its variation over $\mathbb{T}$ is given by $Vg=\sup\sum|g(s_i)-g(t_i)|$ where the supremum is taken over all disjoint intervals $\{(s_i,t_i)\}\subset(-\pi,\pi)$. We write $\mathcal{BV}(\mathbb{T})$ for the periodic functions with finite variation. This is a Banach space under the norm $\|g\|_{\mathcal{BV}}=\|g\|_\infty+Vg$. If g is complex-valued with real and imaginary parts g_r and g_i, then $Vg=\sqrt{(Vg_r)^2+(Vg_i)^2}$. If $f\in\mathcal{A}_c(\mathbb{T})$ with primitive $F\in\mathcal{B}_c(\mathbb{T})$ and $g\in\mathcal{BV}(\mathbb{T})$ then the integral of fg is defined using a Riemann–Stieltjes integral,

$$H(x)=\int_{-\pi}^x fg=F(x)g(x)-\int_{-\pi}^x F(t)\,dg(t),\quad x\in[-\pi,\pi). \tag{1}$$

Extension of H outside this interval using $H(x)=(x-x\bmod 2\pi)H(\pi)/(2\pi)+H(x\bmod 2\pi)$ yields an element of $\mathcal{B}_c(\mathbb{T})$ whose derivative is then interpreted as $fg\in\mathcal{A}_c(\mathbb{T})$. Note that $\mathcal{BV}(\mathbb{T})\subset L^1(\mathbb{T})\subset\mathcal{A}_c(\mathbb{T})$.

Growth estimates and other basic properties of Fourier coefficients are proved in Theorem 2. Let $\hat{f}(n)=\int_{-\pi}^{\pi}f(t)e^{-int}\,dt$ denote the Fourier coefficient of $f\in\mathcal{A}_c(\mathbb{T})$. The Riemann–Lebesgue lemma takes the form $\hat{f}(n)=o(n)$ as $|n|\to\infty$. In Theorem 4, the convolution $f*g(x)=\int_{-\pi}^{\pi}f(x-t)g(t)\,dt$ is defined for $f\in\mathcal{A}_c(\mathbb{T})$ and $g\in\mathcal{BV}(\mathbb{T})$. The convolution is then continuous, commutes with translations and is commutative and associative. There is the estimate $\|f*g\|_\infty\leq\|f\|_{\mathbb{T}}\|g\|_{\mathcal{BV}}$. As well, $\widehat{f*g}(n)=\hat{f}(n)\hat{g}(n)$. The integral $\int_{-\pi}^{\pi}f(x-t)g(t)\,dt$ need not exist for $f\in\mathcal{A}_c(\mathbb{T})$ and $g\in L^1(\mathbb{T})$. But using the density of $L^1(\mathbb{T})$ in $\mathcal{A}_c(\mathbb{T})$ and the density of $\mathcal{BV}(\mathbb{T})$ in $L^1(\mathbb{T})$ we can define the convolution for $f\in\mathcal{A}_c(\mathbb{T})$ and $g\in L^1(\mathbb{T})$ as the limit of a sequence of convolutions f_k*g for $f_k\in L^1(\mathbb{T})$ or as the limit of $f*g_k$ for $g_k\in\mathcal{BV}(\mathbb{T})$ (Theorem 7). The usual properties of convolution continue to hold. Now we have $\|f*g\|_{\mathbb{T}}\leq\|f\|_{\mathbb{T}}\|g\|_1$. Theorem 8 gives a version of the Salem–Zygmund–Rudin–Cohen factorization theorem, $\mathcal{A}_c(\mathbb{T})=L^1(\mathbb{T})*\mathcal{A}_c(\mathbb{T})$. Using the Fejér kernel it is shown that the trigonometric polynomials are dense in $\mathcal{A}_c(\mathbb{T})$. There is the uniqueness result that if $\hat{f}=\hat{g}$ then $f=g$ as distributions in $\mathcal{A}_c(\mathbb{T})$. As well, the convolution of f with a sequence of Fejér kernels converges to f in the Alexiewicz norm (Theorem 13). Let D_n be the Dirichlet kernel and let $f\in L^1(\mathbb{T})$. Then $\|D_n*f-f\|_{\mathbb{T}}\to 0$ as $n\to\infty$ (Theorem 15). Example 16 shows there is $f\in\mathcal{A}_c(\mathbb{T})$ such that $\|D_n*f-f\|_{\mathbb{T}}\not\to 0$. Proposition 17 is a version of Fejér's lemma and Theorem 18 is a type of Parseval equality. Theorem 19 gives a characterization of Fourier coefficients of functions in $\mathcal{BV}(\mathbb{T})$. The Appendix contains a type of Fubini theorem.

We will use the following version of the Hölder inequality from the Appendix of [14].

Proposition 1 (Hölder inequality) *Let $f \in \mathcal{A}_c(\mathbb{T})$. If $g \in \mathcal{BV}(\mathbb{T})$ then* $|\int_{-\pi}^{\pi} fg| \le |\int_{-\pi}^{\pi} f| \inf|g| + \|f\|_{\mathbb{T}} Vg \le \|f\|_{\mathbb{T}}\|g\|_{\mathcal{BV}}$.

Distributions in $\mathcal{A}_c(\mathbb{T})$ are continuous in the Alexiewicz norm. This means that if $f \in \mathcal{A}_c(\mathbb{T})$ then $\|f - \tau_s f\|_{\mathbb{T}} \to 0$ as $s \to 0$. See [15, Theorem 28] for a proof.

A function on the real line is called *regulated* if it has a left limit and a right limit at each point. The *regulated primitive integral* integrates those distributions that are the distributional derivative of a regulated function. Analogous to $\mathcal{A}_c(\mathbb{T})$, the space of integrable distributions is a Banach space. This space includes $\mathcal{A}_c(\mathbb{T})$ and also all signed Radon measures. A theory of Fourier series can be obtained as in the present paper. The chief difference between these two integrals is in the integration by parts formula and in the fact that we no longer have continuity in the Alexiewicz norm. See [17].

If u is a periodic distribution then it has a Fourier series given by $u(x) = [1/(2\pi)] \sum_{n\in\mathbb{Z}} \hat{u}_n e^{inx}$. The convergence is in the distributional sense, i.e., weak convergence. There is also a converse, any trigonometric series with coefficients of polynomial growth is the Fourier series of a distribution. See [6, Theorems 8.5.2, 8.5.3]. Other methods of defining Fourier series of distributions are given in [5] and [19]. Since $\mathcal{A}_c(\mathbb{T})$ is a subspace of distributions, all of the results in these works continue to hold. However, $\mathcal{A}_c(\mathbb{T})$ is also a Banach space. We will see below that Fourier series of distributions in $\mathcal{A}_c(\mathbb{T})$ behave more like those for L^1 functions than for general distributions.

In this paper we develop basic properties of Fourier series and convolutions ab initio from the definition of the integral. As the functions of bounded variation are pointwise multipliers for distributions in $\mathcal{A}_c(\mathbb{T})$ it follows that $\mathcal{A}_c(\mathbb{T})$ is a Banach $\mathcal{BV}(\mathbb{T})$-module over the pointwise algebra of $\mathcal{BV}(\mathbb{T})$. And, as is shown in Sect. 3 below, distributions in $\mathcal{A}_c(\mathbb{T})$ can be convolved with functions in $L^1(\mathbb{T})$ such that $\mathcal{A}_c(\mathbb{T})$ is a Banach $L^1(\mathbb{T})$-module over the convolution algebra of $L^1(\mathbb{T})$. Although we have employed a concrete approach here, such a two-module property may allow the abstract methods developed in [1] to be used to deduce some of the theorems below. An anonymous referee suggested Corollary 20 might be proved this way.

