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# Spectral Concentration of Positive Functions on Compact Groups

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**Abstract** The problem of understanding the Fourier-analytic structure of the cone of positive functions on a group has a long history. In this article, we develop the first quantitative spectral concentration results for such functions over arbitrary compact groups. Specifically, we describe a family of finite, positive quadrature rules for the Fourier coefficients of band-limited functions on compact groups. We apply these quadrature rules to establish a spectral concentration result for positive functions: given appropriately nested band limits $\mathcal{A} \subset \mathcal{B} \subset \widehat{G}$, we prove a lower bound on the fraction of $L^2$-mass that any $\mathcal{B}$-band-limited positive function has in $\mathcal{A}$. Our bounds are explicit and depend only on elementary properties of $\mathcal{A}$ and $\mathcal{B}$; they are the first such bounds that apply to arbitrary compact groups. They apply to finite groups as a special case, where the quadrature rule is given by the Fourier transform on the smallest quotient whose dual contains the Fourier support of the function.

**Keywords** Approximation · Positive functions · Band-limited functions · Spectral analysis

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## 1 Introduction

Understanding the Fourier-analytic structure of the cone of positive functions on a group is a long-standing problem inspired by signal processing applications. Bochner's theorem, for instance, characterizes the set of functions on $\mathbb{R}^n$ whose inverse Fourier transforms are positive [13]. In this article, we study the energy distribution of the Fourier transforms of positive functions on arbitrary compact groups, establishing lower bounds for the fraction of $L^2$-mass that a positive band-limited function must have in a given portion of the spectrum. Such bounds have been known in specific cases since 1956, when Erdős and Fuchs [3] showed that the Fourier transform $\phi(\theta) = \sum_{n=0}^{\infty} f(n)e^{in\theta}$ of a nonnegative function $f \in L^1(\mathbb{Z}_{\geq 0})$ satisfies

$$\int_{-\pi/2}^{\pi/2} |\phi(\theta)|^2 d\theta \geq \frac{1}{6}\|\phi\|_2^2; \tag{1.1}$$

thus, the fraction of energy appearing in the "low frequency" end of the spectrum is at least 1/6. This constant was later improved to 1/4 by Logan [10] and Shapiro [16].

In the setting of an arbitrary compact group $G$, the Fourier transform is determined by the irreducible representations of $G$, associating with every square-integrable function $f \in L^2(G)$ a family of linear operators

$$\hat{f}(\rho) = \sqrt{d_\rho}\int_G f(x)\rho(x)^\dagger \, d\mu(x),$$

one for each irreducible representation $\rho$ of $G$. This transformation is unitary in the sense that $\|f\|_2^2$ is equal to the sum of the squared Hilbert-Schmidt norms $\|\hat{f}(\rho)\|_{\mathrm{HS}}^2$. In this setting, there is no canonical notion of "low frequency"; however, nested families of "band limits," under mild conditions, can serve as a remarkable proxy. By *band limit*, we mean a finite collection $\mathcal{B}$ of irreducible representations of $G$. For a representation $\rho$ of degree $d_\rho$, we reserve the notation

$$\dim_{\mathrm{I}} \rho \triangleq d_\rho^2$$

to denote the dimension of the bi-invariant subspace of $L^2(G)$ associated with $\rho$. We extend this notation to band limits: $\dim_{\mathrm{I}} \mathcal{B} \triangleq \sum_{\rho\in\mathcal{B}} \dim_{\mathrm{I}} \rho$; note that this is precisely the dimension of the space of $\mathcal{B}$-band-limited functions, that is, those functions whose Fourier transforms are supported only on $\mathcal{B}$. In general, given a function $f \in L^2(G)$ and a band limit $\mathcal{B}$, we denote the energy $f$ has in $\mathcal{B}$ by

$$\|\hat{f}(\mathcal{B})\|_2^2 \triangleq \sum_{\rho\in\mathcal{B}} \|\hat{f}(\rho)\|_{\mathrm{HS}}^2.$$

In the special case where $G$ is finite, Kueh et al. [9] proved a natural analogue of (1.1), giving a lower bound for the fraction of energy that a positive function must have in certain band limits. These band limits have a particular representation-theoretic structure, due in part to the positivity constraint. Specifically, given $\mathcal{A} \subset \hat{G}$, let $\mathbf{E}(\mathcal{A})$ denote the set of irreducible representations appearing in the representations $\rho \otimes \sigma^*$, for $\rho, \sigma \in \mathcal{A}$. Considering such band limits leads to a particularly attractive concentration bound.

**Theorem 1** (Theorem 5 of [9]) *Let $f$ be a positive function on a finite group $G$, and let $\mathcal{A} \subset \hat{G}$. Then*

$$\frac{\|\hat{f}(\mathbf{E}(\mathcal{A}))\|_2^2}{\|f\|_2^2} \geq \frac{\dim_{\mathrm{I}} \mathcal{A}}{|G|}.$$

We remark that this bound is tight in the case of cyclic groups, in the sense that the quantity on the right hand side cannot be increased uniformly in the size of the group. Kueh et al. also extended the above theorem to the setting of certain compact groups and band-limited positive functions. However, these results suffered from a drawback: they depended critically on the maximum weight for an associated positive quadrature rule for the underlying group and, in general, no bounds were known for this quantity. Indeed, quadrature rules were only known for specific families of band limits defined on certain Lie and finite groups [11, 12].

The principal contribution of this article is an explicit quantitative lower bound, applying for band limits $\mathbf{E}(\mathcal{A}) \subset \mathcal{B}$ of arbitrary compact groups, on the fraction of energy that positive $\mathcal{B}$-band-limited functions must have in the representations of $\mathcal{A}$. Specifically, we establish the following theorem.

**Theorem 2** *Let $G$ be a compact group with nonempty finite band limits $\mathcal{A}$ and $\mathcal{B}$ satisfying $\mathbf{E}(\mathcal{A}) \subset \mathcal{B}$. Define $d = \max_{\rho \in \mathbf{E}(\mathcal{B})} \dim_{\mathrm{I}} \rho$. Then any positive, $\mathcal{B}$-band-limited function $f$ satisfies*

$$\frac{\|\hat{f}(\mathbf{E}(\mathcal{A}))\|_2^2}{\|f\|_2^2} \geq \frac{\dim_{\mathrm{I}} \mathcal{A}}{8d(3 + \ln d) \cdot \dim_{\mathrm{I}} \mathbf{E}(\mathcal{B})}.$$

The proof relies on establishing quantitative bounds on the weights of a generic family of positive quadrature rules, a notion defined below. Indeed, the quantity $8d(3 + \ln d) \cdot \dim_{\mathrm{I}} \mathbf{E}(\mathcal{B})$ appearing in the estimate arises as the cardinality of the support of a quadrature rule for the space of $\mathcal{B}$-band-limited functions. We remark that for a finite group $G$, uniform weights on the entire set $G$ form a quadrature rule for any band limit which recovers Theorem 1 as a consequence of Theorem 2.

Given a band limit $\mathcal{B}$, a *quadrature rule* is a finite set of sample points $X \subset G$, coupled with weights $w : X \to \mathbb{C}$, which allow one to calculate the Fourier transform of any $\mathcal{B}$-band-limited function at a representation $\rho \in \mathcal{B}$ via the finite sum:

$$\hat{f}(\rho) = \sqrt{d_\rho} \sum_{x \in X} w(x) f(x) \rho(x)^\dagger.$$

A simple example of a quadrature rule can be obtained by considering a function $f : G \to \mathbb{C}$ that is bi-invariant under the action of a normal subgroup $H$ of finite index. In this case, the Fourier transform of $f$ is supported on $\widehat{G/H} \subset \widehat{G}$ and, furthermore, the nontrivial Fourier coefficients are uniquely determined by the value taken by $f$ on any set of coset representatives.

