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Weak-Type Inequality for Conjugate First Order Riesz-Laguerre Transforms

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Abstract In this paper we introduce a conjugate class of Riesz transforms in the context of Laguerre polynomials. We prove their weak-type $(1, 1)$ and L^p, $1 < p < \infty$, boundedness with respect to the Laguerre measure. A similar result is known in the Hermite context, see Aimar et al. (Trans. Am. Math. Soc. 359(5), 2137–2154, 2007).

Keywords Laguerre expansions · Conjugate Riesz-Laguerre transforms · Conjugate Poisson integrals

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1 Introduction: Conjugate Riesz-Laguerre Transforms of Order 1

Singular integrals, specially Riesz transforms, associated to orthogonal systems have been a main topic of research since the 60's. These orthogonal systems usually are eigenfunctions of a Sturm-Liouville differential operator.

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In 2006, Nowak and Stempak [18] proposed a fairly general unified approach to the theory of Riesz transforms and their conjugate in the setting of multi-dimensional orthogonal expansions. In that paper they consider a second order differential operator $\mathcal{L}$ self-adjoint with respect to a measure $\mathrm{d}\mu$ such that it has a decomposition of the form

$$\mathcal{L}=\sum_{i=1}^{d}\delta_i^*\delta_i,$$

being δ_i and δ_i^* differential operators of first order. They assume that the spectrum of $\mathcal{L}$ on $L^2(\mathrm{d}\mu)$ consists of a discrete set of eigenvalues $\{\Lambda_i\}_{i=0}^{\infty}$ with $0\le\Lambda_0<\Lambda_1<\cdots$ and $\lim_{i\to\infty}\Lambda_i=\infty$. In order to define the conjugate Riesz transforms they introduce, for $i=1,\dots,d$, the operator $\mathcal{M}_i$, as being

$$\mathcal{M}_i=\mathcal{L}+[\delta_i,\delta_i^*]$$

where $[\cdot,\cdot]$ is the commutator defined as

$$[\delta_i,\delta_i^*]=\delta_i\delta_i^*-\delta_i^*\delta_i.$$

Then they define in L^2 the Riesz transforms of order 1 associated to δ_i and their conjugates, the Riesz transforms associated with δ_i^*, respectively as

$$R_i=\delta_i\mathcal{L}^{-1/2}\Pi_i,\quad \text{for } i=1,\dots,d$$

and

$$R_i^*=\delta_i^*\mathcal{M}_i^{-1/2}\Pi_i,\quad \text{for } i=1,\dots,d; \tag{1.1}$$

where Π_i denotes the orthogonal projection onto the closed subspace spanned by the eigenfunctions that are not in the kernel of either $\mathcal{L}$ or $\mathcal{M}_i$.

Under broad assumptions they proved the boundedness of those operators on $L^2(\mathrm{d}\mu)$. The theory includes the already classical expansions with Hermite, Laguerre, Jacobi polynomials among others. The L^p, $1<p<\infty$, boundedness and the weak-L^1 boundedness of the Riesz transforms associated to δ_i were already very well known for some systems. In particular, for Hermite polynomial expansions, the boundedness of the Riesz transforms associated to δ_i were known and extended to higher order Riesz transforms, see [3, 4, 6, 8–14, 19–21, 26], and [5]. The same is known for the Laguerre polynomial expansions, see [15, 16, 23], and [7]. For Jacobi polynomial expansions, only the L^p, $1<p<\infty$, boundedness is known, see [17].

For the conjugate Riesz transforms in the Hermite context the only known result for weak type $(1,1)$ is due to Aimar, Forzani, and Scotto [2] where they showed that the conjugate Riesz transforms of all orders are weak-type $(1,1)$ and this came as a surprise since this is not the case with the higher order Riesz transforms in the same context. In fact, these last ones are weak-type $(1,1)$ if and only if their order is at most 2.

In this paper we will investigate the boundedness of these conjugate Riesz transforms in the context of Laguerre polynomial expansions. For $\alpha=(\alpha_1,\dots,\alpha_d)$ with

$\alpha_i > -1$ for all $i = 1, \ldots, d$ the Laguerre operator

$$\mathcal{L}_\alpha = -\sum_{i=1}^{d} \left(x_i \frac{\partial^2}{\partial x_i^2} + (\alpha_i + 1 - x_i) \frac{\partial}{\partial x_i} \right)$$

is self-adjoint with respect to $d\mu_\alpha$,

$$d\mu_\alpha(x) = \prod_{i=1}^{d} x_i^{\alpha_i} e^{-x_i}\, dx,$$

and it can be written as the sum of the composition of two first order differential operators, which are in duality with respect to the Laguerre measure $d\mu_\alpha$, that is

$$\mathcal{L}_\alpha = \sum_{i=1}^{d} \delta_i^* \delta_i,$$

with $\delta_i = \sqrt{x_i} \frac{\partial}{\partial x_i}$ and $\delta_i^* = -\delta_i - \frac{\alpha_i + \frac{1}{2}}{\sqrt{x_i}} + \sqrt{x_i}$. From [18, p. 690] we have

$$\delta_i^* \delta_i L_k^\alpha = k_i L_k^\alpha, \tag{1.2}$$

where $L_k^\alpha(x) = \prod_{i=1}^{d} L_{k_i}^{\alpha_i}(x_i)$ is the d-dimensional Laguerre polynomial of degree $|k| = \sum_{i=1}^{d} k_i$ and of order α (see [25, p. 100]). Hence

$$\mathcal{L}_\alpha L_k^\alpha = |k| L_k^\alpha.$$

From [18, Lemma 5, p. 683], we also have

$$\mathcal{M}_i(\delta_i L_k^\alpha) = |k| \delta_i L_k^\alpha, \tag{1.3}$$

and the definition of the conjugate Riesz transforms of order 1 in $L^2(d\mu_\alpha)$ is given by (1.1) and Π_i, in this case, denotes the orthogonal projection onto the closed subspace spanned by the system $\{\delta_i L_k^\alpha\}_{\{k \in \mathbb{Z}^d_{\geq 0} : k_i > 0\}}$.

In order to extend this definition to $L^p(d\mu_\alpha)$ and prove the corresponding boundedness of these Riesz-Laguerre transforms we need to define the conjugate Poisson integrals associated with R_i^* as:

$$\widetilde{U}_t^i = e^{-\mathcal{L}_\alpha{}^{1/2} t} R_i^*, \quad i = 1, \ldots, d.$$

For $f \in L^2(d\mu_\alpha)$ they satisfy the following Cauchy-Riemann type differential system:

$$\delta_i^* e^{-\mathcal{M}_i{}^{1/2} t} \Pi_i f = -\frac{\partial}{\partial t} \widetilde{U}_t^i f, \quad i = 1, \ldots, d; \tag{1.4}$$

see [18, Sects. 6, 7.10].

Let us call

$$\widetilde{U}^i_* f = \sup_{t>0} \left| \widetilde{U}^i_t f \right| = \sup_{t>0} \left| \int_t^\infty \delta_i^* e^{-\mathcal{M}_i{}^{1/2} l} \Pi_i f \, \mathrm{d}l \right|,$$

where the second equality follows from (1.4).

For every multi-index α such as $\alpha_i \geq 0 \;\; \forall i = 1, \ldots, d$ the main results of this paper are:

Theorem 1.1 *For every $i = 1, \ldots, d$, $\widetilde{U}^i_*$ is bounded from $L^1(\mathrm{d}\mu_\alpha)$ into $L^{1,\infty}(\mathrm{d}\mu_\alpha)$.*

Theorem 1.2 *For every $i = 1, \ldots, d$, $\widetilde{U}^i_*$ is bounded on $L^p(\mathrm{d}\mu_\alpha)$ for $1 < p < \infty$.*

Now for $f \in L^1(\mathrm{d}\mu_\alpha)$ we define

$$\widetilde{R}^*_i f = \lim_{t \to 0^+} \widetilde{U}^i_t f,$$

which exists almost everywhere. Indeed, the claim follows taking into account Theorem 1.1 and the facts that for $k_i > 0$, there exists

$$\begin{aligned} \lim_{t\to 0} \widetilde{U}^i_t (\delta_i L^\alpha_k)(x) &= \lim_{t\to 0} \int_t^\infty \delta_i^* e^{-|k|^{1/2} l} \delta_i L^\alpha_k(x) \mathrm{d}l \\ &= \lim_{t\to 0} e^{-|k|^{1/2} t} \frac{\delta_i^* \delta_i L^\alpha_k(x)}{|k|^{1/2}} \\ &= \frac{k_i}{|k|^{1/2}} L^\alpha_k(x), \end{aligned}$$

and the span of $\{\delta_i L^\alpha_k\}_{\{k \in \mathbb{Z}^d_{\geq 0} : k_i > 0\}}$ is dense in $L^1(\mathrm{d}\mu_\alpha)$. The boundedness properties of $\widetilde{R}^*_i$ are given in the following corollary:

Corollary 1.1 *For every $i = 1, \ldots, d$, $\widetilde{R}^*_i = R^*_i$ on $L^2(\mathrm{d}\mu_\alpha)$, $\widetilde{R}^*_i$ is weak-type $(1,1)$, and it is bounded on $L^p(\mathrm{d}\mu_\alpha)$ for $1 < p < \infty$.*

In order to achieve the first result, we prove the $L^2(\mathrm{d}\mu_\alpha)$-boundedness of $\widetilde{U}^i_*$ in Sect. 2 and we simplify the notation and give an expression for the kernel in L^2 in Sect. 3. Then the proof of Theorem 1.1 is based upon the *usual* decomposition of the kernel of the singular operator into a local part, where we can use an adaptation of the Calderón-Zygmund theory to the Laguerre measure (Sect. 5), and in a global part, where we obtain good estimates of the kernel (Sect. 4). Section 6 will be devoted to the proof of Theorem 1.2 and within the proof of this theorem we will also have the proof of Corollary 1.1. Some proofs are given in the Appendix.