2 Fourier Coefficients

Let $e_n(t) = e^{int}$. If $f \in \mathcal{A}_c(\mathbb{T})$ then the Fourier coefficients of f are $\hat{f}(n) = \langle f, e_{-n}\rangle = \int_{-\pi}^{\pi} fe_{-n} = \int_{-\pi}^{\pi} f(t)e^{-int}\,dt$, where $n \in \mathbb{Z}$. Since the functions e_n and $1/e_n$ are in $\mathcal{BV}(\mathbb{T})$ for each $n \in \mathbb{Z}$, the Fourier coefficients exist on $\mathbb{Z}$ as continuous primitive integrals if and only if $f \in \mathcal{A}_c(\mathbb{T})$. Let $F(x) = \int_{-\pi}^{x} f$ be the primitive of f. Integrating by parts as in (1) gives

$$\hat{f}(n) = (-1)^n F(\pi) + in \int_{-\pi}^{\pi} F(t)e^{-int}\,dt. \tag{2}$$

This last integral is the Riemann integral of a continuous function. Formula (2) can be used as an alternative definition of $\hat{f}(n)$. Note also that $\hat{f}(n) = \int_{\alpha}^{\alpha+2\pi} f(t)e^{-int}\,dt$ for each $\alpha \in \mathbb{R}$. The following properties of the Fourier coefficients follow easily from the linearity of the integral and from (2). The complex conjugate is denoted $\overline{x+iy}=x-iy$ for $x,y\in\mathbb{R}$. We will take f to be real-valued but only trivial changes are required for complex-valued distributions.

Theorem 2 *Let* $f,g\in\mathcal{A}_c(\mathbb{T})$. *Then* (a) $\widehat{f+g}(n)=\hat{f}(n)+\hat{g}(n)$; (b) *if* $\alpha\in\mathbb{C}$ *then* $(\widehat{\alpha f})(n)=\alpha\hat{f}(n)$; (c) $\overline{\hat{f}(n)}=\hat{f}(-n)$; (d) *if* $s\in\mathbb{R}$ *then* $\widehat{\tau_s f}(n)=\hat{f}(n)e^{-ins}$; (e) $|\hat{f}(n)|\le|F(\pi)|+|n|\int_{-\pi}^{\pi}|F|$ *where* $F(x)=\int_{-\pi}^{x}f$; (f) *for* $n\neq 0$, $|\hat{f}(n)|\le 4\sqrt{2}\,|n|\|f\|_{\mathbb{T}}$; (g) $\hat{f}(n)=o(n)$ *as* $|n|\to\infty$ *and this estimate is sharp*; (h) *for* $n\neq 0$ *we have* $|\hat{f}(n)|\le 2\sqrt{2}\,|n|\|f-\tau_{\pi/n}f\|_{\mathbb{T}}$; (i) *if* $F\in C^0(\mathbb{T})$ *then* $\widehat{F'}(n)=in\hat{F}(n)$; (j) *if* $F\in C^{k-1}(\mathbb{T})$ *for some* $k\in\mathbb{N}$ *then for* $n\neq 0$ *and each* $0\le\ell\le k$, $\hat{F}(n)=(in)^{-\ell}\widehat{F^{(\ell)}}(n)$ *and*

$$|\hat{F}(n)|\le 4\sqrt{2}\min_{0\le\ell\le k}\frac{\|F^{(\ell)}\|_{\mathbb{T}}}{|n|^{\ell-1}}.$$

As $|n|\to\infty$, $\hat{F}(n)=o(n^{1-k})$.

Part (g) is a version of the Riemann–Lebesgue lemma for the continuous primitive integral. When $f\in L^1(\mathbb{T})$ then $\hat{f}(n)=o(1)$ as $|n|\to\infty$. This estimate is sharp in the sense that if $\psi:\mathbb{N}\to(0,\infty)$ and $\psi(n)=o(1)$ as $n\to\infty$ then there is a function $f\in L^1(\mathbb{T})$ such that $\hat{f}(n)\neq o(\psi(n))$ as $|n|\to\infty$. Estimates similar to those in (j) appear in [11, I 4.4] for $F^{(k-1)}$ absolutely continuous.

Proof To prove (f), apply the Hölder inequality (Proposition 1) as follows. Notice that the minimum of $|\sin(nt)|$ and $|\cos(nt)|$ for $|t|\le\pi$ are both zero. Then

$$|\hat{f}(n)|\le\|f\|_{\mathbb{T}}\sqrt{\left[\int_{-\pi}^{\pi}|n\sin(nt)|\,dt\right]^2+\left[\int_{-\pi}^{\pi}|n\cos(nt)|\,dt\right]^2}=4\sqrt{2}\,|n|\|f\|_{\mathbb{T}}.$$

Part (g) follows upon integrating by parts and using the L^1 form of the Riemann–Lebesgue lemma on the integral $\int_{-\pi}^{\pi}F(t)e^{-int}\,dt$. The estimate was proved sharp by Titchmarsh [18]. To prove (h), use a linear change of variables to write $\hat{f}(n)=(1/2)\int_{-\pi}^{\pi}[f(t)-f(t-\pi/n)]e^{-int}\,dt$. Then proceed as in (f) to get $|\hat{f}(n)|\le 2\sqrt{2}\,|n|\|f-\tau_{\pi/n}f\|_{\mathbb{T}}$. Since f is continuous in the Alexiewicz norm this also gives the little oh estimate in (g). Part (i) follows from integrating by parts and then (j) is obtained using (i) with the estimates in (f) and (g). □

The quantity $\omega(f,\delta)=\sup_{|t|<\delta}\|f-\tau_t f\|_{\mathbb{T}}$ is known as the *modulus of continuity* of f in the Alexiewicz norm. Part (h) gives $|\hat{f}(n)|\le 2\sqrt{2}\,|n|\omega(f,\pi/|n|)$.

Katznelson [11] gives other estimates for $\hat{f}(n)$ under such assumptions as f is of bounded variation, absolutely continuous, Lipschitz continuous or in $L^p(\mathbb{T})$.

The next theorem shows that when we have a sequence converging in the Alexiewicz norm, the Fourier coefficients also converge.

Theorem 3 *For $j \in \mathbb{N}$, let $f, f_j \in \mathcal{A}_c(\mathbb{T})$ such that $\|f_j - f\|_{\mathbb{T}} \to 0$ as $j \to \infty$. Then for each $n \in \mathbb{Z}$ we have $\hat{f}_j(n) \to \hat{f}(n)$ as $j \to \infty$. The convergence need not be uniform in $n \in \mathbb{Z}$.*

Proof If $n = 0$ then $|\hat{f}_j(0) - \hat{f}(0)| = |\int_{-\pi}^{\pi}[f_j(t) - f(t)]\,dt| \le \|f_j - f\|_{\mathbb{T}}$. If $n \neq 0$ then

$$
\begin{aligned}
|\hat{f}_j(n) - \hat{f}(n)| &= \left|\int_{-\pi}^{\pi}[f_j(t) - f(t)]e^{-int}\,dt\right| \\
&\le 4\sqrt{2}\,|n|\|f_j - f\|_{\mathbb{T}} \to 0 \quad \text{as } j \to \infty. \qquad (3)
\end{aligned}
$$

Theorem 2(f) is used in (3).

To show the convergence need not be uniform, let $f_j(t) = e^{ijt}$ and $f = 0$. Then $\|f_j\|_{\mathbb{T}} = \sup_{|\alpha-\beta|\le 2\pi} |\int_{\alpha}^{\beta} e^{ijt}\,dt|$. We have

$$
\left|\int_{\alpha}^{\beta} e^{ijt}\,dt\right| = \frac{1}{j}\left|e^{ij(\beta-\alpha)} - 1\right| \le \frac{2}{j}.
$$

Equality is realized when $\beta = \alpha + \pi/j$. Hence, $\|f_j\|_{\mathbb{T}} = 2/j \to 0$ as $j \to \infty$. But, $\hat{f}_n(n) = \int_{-\pi}^{\pi} dt = 2\pi \not\to 0$. □

This is different from the case $f, f_j \in L^1(\mathbb{T})$. There, if $\{f_j\}$ converges to f in the L^1 norm then $\hat{f}_j(n)$ converges to $\hat{f}(n)$ uniformly in n as $j \to \infty$. See [11, I Corollary 1.5].