Quadrature rules are basic tools for numerical integration [2] and can be formulated in a variety of settings. In recent work, Schmid, Gräf and Potts [8, 14] established necessary and sufficient conditions for positive quadrature rules on the rotation group $\mathsf{SO}(3)$. In a quite different setting, Filbir and Mhaskar [4] have established the existence of quadrature rules for integrating diffusion polynomials on compact Riemannian manifolds. Quadrature rules are also an essential ingredient for Fast Fourier Transform algorithms for certain families of compact Lie groups [11]. In that setting, the rules in question depend heavily on the structure of the particular group, and are tuned to specific families of band limits; this is a requirement imposed by the algorithmic application. For our purposes, we only require that the quadrature rule have positive weights, and we prefer rules with small sample sets. This flexibility allows for a (nonconstructive) probabilistic proof for the existence of quadrature rules that applies to any band limit on any compact group.

In the next section, we review some basic notions about representation theory, nonabelian Fourier analysis, and quadrature rules on compact groups. In Sect. 3, we construct new quadrature rules and prove lower bounds on their maximum weights. In the last section, we apply these rules to prove our main result, a spectral concentration theorem for band-limited positive functions.

## 2 Preliminaries

### 2.1 Fourier Analysis on Compact Groups

We briefly recall some basic facts from representation theory and nonabelian Fourier analysis on compact groups, following Folland [5]. Our purpose here is primarily to set down notation and establish normalizations for various Fourier-analytic equalities.

We define a *representation* of a compact group $G$ to be a continuous homomorphism $\rho : G \to \mathsf{U}(\mathcal{H}_\rho)$ from $G$ to the group of unitary operators of some nonzero Hilbert space $\mathcal{H}_\rho$. Continuity is the condition that the matrix elements $\rho_{uv} : x \mapsto \langle \rho(x)u, v\rangle$ of $\rho$ are continuous maps for every $u, v \in \mathcal{H}_\rho$. If the dimension $d_\rho$ of $\mathcal{H}_\rho$ is finite, then any orthonormal basis $\{e_i\}$ allows us to express $\rho(x)$ as a complex, unitary $d_\rho \times d_\rho$ matrix whose $i, j$-th entry is $\rho_{e_i e_j}(x)$; when there is no ambiguity, we will simply write $\rho_{ij}(x)$. We refer to $d_\rho$ as the *dimension* of $\rho$.

A subspace $W$ of $\mathcal{H}_\rho$ is said to be $G$-*invariant* if $\rho(g)W \subset W$ for all $g \in G$. In this case, the restriction $\rho|_W$ of $\rho$ to $W$ is a representation, called a *subrepresentation* of $\rho$; we write $\rho|_W \prec \rho$. A representation is *irreducible* if it has no proper subrepresentations. In this compact case, every irreducible representation (or irrep, for short) is finite-dimensional, and every representation decomposes into a direct sum of irreducible ones. Two representations $\rho$ and $\sigma$ are *equivalent* if they differ by a change of basis. We let $\hat{G}$ denote the set of equivalence classes of irreps of $G$.

The representations of a compact group $G$ can be given natural ring structure over the operations of direct sum and tensor product. We let $\mathbf{1} : G \to \mathsf{U}(\mathbb{C})$ denote the one-dimensional *trivial* representation, $\mathbf{1} : g \mapsto 1$. With two representations

$\rho : G \to \mathsf{U}(\mathcal{H}_\rho)$ and $\sigma : G \to \mathsf{U}(\mathcal{H}_\sigma)$, we may naturally define a representation acting on $\mathcal{H}_\rho \oplus \mathcal{H}_\sigma$ by the rule $g \mapsto \rho(g) \oplus \sigma(g)$; this representation we denote $\rho \oplus \sigma$. The space $\mathcal{H}_\rho \otimes \mathcal{H}_\sigma$ can likewise be given the structure of a representation by the rule $g \mapsto \rho(g) \otimes \sigma(g)$ [18]. In general, the resulting representation, which we denote $\rho \otimes \sigma$, is not irreducible even when both $\rho$ and $\sigma$ are. These operations define the *representation ring* of $G$. The additive structure is given by the free abelian group generated by the (isomorphism classes of) finite-dimensional representations of $G$ under the relation $\rho + \sigma - \rho \oplus \sigma$; multiplication is determined by the tensor product. Finally, we remark that dual spaces can also be given the structure of a representation: with each representation $\rho : G \to \mathsf{U}(\mathcal{H})$, we may associate the *contragredient* representation $\rho^* : G \to \mathsf{U}(\mathcal{H}^*)$ by the rule $\rho^*(g) = \rho(g^{-1})^*$ where $A^*$, for a linear operator $A$, denotes the dual operator on $\mathcal{H}^*$ given by $A^* : \phi \mapsto \phi \circ A$.

Every compact group admits a nonnegative bi-invariant Radon measure $\mu$ of total mass 1, called *Haar measure*. This allows us to define the inner product

$$\langle f, g \rangle = \int_G f(x)\overline{g(x)}\, d\mu(x)$$

and the associated $L^2$-norm $\|f\|_2^2 = |\langle f, f \rangle|$ for complex-valued functions on $G$. The (typically infinite-dimensional) space $L^2(G)$, consisting of square-integrable complex-valued functions on $G$, is naturally a representation of $G$ under the left (or right) multiplication action $[x \cdot f](y) = f(x^{-1}y)$. The Peter-Weyl theorem asserts that this representation decomposes into a direct sum of irreps of $G$, each appearing with multiplicity equal to its dimension. Concretely, if we decompose $L^2(G)$ into subspaces invariant under both left and right translation by elements of $G$, then

$$L^2(G) \cong \bigoplus_{\rho \in \hat{G}} \mathcal{E}_\rho,$$

where $\mathcal{E}_\rho$ is the span of the matrix elements of $\rho$, called the $\rho$-*isotypic* subspace of $L^2(G)$. The spaces $\mathcal{E}_\rho$ decompose into irreducible representations, according to the left (or right) action, in a particularly attractive way. Given an orthonormal basis $\{e_i\}$ for $\mathcal{H}_\rho$, the Schur orthogonality relations imply that the $d_\rho^2$ (scaled) matrix elements $\sqrt{d_\rho}\rho_{e_i e_j}$ for $1 \le i, j \le d_\rho$ form an orthonormal basis for $\mathcal{E}_\rho$. The union of all the bases of the $\mathcal{E}_\rho$ is then an orthonormal basis for $L^2(G)$, a *Fourier basis*. For a particular $\mathcal{E}_\rho$, each "row" $\{\rho_{e_i e_j} : 1 \le j \le d_\rho\}$ spans a right-$G$-invariant subspace isomorphic to $\rho$; similarly, each "column" $\{\rho_{e_i e_j} : 1 \le i \le d_\rho\}$ spans a left-$G$-invariant subspace isomorphic to $\rho^*$.

With a finite-dimensional representation $\rho$ we associate the function $\chi_\rho(g) = \mathbf{tr}\,\rho(g)$, the *character* of $\rho$. The function $\chi_\rho$ is invariant under conjugation and hence constant on conjugacy classes of $G$. In fact, the set $\{\chi_\rho : \rho \in \hat{G}\}$ of characters of irreps forms an orthonormal basis for the subspace in $L^2(G)$ of functions invariant under conjugation (the *class functions*). The multiplicity of an irrep $\rho$ in a representation $\sigma$ is given by the inner product $\langle \chi_\rho, \chi_\sigma \rangle$ of their characters.