Throughout this paper the symbol $a \lesssim b$ means $a \leq Cb$ for some constant C that may be different on each occurrence. And we will write $a \sim b$ whenever $a \lesssim b$ and $b \lesssim a$.

2 L^2-Boundedness

Let us prove the $L^2(\mathrm{d}\mu_\alpha)$-boundedness of $\widetilde{U}_*^i$.

Proposition 2.1 *For every* $i = 1, \dots, d$, $\widetilde{U}_*^i$ *is bounded on* $L^2(\mathrm{d}\mu_\alpha)$.

Proof The boundedness

$$\|R_i^* f\|_{L^2(\mathrm{d}\mu_\alpha)} \le \|f\|_{L^2(\mathrm{d}\mu_\alpha)} \tag{2.1}$$

follows from [18].

Now taking into account that $\widetilde{U}_*^i f = \sup_{t>0} |e^{-\mathcal{L}_\alpha^{1/2} t} R_i^* f|$ then from the Maximal Theorem on Semigroups proved in [24, p. 73] and from (2.1) we get that

$$\|\widetilde{U}_*^i f\|_{L^2(\mathrm{d}\mu_\alpha)} = \|\sup_{t>0} |e^{-\mathcal{L}_\alpha^{1/2} t} R_i^* f|\|_{L^2(\mathrm{d}\mu_\alpha)} \lesssim \|R_i^* f\|_{L^2(\mathrm{d}\mu_\alpha)} \lesssim \|f\|_{L^2(\mathrm{d}\mu_\alpha)}. \qquad \square$$

3 Kernel of $\widetilde{U}_t^i f$ in $L^2(\mathrm{d}\mu_\alpha)$

Let $f \in L^2(\mathrm{d}\mu_\alpha)$, then $\widetilde{U}_t^i f$ satisfies the Cauchy-Riemann type differential system (1.4) and therefore

$$\widetilde{U}_t^i f(x) = \int_t^\infty \delta_i^* e^{-\mathcal{M}_i{}^{1/2} l} \Pi_i f(x)\, \mathrm{d}l.$$

As a consequence,

$$\widetilde{U}_t^i f(x) = \int_{\mathbb{R}_+^d} \int_0^1 \frac{e^{\frac{t^2}{4\log r}}}{\sqrt{\pi}} w(r) \widetilde{U}_i^\alpha(x, y, r) \frac{\mathrm{d}r}{\sqrt{r}} f(y)\, \mathrm{d}m_\alpha(y), \tag{3.1}$$

with $w(r) = (\frac{1-r}{-\log r})^{1/2}$,

$$\begin{aligned} \widetilde{U}_i^\alpha(x, y, r) &= \frac{1}{\alpha_i + 1/2} \int_{[-1,1]^d} \Bigg[\frac{\sqrt{r x_i y_i}(\sqrt{x_i} - \sqrt{r y_i} s_i)}{(1-r)^{|\alpha|+d+5/2}} \\ &\quad - \frac{(\alpha_i + 1)\sqrt{r y_i}}{(1-r)^{|\alpha|+d+3/2}} \Bigg] (1 - s_i^2) e^{-\frac{q_-(rx,y,s)}{1-r}} \Pi_\alpha(s)\, \mathrm{d}s \\ &:= \int_{[-1,1]^d} \widetilde{U}_i^\alpha(x, y, r, s) \Pi_\alpha(s)\, \mathrm{d}s \end{aligned} \tag{3.2}$$

being $q_-(x, y, s) = \sum_{j=1}^d (x_j + y_j - 2\sqrt{x_j y_j} s_j)$, $\Pi_\alpha(s) = \prod_{j=1}^d \frac{(1-s_j^2)^{\alpha_j - 1/2}}{\Gamma(\alpha_j + 1/2)\Gamma(1/2)}$, and $\mathrm{d}m_\alpha(y) = \prod_{j=1}^d e^{y_j} \mathrm{d}\mu_\alpha(y)$. See the Appendix for the proof of how to obtain kernel (3.2).

Thus (3.1) can be written in the following way

$$\widetilde{U}_t^i f(x) = \int_{\mathbb{R}_+^d} \int_{[-1,1]^d} \left(\int_0^1 \frac{e^{\frac{t^2}{4\log r}}}{\sqrt{\pi}} w(r) \widetilde{U}_i^\alpha(x,y,r,s) \frac{dr}{\sqrt{r}} \right) f(y) \Pi_\alpha(s)\, \mathrm{d}s\, \mathrm{d}m_\alpha(y). \tag{3.3}$$

In order to pass from the Laguerre context into a Gaussian-like context, we are going to perform a change of coordinates in $\mathbf{R}_+^d$. Let $\Psi : \mathbf{R}_+^d \to \mathbf{R}_+^d$ be defined as $\Psi(x) = x^2$, where for $x \in \mathbf{R}_+^d$, $x = (x_1, \dots, x_d)$, $x^2 = (x_1^2, \dots, x_d^2)$. Let $\mathrm{d}\tilde{\mu}_\alpha = \mathrm{d}\mu_\alpha \circ \Psi^{-1}$ be the pull-back measure from $\mathrm{d}\mu_\alpha$. Then the modified Laguerre measure $\mathrm{d}\tilde{\mu}_\alpha$ is given by

$$\mathrm{d}\tilde{\mu}_\alpha(x) = e^{-|x|^2} \mathrm{d}\widetilde{m}_\alpha(x),$$

where $\widetilde{m}_\alpha$ is the polynomial measure on $\mathbf{R}_+^d$ defined as

$$\mathrm{d}\widetilde{m}_\alpha(x) = 2^d \prod_{j=1}^d x_j^{2\alpha_j+1} \mathrm{d}x.$$

The map $f \to \mathcal{U}_\Psi f = f \circ \Psi$ is an isometry from $L^q(\mathrm{d}\mu_\alpha)$ onto $L^q(\mathrm{d}\tilde{\mu}_\alpha)$ and from $L^{q,\infty}(\mathrm{d}\mu_\alpha)$ onto $L^{q,\infty}(\mathrm{d}\tilde{\mu}_\alpha)$, for every q in $[1,\infty]$. So we may reduce the problem of studying the weak-type (1, 1) of $\widetilde{U}_*^i$ to the study of the same boundedness for the modified maximal conjugate Poisson integrals $U_*^i = \mathcal{U}_\Psi \widetilde{U}_*^i \mathcal{U}_\Psi^{-1}$ with respect to the measure $\mathrm{d}\tilde{\mu}_\alpha$. Thus

$$U_*^i f(x) = \sup_{t>0} \left| \int_{\mathbb{R}_+^d} \int_{[-1,1]^d} \left(\int_0^1 \frac{e^{\frac{t^2}{4\log r}}}{\sqrt{\pi}} w(r) U_i^\alpha(x,y,r,s) \frac{dr}{\sqrt{r}} \right) \Pi_\alpha(s) \mathrm{d}s f(y) \mathrm{d}\widetilde{m}_\alpha(y) \right|$$

with

$$\begin{aligned} U_i^\alpha(x,y,r,s) &= \widetilde{U}_i^\alpha(x^2,y^2,r,s) \\ &= C_\alpha \left[\frac{\sqrt{r} x_i y_i (x_i - \sqrt{r} y_i s_i)}{(1-r)^{|\alpha|+d+5/2}} - \frac{(\alpha_i+1)\sqrt{r} y_i}{(1-r)^{|\alpha|+d+3/2}} \right] \\ &\quad \times (1-s_i^2) e^{-\frac{q_-(rx^2,y^2,s)}{1-r}}. \end{aligned} \tag{3.4}$$

Due to the isometry $\mathcal{U}_\Psi$ the proof of Theorem 1.1 is a consequence of

Theorem 3.1 *The conjugate Poisson integrals U_*^i are bounded from $L^1(\mathrm{d}\tilde{\mu}_\alpha)$ into $L^{1,\infty}(\mathrm{d}\tilde{\mu}_\alpha)$.*

In order to prove Theorem 3.1 we need to apply the usual technique in this field: first we define the local region

$$N_\tau = \left\{(x,y,s) \in \mathbf{R}_+^d \times \mathbf{R}_+^d \times [-1,1]^d : q_-^{1/2}(x^2,y^2,s) \le \frac{C\tau}{1+|x|}\right\}, \quad \tau > 0$$

and then we divide the operator into the local and global operators by using a cut-off function

$$\varphi(x,y,s) = \begin{cases} 1 & \text{if } (x,y,s) \in N_1 \\ 0 & \text{if } (x,y,s) \notin N_2 \end{cases}$$

with $\varphi \in C^\infty$, $0 \le \varphi \le 1$, and $|\nabla_{(x,y)}\varphi(x,y,s)| \lesssim \frac{1}{q_-^{1/2}(x^2,y^2,s)}$.