3 Convolution

The convolution is one of the most important operations in analysis, with applications to differential equations, integral equations and approximation of functions. For $f \in \mathcal{A}_c(\mathbb{T})$ and $g \in \mathcal{BV}(\mathbb{T})$ the convolution is $\int_{-\pi}^{\pi}(f \circ r_x)g$ where $r_x(t) = x - t$. We write this as $f * g(x) = \int_{-\pi}^{\pi} f(x-t)g(t)\,dt$. This integral exists for all such f and g. The convolution inherits smoothness properties from f and g. We will also use a limiting process to define the convolution for $g \in L^1(\mathbb{T})$. This then makes $\mathcal{A}_c(\mathbb{T})$ into an $L^1(\mathbb{T})$-module over the $L^1(\mathbb{T})$ convolution algebra. See [9, 32.14] for the definition. The convolution was considered for the continuous primitive integral on the real line in [16]. Many of the results of that paper are easily adapted to the setting of $\mathbb{T}$, especially differentiation and integration theorems which we do not reproduce here.

When f and g are in $L^1(\mathbb{T})$ the convolution $f * g$ is commutative and associative. The estimate $\|f * g\|_1 \le \|f\|_1\|g\|_1$ shows the convolution is a bounded linear operator $* : L^1(\mathbb{T}) \times L^1(\mathbb{T}) \to L^1(\mathbb{T})$ and $L^1(\mathbb{T})$ is a Banach algebra under convolution. See [11] for details.

Since $\mathcal{BV}(\mathbb{T})$ is the dual of $\mathcal{A}_c(\mathbb{T})$, many of the usual properties of convolutions hold when it is defined on $\mathcal{A}_c(\mathbb{T}) \times \mathcal{BV}(\mathbb{T})$.

Theorem 4 *Let $f\in\mathcal{A}_c(\mathbb{T})$ and let $g\in\mathcal{BV}(\mathbb{T})$. Then* (a) $f*g\in C^0(\mathbb{T})$; (b) $f*g=g*f$; (c) $\|f*g\|_\infty\le\|f\|_{\mathbb{T}}\|g\|_{\mathcal{BV}}$; (d) *for $y\in\mathbb{R}$ we have $\tau_y(f*g)=(\tau_y f)*g=f*(\tau_y g)$.* (e) *If $h\in L^1(\mathbb{T})$ then $f*(g*h)=(f*g)*h\in C^0(\mathbb{T})$.* (f) *For each $f\in\mathcal{A}_c(\mathbb{T})$, define $\Phi_f:\mathcal{BV}(\mathbb{T})\to C^0(\mathbb{T})$ by $\Phi_f[g]=f*g$. Then Φ_f is a bounded linear operator and $\|\Phi_f\|\le\|f\|_{\mathbb{T}}$. For each $g\in\mathcal{BV}(\mathbb{T})$, define $\Psi_g:\mathcal{A}_c(\mathbb{T})\to C^0(\mathbb{T})$ by $\Psi_g[f]=f*g$. Then Ψ_f is a bounded linear operator and $\|\Psi_g\|\le\|g\|_{\mathcal{BV}}$.* (g) *We have $\widehat{f*g}(n)=\hat f(n)\hat g(n)$ for all $n\in\mathbb{Z}$.* (h) $\|f*g\|_{\mathbb{T}}\le\|f\|_{\mathbb{T}}\|g\|_1$. (i) *Let $f,g\in L^1(\mathbb{T})$. Then $\|f*g\|_{\mathbb{T}}\le\|f\|_{\mathbb{T}}\|g\|_1\le\|f\|_1\|g\|_1$.*

Proof The proofs of (a) through (e) are essentially the same as for [16, Theorem 1]. Proposition 1 is used in the proof of (c). Theorem 21 is used in the proof of (e). Part (f) follows from part (c). To prove (g), write $\widehat{f*g}(n)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x-t)e^{-in(x-t)}g(t)e^{-int}\,dt\,dx$. By Theorem 21 we can interchange the orders of integration. The result then follows upon a change of variables. To prove (h), use Theorem 21 and a linear change of variables to write

$$\begin{aligned}\int_\alpha^\beta f*g(x)\,dx&=\int_\alpha^\beta\int_{-\pi}^{\pi}f(x-t)g(t)\,dt\,dx=\int_{-\pi}^{\pi}\int_\alpha^\beta f(x-t)g(t)\,dx\,dt\\&=\int_{-\pi}^{\pi}g(t)\int_{\alpha-t}^{\beta-t}f(x)\,dx\,dt.\end{aligned}$$

Then

$$\left|\int_\alpha^\beta f*g(x)\,dx\right|\le\sup_{u<v}\left|\int_u^v f\right|\|g\|_1\le\|f\|_{\mathbb{T}}\|g\|_1.$$

The proof of (i) is similar but now the usual Fubini theorem is used. □

Using two equivalent norms, we can have equality in part (f). Define $\|f\|'_{\mathbb{T}}=\sup_{-\pi\le x\le\pi}|\int_{-\pi}^x f|$ for $f\in\mathcal{A}_c(\mathbb{T})$ and define $\|g\|'_{\mathcal{BV}}=|g(-\pi)|+0.5Vg$ for $g\in\mathcal{BV}(\mathbb{T})$. These norms are equivalent to $\|\cdot\|_{\mathbb{T}}$ and $\|\cdot\|_{\mathcal{BV}}$, respectively. Given $f\in\mathcal{A}_c(\mathbb{T})$ with $f\ne0$ there is $\alpha\in(-\pi,\pi]$ such that $\|f\|'_{\mathbb{T}}=|\int_{-\pi}^\alpha f|$. Define $g\in\mathcal{BV}(\mathbb{T})$ by $g(t)=\chi_{(-\alpha,\pi)}(t)$ for $t\in[-\pi,\pi)$ and extend periodically. Then $\|g\|'_{\mathcal{BV}}=1$ and $|f*g(0)|=\|f\|'_{\mathbb{T}}$. With these norms, $\|\Phi_f\|=\|f\|'_{\mathbb{T}}$. However, we can have $\|\Psi_g\|<\|g\|_{\mathcal{BV}}$. Let $g(t)=(1/3)\chi_{\{0\}}(t)$ for $t\in[-\pi,\pi)$ and extend periodically. Then $\|g\|_{\mathcal{BV}}=1$ but $f*g=0$ for each $f\in\mathcal{A}_c(\mathbb{T})$. Hence, $\|\Psi_g\|=0$. This problem goes away if we replace $\mathcal{BV}(\mathbb{T})$ with functions of normalized bounded variation. Fix $0\le\lambda\le1$. A function $g\in\mathcal{BV}(\mathbb{T})$ is of normalized bounded variation if $g(x)=(1-\lambda)g(x-)+\lambda g(x+)$ for each $x\in[-\pi,\pi)$. The case $\lambda=0$ corresponds to left continuity and $\lambda=1$ corresponds to right continuity. See [17] for details.

Linearity in each component, associativity (e) and inequality (c) show that $\mathcal{A}_c(\mathbb{T})$ is a $\mathcal{BV}(\mathbb{T})$-module. Note that $\mathcal{BV}(\mathbb{T})$ is a Banach algebra under pointwise operations.

Example 5 Note that $f*g$ need not be of bounded variation and hence need not be absolutely continuous. For example, let g be the periodic extension of $\chi_{(0,\pi)}$. Then $f*g(x)=F(x)-F(x-\pi)$ where $F\in\mathcal{B}_c(\mathbb{T})$ is the primitive of f. Since F

need not be of bounded variation, the same can be said for $f * g$. For instance, take $F(x) = x\sin(x^{-2})\chi_{[0,\pi)}(x)$ on $[-\pi,\pi)$ with $F(0)=0$.

Using the estimate $\|f * g\|_{\mathbb{T}} \le \|f\|_{\mathbb{T}}\|g\|_1$ (Theorem 4(h)) and the fact that $\mathcal{BV}(\mathbb{T})$ is dense in $L^1(\mathbb{T})$ we can define $f * g$ for $f \in \mathcal{A}_c(\mathbb{T})$ and $g \in L^1(\mathbb{T})$. Since $L^1(\mathbb{T})$ is dense in $\mathcal{A}_c(\mathbb{T})$ we can also define the convolution using a sequence in $L^1(\mathbb{T})$ with the inequality in Theorem 4(i).