The *Fourier transform* of a function $f \in L^2(G)$ at an irreducible representation $\rho$ is given by

$$\hat{f}(\rho) \triangleq \sqrt{d_\rho} \int_G f(x)\rho(x)^\dagger \, d\mu(x).$$

Here the notation $\cdot^\dagger$ denotes the Hilbert dual: for an operator $A$ on $\mathcal{H}$, the Hilbert dual $A^\dagger : \mathcal{H} \to \mathcal{H}$ has the defining property that $\langle v, Aw\rangle = \langle A^\dagger v, w\rangle$ for all $v, w \in \mathcal{H}$. Given an orthonormal basis $\{e_i\}$ for the space of $\rho$, one can view $\hat{f}(\rho)$ as the matrix whose $i, j$-th entry is given by $\sqrt{d_\rho}\langle f, \rho_{e_i e_j}\rangle$; indeed, this is precisely the definition of the operator-valued Haar integral appearing in the transform. The Fourier inversion formula allows us to write $f$ in terms of its transform:

$$f(x) = \sum_{\rho \in \hat{G}} \sqrt{d_\rho}\, \mathbf{tr}[\hat{f}(\rho)\rho(x)], \tag{2.1}$$

where the series converges in $L^2$ so long as $f \in L^2(G)$. If we fix a basis for each $\rho \in \hat{G}$, then (2.1) yields an expansion of $f$ in the resulting (orthonormal) Fourier basis for $L^2(G)$. Taken together, the inversion formula and the Peter-Weyl theorem imply the Plancherel equality:

$$\|f\|_2^2 = \sum_{\rho \in \hat{G}} \|\hat{f}(\rho)\|_{\mathrm{HS}}^2; \tag{2.2}$$

indeed, with appropriate normalizations, the Fourier transform is unitary.

Finally, we remark that the space $L^2(G)$ is afforded an algebra structure by the convolution product:

$$[f \star g](x) = \int f(y)g(y^{-1}x)\, d\mu(y).$$

This product is the linearization of the group action of $G$ on $L^2(G)$. As a result, the Fourier transform carries convolution to matrix product. Specifically,

$$\widehat{f \star g}(\rho) = \frac{1}{\sqrt{d_\rho}} \hat{g}(\rho)\hat{f}(\rho).$$

## 2.2 Quadrature Rules

Let $\mathcal{B}$ be a finite collection of irreducible representations of a compact group $G$. A function $f \in L^2(G)$ is said to be *$\mathcal{B}$-band-limited* if the support of $\hat{f}$ is contained in $\mathcal{B}$. The space of such functions is spanned by the matrix entries of the representations in $\mathcal{B}$, a subspace of $L^2(G)$ of dimension $\dim_{\mathrm{I}} \mathcal{B} < \infty$. By analogy with the finite group case, where $L^2(G)$ is itself finite-dimensional, one might expect that the Fourier transform of a $\mathcal{B}$-band-limited function could be determined via a finite sum rather than an operator-valued Haar integral. This is indeed the case, a phenomenon expressed by a *quadrature rule* for $\mathcal{B}$. Such a rule specifies a finitely-supported measure $\nu$ (that is, a finite set $X = \mathbf{supp}\,\nu \subset G$ along with complex weights $\nu(x)$ associated to each $x \in X$) with the promise that the Fourier coefficients of any $\mathcal{B}$-band limited function $f$ satisfy

$$\hat{f}(\rho) = \sqrt{d_\rho} \sum_{x \in \mathbf{supp}\,\nu} \nu(x) f(x) \rho(x)^\dagger \quad \text{for all } \rho \in \mathcal{B}. \tag{2.3}$$

We remark that as $f$ is $\mathcal{B}$-band-limited, $\hat{f}(\rho) = 0$ for those $\rho \notin \mathcal{B}$; we do not insist, however, that the corresponding finite sum (2.3) for such $\rho$ evaluate to zero. Observe that the quadrature rule, coupled with the Fourier inversion formula, yields a description of $f$ as a finite sum:

$$\begin{aligned} f(y) &= \sum_{\rho \in \mathcal{B}} d_\rho \sum_{x \in \mathbf{supp}\,\nu} \mathbf{tr}[\nu(x) f(x) \rho(x)^\dagger \rho(y)] \\ &= \sum_{\substack{x \in \mathbf{supp}\,\nu \\ \rho \in \mathcal{B}}} d_\rho \nu(x) f(x) \chi_\rho(x^{-1} y). \end{aligned}$$

We remark that the uncertainty principle places certain restrictions on the families of functions that can be reconstructed via a quadrature rule. One such principle for compact groups [1] states that any nonzero $f \in L^2(G)$ satisfies $\mu(\mathbf{supp}\, f) \sum_{\rho \in \hat{G}} d_\rho \, \mathbf{rk}\, \hat{f}(\rho) \geq 1$. It immediately follows that the measure of the support of any $\mathcal{B}$-band-limited function $f$ must be at least $1/\dim_{\mathrm{I}} \mathcal{B}$.

The following proposition gives an equivalent condition for a finitely supported measure to determine a quadrature rule for a band limit $\mathcal{B}$; a similar statement, with some additional restrictions on $\mathcal{B}$, appears as Lemma 1 in [11]. Recall from above the notation $\mathbf{E}(\cdot)$:

$$\mathbf{E}(\mathcal{B}) \triangleq \{\tau \in \hat{G} : \tau \prec \rho \otimes \sigma^* \text{ for some } \rho, \sigma \in \mathcal{B}\}.$$

**Proposition 3** *Let $G$ be a compact group and $\mathcal{B} \subset \hat{G}$ a finite, nonempty band limit. A finitely supported measure $\nu$ is a quadrature rule for $\mathcal{B}$ if and only if for all $\rho \in \mathbf{E}(\mathcal{B})$,*

$$\sum_{x \in \mathbf{supp}\,\nu} \nu(x) \rho(x)^\dagger = \begin{cases} \mathbb{1} & \text{if } \rho \text{ is trivial, and} \\ 0 & \text{if } \rho \text{ is nontrivial.} \end{cases}$$

*Proof* We first observe that, given a choice of basis for every irrep in $\mathcal{B}$, $\nu$ gives a quadrature rule for $\mathcal{B}$ if and only if the Fourier basis functions in $\mathcal{B}$ are orthonormal when computed over $\nu$. Specifically, for every $\rho, \sigma \in \mathcal{B}$ and $1 \leq i, j \leq d_\rho$ and $1 \leq k, \ell \leq d_\sigma$, we have

$$\sum_{x \in \mathbf{supp}\,\nu} \nu(x) \rho_{ij}(x) \overline{\sigma_{k\ell}(x)} = \begin{cases} \frac{1}{d_\rho} & \text{if } \rho \cong \sigma, i = k, j = \ell; \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Since $\sqrt{d_\sigma} \langle \rho_{ij}, \sigma_{k\ell} \rangle = \widehat{\rho_{ij}}(\sigma)_{k\ell}$, a quadrature rule $\nu$ for $\mathcal{B}$ will imply the above by definition; the other direction of the equivalence follows directly from the Peter-Weyl theorem:

$$\begin{aligned}\sqrt{d_\rho}\sum_{x\in\mathbf{supp}\,\nu} f(x)\rho(x)^\dagger\nu(x) &= \sqrt{d_\rho}\sum_{x\in\mathbf{supp}\,\nu}\sum_{\substack{\sigma\in\mathcal{B}\\1\le k,\ell\le d_\sigma}}\sqrt{d_\sigma}\hat{f}(\sigma)_{\ell k}\sigma_{k\ell}(x)\rho(x)^\dagger\nu(x)\\ &= \sqrt{d_\rho}\sum_{\substack{\sigma\in\mathcal{B}\\1\le k,\ell\le d_\sigma}}\hat{f}(\sigma)_{\ell k}\sqrt{d_\sigma}\sum_{x\in\mathbf{supp}\,\nu}\sigma_{k\ell}(x)\rho(x)^\dagger\nu(x)\\ &= \sum_{k,\ell} d_\rho\hat{f}(\rho)_{\ell k}\frac{e_{\ell k}}{d_\rho}\\ &= \hat{f}(\rho),\end{aligned}$$

where $e_{\ell k}$ denotes the $d_\rho \times d_\rho$ matrix with a 1 in the $\ell, k$-th entry and zeroes elsewhere.