We define the global operator as being:

$$U^i_{*,global} f(x) = \sup_{t>0} \left| \int_{\mathbb{R}_+^d} \int_{[-1.1]^d} \overline{\mathcal{K}}_i(x,y,s,t)(1-\varphi(x,y,s))\Pi_\alpha(s) \mathrm{d}s f(y) \mathrm{d}\widetilde{m}_\alpha(y) \right|$$

with

$$\overline{\mathcal{K}}_i(x,y,s,t) = \int_0^1 \frac{e^{\frac{t^2}{4\log r}}}{\sqrt{\pi}} w(r)\, U_i^\alpha(x,y,r,s) \frac{dr}{\sqrt{r}}. \tag{3.5}$$

The local operator will be:

$$U^i_{*,local} f(x) = \sup_{t>0} \left| \int_{\mathbb{R}_+^d} \int_{[-1.1]^d} \overline{\mathcal{K}}_i(x,y,s,t)\varphi(x,y,s)\Pi_\alpha(s) \mathrm{d}s f(y) \mathrm{d}\widetilde{m}_\alpha(y) \right|,$$

(see for example [23]). Hence

$$U^i_* f(x) \le U^i_{*,local} f(x) + U^i_{*,global} f(x), \tag{3.6}$$

and also

$$U^i_{*,local} f(x) \le U^i_* f(x) + U^i_{*,global} f(x). \tag{3.7}$$

From inequality (3.6) Theorem 3.1 will be a consequence of Lemmas 3.1 and 3.2 that deal with the boundedness of the operator on each part.

Lemma 3.1 *The operator $U^i_{*,global}$ is bounded from $L^1(\mathrm{d}\tilde{\mu}_\alpha)$ into $L^{1,\infty}(\mathrm{d}\tilde{\mu}_\alpha)$.*

and

Lemma 3.2 *The operator $U^i_{*,local}$ is bounded from $L^1(\mathrm{d}\tilde{\mu}_\alpha)$ into $L^{1,\infty}(\mathrm{d}\tilde{\mu}_\alpha)$.*

These lemmas will be proved in Sects. 4 and 5 respectively.

Remark 3.8 Let us recall that $q_\pm(x,y,s) = \sum_{j=1}^d (x_j + y_j \pm 2\sqrt{x_j y_j} s_j)$ and since there will be some calculations involving $q_\pm(x^2,y^2,s)$ it is convenient to think of this expression as being a distance between vectors in a space of a higher dimension. Namely, let $n=(n_1,\dots,n_d)\in\mathbf{N}^d$ and let $X=(X_1,\dots,X_d)$ and $Y=(Y_1,\dots,Y_d)$ be two vectors in $\mathbf{R}^{|n|}$ with $X_i, Y_i\in\mathbf{R}^{n_i}$ such that $|X_i|=x_i$ and $|Y_i|=y_i$, for every $i=1,\dots,d$. Let φ_i be the angle between X_i and Y_i. Calling $s_i=\cos\varphi_i$, we have

$$\begin{aligned}q_\pm(x^2,y^2,s)&=|X\pm Y|^2\\&=|x|^2+|y|^2\pm 2|x||y|\cos\theta\end{aligned}$$

where the symbol in the first equality stands for the Euclidean distance in $\mathbf{R}^{|n|}$ and $|x||y|\cos\theta=\sum_{j=1}^d x_j y_j s_j$.

In the proof of the lemmas above we will use some inequalities, some of them were already proven in [7]. For the sake of completeness we will write them all.

$$\frac{1-r}{-\log r}\le C \quad \text{on the interval } (0,1), \tag{3.9}$$

$$|x_i-\sqrt{r}y_is_i|\le q_-^{1/2}(x^2,ry^2,s), \tag{3.10}$$

$$\sqrt{r}x_iy_i(1-s_i^2)\le q_-(x^2,ry^2,s), \tag{3.11}$$

$$\sqrt{r}y_i(1-s_i^2)\le\sqrt{r}y_i\sqrt{(1-s_i^2)}\le q_-^{1/2}(x^2,ry^2,s). \tag{3.12}$$

Let us prove (3.10). $q_-(x^2,ry^2,s)\ge x_i^2+ry_i^2-2\sqrt{r}x_iy_is_i\ge x_i^2+rs_i^2y_i^2-2\sqrt{r}x_iy_is_i=|x_i-\sqrt{r}y_is_i|^2$. Here the first inequality is immediate from the definition of q_- and the second one is due to $s_i\in[-1,1]$.

Let us prove now (3.11): $q_-(x^2,ry^2,s)=|x-\sqrt{r}y|^2+2\sqrt{r}\sum_{j=1}^d x_jy_j(1-s_j)\ge\sqrt{r}x_iy_i2(1-s_i)\ge\sqrt{r}x_iy_i(1-s_i^2)$, since $x_i, y_i>0$.

Finally let us prove (3.12): $q_-(x^2,ry^2,s)=|x|^2+r|y|^2-\sum_{j=1}^d 2x_j\sqrt{r}y_js_j\ge |x|^2+r|y|^2-(|x|^2+r\sum_{j=1}^d y_j^2s_j^2)=r\sum_{j=1}^d y_j^2(1-s_j^2)\ge ry_i^2(1-s_i^2)$.

Now we are going to define the global region and write out inequalities on every subregion in which this global region is split. These inequalities will be used in the proof of $\widetilde{U}^i_{*,global}$-boundedness.

Let us call $G=\mathbf{R}_+^d\times[-1,1]^d\setminus N_1^x$ the global region, being N_1^x the section of N_1 at a fixed level x, i.e. $N_1^x=\{(y,s):\ (x,y,s)\in N_1\}$.

We divide the global region as $G=R_1\cup R_2\cup R_3\cup R_4$, with

$$\begin{aligned}&R_1=\{(y,s)\notin N_1^x:\cos\theta<0\},\\&R_2=\{(y,s)\notin N_1^x:\cos\theta\ge 0, |y|\le|x|\},\\&R_3=\{(y,s)\notin N_1^x:\cos\theta\ge 0, |x|\le|y|\le 2|x|\},\\&R_4=\{(y,s)\notin N_1^x:\cos\theta\ge 0, |y|\ge 2|x|\}.\end{aligned}$$

For $0<l\le 1$ let us define

$$u(l)=\frac{q_-((1-l)x^2,y^2,s)}{l}.$$

Then

$$u(l) = \frac{q_-((1-l)x^2, y^2, s)}{l} \tag{3.13}$$

$$= \frac{|x|^2+|y|^2}{l} - \frac{\sqrt{1-l}}{l} 2|x||y|\cos\theta - |x|^2 \tag{3.14}$$

$$= \frac{q_-(x^2, (1-l)y^2, s)}{l} - |x|^2 + |y|^2. \tag{3.15}$$

By doing the same computations as in [12, p. 849, Proposition 2.1, with $t = l$ and $t_0 = l_0$] we have on G that

$$\sup_{0<l<1} \frac{e^{-u(l)}}{l^{|\alpha|+d}} \sim \frac{e^{-u_0}}{l_0^{|\alpha|+d}} \tag{3.16}$$

with $u_0 = u(l_0)$ and $l_0 \in (0, 1]$. By setting $a = |x|^2 + |y|^2$ and $b = 2|x||y|\cos\theta$ and considering $b \geq 0$ we obtain that $l_0 \sim \frac{\sqrt{a^2-b^2}}{a}$. Since $q_- = a - b$, $q_+ = a + b$, then $a \sim q_+$ and

$$l_0 \sim \sqrt{\frac{q_-}{q_+}}. \tag{3.17}$$

Moreover, since $\sqrt{q_+q_-} \geq ||x|^2 - |y|^2|$,

$$\begin{aligned} u(l) \geq u_0 = u(l_0) &= \frac{|y|^2 - |x|^2}{2} + \frac{(q_+q_-)^{1/2}}{2} \\ &= |y|^2 - |x|^2 + \frac{(q_+q_-)^{1/2} + |x|^2 - |y|^2}{2} \\ &\geq |y|^2 - |x|^2, \end{aligned} \tag{3.18}$$

and also

$$u_0 \leq (q_+q_-)^{1/2}. \tag{3.19}$$

And from [5] we will use inequality (10):

$$\int_0^1 u^{1/2}(l) e^{-\nu u(l)} \frac{dl}{l^{3/2}\sqrt{1-l}} \lesssim \frac{e^{-\nu u_0}}{l_0^{1/2}}. \tag{3.20}$$

In region R_1

$$u(l) \geq \frac{a}{l} - |x|^2 \quad \text{and} \quad a \geq c. \tag{3.21}$$

Indeed, the first inequality follows from (3.14) since $\cos\theta < 0$ on R_1. On the other hand, taking into account again that $\cos\theta < 0$ on R_1, the second inequality follows from the reasoning below

$$2a = |x|^2 + |y|^2 + |x|^2 + |y|^2 \geq |x|^2 + |y|^2 - 2|x||y|\cos\theta = q_-(x^2, y^2, s)$$
$$\geq \frac{C^2}{(1+|x|)^2} \geq \frac{C^2}{2(1+|x|^2)} \geq \frac{C^2}{2(1+a)}.$$

And from this $a \geq c$.