Definition 6 Let $f \in \mathcal{A}_c(\mathbb{T})$ and $g \in L^1(\mathbb{T})$. (a) Let $\{g_k\} \subset \mathcal{BV}(\mathbb{T})$ such that $\|g_k - g\|_1 \to 0$. Then $f * g$ is the unique element of $\mathcal{A}_c(\mathbb{T})$ such that $\|f * g_k - f * g\|_{\mathbb{T}} \to 0$. (b) Let $\{f_k\} \subset L^1(\mathbb{T})$ such that $\|f_k - f\|_{\mathbb{T}} \to 0$. Then $f * g$ is the unique element of $\mathcal{A}_c(\mathbb{T})$ such that $\|f_k * g - f * g\|_{\mathbb{T}} \to 0$.

See [16] for a proof that (a) defines a unique element of $\mathcal{A}_c(\mathbb{T})$. The validity of (b) and the equality of definitions (a) and (b) is proved following the proof of Theorem 7.

Theorem 7 *Let $f \in \mathcal{A}_c(\mathbb{T})$ and $g \in L^1(\mathbb{T})$. Then* (a) $f * g \in \mathcal{A}_c(\mathbb{T})$; (b) $\|f * g\|_{\mathbb{T}} \le \|f\|_{\mathbb{T}}\|g\|_1$; (c) *for $y \in \mathbb{R}$ we have $\tau_y(f * g) = (\tau_y f) * g = f * (\tau_y g)$.* (d) *If $h \in L^1(\mathbb{T})$ then $f * (g * h) = (f * g) * h \in \mathcal{A}_c(\mathbb{T})$.* (e) *For each $f \in \mathcal{A}_c(\mathbb{T})$, define $\Phi_f : L^1(\mathbb{T}) \to \mathcal{A}_c(\mathbb{T})$ by $\Phi_f[g] = f * g$. Then Φ_f is a bounded linear operator and $\|\Phi_f\| = \|f\|_{\mathbb{T}}$. For each $g \in L^1(\mathbb{T})$, define $\Psi_g : \mathcal{A}_c(\mathbb{T}) \to \mathcal{A}_c(\mathbb{T})$ by $\Psi_g[f] = f * g$. Then Ψ_g is a bounded linear operator and $\|\Psi_g\| = \|g\|_1$.* (f) *We have $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$ for all $n \in \mathbb{Z}$.*

Proof Using Theorem 21, the proofs of (a) through (d) are essentially the same as for [16, Theorem 3], taking Theorem 4 into account. To proof (e), let $f \in \mathcal{A}_c(\mathbb{T})$. Then $\|\Phi_f\| = \sup_{\|g\|_1=1} \|f * g\|_{\mathbb{T}} \le \sup_{\|g\|_1=1} \|f\|_{\mathbb{T}}\|g\|_1 = \|f\|_{\mathbb{T}}$. To show we have equality, let $k_n \in \mathcal{BV}(\mathbb{T})$ be a sequence of non-negative summability kernels as in Definition 10. Using Theorem 11, $\|\Phi_f\| \ge \|f * k_n\|_{\mathbb{T}} \to \|f\|_{\mathbb{T}}$ as $n \to \infty$. The inequality $\|\Psi_g\| \le \|g\|_1$ follows similarly. To prove we have equality, note that $C^0(\mathbb{T})$ is dense in $L^1(\mathbb{T})$ and such functions are uniformly continuous, so for each $g \in L^1(\mathbb{T})$ and each $\epsilon > 0$ there is a step function $\sigma(x) = \sum_{j=1}^n \sigma_j \chi_{(a_{j-1},a_j)}(x)$ defined by a uniform partition $-\pi = a_0 < a_1 < \cdots < a_n = \pi$, $a_j = -\pi + 2\pi j/n$, for which $\|g - \sigma\|_1 < \epsilon$. It then suffices to find a sequence $\{f_m\} \subset \mathcal{A}_c(\mathbb{T})$ with $\|f_m\|_{\mathbb{T}} = 1$ and $\|f_m * \sigma\|_{\mathbb{T}} \to \|\sigma\|_1$ as $m \to \infty$. Define $f_m(x) = \sum_{i=1}^n \epsilon_i p_m(x + a_{i-1})$ where $\epsilon_i = \operatorname{sgn}(\sigma_i)$ and p_m is a delta sequence. This is a sequence of continuous functions $p_m \ge 0$ such that there is a sequence of real numbers $\delta_m \downarrow 0$ with $\operatorname{supp}(p_m) \subset (-\delta_m, \delta_m)$ and $\int_{-\delta_m}^{\delta_m} p_m(x)\,dx = 1$. Take m large enough so that $\delta_m < \pi/n$. With the Fubini theorem we have

$$\|f_m * \sigma\|_{\mathbb{T}} \ge \int_0^{2\pi/n} f_m * \sigma(x)\,dx = \sum_{i,j=1}^n \epsilon_i \sigma_j \int_{a_{j-1}}^{a_j} \int_{a_{i-1}-t}^{a_i - t} p_m(x)\,dx\,dt. \tag{4}$$

The non-zero terms in (4) occur when $j = i-1, i, i+1$. The $j = i$ term yields

$$\begin{aligned}
&\sum_{i=1}^{n}|\sigma_i|\int_{a_{i-1}}^{a_i}\int_{a_{i-1}-t}^{a_i-t}p_m(x)\,dx\,dt\\
&\quad\geq\sum_{i=1}^{n}|\sigma_i|\int_{a_{i-1}+\delta_m}^{a_i-\delta_m}\int_{-\delta_m}^{\delta_m}p_m(x)\,dx\,dt\\
&\quad=\sum_{i=1}^{n}|\sigma_i|\left(\frac{2\pi}{n}-2\delta_m\right)\to\|\sigma\|_1\quad\text{as } m\to\infty.
\end{aligned}$$

When $j=i+1$, (4) gives

$$\begin{aligned}
&\left|\sum_{i=1}^{n-1}\epsilon_i\sigma_{i+1}\int_{a_i}^{a_{i+1}}\int_{a_{i-1}-t}^{a_i-t}p_m(x)\,dx\,dt\right|\\
&\quad=\left|\sum_{i=1}^{n-1}\epsilon_i\sigma_{i+1}\int_{a_i}^{a_i+\delta_m}\int_{-\delta_m}^{a_i-t}p_m(x)\,dx\,dt\right|\leq n\delta_m\|\sigma\|_1/(2\pi)\to 0\quad\text{as } m\to\infty.
\end{aligned}$$

Similarly with the $j=i-1$ term in (4). To prove (f), consider a sequence $\{g_n\}\subset\mathcal{BV}(\mathbb{T})$ such that $\|g_k-g\|_1\to 0$ as $k\to\infty$. From (g) in Theorem 4 we have $\widehat{f*g_k}(n)=\hat{f}(n)\hat{g_k}(n)$. But $\{\widehat{g_k}\}$ converges to $\hat{g}$ as $k\to\infty$, uniformly on $\mathbb{Z}$ ([11, I Corollary 1.5]) so we can take the limit $k\to\infty$ to complete the proof. □

To see that (b) of Definition 6 makes sense, take a sequence $\{f_k\}\subset L^1(\mathbb{T})$ such that $\|f_k-f\|_{\mathbb{T}}\to 0$. The estimate $\|f_k*g-f_l*g\|_{\mathbb{T}}\leq\|f_k-f_l\|_{\mathbb{T}}\|g\|_1$ from Theorem 4(i) shows $\{f_k*g\}$ converges to a unique element of $\mathcal{A}_c(\mathbb{T})$. It is easy to see that this does not depend on the choice of sequence $\{f_k\}$. To see that (a) and (b) agree, take $\{f_k\}$ and $\{g_k\}$ as in Definition 6. Then

$$\begin{aligned}
\|f_k*g-f*g_k\|_{\mathbb{T}}&\leq\|f_k*g-f_k*g_k\|_{\mathbb{T}}+\|f*g_k-f_k*g_k\|_{\mathbb{T}}\\
&\leq\|f_k\|_{\mathbb{T}}\|g_k-g\|_1+\|f_k-f\|_{\mathbb{T}}\|g_k\|_1.
\end{aligned}$$

Since $\{\|f_k\|_{\mathbb{T}}\}$ and $\{\|g_k\|_1\}$ are bounded, letting $k\to\infty$ shows that $f*g$ as defined by (a) and (b) are the same.