Note that the matrix entries of the operator $\sqrt{d_\rho d_\sigma}\int_{x\in G}[\rho\otimes\sigma^*](x)\,d\mu(x)$ are precisely the inner products of the various Fourier basis functions belonging to $\rho$ and $\sigma$. By the equivalence shown above, it follows that $\nu$ gives a quadrature rule for $\mathcal{B}$ if and only if, for every $\rho, \sigma \in \mathcal{B}$, we have

$$\sum_{x\in\mathbf{supp}\,\nu}\nu(x)[\rho\otimes\sigma^*](x) = \int_{x\in G}[\rho\otimes\sigma^*](x)\,d\mu(x).$$

If we consider the orthogonal decomposition of each tensor product $\rho\otimes\sigma^*$ into irreducible subrepresentations $\tau \prec \rho\otimes\sigma^*$, the above amounts to requiring that

$$\sum_{x\in\mathbf{supp}\,\nu}\nu(x)\tau(x) = \int_{x\in G}\tau(x)\,d\mu(x)$$

for every $\tau \in \mathbf{E}(\mathcal{B})$. When $\tau$ is trivial, the right hand side is equal to 1; otherwise, by Schur's Lemma (see p. 13 of [15]), it is the zero operator. □

We remark that the condition of Proposition 3 referring to the trivial representation is relevant regardless of the structure of $\mathcal{B}$. Indeed, for any pair of irreps $\rho$ and $\sigma$, the multiplicity with which the trivial representation appears in $\rho\otimes\sigma^*$ is given by $\langle \mathbf{1}, \chi_{\rho\otimes\sigma^*}\rangle = \langle\chi_\rho, \chi_\sigma\rangle$, and is equal to one when $\rho \cong \sigma$ and zero otherwise. Since $\mathcal{B}$ is nonempty, this implies that the trivial representation always appears in $\mathbf{E}(\mathcal{B})$.

## 3 Quadrature Rules for Arbitrary Compact Groups

### 3.1 Representations Evaluated at Random Group Elements

The goal of this section is to establish the existence of nonnegative quadrature rules for arbitrary compact groups. The construction proceeds by demonstrating that, with nonzero probability, a sufficiently large family of independent samples according to

Haar measure can be given weights that yield a positive quadrature rule. In particular, we establish that for appropriate representations $\rho$ of $G$, the convex hull of the operators $\rho(x)$ contains zero with nonzero probability. Then any convex combination of these operators that equals zero yields weights which satisfy the quadrature rule conditions given by Proposition 3.

For a finite-dimensional complex Hilbert space $\mathcal{H}$ we let $\mathrm{End}(\mathcal{H})$ denote the Hilbert space of linear operators on $\mathcal{H}$ with the inner product $\langle A, B\rangle = \mathbf{tr}(A^\dagger B)$. As real weights play a special role in our application, we shall focus on real subspaces of $\mathrm{End}(\mathcal{H})$. Specifically, define $\mathrm{End}_{\mathbb{R}}(\mathcal{H})$ to be the real Hilbert space obtained from $\mathrm{End}(\mathcal{H})$ by restriction of scalars to $\mathbb{R}$ and adoption of the real inner product

$$\langle A, B\rangle = \mathrm{Re}\big(\mathbf{tr}(A^\dagger B)\big).$$

If $\rho$ is an irreducible, finite-dimensional unitary representation $\rho: G \to \mathsf{U}(\mathcal{H})$ the linear operators $\{\rho(g) : g \in G\}$ span $\mathrm{End}(\mathcal{H})$. As vectors in $\mathrm{End}_{\mathbb{R}}(\mathcal{H})$, however, they may span a strict subspace, which we denote $\mathbf{span}_{\mathbb{R}}\,\rho$.

We naturally extend the notation $\mathbf{span}_{\mathbb{R}}\,\rho$ to general finite-dimensional representations $\rho$ (which may not be irreducible). The first step in our proof is to show that when $x$ is drawn from the Haar measure, the random variable $\rho(x)$ is evenly distributed in some approximate sense. More precisely, let $N$ be a vector in $\mathbf{span}_{\mathbb{R}}\,\rho$; with the inner product above, $N$ determines a hyperplane in $\mathbf{span}_{\mathbb{R}}\,\rho$. We will see that, for any $N$, the fraction of operators $\rho(x)$ lying in either of the (open) half-spaces induced by the hyperplane is not too small.

**Proposition 4** *Let $G$ be a compact group and $\mathcal{K}$ a finite subset of $\hat{G}$ closed under taking contragredients and not containing the trivial representation. Define $\mathcal{K}(g) = \bigoplus_{\tau\in\mathcal{K}} \tau(g)$ and suppose $N \in \mathbf{span}_{\mathbb{R}}\,\mathcal{K}$. If $x$ is drawn from the Haar measure on $G$, then*

$$\Pr[\mathrm{Re}\,(\mathbf{tr}\,(\mathcal{K}(x)N)) > 0] \geq \frac{1}{2\cdot \max_{\tau\in\mathcal{K}} \dim_{\mathrm{I}} \tau}.$$

*Proof* Observe that $\mathbb{E}[\tau(x)] = \int \tau(x)\, d\mu(x) = 0$ for each $\tau \in \mathcal{K}$, so that

$$\mathbb{E}\,[\mathrm{Re}(\mathbf{tr}(\mathcal{K}(x)N))] = 0. \tag{3.1}$$

Both $\mathcal{K}(x)$ and $N$ can be viewed as block diagonal operators, each block corresponding to an irrep $\tau$ in $\mathcal{K}$. We let $N_\tau$ denote the projection of $N$ into the subspace spanned by matrices of $\tau$; the projection of $\mathcal{K}(x)$ into that subspace is simply $\tau(x)$. By Cauchy-Schwarz, the triangle inequality, and the unitarity of each $\tau$, we now have

$$\begin{aligned} |\mathrm{Re}\,\mathbf{tr}(\mathcal{K}(x)N)| &\leq \sum_{\tau\in\mathcal{K}} \|\tau(x)\|_{\mathrm{HS}} \|N_\tau\|_{\mathrm{HS}} = \sum_{\tau\in\mathcal{K}} \|N_\tau\|_{\mathrm{HS}} \sqrt{d_\tau} \\ &\leq \|N\|_{\mathrm{HS}} \sqrt{\max_{\tau\in\mathcal{K}} d_\tau}. \end{aligned} \tag{3.2}$$

The next step is to control the second moment. It can be expanded as follows:

$$\begin{aligned}\mathbb{E}\big[|\mathrm{Re}\,\mathbf{tr}(\mathcal{K}(x)N)|^2\big] &= \frac{1}{4}\mathbb{E}\big[\mathbf{tr}(\mathcal{K}(x)N)^2 + 2\,\mathbf{tr}(\mathcal{K}(x)N)\overline{\mathbf{tr}(\mathcal{K}(x)N)} + \overline{\mathbf{tr}(\mathcal{K}(x)N)}^2\big] \\ &= \frac{1}{4}\sum_{\rho,\sigma\in\mathcal{K}} \mathbb{E}\big[\mathbf{tr}(\rho(x)N_\rho)\,\mathbf{tr}(\sigma(x)N_\sigma) \\ &\qquad + 2\,\mathbf{tr}(\rho(x)N_\rho)\overline{\mathbf{tr}(\sigma(x)N_\sigma)} \\ &\qquad + \overline{\mathbf{tr}(\rho(x)N_\rho)\,\mathbf{tr}(\sigma(x)N_\sigma)}\big].\end{aligned}$$

Each term inside the expectation above then involves a sum of products of Fourier basis functions. When averaged over the group, the second term is zero except when $\rho \cong \sigma$; the first and third terms are zero except when $\rho^* \cong \sigma$. The expectation above can then be expressed as

$$\begin{aligned}&\frac{1}{4}\sum_{\tau\in\mathcal{K}} \mathbb{E}\big[\mathbf{tr}(\tau(x)N_\tau)\,\mathbf{tr}(\tau^*(x)N_{\tau^*}) + 2\,\mathbf{tr}(\tau(x)N_\tau)\overline{\mathbf{tr}(\tau(x)N_\tau)} \\ &\qquad + \overline{\mathbf{tr}(\tau(x)N_\tau)\,\mathbf{tr}(\tau^*(x)N_{\tau^*})}\big].\end{aligned}$$