In Region R_2, from (1) of [7, p. 263] we know that

$$|x| \geq c > 0, \qquad q_+ \sim |x|^2, \quad \text{and} \quad (q_+/q_-)^{\frac{1}{2}} \lesssim |x|^2. \tag{3.22}$$

On the other hand from (3.18) we obtain that

$$u_0 \geq |y|^2 - |x|^2 + \frac{(q_+q_-)^{1/2}}{2}. \tag{3.23}$$

In Region R_3 we have

$$u_0 \geq |y|^2 - |x|^2 + \frac{c|x|^4 \sin^2\theta}{|y|^2 - |x|^2 + |x||y|\sin\theta}, \quad u_0 \geq c > 0, \tag{3.24}$$

(see (3.4) of [7]) and

$$(q_+q_-)^{1/2} \sim |y|^2 - |x|^2 + |x||y|\sin\theta \geq |y|^2 - |x|^2 + |x|^2\sin\theta, \tag{3.25}$$

(see (3.3) of [7]). Using the fact that in R_3, $|x| \leq |y| \leq 2|x|$, (3.19), and the second inequality of (3.24) we get

$$q_+ \leq C|x|^2 \quad \text{and} \quad q_+q_- \geq c > 0. \tag{3.26}$$

In R_4 the following inequalities hold, their proofs can be found in page 265 of [7]:

$$q_- \geq c|y|^2, \qquad q_+ \lesssim |y|^2, \qquad u_0 \geq c > 0, \quad \text{and} \quad u_0 \geq |y|^2 - |x|^2 + \frac{\sin^2\theta}{2}|x|^2. \tag{3.27}$$

4 Boundedness of $\widetilde{U}^i_{*,global}$

In this section we are going to prove Lemma 3.1 and the L^p-boundedness of $\widetilde{U}^i_{*,global}$ for $1 < p < \infty$.

Let us observe that

$$U^i_{*,global} f(x) \lesssim \int_{\mathbb{R}^d_+} \int_{[-1.1]^d} \chi_G(y,s) |\overline{\mathcal{K}}_i(x,y,s,t)| \Pi_\alpha(s) ds |f(y)| d\widetilde{m}_\alpha(y),$$

then Lemma 3.1 is a consequence of the following propositions:

Proposition 4.1

$$|\bar{\mathcal{K}}_i(x,y,s,t)| \lesssim e^{|x|^2}\mathcal{K}^*(x,y,s)e^{-|y|^2}$$

on the global region G with

$$\mathcal{K}^*(x,y,s)=\begin{cases} e^{-\frac{|x|^2}{2}}, & (y,s)\in R_1,\\ |x|^{2|\alpha|+2d}e^{-c|x|q_-^{1/2}(x^2,y^2,s)}, & (y,s)\in R_2,\\ |x|^{2|\alpha|+2d}\left(1\wedge \dfrac{e^{-\frac{c|x|^4\sin^2\theta}{|y|^2-|x|^2+\sin\theta|x|^2}}}{(|y|^2-|x|^2+\sin\theta|x|^2)^{\frac{2|\alpha|+2d-1}{2}}}\right), & (y,s)\in R_3,\\ (1+|x|)e^{-\sin^2\theta|x|^2}, & (y,s)\in R_4. \end{cases}$$

Proposition 4.2 *The operator* $\mathcal{K}^*$ *defined as*

$$\mathcal{K}^*f(x)=e^{|x|^2}\int_{\mathbf{R}^d_+}\int_{[-1,1]^d}\chi_G(y,s)\mathcal{K}^*(x,y,s)\Pi_\alpha(s)\mathrm{d}s|f(y)|\mathrm{d}\tilde{\mu}_\alpha(y),$$

is of weak type $(1,1)$ *with respect to the measure* $\tilde{\mu}_\alpha$.

The proof of Proposition 4.2 is given in [7, Proposition 2.2], except for the region R_1 but the operator on this region turns out to be bounded on $L^1(\mathrm{d}\tilde{\mu}_\alpha)$ as can be verified easily.

Proof of Proposition 4.1 By definition (3.5) of $\overline{\mathcal{K}}_i(x,y,s,t)$, definition (3.4) of $U_i^\alpha(x,y,s,t)$ and taking into account inequalities (3.9)–(3.12) we obtain that

$$\begin{aligned}|\overline{\mathcal{K}}_i(x,y,s,t)| &\lesssim \int_0^1 \sqrt{r}y_ix_i(1-s_i^2)|x_i-\sqrt{r}y_is_i|\frac{e^{-\frac{q_-(rx^2,y^2,s)}{1-r}}}{(1-r)^{|\alpha|+d+5/2}}\frac{\mathrm{d}r}{\sqrt{r}}\\ &\quad+\int_0^1\sqrt{r}y_i(1-s_i^2)\frac{e^{-\frac{q_-(rx^2,y^2,s)}{1-r}}}{(1-r)^{|\alpha|+d+3/2}}\frac{\mathrm{d}r}{\sqrt{r}}\\ &\lesssim \int_0^1 (q_-(x^2,ry^2,s))^{3/2}\frac{e^{-\frac{q_-(rx^2,y^2,s)}{1-r}}}{(1-r)^{|\alpha|+d+5/2}}\frac{\mathrm{d}r}{\sqrt{r}}\\ &\quad+\int_0^1 (q_-(x^2,ry^2,s))^{1/2}\frac{e^{-\frac{q_-(rx^2,y^2,s)}{1-r}}}{(1-r)^{|\alpha|+d+3/2}}\frac{\mathrm{d}r}{\sqrt{r}}.\end{aligned}$$

In these last integrals we make the change of variables $l=1-r$ and taking into account (3.13) and (3.15) we have

$$|\overline{\mathcal{K}}_i(x,y,s,t)|\lesssim\int_0^1(1\vee(u(l)+|x|^2-|y|^2)^{3/2})\frac{e^{-u(l)}}{l^{|\alpha|+d+1}}\frac{\mathrm{d}l}{\sqrt{1-l}}. \tag{4.1}$$

Now, we have the following lemmas, whose proofs can be found in the Appendix.

Lemma 4.3 *For $\nu > 0$ and $m > 0$ we have*

$$(1 \vee (u(l) + |x|^2 - |y|^2)^m) e^{-\nu u(l)} \lesssim e^{\nu |x|^2} e^{-\nu |y|^2}.$$

Lemma 4.4 *For $0 < \eta < 1$, $q_-^{1/2}(x^2, y^2, s) > \frac{C}{1+|x|}$, $\cos\theta \ge 0$ and $|x| \ge |y|$, we have*

$$\int_0^1 \frac{e^{-\eta u(l)}}{l^{|\alpha|+d+1}} \frac{dl}{\sqrt{1-l}} \lesssim |x|^{2|\alpha|+2d}.$$

Lemma 4.5 *For u_0 defined in (3.18) with $u_0 \gtrsim 1$, we have*

$$\int_0^1 (1 \vee (u(l) + |x|^2 - |y|^2)^{3/2}) \frac{e^{-u(l)}}{l^{|\alpha|+d+1}} \frac{dl}{\sqrt{1-l}} \lesssim e^{\nu |x|^2} e^{-\nu |y|^2} \left(\frac{q_+}{q_-}\right)^{(|\alpha|+d)/2} e^{-(1-\nu)u_0},$$

with $\nu = \frac{1}{4(|\alpha|+d)}$.

We prove now the boundedness of $|\overline{\mathcal{K}}_i(x, y, s, t)|$ on each region R_j, $j = 1, 2, 3, 4$.