The factorization theorem of Salem–Zygmund–Rudin–Cohen states that if E is any of the spaces $L^p(\mathbb{T})$ for $1\leq p<\infty$ or any of the spaces $C^k(\mathbb{T})$ for $0\leq k<\infty$ then $E=L^1(\mathbb{T})*E$, i.e., for each $f\in E$ there exist $g\in L^1(\mathbb{T})$ and $h\in E$ such that $f=g*h$. See [4, 7.5.1]. We have a similar result in $\mathcal{A}_c(\mathbb{T})$.

Theorem 8 $\mathcal{A}_c(\mathbb{T})=L^1(\mathbb{T})*\mathcal{A}_c(\mathbb{T})$.

Proof Let $f\in\mathcal{A}_c(\mathbb{T})$. Its primitive in $\mathcal{B}_c(\mathbb{T})$ is given by $F(x)=\int_{-\pi}^{x}f$. Write $\tilde{f}=f-F(\pi)/(2\pi)$ and $\tilde{F}(x)=\int_{-\pi}^{x}\tilde{f}$. Then $\tilde{f}\in\mathcal{A}_c(\mathbb{T})$ and $\tilde{F}\in C^0(\mathbb{T})$ since $\int_{-\pi}^{\pi}\tilde{f}=0$. As $C^0(\mathbb{T})=L^1(\mathbb{T})*C^0(\mathbb{T})$ there exist $g\in L^1(\mathbb{T})$ and $H\in C^0(\mathbb{T})$ such that $\tilde{F}=g*H$. Differentiating both sides [16, Theorem 12] gives $\tilde{f}=g*H'$. Now let c_1 and c_2 be constants. Then $(g+c_1)*(H'+c_2) = f - F(\pi)/(2\pi) + c_2\int_{-\pi}^{\pi} g + 2\pi c_1 c_2$. Let $c_2 = 1/(2\pi)$ and $c_1 = (F(\pi) - \int_{-\pi}^{\pi} g)/(2\pi)$ to complete the proof. □

This theorem also follows from Theorem 22 and Note 25a in [9, Sect. 32] since the approximate unit for $L^1(\mathbb{T})$ is also an approximate unit for $\mathcal{A}_c(\mathbb{T})$. This is a sequence $\{k_n\} \subset L^1(\mathbb{T})$ such that $\|k_n\|_1 < M$ and $\|f * k_n - f\|_1 \to 0$ as $n \to \infty$, for each $f \in L^1(\mathbb{T})$. See [9, 28.51] and Theorem 11 below. This connection was pointed out by an anonymous referee.

Example 9 Using the method of Definition 6, it does not seem possible to define the convolution on $\mathcal{A}_c(\mathbb{T}) \times \mathcal{A}_c(\mathbb{T})$. The following example shows there is no $k \in \mathbb{R}$ such that $\|f * g\|_{\mathbb{T}} \le k\|f\|_{\mathbb{T}}\|g\|_{\mathbb{T}}$ for all $f, g \in \mathcal{A}_c(\mathbb{T})$. Let $f(t) = t^{-3}\sin(t^{-4})$ for $t \in (0,\pi)$, let $f(t) = 0$ for $t \in [-\pi, 0]$ and extend f periodically. The primitive is

$$F(x) = \int_{-\pi}^{x} f = \begin{cases} 0, & -\pi \le x \le 0, \\ \frac{x^2}{4}\cos(x^{-4}) - \frac{1}{2}\int_0^x t\cos(t^{-4})\,dt, & 0 < x < \pi, \end{cases}$$

extended outside $[-\pi,\pi)$ so that $F \in \mathcal{B}_c(\mathbb{T})$. Let $f_n(t) = t^{-3}\sin(t^{-4})\chi_{((n\pi)^{-1/4},\pi)}(t)$. Extend periodically outside $[-\pi,\pi)$ then $f_n \in \mathcal{BV}(\mathbb{T})$. Now,

$$\begin{aligned}\|f - f_n\|_{\mathbb{T}} &= \sup_{|\alpha-\beta|<2\pi}\left|\int_\alpha^\beta f - f_n\right| = \max_{x,y\in[0,(n\pi)^{-1/4}]}|F(x) - F(y)| \\ &\le (n\pi)^{-1/2} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Define $G(t) = t\sin(t^{-4})$ for $t \in [-\pi, 0)$, $G(t) = 0$ for $t \in [0,\pi)$ and extend G so that $G \in \mathcal{B}_c(\mathbb{T})$. Let $g = G'$. Then $g * f_n(x) = \int_{(n\pi)^{-1/4}}^{\pi} g(x-t)f(t)\,dt$. Using Theorem 21,

$$\left|\int_0^\pi g * f_n(x)\,dx\right| = \int_{(n\pi)^{-1/4}}^{\pi} \sin^2(t^{-4})\frac{dt}{t^2} = \frac{1}{4}\int_{\pi^{-4}}^{n\pi} x^{-3/4}\sin^2(x)\,dx.$$

Hence, $\|g * f_n\|_{\mathbb{T}} \to \infty$ as $n \to \infty$.

Since the convolution is linear in both arguments, associative over $L^1(\mathbb{T})$ and satisfies the inequality $\|f * g\|_{\mathbb{T}} \le \|f\|_{\mathbb{T}}\|g\|_1$, the convolution maps $* : \mathcal{A}_c(\mathbb{T}) \times L^1(\mathbb{T}) \to \mathcal{A}_c(\mathbb{T})$ and $\mathcal{A}_c(\mathbb{T})$ is an $L^1(\mathbb{T})$-module over the $L^1(\mathbb{T})$ convolution algebra. Trigonometric polynomials are dense in $\mathcal{A}_c(\mathbb{T})$ (Lemma 12) so $\mathcal{A}_c(\mathbb{T})$ is an essential Banach module. See [3] for the definitions.

A Segal algebra is a subalgebra of $L^1(\mathbb{T})$ that is dense, translation invariant, and continuous in norm. See [12]. Since $L^1(\mathbb{T})$ is a subalgebra of $\mathcal{A}_c(\mathbb{T})$ the roles of the spaces are reversed. For each $x \in \mathbb{R}$ and $f \in L^1(\mathbb{T})$ we have $\|\tau_x f\|_1 = \|f\|_1$ and there is continuity in $\|\cdot\|_1$. Some properties of Segal algebras hold in this case. For example, if $f \in L^1(\mathbb{T})$ then $\|f\|_{\mathbb{T}} \le \|f\|_1$ [12, Proposition 6.2.3]. However, it follows from Theorem 8 that $L^1(\mathbb{T})$ is not an ideal of $\mathcal{A}_c(\mathbb{T})$ [12, Proposition 6.2.4].

4 Convergence

The series $\sum_{-\infty}^{\infty}\hat{f}(n)e^{int}$ is known as the Fourier series of f. If f is a smooth enough function then the Fourier series of f converges to f. There is a substantial literature on pointwise convergence of Fourier series. For example, if the pointwise derivative $f'(x)$ exists then the Fourier series converges to f at x [7, Corollary 3.3.9]. It is a celebrated result of A.N. Kolmogorov that there exists a function $f\in L^1(\mathbb{T})$ such that for each $t\in\mathbb{T}$ the sequence $\sum_{-N}^{N}\hat{f}(n)e^{int}$ diverges as $N\to\infty$ [11, p. 80]. L. Carleson and R.A. Hunt have proved that if $f\in L^p(\mathbb{T})$ for some $1<p<\infty$ these symmetric partial sums (given by convolution of f with the Dirichlet kernel) converge to f almost everywhere. For a proof see [7, Sect. 3.6] together with [8, Chap. 11]. On the one-dimensional torus, convergence of these symmetric partial sums to f in the p-norm is equivalent to $L^p(\mathbb{T})$ boundedness of the conjugate function. M. Riesz has shown that the conjugate function is bounded for $1<p<\infty$. See [7, Sect. 3.5]. We will see below that these symmetric partial sums converge to $f\in L^1(\mathbb{T})$ in the Alexiewicz norm. For $f\in\mathcal{A}_c(\mathbb{T})$ we will show that the Fourier series converges in the Alexiewicz norm with an appropriate summability factor.

First we consider summability kernels.

Definition 10 A summability kernel is a sequence $\{k_n\}\subset\mathcal{BV}(\mathbb{T})$ such that $\int_{-\pi}^{\pi}k_n=1$, $\lim_{n\to\infty}\int_{|s|>\delta}|k_n(s)|\,ds=0$ for each $0<\delta\leq\pi$ and there is $M\in\mathbb{R}$ so that $\|k_n\|_1\leq M$ for all $n\in\mathbb{N}$.