The second term is

$$\sum_{i,j,k,\ell} \mathbb{E}\big[\tau(x)_{ij}(N_\tau)_{ji}\overline{\tau(x)_{k\ell}(N_\tau)_{\ell k}}\big] = \frac{1}{d_\tau}\sum_{i,j}(N_\tau)_{ji}\overline{(N_\tau)_{ji}} = \frac{\|N_\tau\|_{\mathrm{HS}}^2}{d_\tau}.$$

To simplify the first and third term, we distinguish between three cases: when $\tau$ is real, complex, and quaternionic (see Sect. III.5 in [17].) If $\tau$ is not isomorphic to its contragredient $\tau^*$, then we say that $\tau$ is complex. If $\tau$ is isomorphic to $\tau^*$, then either $\tau$ is real (meaning that $\tau = \tau^*$), or it is quaternionic (meaning that there exists a nontrivial unitary operator $U$ such that $\tau = U\tau^*U^\dagger$). Recalling that $\mathcal{K}$ is closed under taking contragredients, we see that each complex $\tau$ will contribute

$$\mathbb{E}\big[\mathbf{tr}(\tau(x)N_\tau)\,\mathbf{tr}(\tau^*(x)N_{\tau^*})\big] = \mathbb{E}\big[\mathbf{tr}(\tau(x)N_\tau)\overline{\mathbf{tr}(\tau(x)N_\tau)}\big] = \frac{\|N_\tau\|_{\mathrm{HS}}^2}{d_\tau},$$

while, likewise, each real $\tau$ contributes

$$\mathbb{E}\big[\mathbf{tr}(\tau(x)N_\tau)\,\mathbf{tr}(\tau(x)N_\tau)\big] = \mathbb{E}\big[\mathbf{tr}(\tau(x)N_\tau)\overline{\mathbf{tr}(\tau(x)N_\tau)}\big] = \frac{\|N_\tau\|_{\mathrm{HS}}^2}{d_\tau}.$$

In the quaternionic case, letting $U$ be the unitary operator that satisfies $\tau = U\tau^*U^\dagger$, we have

$$\begin{aligned}\mathbb{E}\big[\mathbf{tr}(\tau(x)N_\tau)\,\mathbf{tr}(\tau(x)N_\tau)\big] &= \mathbb{E}\big[\mathbf{tr}(\tau(x)N_\tau)\overline{\mathbf{tr}(U\tau^*(x)U^\dagger N_\tau)}\big] \\ &= \mathbf{tr}\big[N_\tau(U^\dagger N_\tau U)^\perp\big].\end{aligned}$$

It is easy to check that for a quaternionic irrep, the Fourier transform of a real element of the group algebra commutes with conjugation, up to unitary equivalence; specifically, $U(N_\tau)^* U^\dagger = N_\tau$. Substituting into the above, we now have

$$\mathbb{E}\big[\mathbf{tr}(\tau(x)N_\tau)\,\mathbf{tr}(\tau(x)N_\tau)\big] = \mathbf{tr}\big[N_\tau(U^\dagger U N_\tau^* U^\dagger U)^\perp\big] = \frac{\|N_\tau\|_{\mathrm{HS}}^2}{d_\tau}$$

for every quaternionic $\tau$. We conclude that

$$\mathbb{E}\big[|\mathrm{Re}(\mathbf{tr}(\mathcal{K}(x)N))|^2\big] = \sum_{\tau\in\mathcal{K}} \frac{\|N_\tau\|_{\mathrm{HS}}^2}{d_\tau} \geq \frac{\|N\|_{\mathrm{HS}}^2}{\max_{\tau\in\mathcal{K}} d_\tau}. \tag{3.3}$$

Now let $X$ be the random variable $\mathrm{Re}(\mathbf{tr}(\mathcal{K}(x)N))$, where $x$ is chosen from Haar measure on $G$, and consider (3.1), (3.2) and (3.3). Let $A^+$ and $A^-$ be the subsets of $G$ where $X$ takes positive and negative values, respectively. Then

$$\begin{aligned}\frac{\|N\|_{\mathrm{HS}}^2}{\max_{\tau\in\mathcal{K}} d_\tau} &\leq \mathbb{E}_{A^+}[X^2] + \mathbb{E}_{A^-}[X^2] \leq 2\sqrt{\max_{\tau\in\mathcal{K}} d_\tau}\,\|N\|_{\mathrm{HS}}\mathbb{E}_{A^+}[X]\\ &\leq 2\max_{\tau\in\mathcal{K}} d_\tau\,\|N\|_{\mathrm{HS}}^2\mu(A^+),\end{aligned}$$

and the result follows. □

*Remark 5* The bound given in the proposition above is nearly tight. To see this, let $p$ be prime and let $\mathbb{F}_p$ denote the finite field with $p$ elements. Consider $A_p$, the group of affine transformations of $\mathbb{F}_p$ of the form $\alpha_{a,b} : x \mapsto ax + b$ with $a \in \mathbb{F}_p^*$ and $b \in \mathbb{F}_p$. We view $A_p$ as a subgroup of the symmetric group $S_p$, and consider its action in the standard representation $\sigma$ (with $d_\sigma = p - 1$) of $S_p$. Recall that the character $\chi_\sigma(\pi) = F(\pi) - 1$, where $F(\pi) = |\{x \mid \pi(x) = x\}|$ is the number of fixed points of the permutation $\pi$. Let $N$ be the identity operator and consider $\mathbf{tr}(\sigma(\alpha_{a,b})N) = \chi_\sigma(\alpha_{a,b})$ for uniformly random $\alpha_{a,b} \in A_p$. This is $p - 1$ with probability $1/p(p - 1) = 1/(d_\sigma^2 + d_\sigma)$, $-1$ with probability $1/p = 1/(d_\sigma + 1)$, and zero otherwise. Note that the probability that $\chi_\sigma(a, b)$ is positive is within a constant of the bound given by Proposition 4.

## 3.2 Generic Quadrature Rules

We establish the existence of our quadrature rules by observing that, with nonzero probability, a sufficiently large family of samples drawn from the Haar measure can be given appropriate positive weights to form a quadrature rule.

**Theorem 6** *Let $G$ be a compact group and $\mathcal{B} \subset \widehat{G}$ a finite band limit closed under taking contragredients. Set $n = \dim_{\mathrm{I}} \mathbf{E}(\mathcal{B})$ and $d = \max_{\tau\in\mathbf{E}(\mathcal{B})} \dim_{\mathrm{I}} \tau$. Then there exists a quadrature rule for $\mathcal{B}$ given by a nonnegative measure $\nu$ with $|\mathbf{supp}\,\nu| \leq 8nd(3 + \ln d)$.*

*Proof* We first define the representation

$$\mathcal{K}(x) = \bigoplus_{\tau \in \mathbf{E}(\mathcal{B}); \tau \not\cong \mathbf{1}} \tau(x).$$

Fixing an index set $I$ of size $4d(2n-3)\ln(e^3 \cdot d) \leq 8nd(3 + \ln(d))$, let $X = \{x_i : i \in I\}$ be a family of independent random variables distributed according to Haar measure on $G$. We shall see that the convex hull of the set

$$\mathcal{K}(X) \triangleq \{\mathcal{K}(x_i) : i \in I\} \subset \operatorname*{\mathbf{span}}_{\mathbb{R}} \mathcal{K}$$

contains the vector 0 with probability bounded away from zero; in this case, there exists a nonnegative linear combination

$$\sum_i \nu_i \cdot \mathcal{K}(x_i) = 0 \quad \text{with} \quad \sum_i \nu_i = 1.$$

Proposition 3 then guarantees that the finitely-supported measure $\nu$ determined by these weights on $X$ is a positive quadrature rule for $\mathcal{B}$.