Region R_1: Using Lemma 4.3 with $m = 3/2$ and $\nu = 1/2$ and taking into account (3.21): $\frac{a}{l} - |x|^2 \le u(l)$ and $a \ge c$, we have

$$\begin{aligned} &\int_0^1 (1 \vee (u(l) + |x|^2 - |y|^2)^{3/2}) \frac{e^{-u(l)}}{l^{|\alpha|+d+1}} \frac{dl}{\sqrt{1-l}} \\ &\lesssim e^{\frac{|x|^2}{2}} \int_0^1 \frac{e^{-\frac{a}{2l} + \frac{|x|^2}{2}}}{l^{|\alpha|+d+1}} \frac{dl}{\sqrt{1-l}} e^{\frac{-|y|^2}{2}} \\ &\lesssim e^{|x|^2} e^{-\frac{a}{2}} \int_0^\infty e^{-v} \left(\frac{2v}{a} + 1\right)^{|\alpha|+d} \frac{dv}{\sqrt{v + \frac{a}{2}}\sqrt{v}} e^{\frac{-|y|^2}{2}} \lesssim e^{\frac{|x|^2}{2}} e^{-|y|^2}, \end{aligned} \tag{4.2}$$

where we made the change of variables $v = \frac{a}{2l} - \frac{a}{2}$. Therefore for $(y, s) \in R_1$ we have

$$|\mathcal{K}(x, y, s)| \lesssim e^{|x|^2} \mathcal{K}^*(x, y, s) e^{-|y|^2}.$$

Region R_2: In order to prove the boundedness in R_2 we need to consider the cases $u_0 \lesssim 1$ and $u_0 \gtrsim 1$ separately.

If $u_0 \lesssim 1$, taking into account Lemma 4.3 with $m = 3/2$, $0 < \nu < 1$ and Lemma 4.4 with $\eta = 1 - \nu$, (3.22) and (3.23) we obtain that

$$\begin{aligned} &\int_0^1 (1 \vee (u(l) + |x|^2 - |y|^2)^{3/2}) \frac{e^{-u(l)}}{l^{|\alpha|+d+1}} \frac{dl}{\sqrt{1-l}} \\ &\lesssim e^{\nu |x|^2} e^{-\nu |y|^2} \int_0^1 \frac{e^{-(1-\nu)u(l)}}{l^{|\alpha|+d+1}} \frac{dl}{\sqrt{1-l}} \end{aligned}$$

$$\begin{aligned}
&\lesssim e^{\nu|x|^2}e^{-\nu|y|^2}|x|^{2|\alpha|+2d}\\
&\lesssim e^{\nu|x|^2}e^{-\nu|y|^2}|x|^{2|\alpha|+2d}e^{-(1-\nu)u_0}\\
&\lesssim |x|^{2|\alpha|+2d}e^{|x|^2}e^{-|y|^2}e^{-\frac{(1-\nu)(q_+q_-)^{1/2}}{2}}\\
&\lesssim |x|^{2|\alpha|+2d}e^{|x|^2}e^{-|y|^2}e^{-c|x|q_-^{1/2}}.
\end{aligned}$$

From inequality (4.1) and this last estimate, $|\overline{\mathcal{K}}_i(x,y,s,t)|\lesssim e^{|x|^2}\mathcal{K}^*(x,y,s)e^{-|y|^2}$.

If $u_0\gtrsim 1$, from inequality (4.1), Lemma 4.5, (3.22) and (3.23) we get

$$\begin{aligned}
|\overline{\mathcal{K}}_i(x,y,s,t)| &\lesssim e^{\nu|x|^2}e^{-\nu|y|^2}|x|^{2|\alpha|+2d}e^{-(1-\nu)u_0}\\
&\lesssim |x|^{2|\alpha|+2d}e^{|x|^2}e^{-|y|^2}e^{-c(q_+q_-)^{1/2}}\\
&\lesssim |x|^{2|\alpha|+2d}e^{|x|^2}e^{-|y|^2}e^{-c|x|q_-^{1/2}} = e^{|x|^2}\mathcal{K}^*(x,y,s)e^{-|y|^2}.
\end{aligned}$$

Region R_3: Taking into account inequality (4.1), Lemma 4.5, (3.24), (3.26) and (3.25), we have

$$\begin{aligned}
|\overline{\mathcal{K}}_i(x,y,s,t)| &\lesssim e^{\nu|x|^2}e^{-\nu|y|^2}\left(\frac{q_+}{q_-}\right)^{(|\alpha|+d)/2} e^{-(1-\nu)\left(|y|^2-|x|^2+\frac{c|x|^4\sin^2\theta}{|y|^2-|x|^2+|x||y|\sin\theta}\right)}\\
&= e^{|x|^2}e^{-|y|^2}\left(\frac{q_+}{q_-}\right)^{(|\alpha|+d)/2} e^{-\frac{\tilde{c}|x|^4\sin^2\theta}{|y|^2-|x|^2+|x||y|\sin\theta}}\\
&= e^{|x|^2}e^{-|y|^2}\frac{q_+^{|\alpha|+d}}{(q_+q_-)^{1/4}}\frac{e^{-\frac{\tilde{c}|x|^4\sin^2\theta}{|y|^2-|x|^2+|x||y|\sin\theta}}}{(q_+q_-)^{\frac{2|\alpha|+2d-1}{4}}}\\
&\lesssim e^{|x|^2}e^{-|y|^2}|x|^{2|\alpha|+2d}\frac{e^{-\frac{\tilde{c}|x|^4\sin^2\theta}{|y|^2-|x|^2+|x||y|\sin\theta}}}{\left(|y|^2-|x|^2+\sin\theta|x|^2\right)^{\frac{2|\alpha|+2d-1}{2}}}.
\end{aligned}$$

On the other hand taking into account the first inequality above and (3.26), it is immediate this other inequality:

$$|\overline{\mathcal{K}}_i(x,y,s,t)|\lesssim |x|^{2|\alpha|+2d}e^{|x|^2}e^{-|y|^2}.$$

Thus, for $(y,s)\in R_3$, $|\overline{\mathcal{K}}_i(x,y,s,t)|\lesssim e^{|x|^2}\mathcal{K}^*(x,y,s)e^{-|y|^2}$.

Region R_4: Using Lemma 4.5 and (3.27) we get

$$\begin{aligned}
|\overline{\mathcal{K}}_i(x,y,s)| &\lesssim e^{\nu|x|^2}e^{-\nu|y|^2}\left(\frac{q_+}{q_-}\right)^{(|\alpha|+d)/2} e^{-(1-\nu)u_0}\\
&\lesssim e^{|x|^2}e^{-|y|^2}e^{-c\sin^2\theta|x|^2} = e^{|x|^2}\mathcal{K}^*(x,y,s)e^{-|y|^2}. \qquad \square
\end{aligned}$$

Now we prove the L^p-boundedness of $\widetilde{U}^i_{*,global}$.

Theorem 4.1 *For* $1 < p < \infty$, *the operator* $U^i_{*,global}$ *is bounded on* $L^p(\mathrm{d}\tilde{\mu}_\alpha)$.

Proof *Region* R_1: By Lemma 4.3 with $m = \frac{3}{2}$ and $0 < \nu < 1/p$

$$\begin{aligned}
&\int_0^1 (1 \vee (u(l) + |x|^2 - |y|^2)^{3/2}) \frac{e^{-u(l)}}{l^{|\alpha|+d+1}} \frac{\mathrm{d}l}{\sqrt{1-l}} \\
&\quad = \int_0^1 (1 \vee (u(l) + |x|^2 - |y|^2)^{3/2}) \frac{e^{-\nu u(l)}}{l^{|\alpha|+d+1}} e^{-(1-\nu)u(l)} \frac{\mathrm{d}l}{\sqrt{1-l}} \\
&\quad \lesssim e^{\nu|x|^2 - \nu|y|^2} \int_0^1 \frac{e^{-(1-\nu)u(l)}}{l^{|\alpha|+d+1}} \frac{\mathrm{d}l}{\sqrt{1-l}}.
\end{aligned}$$

By using that $\frac{a}{l} - |x|^2 \le u(l)$ and then the change of variables $v = \frac{(1-\nu)a}{l} - (1-\nu)a$, the fact that $a \ge c$, and following similar estimates as the ones done in (4.2) we obtain that

$$e^{\nu|x|^2 - \nu|y|^2} \int_0^1 \frac{e^{-(1-\nu)u(l)}}{l^{|\alpha|+d+1}} \frac{\mathrm{d}l}{\sqrt{1-l}} \lesssim e^{\nu|x|^2} e^{-|y|^2}.$$

So on R_1

$$|\overline{\mathcal{K}}_i(x, y, s, t)| \lesssim e^{|x|^2} \left(e^{-(1-\nu)|x|^2} \right) e^{-|y|^2},$$

and the $L^p(\mathrm{d}\tilde{\mu}_\alpha)$-boundedness of $U^i_{*,global}$ on region R_1 follows. Indeed, for $f \in L^p(\mathrm{d}\tilde{\mu}_\alpha)$, since $p\nu - 1 < 0$, we obtain that

$$\|U^i_{*,global}(\chi_{R_1} f)\|_{L^p(\mathrm{d}\tilde{\mu}_\alpha)} \lesssim \|e^{\nu|\cdot|^2}\|_{L^p(\mathrm{d}\tilde{\mu}_\alpha)} \|f\|_{L^p(\mathrm{d}\tilde{\mu}_\alpha)} \lesssim \|f\|_{L^p(\mathrm{d}\tilde{\mu}_\alpha)}.$$