Theorem 11 *Let $f\in\mathcal{A}_c(\mathbb{T})$. Let k_n be a summability kernel. Then $\|f*k_n-f\|_{\mathbb{T}}\to 0$ as $n\to\infty$.*

Proof Let $-\pi\leq\alpha<\beta\leq\pi$. Then

$$\begin{aligned}
&\left|\int_\alpha^\beta\left[f*k_n(t)-f(t)\right]dt\right|\\
&\quad=\left|\int_\alpha^\beta\left[\int_{-\pi}^{\pi}k_n(s)f(t-s)\,ds-f(t)\int_{-\pi}^{\pi}k_n(s)ds\right]dt\right|\\
&\quad=\left|\int_{-\pi}^{\pi}k_n(s)\int_\alpha^\beta\left[f(t-s)-f(t)\right]dt\,ds\right|\\
&\quad\leq\sup_{|s|<\delta}\|f-\tau_sf\|_{\mathbb{T}}\int_{|s|<\delta}|k_n(s)|\,ds+2\|f\|_{\mathbb{T}}\int_{\delta<|s|<\pi}|k_n(s)|\,ds. \qquad (5)
\end{aligned}$$

The interchange of integrals in (5) is accomplished using Theorem 21 in the Appendix. Due to continuity in the Alexiewicz norm, given ϵ, we can take $0<\delta<\pi$ small enough so that $\|f-\tau_sf\|_{\mathbb{T}}<\epsilon$ for all $|s|<\delta$. Hence, $\|f*k_n-f\|_{\mathbb{T}}<M\epsilon+2\|f\|_{\mathbb{T}}\int_{\delta<|s|<\pi}|k_n(s)|\,ds$. Letting $n\to\infty$ completes the proof. □

A commonly used summability kernel is the Fejér kernel,

$$k_n(t) = \frac{1}{2\pi}\sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)e^{ikt} = \frac{1}{2\pi(n+1)}\left[\frac{\sin((n+1)t/2)}{\sin(t/2)}\right]^2.$$

See [11] for this and other summability kernels. The classical summability kernels (de la Vallée Poussin, Poisson, Jackson) all satisfy the conditions of Theorem 11, which differ from Lebesgue integral conditions by requiring the kernels be of bounded variation. A sequence (or net) of functions satisfying the conclusion of Theorem 11 is also called an approximate unit when its Fourier series consists of a finite number of terms. See [12]. The approximate units for $L^1(\mathbb{T})$ are then approximate units for $\mathcal{A}_c(\mathbb{T})$.

Lemma 12 *Let $f \in \mathcal{A}_c(\mathbb{T})$. Then $f * e_n(x) = \hat{f}(n)e^{inx}$. Let $g(t) = \sum_{-n}^{n} a_k e_k(t)$ for a sequence $\{a_k\} \subset \mathbb{R}$. Then $f * g(x) = \sum_{-n}^{n} a_k \hat{f}(k)e^{ikx}$.*

The proof follows from the identity $e_n(x-t) = e_n(x)e_n(-t)$ and linearity of the integral.

The lemma allows us to prove that trigonometric polynomials are dense in $\mathcal{A}_c(\mathbb{T})$ and gives a uniqueness result. Let k_n be the Fejér kernel and define $\sigma_n[f] = k_n * f$. From Theorem 11 we have $\sigma_n[f] \to f$ in the Alexiewicz norm. The lemma shows $\sigma_n[f]$ is a trigonometric polynomial. Hence, the trigonometric polynomials are dense in $\mathcal{A}_c(\mathbb{T})$.

Theorem 13 *Let $f \in \mathcal{A}_c(\mathbb{T})$. The trigonometric polynomials are dense in $\mathcal{A}_c(\mathbb{T})$;*

$$\sigma_n[f](t) = \frac{1}{2\pi}\sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right)\hat{f}(k)e^{ikt} \quad \text{and} \quad \lim_{n\to\infty}\|f - \sigma_n[f]\|_{\mathbb{T}} = 0. \tag{6}$$

If $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$ then $f = 0$.

Define the space of doubly indexed sequences converging to 0 by $c_0 = \{\sigma : \mathbb{Z} \to \mathbb{R} \mid \sigma_n = o(1) \text{ as } |n| \to \infty\}$. Then c_0 is a Banach space under the uniform norm. Distributions whose sequence of Fourier coefficients are in c_0 are known as pseudofunctions. Let $d = \{\sigma : \mathbb{Z} \to \mathbb{R} \mid \sigma_n = o(n) \text{ as } |n| \to \infty\}$. Then d is a Banach space under the norm $\|\sigma\|_d = \sup_{n\in\mathbb{Z}} |\sigma_n|/(|n|+1)$. In fact, c_0 and d are isometrically isomorphic, a linear isometry being given by $\sigma_n \mapsto \sigma_n/(|n|+1)$. Note that a corollary to Theorem 3 is that if $\|f_j - f\|_{\mathbb{T}} \to 0$ then $\|\hat{f}_j - \hat{f}\|_d \to 0$. The following theorem summarizes the properties of the mapping $f \mapsto \hat{f}$ for $f \in \mathcal{A}_c(\mathbb{T})$.

Theorem 14 *Define $\mathcal{F} : \mathcal{A}_c(\mathbb{T}) \to d$ by $\mathcal{F}[f] = \hat{f}$. Then $\mathcal{F}$ is a bounded linear transformation that is injective but not surjective.*

Proof Linearity is given in Theorem 2(a), (b). Part (f) of the same theorem shows $\mathcal{F}$ is bounded. The uniqueness theorem (Theorem 13) shows $\mathcal{F}$ is an injection. If $\mathcal{F}$ were also a surjection then a consequence of the Open Mapping Theorem is that there is $\delta > 0$ such that $\|\hat{f}\|_d \geq \delta \|f\|_{\mathbb{T}}$ for all $f \in \mathcal{A}_c(\mathbb{T})$. See [13, Theorem 5.10]. For each $\alpha \in \mathbb{R}$ let $f_\alpha(t) = |t|^{-\alpha}\operatorname{sgn}(t)$ on $[-\pi, \pi)$. Then $f_\alpha \in \mathcal{A}_c(\mathbb{T})$ if and only if $\alpha < 1$. We have $\hat{f}_\alpha(0) = 0$ and for $n \neq 0$ we get $|\hat{f}_\alpha(n)| \leq 2|n| \int_0^\pi t^{1-\alpha}\,dt = 2|n|\pi^{2-\alpha}/(2-\alpha)$. Then $\|\hat{f}_\alpha\|_d \leq 2\pi^{2-\alpha}/(2-\alpha) \to 2\pi$ as $\alpha \to 1^-$. And, $\|f_\alpha\|_{\mathbb{T}} = \pi^{1-\alpha}/(1-\alpha) \to \infty$ as $\alpha \to 1^-$. Hence, $\mathcal{F}$ cannot be surjective. □

For L^1 Fourier series, $\hat{f}(n) = o(1)$ but the transformation $f \mapsto \hat{f}$ is not onto c_0. See [13, Theorem 5.15].

For $n \geq 0$ define the Dirichlet kernel $D_n(t) = \sum_{-n}^{n} e^{ikt} = \sin[(n+1/2)t]/\sin(t/2)$. Notice that according to the definition in Theorem 11, D_n is not a summability kernel. In fact, $\|D_n\|_1 \sim (4/\pi^2)\log(n)$ as $n \to \infty$. See [11, p. 71]. However, $\|D_n\|_{\mathbb{T}}$ are bounded. This shows that $D_n * f$ converges to f in $\|\cdot\|_{\mathbb{T}}$ for $f \in L^1(\mathbb{T})$.