To set down notation, let $n_{\mathbb{R}}$ denote the real dimension of $\mathbf{span}_{\mathbb{R}} \mathcal{K}$. For a finite set of points $Z \subset \mathbf{span}_{\mathbb{R}} \mathcal{K}$, we let

$$\mathbf{hull}(Z) = \left\{ \sum_{z \in Z} \lambda_z \cdot z : \lambda_z \geq 0, \sum_{z \in Z} \lambda_z = 1 \right\}$$

denote their convex hull. We say that a hyperplane $H$ *bounds* a convex set if the set is entirely contained in one of the closed half-spaces determined by $H$. A *convex polytope* $P$ is the convex hull of a finite set of vectors in a real vector space; the *dimension* of such a polytope is the dimension of the corresponding space spanned by the elements $\{x - y \mid x, y \in P\}$. The intersection of $P$ with a bounding hyperplane is a convex set we call a *face* of $P$; if $P$ is $d$-dimensional, a face of dimension $d-1$ is called a *facet*. We remark that any hyperplane containing a facet bounds the polytope.

Returning to the set $\mathcal{K}(X)$, observe that if $0 \notin \mathbf{hull}(\mathcal{K}(X))$, there is a facet of $\mathbf{hull}(\mathcal{K}(X) \cup \{0\})$ containing 0; in this case, regardless of the dimension of $\mathbf{hull}(\mathcal{K}(X))$, the offending facet lies in the subspace spanned by some $n_{\mathbb{R}} - 1$ elements of $\mathcal{K}(X)$. Any hyperspace containing this subspace evidently bounds $\mathbf{hull}(\mathcal{K}(X))$. For a fixed subset $F \subset I$ of size $n_{\mathbb{R}} - 1$, let $B_F$ denote the bad event that every hyperspace $H$ containing $\{\mathcal{K}(x_i) : i \in F\}$ bounds $\mathbf{hull}(\mathcal{K}(X))$; in particular, for each such hyperspace $H$, the remaining elements $\{\mathcal{K}(x_i) : i \notin F\}$ of $\mathcal{K}(X)$ collectively lie in one of the closed half-spaces determined by $H$. With these events defined, we have

$$\Pr[0 \notin \mathbf{hull}(\mathcal{K}(X))] \leq \Pr\left[\bigcup_F B_F\right] \leq \sum_F \Pr[B_F],$$

where both the union and sum are extended over all $F \subset I$ of size $n_{\mathbb{R}} - 1$. Recalling that the trivial representation has been excluded from $\mathcal{K}$, the real dimension $n_{\mathbb{R}}$ of

$\mathbf{span}_{\mathbb{R}}\,\mathcal{K}$ is no more than $2(n-1)$. As the $x_i$ are independent and identically distributed (and, specifically, invariant under permutation), $\Pr[B_F] = \Pr[B_{F'}]$ for any pair $F, F'$ and we conclude that

$$\Pr[0 \notin \mathbf{hull}(\mathcal{K}(X)] \leq \binom{|X|}{n_{\mathbb{R}}-1}\Pr[B_{F_0}] \leq \binom{|X|}{2n-3}\Pr[B_{F_0}],$$

for any fixed $F_0$.

Consider a collection $F \subset I$ of $n_{\mathbb{R}} - 1$ indices; to estimate $\Pr[B_F]$, we shall analyze the event conditioned on a fixed, but arbitrary, assignment $a : F \to G$ for the variables $\{x_i : i \in F\}$. Fixing a vector $N_a$ normal to the subspace spanned by the elements $\{\mathcal{K}(a(i)) : i \in F\}$, the inner product of an element $\mathcal{K}(z)$ with $N_a$ determines the half-space of the hyperplane orthogonal to $N_a$ in which $\mathcal{K}(z)$ lies. As the $x_i$ are independent,

$$\begin{aligned}\Pr[B_F \mid x_i = a(i), i \in F] &\leq \prod_{i \notin F} \Pr\big[\mathrm{Re}\big(\mathbf{tr}[\mathcal{K}(x_i)N_a^\dagger]\big) \leq 0\big] \\ &\quad + \prod_{i \notin F} \Pr\big[\mathrm{Re}\big(\mathbf{tr}[\mathcal{K}(x_i)N_a^\dagger]\big) \geq 0\big] \\ &\leq 2 \cdot \sup_N \Big(\Pr_x\big[\mathrm{Re}\big(\mathbf{tr}[\mathcal{K}(x)N^\dagger]\big) \leq 0\big]\Big)^{|I \setminus F|},\end{aligned}$$

where $x$ is distributed according to Haar measure and the supremum is taken over all unit length vectors $N \in \mathbf{span}_{\mathbb{R}}\,\mathcal{K}$. Hence

$$\Pr[B_F] \leq \sup_a \Pr[B_F \mid x_i = a(i), i \in F] \leq 2 \cdot \sup_N \Big(\Pr_x\big[\mathrm{Re}\big(\mathbf{tr}[\mathcal{K}(x)N^\dagger]\big) \leq 0\big]\Big)^{|I \setminus F|}.$$

In light of Proposition 4 and the fact that $\binom{n}{k} \leq (ne/k)^k$ we infer that

$$\begin{aligned}\Pr\big[0 \notin \mathbf{hull}(X)\big] &\leq 2\binom{|X|}{2n-3}\Big(\sup_N\Big(\Pr_x\big[\mathrm{Re}(\mathbf{tr}[\mathcal{K}(x)N^\dagger]) \leq 0\big]\Big)\Big)^{|X|-(2n-3)} \\ &\leq 2\left(\frac{e|X|}{2n-3}\right)^{2n-3}\left(1 - \frac{1}{2d}\right)^{|X|-(2n-3)}.\end{aligned}$$

Recalling that $|X| = 4d(2n-3)\ln(e^3 \cdot d)$ and that $(1-x) \leq e^{-x}$ for $x \in [0, \infty)$,

$$\Pr[0 \notin \mathbf{hull}(X)] \leq \left(\frac{2e|X|}{2n-3}\right)^{2n-3} e^{-(|X|-(2n-3))/2d} = \left(\frac{8\ln(e^3 d)}{e^5 d} \cdot e^{\frac{1}{2d}}\right)^{2n-3}.$$

Observing that the derivative of $\ln(e^3 \cdot d)/d$ is negative in $[1, \infty)$ and that $1 \leq d < n$, we conclude that

$$\left(\frac{8\ln(e^3 d)}{e^5 d} \cdot e^{\frac{1}{2d}}\right)^{2n-3} \leq \left(\frac{24}{e^5}\sqrt{e}\right)^{2n-3} \leq \left(\frac{24}{e^{4.5}}\right) < 1,$$

as desired. □

*Remark 7* The argument above can yield slightly stronger constants than those claimed. In particular, a more involved computation shows there is a function $f: \mathbb{R}^+ \to \mathbb{R}^+$ so that for any $\epsilon > 0$ there is a quadrature rule of size $4(1+\epsilon)nd(\ln(d) + f(\epsilon))$.

## 4 Spectral Concentration of Positive Functions

### 4.1 The General Approach, and Previous Results

Let $\mathcal{A} \subset \mathcal{B}$ be two band limits of a compact group $G$, with $\mathcal{B}$ closed under contragredients. Our goal is to prove a lower bound for the fraction of spectral mass that any positive $\mathcal{B}$-band-limited function must have in $\mathcal{A}$. The insight of Kueh et al. [9] in bounding this quantity was to study the function

$$f^c = \tilde{f} \star f,$$

where $\tilde{f}$ is defined by $\tilde{f}(x) = f(x^{-1})$. Note that $\hat{f}$ and $\hat{f^c}$ have the same Fourier support. The essential property of $f^c$ for our application is that it is an indicator of the spectral mass of $f$:

$$\langle f^c, \chi_\rho \rangle = \frac{1}{\sqrt{d_\rho}} \mathbf{tr}\big[\widehat{f^c}(\rho)\big] = \frac{1}{d_\rho} \mathbf{tr}\big[\hat{f}(\rho)^\dagger \hat{f}(\rho)\big] = \frac{\|\hat{f}(\rho)\|_{\mathrm{HS}}^2}{d_\rho}.$$