Region $G \setminus R_1$: From the proof of Proposition 4.1 we get the following inequality

$$\begin{aligned}
|\overline{\mathcal{K}}_i(x, y, s, t)| &\lesssim \int_0^1 \sqrt{r} y_i x_i (1 - s_i^2) |x_i - \sqrt{r} y_i s_i| \frac{e^{-\frac{q_-(rx^2, y^2, s)}{1-r}}}{(1-r)^{|\alpha|+d+5/2}} \frac{\mathrm{d}r}{\sqrt{r}} \\
&\quad + \int_0^1 \sqrt{r} y_i (1 - s_i^2) \frac{e^{-\frac{q_-(rx^2, y^2, s)}{1-r}}}{(1-r)^{|\alpha|+d+3/2}} \frac{\mathrm{d}r}{\sqrt{r}} \\
&\lesssim \int_0^1 (q_-(x^2, ry^2, s))^{3/2} \frac{e^{-\frac{q_-(rx^2, y^2, s)}{1-r}}}{(1-r)^{|\alpha|+d+5/2}} \frac{\mathrm{d}r}{\sqrt{r}} \\
&\quad + \int_0^1 (q_-(x^2, ry^2, s))^{1/2} \frac{e^{-\frac{q_-(rx^2, y^2, s)}{1-r}}}{(1-r)^{|\alpha|+d+3/2}} \mathrm{d}r \\
&= \int_0^1 (q_-(x^2, (1-l)y^2, s))^{3/2} \frac{e^{-\frac{q_-((1-l)x^2, y^2, s)}{l}}}{l^{|\alpha|+d+5/2}} \frac{\mathrm{d}l}{\sqrt{1-l}}
\end{aligned}$$

$$+\int_0^1 (q_-(x^2,(1-l)y^2,s))^{1/2}\frac{e^{-\frac{q_-((1-l)x^2,y^2,s)}{l}}}{l^{|\alpha|+d+3/2}}\frac{\mathrm{d}l}{\sqrt{1-l}}$$
$$= I + II.$$

Let us set

$$\tilde{u}(l)=\frac{q_-(x^2,(1-l)y^2,s)}{l}.$$

From (3.15) $\tilde{u}(l)=u(l)+|x|^2-|y|^2$, we can rewrite I and II as

$$I=\int_0^1(\tilde{u}(l))^{3/2}\frac{e^{-\tilde{u}(l)}}{l^{|\alpha|+d+1}}\frac{\mathrm{d}l}{\sqrt{1-l}}\,e^{|x|^2-|y|^2},$$
$$II=\int_0^1(\tilde{u}(l))^{1/2}\frac{e^{-\tilde{u}(l)}}{l^{|\alpha|+d+1}}\frac{\mathrm{d}l}{\sqrt{1-l}}\,e^{|x|^2-|y|^2}.$$

We bound I and II separately.

Let us observe that by an adaptation of the proof of Lemma 4.3 in [12] with $m=3$ (exchanging x with y) we get

$$I\lesssim e^{|x|^2-|y|^2}e^{-u_0(y,x,s)}\frac{1+u_0(y,x,s)}{l_0^{|\alpha|+d}}$$

with $u_0(x,y,s)=\frac{|y|^2-|x|^2}{2}+\frac{(q_+q_-)^{1/2}}{2}$. By using (3.17), (3.19) and the fact that for $\cos\theta\ge 0$, $q_+q_-\ge c$ (see [12, p. 863]) we obtain that

$$\begin{aligned}I&\lesssim\left(\frac{q_+}{q_-}\right)^{\frac{|\alpha|+d}{2}}(q_+q_-)^{1/2}e^{-(\frac{|y|^2-|x|^2}{2}+\frac{(q_+q_-)^{1/2}}{2})}\\&\lesssim q_+^{|\alpha|+d}(q_+q_-)^{1/2}e^{-(\frac{|y|^2-|x|^2}{2}+\frac{(q_+q_-)^{1/2}}{2})}.\end{aligned}\tag{4.3}$$

On the other hand by an adaptation of the proof of Proposition 2.2 in [12] (exchanging x with y) we have

$$\begin{aligned}II&\lesssim\left(\frac{q_+}{q_-}\right)^{\frac{|\alpha|+d}{2}}e^{-(\frac{|y|^2-|x|^2}{2}+\frac{(q_+q_-)^{1/2}}{2})}\\&\lesssim q_+^{|\alpha|+d}e^{-(\frac{|y|^2-|x|^2}{2}+\frac{(q_+q_-)^{1/2}}{2})}.\end{aligned}$$

Taking into account that $q_+q_-\ge c$ on $G\setminus R_1$ we conclude that this last kernel is bounded also by the kernel (4.3).

Therefore both integrals I and II are bounded by the kernel $\tilde{\tilde{\mathcal{K}}}^m$ for $m=3$ of [7] (kernel defined within proof of Theorem 8). So by Theorem 8 of the same paper, we have the L^p-boundedness, for $1<p<\infty$ with respect to the measure $\tilde{\mu}_\alpha$. □

5 Local Part—Proof of Lemma 3.2

In order to prove the weak-type $(1,1)$ of $U^i_{*,local}$, all we need to know is the $L^2(\mathrm{d}\tilde{\mu}_\alpha)$-boundedness of $U^i_{*,local}$ and suitable gradient estimates for the kernel of $U^i_{*,local}$, $\overline{\mathcal{K}}_i(x,y,s,t)\varphi(x,y,s)$.

Indeed, the weak-type $(1,1)$ of $U^i_{*,local}$ with respect to the measure $\tilde{\mu}_\alpha$ follows from the proof of Theorem 5 in [23] where in that proof we have to replace Proposition 6 of [22] by

Proposition 5.1 *Let us assume α is a multi-index such that $\alpha_i \ge 0$ for all $i = 1,\dots,d$. Let $\{T_t\}_{t>0}$ be a family of linear operators defined on $L^2(\mathbb{R}^d_+, \mathrm{d}\widetilde{m}_\alpha)$ such that $T_* f(x) = \sup_{t>0}|T_t f(x)|$ is bounded on $L^2(\mathbb{R}^d_+, \mathrm{d}\widetilde{m}_\alpha)$. Let us assume also that there exists a function $k = k(x,y,s,t)$ defined on $\mathbb{R}^d_+ \times \mathbb{R}^d_+ \times [-1,1]^d \times (0,\infty)$ that satisfies*

(i) *k is of class C^1 on the variables x and y and there exists $C > 0$ such that*

$$|\nabla_{(x,y)} k(x,y,s,t)| \le \frac{C}{q_-^{|\alpha|+d+\frac{1}{2}}(x^2,y^2,s)}.$$

(ii)

$$T_t f(x) = \int_{\mathbb{R}^d_+}\int_{[-1,1]^d} k(x,y,s,t)\Pi_\alpha(s)\mathrm{d}s f(y)\mathrm{d}\widetilde{m}_\alpha(y).$$

Then T_* can be extended to a bounded operator on $L^p(\mathbb{R}^d_+, \mathrm{d}\widetilde{m}_\alpha)$ for $1 < p < \infty$ and of weak-type $(1,1)$.

The proof of Proposition 5.1 follows essentially from the technique developed by E. Sasso in [22, Proposition 6].

In our case we have to take $k(x,y,s,t) = \overline{\mathcal{K}}_i(x,y,s,t)\varphi(x,y,s)$. Now we are going to prove the claims stated at the beginning of this section.

From Sect. 2 and taking into account the isometry $\mathcal{U}_\Psi$ defined in Sect. 3 we get the boundedness of U^i_* on $L^2(\mathrm{d}\tilde{\mu}_\alpha)$. The $L^2(\mathrm{d}\tilde{\mu}_\alpha)$-boundedness of $U^i_{*,global}$ is a consequence of Theorem 4.1. Therefore from (3.7) the $L^2(\mathrm{d}\tilde{\mu}_\alpha)$-boundedness of $U^i_{*,local}$ follows.

Finally let us obtain suitable estimates on the local region N^x_2 for both the kernel $\overline{\mathcal{K}}_i(x,y,s,t)\varphi(x,y,s)$ and its gradients with respect to x and y.