Theorem 15 *The sequence $\|D_n\|_{\mathbb{T}}$ is bounded. Let $f \in L^1(\mathbb{T})$. Then $\|D_n * f - f\|_{\mathbb{T}} \to 0$ as $n \to \infty$.*

Proof Fix $n \in \mathbb{N}$. Since the function $t \mapsto \sin(t/2)$ is increasing on $[0, \pi]$ we have $|\int_{k\pi/(n+1/2)}^{(k+1)\pi/(n+1/2)} D_n(t)\,dt| \geq |\int_{(k+1)\pi/(n+1/2)}^{(k+2)\pi/(n+1/2)} D_n(t)\,dt|$ for each integer $k \geq 0$. We then have

$$\begin{aligned}
\|D_n\|_{\mathbb{T}} &= 2\int_0^{2\pi/(2n+1)} \frac{\sin[(n+1/2)t]}{\sin(t/2)}\,dt = 2\int_0^{2\pi/(2n+1)} \sum_{k=-n}^{n} e^{ikt}\,dt \\
&= \frac{4\pi}{2n+1} + 4\sum_{k=1}^{n} \frac{1}{k}\sin\left(\frac{2\pi k}{2n+1}\right) \\
&\leq \frac{4\pi}{2n+1} + \frac{8\pi n}{2n+1} = 4\pi.
\end{aligned}$$

Let $f \in L^1(\mathbb{T})$ and let $\epsilon > 0$. There is a trigonometric polynomial p such that $\|f - p\|_1 < \epsilon/(4\pi + 1)$. Let n be greater than the degree of p. Then from Lemma 12 we have $D_n * p = p$. Using the estimate in Theorem 7(b)

$$\begin{aligned}
\|D_n * f - f\|_{\mathbb{T}} &= \|D_n * (f - p) + p - f\|_{\mathbb{T}} \\
&\leq \|D_n\|_{\mathbb{T}}\|f - p\|_1 + \|f - p\|_{\mathbb{T}} \\
&\leq (4\pi + 1)\|f - p\|_1 < \epsilon.
\end{aligned}$$
□

Example 16 Since the Dirichlet kernels are not uniformly bounded in the L^1 norm there is a function $f \in \mathcal{A}_c(\mathbb{T})$ such that $\|D_n * f - f\|_{\mathbb{T}} \not\to 0$. To see this, for each $n \in \mathbb{N}$ define

$$F_n(t) = \begin{cases} 0, & 0 \leq t \leq \pi \\ -\sin[(n+1/2)t], & -n\pi/(n+1/2) \leq t \leq 0 \\ 0, & -\pi \leq t \leq -n\pi/(n+1/2), \end{cases}$$

with F_n extended periodically. Then $F_n \in \mathcal{B}_c(\mathbb{T})$ so $F_n' \in \mathcal{A}_c(\mathbb{T})$.

We have $D_n * F_n' = (D_n * F_n)'$ [16, Proposition 4.2]. Therefore, $\|D_n * F_n'\|_{\mathbb{T}} = \max_{x,y\in[-\pi,\pi]} |D_n * F_n(y) - D_n * F_n(x)|$. Note that

$$D_n * F_n(0) = \frac{2}{2n+1}\int_0^{n\pi} \frac{\sin^2(t)}{\sin[t/(2n+1)]}\,dt \geq 2\int_\pi^{n\pi} \sin^2(t)\,\frac{dt}{t}$$
$$= \log(n) - \int_\pi^{n\pi} \cos(2t)\,\frac{dt}{t}.$$

Hence, $D_n * F_n(0) \geq 0.5\log(n)$ for large enough n. As well,

$$D_n * F_n(\pi) = \frac{(-1)^n}{2n+1}\int_0^{n\pi} \frac{\sin(2t)}{\cos[t/(2n+1)]}\,dt.$$

Since the function $t \mapsto \sec[t/(2n+1)]$ is positive and increasing on $[0, n\pi]$ we have $\int_0^{n\pi} \sin(2t)\sec[t/(2n+1)]\,dt < 0$.

We now have $\max_{x,y\in[-\pi,\pi]} |D_{2n} * F_{2n}(y) - D_{2n} * F_{2n}(x)| \geq 0.5\log(n)$ for large enough n. And, $\|F_{2n}'\|_{\mathbb{T}} = 2$. Hence, $\|D_n\| = \sup_{\|f\|_{\mathbb{T}}=1} \|D_n * f\|_{\mathbb{T}}$ are not uniformly bounded. By the Uniform Boundedness Principle, there exists $f \in \mathcal{A}_c(\mathbb{T})$ such that $\|D_n * f\|_{\mathbb{T}}$ is not bounded as $n \to \infty$. Therefore, $\|D_n * f - f\|_{\mathbb{T}} \not\to 0$.

Note that if $f \in L^1(\mathbb{T})$ then $\|D_n * f - f\|_1$ need not tend to zero. See [11, p. 68].

If $f \in L^1(\mathbb{T})$ and $g \in L^\infty(\mathbb{T})$ then Fejér's lemma states that $\int_{-\pi}^{\pi} f(t)g(nt)\,dt$ has limit $\hat{f}(0)\hat{g}(0)/(2\pi)$ as $n \to \infty$. Since the multipliers for $\mathcal{A}_c(\mathbb{T})$ are the functions of bounded variation we have the following version for the continuous primitive integral.

Proposition 17 *Let $f \in \mathcal{A}_c(\mathbb{T})$ and $g \in \mathcal{BV}(\mathbb{T})$. Then $\int_{-\pi}^{\pi} f(t)g(nt)\,dt = o(n)$ as $n \to \infty$. The order estimate is sharp.*

Proof The trigonometric polynomials are dense in $\mathcal{A}_c(\mathbb{T})$ (Lemma 12) so there are sequences of trigonometric polynomials $\{p_\ell\}$ and $\{q_m\}$ such that $\|f - p_\ell\|_{\mathbb{T}} \to 0$ and $\|g - q_m\|_{\mathbb{T}} \to 0$ as $\ell, m \to \infty$. Write

$$\int_{-\pi}^{\pi} f(t)g(nt)\,dt = \int_{-\pi}^{\pi} [f(t) - p_\ell(t)]g(nt)\,dt + \int_{-\pi}^{\pi} p_\ell(t)[g(nt) - q_m(nt)]\,dt$$
$$+ \int_{-\pi}^{\pi} p_\ell(t)q_m(nt)\,dt.$$

Use the Hölder inequality, Proposition 1. Then $|\int_{-\pi}^{\pi}[f(t) - p_\ell(t)]g(nt)\,dt| \leq n\|f - p_\ell\|_{\mathbb{T}}\|g\|_{\mathcal{BV}}$. Take $\ell \in \mathbb{N}$ large enough so that $\|f - p_\ell\|_{\mathbb{T}}\|g\|_{\mathcal{BV}} < \epsilon$. And, $|\int_{-\pi}^{\pi} p_\ell(t)[g(nt) - q_m(nt)]\,dt| \leq \|p_\ell\|_{\mathcal{BV}}\|g - q_m\|_{\mathbb{T}}$. Take $m \in \mathbb{N}$ large enough so that $\|p_\ell\|_{\mathcal{BV}}\|g - q_m\|_{\mathbb{T}} < \epsilon$. Now write $q_m = \sum_{-m}^{m} a_k e_{-k}$. Then $\int_{-\pi}^{\pi} p_\ell(t)q_m(nt)\,dt = \sum_{-m}^{m} a_k \hat{p}(nk)$. By the Riemann–Lebesgue lemma this tends to $a_0\hat{p}(0)$ as $n \to \infty$. Hence, $\int_{-\pi}^{\pi} f(t)g(nt)\,dt = o(n)$. The estimate is sharp by Theorem 2(g). □

Since the topological dual of $\mathcal{A}_c(\mathbb{T})$ is $\mathcal{BV}(\mathbb{T})$ we can view functions of bounded variation as continuous linear functionals on $\mathcal{A}_c(\mathbb{T})$. For $f \in \mathcal{A}_c(\mathbb{T})$ and $g \in \mathcal{BV}(\mathbb{T})$

we define a linear functional $g : \mathcal{A}_c(\mathbb{T}) \to \mathbb{R}$ by $g[f] = \int_{-\pi}^{\pi} fg$. If $\{f_n\} \subset \mathcal{A}_c(\mathbb{T})$ such that $\|f_n - f\|_{\mathbb{T}} \to 0$ then by the Hölder inequality

$$|g[f_n] - g[f]| = \left| \int_{-\pi}^{\pi} (f_n - f)g \right| \leq \|f_n - f\|_{\mathbb{T}} \|g\|_{\mathcal{BV}} \to 0.$$

Hence g is a continuous linear functional on $\mathcal{A}_c(\mathbb{T})$.