Now consider a positive, $\mathcal{A}$-band-limited class function $\psi$. Computing the inner product of $f^c$ with $\psi$ in the Fourier basis, we have

$$\langle f^c, \psi \rangle = \sum_{\rho \in \hat{G}} \langle \chi_\rho, \psi \rangle \frac{\|\hat{f}(\rho)\|_{\mathrm{HS}}^2}{d_\rho} \le N(\psi) \sum_{\rho \in \mathcal{A}} \|\hat{f}(\rho)\|_{\mathrm{HS}}^2 = N(\psi)\|\hat{f}(\mathcal{A})\|_2^2, \quad (4.1)$$

where $N(\psi) = \max_{\rho \in \mathcal{A}} \langle \chi_\rho, \psi \rangle / d_\rho$. Since both $f^c$ and $\psi$ are $\mathcal{B}$-band-limited, their inner product can be viewed as a finite linear combination of Fourier coefficients associated with representations from $\mathcal{B}$ only; by the Peter-Weyl theorem, this is also true for left translations of $f^c$ and $\psi$. We can thus compute $\langle f^c, \psi \rangle$ via a quadrature rule for $\mathcal{B}$. Let $\nu$ be a quadrature rule for $\mathcal{B}$ with positive weights, and let $x_0 \in G$ be the element where $\nu$ achieves its maximum $\|\nu\|_\infty$. Then, by the positivity of $f^c$ and $\psi$,

$$\begin{aligned}
\langle f^c, \psi \rangle &= \int f^c(x_0^{-1}x)\psi(x_0^{-1}x)\, d\mu(x) \\
&= \sum_{x \in \mathbf{supp}\,\nu} \nu(x) f^c(x_0^{-1}x)\psi(x_0^{-1}x) \\
&\ge f^c(1)\psi(1)\nu(x_0) \\
&= \|f\|_2^2 \psi(1)\|\nu\|_\infty.
\end{aligned}$$

Combining with (4.1), we now have

$$\frac{\|\hat{f}(\mathcal{A})\|_2^2}{\|f\|_2^2} \geq \|\nu\|_\infty \frac{\psi(1)}{N(\psi)}, \tag{4.2}$$

a lower bound for the fraction of $L^2$-mass $f$ has in $\mathcal{A}$ (recall the Plancherel identity (2.2).) Clearly, the quality of this bound depends critically on our choice of quadrature rule $\nu$ and "test function" $\psi$. Our quadrature rules will be provided by the results of Sect. 3.2; we will discuss several possible choices for $\psi$ below.

In [9], the above technique is applied to both finite and compact groups. Given two band limits $\mathcal{A}$ and $\mathcal{B}$ satisfying $\mathbf{E}(\mathcal{A}) \subset \mathcal{B}$, they provide a lower bound for the fraction of energy (that is, $\|\cdot\|_2$-mass) which any $\mathcal{B}$-band-limited function must have in $\mathbf{E}(\mathcal{A})$. This is accomplished by setting $\phi_\mathcal{A} = \sum_{\rho\in\mathcal{A}} \chi_\rho$ and applying (4.2) to the test function $\psi = \phi_\mathcal{A}\overline{\phi_\mathcal{A}}$. The resulting bound (Theorem 14 in [9]) is

$$\frac{\|\hat{f}(\mathbf{E}(\mathcal{A}))\|_2^2}{\|f\|_2^2} \geq \|\nu\|_\infty \frac{(\sum_{\rho\in\mathcal{A}} d_\rho)^2}{N(\phi_\mathcal{A}\overline{\phi_\mathcal{A}})}.$$

The special case of finite groups appears as Theorem 4 in [9].

## 4.2 New Spectral Concentration Bounds

There are a number of ways in which the bounds from the previous section could be improved. Most importantly, prior to the present work no general nonnegative quadrature rules were known for arbitrary compact groups and band limits. The quantity $\|\nu\|_\infty$ was thus only defined for specific families of band limits on particular compact groups where positive quadrature rules had been constructed (see Maslen [12]). In general, no explicit lower bounds were known for $\|\nu\|_\infty$. Secondly, it is unclear if the choice of test function $\psi = \phi_\mathcal{A}\overline{\phi_\mathcal{A}}$ is optimal; different test functions, chosen in a way that maximizes the quantity $\psi(1)/N(\psi)$, may improve the bound. Finally, we may prefer a bound expressed merely in terms of the *dimensions* of the relevant irreducible representations of $G$, rather than "quadratic" features of the band limits such as the multiplicities $N(\psi)$. We address these issues in the following theorem.

**Theorem 8** *Let $G$ be a compact group with nonempty finite band limits $\mathcal{A}$ and $\mathcal{B}$ satisfying $\mathbf{E}(\mathcal{A}) \subset \mathcal{B}$. Define $n = \dim_\mathrm{I} \mathbf{E}(\mathcal{B})$ and $d = \max_{\rho\in\mathbf{E}(\mathcal{B})} \dim_\mathrm{I} \rho$. Then any $\mathcal{B}$-band-limited function $f$ satisfies*

$$\frac{\|\hat{f}(\mathbf{E}(\mathcal{A}))\|_2^2}{\|f\|_2^2} \geq \frac{\dim_\mathrm{I} \mathcal{A}}{8nd(3+\ln d)}.$$

*Proof* The notion of band limit extends immediately to characters: note that if $\mathcal{C}$ is a band limit, a character $\phi$ is *$\mathcal{C}$-band-limited* if and only if

$$\phi = \sum_{\rho\in\mathcal{C}} a_\rho \chi_\rho$$

for a family of non-negative integer coefficients $a_\rho$, $\rho \in \mathcal{C}$. We say that the character $\theta$ is *$\mathcal{C}$-regular* if $\langle \theta, \chi_\rho \rangle = d_\rho$ for all $\rho \in \mathcal{C}$. With this terminology in place, observe that if $\theta$ is $\mathcal{C}$-regular and $\phi = \sum a_\rho \chi_\rho$ is $\mathcal{C}$-band-limited, we have

$$\langle \theta, \phi \rangle = \sum_{\rho \in \mathcal{C}} a_\rho d_\rho = \phi(1).$$

Equivalently, $\langle \theta, \phi \rangle$ is equal to the dimension of any representation $\sigma$ for which $\chi_\sigma = \phi$.

Defining $\phi_{\mathcal{A}} = \sum_{\rho \in \mathcal{A}} d_\rho \chi_\rho$ we shall focus on the character $\psi = \phi_{\mathcal{A}} \bar{\phi}_{\mathcal{A}}$ and, as above, the quantity

$$N(\psi) = N(\phi_{\mathcal{A}} \bar{\phi}_{\mathcal{A}}) = \max_{\rho \in \mathbf{E}(\mathcal{A})} \frac{\langle \chi_\rho, \phi_{\mathcal{A}} \bar{\phi}_{\mathcal{A}} \rangle}{d_\rho}.$$

In order to estimate $N(\phi_{\mathcal{A}} \bar{\phi}_{\mathcal{A}})$, we introduce the band limit

$$\mathcal{A}' = \{\rho \in \hat{G} : \rho \prec \sigma_1 \otimes \sigma_2^* \otimes \sigma_3 \text{ for some } \sigma_1, \sigma_2, \sigma_3 \in \mathcal{A}\}$$

and the $\mathcal{A}'$-regular character $\theta = \sum_{\rho \in \mathcal{A}'} d_\rho \chi_\rho$. Since $\langle \mathbf{1}, \chi_\sigma \chi_{\sigma^*} \rangle = \langle \chi_\sigma, \chi_\sigma \rangle = 1$, the trivial representation is always a direct summand of $\sigma \otimes \sigma^*$; it follows that $\tau \prec \sigma \otimes \sigma^* \otimes \tau$ for any irreps $\sigma, \tau$. As $\mathcal{A}$ is nonempty, we conclude that $\mathcal{A} \subset \mathcal{A}'$. Note, also, that if $\sigma \in \mathbf{E}(\mathcal{A})$ and $\tau \in \mathcal{A}$ we have $\{\rho \in \hat{G} : \rho \prec \sigma \otimes \tau\} \subset \mathcal{A}'$ and, as $\theta$ is $\mathcal{A}'$-regular, that

$$\langle \chi_\sigma \chi_\tau, \theta \rangle = d_\sigma d_\tau \quad \text{and, by linearity,} \quad \langle \chi_\sigma \phi_{\mathcal{A}}, \theta \rangle = d_\sigma \dim_{\mathrm{I}} \mathcal{A}.$$