Lemma 5.2 *For $(y,s) \in N^x_2$, we have the following estimates*

$$|\overline{\mathcal{K}}_i(x,y,s,t)|\varphi(x,y,s) \le |\overline{\mathcal{K}}_i(x,y,s,t)| \le \frac{C}{q_-(x^2,y^2,s)^{|\alpha|+d}},$$
$$|\nabla_{(x,y)}(\overline{\mathcal{K}}_i(x,y,s)\varphi(x,y,s))| \le \frac{C}{q_-(x^2,y^2,s)^{|\alpha|+d+1/2}}.$$

Proof Let us remark that there exists a constant $C>0$ such that $C^{-1}\le e^{|x|^2-|y|^2}\le C$ for every $(y,s)\in N_2^x$. Also for $(y,s)\in N_2^x$

$$e^{-c\frac{q_-(x^2,ry^2,s)}{1-r}}\lesssim e^{-c\frac{q_-(x^2,y^2,s)}{1-r}}.$$

Indeed, according to Remark 3.8

$$\begin{aligned}q_-(x^2,ry^2,s)&=|X-\sqrt{r}Y|^2\\&=|X-Y|^2+(1-\sqrt{r})^2|Y|^2+2(1-\sqrt{r})Y\cdot(X-Y)\\&\ge|X-Y|^2-2(1-r)|Y||X-Y|\\&=q_-(x^2,y^2,s)-2(1-r)|y|q_-^{1/2}(x^2,y^2,s)\\&\ge q_-(x^2,y^2,s)-c(1-r),\end{aligned}$$

since on N_2^x, $|x|\sim|y|$. Again from the proof of Proposition 4.1 and taking into account that $(y,s)\in N_2^x$, we have

$$\begin{aligned}|\overline{\mathcal{K}}_i(x,y,s,t)|&\lesssim\int_0^1(q_-(x^2,ry^2,s))^{3/2}\frac{e^{-\frac{q_-(rx^2,y^2,s)}{1-r}}}{(1-r)^{|\alpha|+d+5/2}}\frac{dr}{\sqrt{r}}\\&\quad+\int_0^1(q_-(x^2,ry^2,s))^{1/2}\frac{e^{-\frac{q_-(rx^2,y^2,s)}{1-r}}}{(1-r)^{|\alpha|+d+3/2}}\frac{dr}{\sqrt{r}}\\&\lesssim\left[\int_0^1\left(\frac{q_-(x^2,ry^2,s)}{1-r}\right)^{3/2}\frac{e^{-\frac{q_-(x^2,ry^2,s)}{1-r}}}{(1-r)^{|\alpha|+d+1}}\frac{dr}{\sqrt{r}}\right.\\&\quad\left.+\int_0^1\left(\frac{q_-(x^2,ry^2,s)}{1-r}\right)^{1/2}\frac{e^{-\frac{q_-(x^2,ry^2,s)}{1-r}}}{(1-r)^{|\alpha|+d+1}}\frac{dr}{\sqrt{r}}\right]e^{|x|^2-|y|^2}\\&\lesssim\int_0^1\frac{e^{-c\frac{q_-(x^2,ry^2,s)}{(1-r)}}}{(1-r)^{|\alpha|+d+1}}\frac{dr}{\sqrt{r}}\,e^{|x|^2-|y|^2}\lesssim\int_0^1\frac{e^{-c\frac{q_-(x^2,y^2,s)}{1-r}}}{(1-r)^{|\alpha|+d+1}}\frac{dr}{\sqrt{r}}\\&\le\frac{C}{q_-(x^2,y^2,s)^{|\alpha|+d}}.\end{aligned}$$

Now, we may consider the derivatives. If $j\ne i$,

$$\begin{aligned}\frac{\partial\overline{\mathcal{K}}_i}{\partial y_j}(x,y,s,t)&=\int_0^1\frac{e^{\frac{t^2}{4\log r}}}{\sqrt{\pi}}w(r)\frac{\partial U_i^\alpha}{\partial y_j}(x,y,r,s)\frac{dr}{\sqrt{r}}\\&=-2\int_0^1\frac{e^{\frac{t^2}{4\log r}}}{\sqrt{\pi}}w(r)U_i^\alpha(x,y,r,s)\frac{y_j-\sqrt{r}x_js_j}{1-r}\frac{dr}{\sqrt{r}}.\end{aligned}$$

Thus, since $|y_j - \sqrt{r}x_js_j| \leq q_-^{1/2}(rx^2, y^2, s)$,

$$|U_i^\alpha(x, y, r, s)| \lesssim \left[\left(\frac{q_-(x^2, ry^2, s)}{1-r}\right)^{3/2} + \left(\frac{q_-(x^2, ry^2, s)}{1-r}\right)^{1/2}\right] \frac{e^{-\frac{q_-(rx^2, y^2, s)}{1-r}}}{(1-r)^{|\alpha|+d+1}}$$

and taking into account the above inequalities, we can conclude as before that

$$\left|\frac{\partial \overline{\mathcal{K}}_i}{\partial y_j}(x, y, s, t)\right| \leq \frac{C}{q_-(x^2, y^2, s)^{|\alpha|+d+1/2}}.$$

On the other hand,

$$\frac{\partial \overline{\mathcal{K}}_i}{\partial y_i}(x, y, s, t) = \int_0^1 \frac{e^{\frac{t^2}{4\log r}}}{\sqrt{\pi}} w(r) \frac{\partial U_i^\alpha}{\partial y_i}(x, y, r, s) \frac{\mathrm{d}r}{\sqrt{r}}$$

with

$$\frac{\partial U_i^\alpha}{\partial y_i}(x, y, r, s) = -2U_i^\alpha(x, y, r, s) \frac{y_i - \sqrt{r}x_is_i}{1-r} + \sqrt{r}\left[\frac{x_i^2 - 2\sqrt{r}x_iy_is_i}{1-r} - (\alpha_i + 1)\right](1 - s_i^2) \frac{e^{-\frac{q_-(rx^2, y^2, s)}{1-r}}}{(1-r)^{|\alpha|+d+3/2}}.$$

Since $|y_j - \sqrt{r}x_js_j| \leq q_-^{1/2}(rx^2, y^2, s)$ and $|x_i^2 - 2\sqrt{r}x_iy_is_i| \leq q_-(x^2, ry^2, s)$, we also obtain that

$$\left|\frac{\partial \overline{\mathcal{K}}_i}{\partial y_i}(x, y, s, t)\right| \leq \frac{C}{q_-(x^2, y^2, s)^{|\alpha|+d+1/2}}.$$

The gradient with respect to x is treated similarly. As for the boundedness of $|\overline{\mathcal{K}}_i(x, y, s, t)|(|\partial_{y_j}\varphi(x, y, s)| + |\partial_{x_j}\varphi(x, y, s)|)$ we use the boundedness of $|\overline{\mathcal{K}}_i(x, y, s, t)|$ together with the assumption on the boundedness of $|\nabla_{(x,y)}\varphi(x, y, s)|$. □

6 Proof of Theorem 1.2 and Proof of Corollary 1.1

For $f \in L^2(\mathrm{d}\mu_\alpha)$, from the Cauchy-Riemann system, $\widetilde{U}_t^i f(x) = e^{-\mathcal{L}_\alpha^{1/2}t} R_i^* f(x)$ for all $x \in \mathbf{R}_+^d$ and from this, $\widetilde{U}_t^i f$ converges to $R_i^* f$ in $L^2(\mathrm{d}\mu_\alpha)$. We also know that $\widetilde{U}_t^i f$ converges to $\widetilde{R}_i^* f$ pointwise. Therefore $\widetilde{R}_i^* = R_i^*$ on $L^2(\mathrm{d}\mu_\alpha)$. On the other hand, from Proposition 2.1, $\widetilde{U}_*^i$ is strong-type $(2, 2)$ and from Theorem 1.1, $\widetilde{U}_*^i$ is weak-type $(1, 1)$ with respect to μ_α, then from Marcinkiewicz's Interpolation Theorem we get that $\widetilde{U}_*^i$ is bounded on $L^p(\mathrm{d}\mu_\alpha)$ for $1 < p \leq 2$. From this result, the pointwise convergence of $\widetilde{U}_t^i f$ and Fatou's Lemma we immediately obtain that $\widetilde{R}_i^*$ is also bounded on $L^p(\mathrm{d}\mu_\alpha)$ for $1 < p \leq 2$.

As for $2 < p < \infty$, let us take $f \in L^p$ $(= L^p \cap L^2)$ and $g \in L^{p'} \cap L^2$, taking into account that

$$|\langle \widetilde{R}_i^* f, g\rangle| = \lim_{t\to 0^+} |\langle \widetilde{U}_t^i f, g\rangle|$$

and

$$\langle \widetilde{U}_t^i f, g\rangle = \langle f, V_t^i g\rangle$$

with $V_t^i g = e^{-\mathcal{M}_i^{1/2} t} R_i g$, $R_i g = \delta_i \mathcal{L}_\alpha^{-1/2} \Pi_0 g$, which is bounded on $L^p(\mathrm{d}\mu_\alpha)$ for every $1 < p < \infty$ (see [16, Theorem 13]) and the Maximal Theorem on Semigroups proved in [24, p. 73], we have

$$\begin{aligned} |\langle \widetilde{R}_i^* f, g\rangle| &\le \| \sup_{t>0} |V_t^i g| \|_{p'} \|f\|_p \\ &\le C_{p'} \|R_i g\|_{p'} \|f\|_p \lesssim \|g\|_{p'} \|f\|_p, \end{aligned}$$

therefore,

$$\|\widetilde{R}_i^* f\|_p \lesssim \|f\|_p.$$

Now taking into account that $\widetilde{U}_t^i f(x) = e^{-\mathcal{L}_\alpha^{1/2} t} \widetilde{R}_i^* f$, and the Maximal Theorem on Semigroups we get again that $\widetilde{U}_*^i$ is bounded on $L^p(\mathrm{d}\mu_\alpha)$ for $p > 2$.