The following Parseval equality states that for every $g \in \mathcal{BV}$ we have $\sigma_n[g] \to g$ in the weak* topology.

Theorem 18 *If $f \in \mathcal{A}_c(\mathbb{T})$ and $g \in \mathcal{BV}(\mathbb{T})$ then*

$$g[f] = \langle f, g \rangle = \int_{-\pi}^{\pi} fg = \lim_{n\to\infty} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \hat{f}(k)\hat{g}(k).$$

The proof is essentially the same as the version for Lebesgue integrals given in [11, p. 37]. Note that for $f \in \mathcal{A}_c(\mathbb{T})$ and $g \in \mathcal{BV}(\mathbb{T})$, the series $\sum_{-\infty}^{\infty} \hat{f}(k)\hat{g}(k)$ need not converge. This follows from the sharp growth estimates $\hat{f}(k) = o(k)$ (Theorem 2(g)) and $\hat{g}(k) = O(1/k)$ [11, I Theorem 4.5]. To see this, let $g(t) = \sqrt{1 - (t/\pi)^2}$ on $(0, \pi)$ and extend g as an odd periodic function. Then

$$\hat{g}(k) = \frac{2i}{k} - \frac{2i}{\pi^2 k} \int_0^{\pi} \frac{t \cos(kt)\, dt}{\sqrt{1 - (\pi/t)^2}}.$$

By the Riemann–Lebesgue lemma, $\hat{g}(k) \sim 2i/k$ as $k \to \infty$.

The Parseval equality lets us characterize sequences of Fourier coefficients of functions in $\mathcal{BV}(\mathbb{T})$.

Theorem 19 *Let $\{a_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{C}$. The following are equivalent*: (a) *There exists $g \in \mathcal{BV}(\mathbb{T})$ and $c \geq 0$ such that $\|g\|_{\mathcal{BV}} \leq c$ and $\hat{g}(n) = a_n$ for each $n \in \mathbb{Z}$.* (b) *For all trigonometric polynomials p we have $|\sum_{-\infty}^{\infty} \hat{p}(n)\overline{a_n}| \leq c\|p\|_{\mathbb{T}}$.*

Corollary 20 *A trigonometric series $S(t) \sim \sum a_n e^{int}$ is the Fourier series of some function $g \in \mathcal{BV}(\mathbb{T})$ with $\|g\|_{\mathcal{BV}} \leq c$ if and only if $\|\sigma_n[S]\|_{\mathcal{BV}} \leq c$ for all $n \in \mathbb{Z}$.*

The proof is essentially the same as that for Theorem 7.3 in [11, p. 39].

Appendix

The following type of Fubini theorem generalizes a similar one for the Henstock–Kurzweil and wide Denjoy integral in [2, Theorem 58].

Theorem 21 *Let $f \in \mathcal{A}_c(\mathbb{T})$. Let $g \in \mathcal{BV}(\mathbb{T})$. If $-\infty < a < b < \infty$ then $\int_a^b \int_{-\pi}^{\pi} f(x-y)g(y)\, dy\, dx = \int_{-\pi}^{\pi} \int_a^b f(x-y)g(y)\, dx\, dy$.*

Proof Let $F \in \mathcal{B}_c(\mathbb{T})$ be the primitive of f. Integrating by parts gives $\int_{-\pi}^{\pi} f(x-y)g(y)\,dy = F(x+\pi)g(-\pi) - F(x-\pi)g(\pi) + \int_{-\pi}^{\pi} F(x-y)\,dg(y)$. Now use the periodicity of g to write

$$\begin{aligned}&\int_a^b \int_{-\pi}^{\pi} f(x-y)g(y)\,dy\,dx\\ &\quad= \left(\int_{a+\pi}^{b+\pi} F - \int_{a-\pi}^{b-\pi} F\right) g(\pi) + \int_a^b \int_{-\pi}^{\pi} F(x-y)\,dg(y)\,dx.\end{aligned}$$

A linear change of variables and integration by parts gives

$$\begin{aligned}&\int_{-\pi}^{\pi} \int_a^b f(x-y)g(y)\,dx\,dy\\ &\quad= \int_{-\pi}^{\pi} g(y) \int_{a-y}^{b-y} f(x)\,dx\,dy\\ &\quad= \int_{-\pi}^{\pi} \big[F(b-y) - F(a-y)\big]\, g(y)\,dy\\ &\quad= \int_{-\pi}^{\pi} [F(b-y) - F(a-y)]\,dy\, g(\pi)\\ &\qquad- \int_{-\pi}^{\pi} \int_{-\pi}^{y} [F(b-z) - F(a-z)]\,dz\,dg(y)\\ &\quad= \left(\int_{b-\pi}^{b+\pi} F - \int_{a-\pi}^{a+\pi} F\right) g(\pi) + \int_{-\pi}^{\pi} \left(\int_a^b F(x-y)\,dx - \int_{a+\pi}^{b+\pi} F\right) dg(y)\\ &\quad= \left(\int_{a+\pi}^{b+\pi} F - \int_{a-\pi}^{b-\pi} F\right) g(\pi) + \int_{-\pi}^{\pi} \int_a^b F(x-y)\,dx\,dg(y) - \int_{a+\pi}^{b+\pi} F \int_{-\pi}^{\pi} dg.\end{aligned}$$

The usual Fubini theorem gives

$$\int_{-\pi}^{\pi} \int_a^b F(x-y)\,dx\,dg(y) = \int_a^b \int_{-\pi}^{\pi} F(x-y)\,dg(y)\,dx$$

since

$$\left|\int_{-\pi}^{\pi} \int_a^b F(x-y)\,dx\,dg(y)\right| \le \max_{x\in[a-\pi,b+\pi]} |F(x)| Vg.$$

As g is periodic, $\int_{-\pi}^{\pi} dg = 0$. □

References

1. Braun, W., Feichtinger, H.G.: Banach spaces of distributions having two module structures. J. Funct. Anal. **51**, 174–212 (1983)
2. Čelidze, V.G., Džvaršeĭšvili, A.G.: The theory of the Denjoy integral and some applications. World Scientific, Singapore (1989) (Trans. P.S. Bullen)

3. Dales, H.G., Lau, A.T.-M.: The second duals of Beurling algebras. Mem. Amer. Math. Soc. **177**(836) (2005)
4. Edwards, R.E.: Fourier Series: A Modern Introduction, vol. I. Springer, New York (1979)
5. Edwards, R.E.: Fourier Series: A Modern Introduction, vol. II. Springer, New York (1982)
6. Friedlander, F.G., Joshi, M.: Introduction to the Theory of Distributions. Cambridge University Press, Cambridge (1999)
7. Grafakos, L.: Classical Fourier Analysis. Springer, New York (2008)
8. Grafakos, L.: Modern Fourier Analysis. Springer, New York (2009)
9. Hewitt, E., Ross, K.A.: Abstract Harmonic Analysis, vol. II. Springer, Berlin (1970)
10. Jeffery, R.L.: The Theory of Functions of a Real Variable. University of Toronto Press, Toronto (1951)
11. Katznelson, Y.: An Introduction to Harmonic Analysis. Cambridge University Press, Cambridge (2004)
12. Reiter, H., Stegeman, J.D.: Classical Harmonic Analysis and Locally Compact Groups. Oxford University Press, Oxford (2000)
13. Rudin, W.: Real and Complex Analysis. McGraw–Hill, New York (1987)
14. Talvila, E.: Henstock–Kurzweil Fourier transforms. Ill. J. Math. **46**, 1207–1226 (2002)
15. Talvila, E.: The distributional Denjoy integral. Real Anal. Exch. **33**, 51–82 (2008)
16. Talvila, E.: Convolutions with the continuous primitive integral. Abstr. Appl. Anal. **2009**, 307404 (2009) 18 pp.
17. Talvila, E.: The regulated primitive integral. Ill. J. Math. **53**, 1187–1219 (2009)
18. Titchmarsh, E.C.: The order of magnitude of the coefficients in a generalised Fourier series. Proc. Lond. Math. Soc. (2) **22**, xxv–xxvi (1923/1924)
19. Zemanian, A.H.: Distribution Theory and Transform Analysis. Dover, New York (1987)