Returning now to estimate $N(\phi_{\mathcal{A}} \bar{\phi}_{\mathcal{A}})$, for any $\sigma \in \mathbf{E}(\mathcal{A})$ we have

$$\frac{\langle \chi_\sigma, \phi_{\mathcal{A}} \bar{\phi}_{\mathcal{A}} \rangle}{d_\sigma} = \frac{\langle \phi_{\mathcal{A}} \chi_\sigma, \phi_{\mathcal{A}} \rangle}{d_\sigma} \le \frac{\langle \phi_{\mathcal{A}} \chi_\sigma, \theta \rangle}{d_\sigma} = \dim_{\mathrm{I}} \mathcal{A},$$

where the inequality follows from the fact that $\langle \chi_\rho, \theta \rangle \ge \langle \chi_\rho, \phi_{\mathcal{A}} \rangle$ for all irreps $\rho \in \hat{G}$.

We now repeat the calculations from the previous section with the test function $\psi = \phi_{\mathcal{A}} \bar{\phi}_{\mathcal{A}}$.

$$\langle f^c, \phi_{\mathcal{A}} \bar{\phi}_{\mathcal{A}} \rangle \le N(\phi_{\mathcal{A}} \bar{\phi}_{\mathcal{A}}) \sum_{\rho \in \mathbf{E}(\mathcal{A})} \|\hat{f}(\rho)\|_{\mathrm{HS}}^2 \le \dim_{\mathrm{I}} \mathcal{A} \|\hat{f}(\mathbf{E}(\mathcal{A}))\|_2^2.$$

Continuing as before, but now with the quadrature rule from Theorem 6, we see that

$$\langle f^c, \phi_{\mathcal{A}} \bar{\phi}_{\mathcal{A}} \rangle \ge \|f\|_2^2 |\phi_{\mathcal{A}}(1)|^2 \|\nu\|_\infty \ge \frac{\|f\|_2^2 (\dim_{\mathrm{I}} \mathcal{A})^2}{8nd(3 + \ln d)}.$$

Combining the last two equations yields the result. □

We remark that for certain choices of $\mathcal{B}$, simple quadrature rules are easy to come by. For instance, suppose $G$ is compact and $H$ is a normal subgroup of finite index in $G$. The functions which are invariant under left translation by elements of

$H$ form a $[G:H]$-dimensional vector subspace of $L^2(G)$. This subspace is also a representation of $G$ under left translation; let $\mathcal{B}$ be the collection of its irreducible subrepresentations. It is easy to check that $\mathcal{B}$ is in fact isomorphic to the dual of the finite quotient group $G/H$. As a result, given any set $\{g_i\}$ of coset representatives, we can compute the Fourier transforms of a $\mathcal{B}$-band-limited function $f$ via the rule

$$\hat{f}(\rho) = \sqrt{d_\rho} \int f(x)\rho(x)^\dagger \, d\mu(x) = \sum_{i=1}^{[G:H]} \mu(g_i H) f(g_i)\rho(g_i)^\dagger$$

$$= \sum_{i=1}^{[G:H]} \frac{1}{[G:H]} f(g_i)\rho(g_i)^\dagger.$$

Hence, given any $\mathcal{A}$ such that $\mathbf{E}(\mathcal{A}) \subset \mathcal{B}$, the proof of Theorem 8 implies that

$$\frac{\|\hat{f}(\mathbf{E}(\mathcal{A}))\|_2^2}{\|f\|_2^2} \geq \frac{\dim_{\mathrm{I}} \mathcal{A}}{[G:H]}.$$

This includes the special case of finite groups and, in particular, Theorem 4 from [9]. For general compact groups and band limits, we would like to find quadrature rules with large weights, as well as a test function $\psi$ which maximizes $N(\psi)$. The positivity constraint makes finding such a function a nontrivial matter.

We point out another natural choice of test function, namely $\psi = \sum_{\rho \in \mathbf{E}(\mathcal{A})} \chi_\rho \bar{\chi}_\rho$. The result is a bound for a different portion of the spectrum; specifically, it controls the fraction of spectral mass appearing in representations $\rho$ which are irreducible summands of $\sigma \otimes \sigma^*$ for some $\sigma \in \mathbf{E}(\mathcal{A})$. It is easy to check that the bound in this case is

$$\frac{\dim_{\mathrm{I}} \mathcal{A}}{8nd(3 + \ln d)N(\psi)};$$

unfortunately, there seems to be no immediate way of explicitly controlling the multiplicity $N(\psi)$.

### 4.3 An Example: The Special Unitary Group

The special unitary group $\mathsf{SU}(2)$ is the set of all unitary transformations of $\mathbb{C}^2$ with determinant one. The representation theory of $\mathsf{SU}(2)$ is particularly attractive and easy to describe. Let $V_n$ denote the vector space of all homogeneous $n$-th degree polynomials over $\mathbb{C}$, in two variables:

$$V_n = \left\{a_0 z^n + a_1 z^{n-1} w + \cdots + a_{n-1} w^{n-1} z + a_n w^n : a_i \in \mathbb{C}\right\}.$$

There is a natural action of $\mathsf{SU}(2)$ on $f \in V_n$, given by $[U \cdot f](\mathbf{z}) = f(U^{-1}\mathbf{z})$ where $\mathbf{z}$ denotes the two-dimensional indeterminate vector $(z, w)$. It is easy to check that each $V_n$ is a representation of $\mathsf{SU}(2)$ under this action; moreover, they are pairwise inequivalent, since $\dim V_n = n + 1$. With some more work, one can show that each $V_n$ is self-dual and irreducible, and that these are the only irreducible representations of SU(2) [6]. We will denote the character of $V_n$ by $\chi_n$; in particular, $\chi_0$ will denote the trivial representation. We will also require the Clebsch-Gordan rule for SU(2), which is given by

$$V_n \otimes V_m = V_{|m-n|} \oplus V_{|m-n|+2} \oplus \cdots \oplus V_{m+n-2} \oplus V_{m+n}. \tag{4.3}$$

We will work with band limits $\mathcal{B}_n = \{V_0, V_1, \ldots, V_{n-1}, V_n\}$. Note that the maximum dimension of any irrep in $\mathcal{B}_n$ is $n+1$, and that the total dimension of the subalgebra of $L^2(\mathsf{SU}(2))$ corresponding to $\mathcal{B}_n$ is equal to the $(n+1)$-st square-pyramidal number (see [7] for a simple geometric proof):

$$P_n = \sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

By the results of Sect. 3, there exist quadrature rules for the space of $\mathcal{B}_n$-band-limited functions involving a number of samples that depends only on the dimensions of irreps in $\mathcal{B}_n \otimes {\mathcal{B}_n}^* = \mathcal{B}_{2n}$. Specifically, we require at most $8 \cdot P_{2n}(2n+1)(\ln(2n+1)+3)$ many samples.

Observe that the set $\mathcal{F}_n$ of positive functions which are band limited to $\mathcal{B}_n$ is nontrivial, since

$$\sum_{0 \le \ell \le n/2} a_\ell \cdot |\chi_\ell|^2 \in \mathcal{F}_n$$

for any non-negative coefficients $a_\ell \ge 0$. Consider now the fraction of energy a function $f \in \mathcal{F}_n$ has in the "low frequency" portion of the spectrum, say in $\mathcal{B}_m$ for some $m < n$. Theorem 8 asserts that

$$\frac{\|\hat{f}(\mathbf{E}(\mathcal{B}_m))\|_2^2}{\|f\|_2^2} \ge \frac{P_{m/2}}{8P_{2n}(2n+1)(\ln(2n+1)+3)},$$

where we assumed $m$ to be even for simplicity.

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