As for the weak-type $(1,1)$ of $\widetilde{R}_i^*$ this follows, for $f \in L^1(\mathrm{d}\mu_\alpha)$, from the inequality $|\widetilde{R}_i^*(x)| \le \widetilde{U}_*^i f(x)$ for a.e. $x \in \mathbb{R}_+^d$ and Theorem 1.1.

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Appendix

In the proof of (3.2) the following estimates will be used:

$$|L_n^\alpha(x)| \le \frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)} \frac{e^{x/2}}{\Gamma(\alpha+1)}, \quad \text{for } \alpha \ge 0, x \ge 0 \text{ and } n = 0, 1, 2, \dots, \tag{7.1}$$

which can be found in [1]; and for $k_j \ge 1$ for all $j = 1, \dots, d$

$$\begin{aligned} &\left(\prod_{j=1}^d \frac{\Gamma(k_j+1)}{\Gamma(\alpha_j+k_j+1)}\right)^{1/2} \left|\frac{\partial}{\partial x_i}(\delta_i L_k^\alpha)(x)\right| \\ &\quad\lesssim (\max(x_i, 1/x_i))^{1/2} \prod_{j=1}^d e^{x_j/2} \frac{k_i}{\alpha_i+1} \left(\prod_{j=1}^d \frac{(1+[\alpha_j]+1)^{[\alpha_j]+1}}{\Gamma(\alpha_j+1)} k_j^{[\alpha_j]+1}\right)^{1/2} \\ &\quad\le C_\alpha(x)|k|^{(|\alpha|+d)/2+1}. \end{aligned} \tag{7.2}$$

This last inequality is a consequence of (7.1) and the inequality $\Gamma(\alpha+n+1) \leq (1+[\alpha]+1)^{[\alpha]+1} n^{[\alpha]+1}\Gamma(n+1)$, for $\alpha \geq 0$ and $n \geq 1$ where this latter inequality is obtained taking into account that the Gamma function is strictly increasing on the interval $[2,\infty)$.

Proof of (3.2) Before getting involved with the calculation of the kernel we are going to write out the kernel of the one-dimensional heat-diffusion semigroup associated with the Laguerre expansions.

$$\begin{aligned}\mathcal{M}_1^a(\eta,\varphi,r) &:= \sum_{n=0}^{\infty} r^n L_n^a(\eta) L_n^a(\varphi) \frac{\Gamma(n+1)}{\Gamma(a+n+1)} \\ &= \frac{(r\eta\varphi)^{-\frac{a}{2}}}{1-r} e^{-\frac{r(\eta+\varphi)}{1-r}} I_a\left(\frac{2\sqrt{r\eta\varphi}}{1-r}\right)\end{aligned}$$

for $0<r<1$, $a>-1$, $\eta,\varphi>0$ and I_a being the modified Bessel function of the first kind.

Taking into account that for $a>-1/2$

$$I_a(z) = \frac{(z/2)^a}{\Gamma(a+1/2)\Gamma(1/2)} \int_{-1}^{1} (1-s^2)^{a-1/2} e^{zs}\, \mathrm{d}s,$$

then

$$\begin{aligned}\mathcal{M}_1^a(\eta,\varphi,r) &= \int_{-1}^{1} \frac{e^{-\frac{r(\eta+\varphi)-2\sqrt{r\eta\varphi}s}{1-r}}}{(1-r)^{1+a}} \Pi_a(s)\, \mathrm{d}s \\ &= \int_{-1}^{1} \frac{e^{-\frac{q_-^1(r\eta,\varphi,s)}{1-r}}}{(1-r)^{1+a}} \Pi_a(s)\, \mathrm{d}s e^{\varphi},\end{aligned}$$

with $q_-^1(\eta,\varphi,s) = \eta+\varphi-2\sqrt{\eta\varphi}s$ and $\Pi_a(s) = \frac{(1-s^2)^{a-1/2}}{\Gamma(a+1/2)\Gamma(1/2)}$.

In order to prove (3.2) let us first calculate the kernel of the integral operator associated with the diffusion semigroup

$$\begin{aligned}e^{-\mathcal{M}_i t}\Pi_i f(x) &= \sum_{k:k_i>0} e^{-|k|t} \frac{\langle f, \delta_i L_k^\alpha\rangle}{\|\delta_i L_k^\delta\|_2^2} \delta_i L_k^\alpha(x) \\ &= \sum_{k:k_i>0} \frac{e^{-|k|t}}{k_i} \prod_{j=1}^{d} \frac{\Gamma(k_j+1)}{\Gamma(\alpha_j+k_j+1)} \langle f, \delta_i L_k^\alpha\rangle \delta_i L_k^\alpha(x) \\ &= \int_{\mathbb{R}_+^d} \mathcal{M}_i(x,y,e^{-t}) f(y)\, \mathrm{d}\mu_\alpha(y),\end{aligned}$$

with

$$\begin{aligned}\mathcal{M}_i(x,y,e^{-t}) &= \prod_{j\neq i}\sum_{k_j=0}^{\infty} e^{-k_j t} L_{k_j}^{\alpha_j}(x_j) L_{k_j}^{\alpha_j}(y_j)\frac{\Gamma(k_j+1)}{\Gamma(\alpha_j+k_j+1)}\\ &\quad\times\left(\sum_{k_i=1}^{\infty} e^{-k_i t}\sqrt{x_i}\sqrt{y_i} L_{k_i-1}^{\alpha_i+1}(x_i) L_{k_i-1}^{\alpha_i+1}(y_i)\frac{\Gamma(k_i)}{\Gamma(\alpha_i+k_i+1)}\right)\\ &= \prod_{j\neq i}\sum_{k_j=0}^{\infty} e^{-k_j t} L_{k_j}^{\alpha_j}(x_j) L_{k_j}^{\alpha_j}(y_j)\frac{\Gamma(k_j+1)}{\Gamma(\alpha_j+k_j+1)}\\ &\quad\times\left(\sum_{k_i=0}^{\infty} e^{-k_i t} e^{-t}\sqrt{x_i y_i} L_{k_i}^{\alpha_i+1}(x_i) L_{k_i}^{\alpha_i+1}(y_i)\frac{\Gamma(k_i+1)}{\Gamma(\alpha_i+1+k_i+1)}\right).\end{aligned}$$

We set $r=e^{-t}$ then for all $\alpha_j > -1/2$,

$$\begin{aligned}\mathcal{M}_i(x,y,r) &= r^{1/2}\prod_{j\neq i}\mathcal{M}_1^{\alpha_j}(x_j,y_j,r)\sqrt{r x_i y_i}\,\mathcal{M}_1^{\alpha_i+1}(x_i,y_i,r)\\ &= r\sqrt{x_i y_i}\prod_{j\neq i}\int_{-1}^{1}\frac{e^{-\frac{q_-^1(rx_j,y_j,s_j)}{1-r}}}{(1-r)^{1+\alpha_j}}\Pi_{\alpha_j}(s_j)\,\mathrm{d}s_j e^{y_j}\\ &\quad\times\int_{-1}^{1}\frac{e^{-\frac{q_-^1(rx_i,y_i,s_i)}{1-r}}}{(1-r)^{2+\alpha_j}}\Pi_{\alpha_i+1}(s_i)\,\mathrm{d}s_i e^{y_i}\\ &= \frac{r\sqrt{x_i y_i}}{\alpha_i+1/2}\int_{[-1,1]^d}(1-s_i^2)\frac{e^{-\frac{q_-(rx,y,s)}{1-r}}}{(1-r)^{|\alpha|+d+1}}\Pi_\alpha(s)\,\mathrm{d}s\prod_{j=1}^{d}e^{y_j}\end{aligned}$$

with $q_-(rx,y,s)=\sum_{j=1}^{d} q_-^1(rx_j,y_j,s_j)$ and $\Pi_\alpha(s)=\prod_{j=1}^{d}\Pi_{\alpha_j}(s_j)$.

Under a subordination formula we can write $e^{-\mathcal{M}_i^{1/2}t}$ in terms of $e^{-\mathcal{M}_i t}=r^{\mathcal{M}_i}$:

$$\begin{aligned}e^{-\mathcal{M}_i^{1/2}t}f(x) &= \frac{t}{2\sqrt{\pi}}\int_0^1\frac{e^{\frac{t^2}{4\log r}}}{r(-\log r)^{3/2}}r^{\mathcal{M}_i}f(x)\,\mathrm{d}r\\ &= \int_{\mathbb{R}_+^d}\int_0^1\frac{t}{2\sqrt{\pi}}\frac{e^{\frac{t^2}{4\log r}}}{r(-\log r)^{3/2}}\mathcal{M}_i(x,y,r)\,\mathrm{d}r f(y)\,\mathrm{d}\mu_\alpha(y).\end{aligned}$$

At this point we are ready to calculate out the kernel associated with the integral operator $\widetilde{U}_t^i$. Indeed,

$$\widetilde{U}_t^i f(x) = \int_t^{\infty}\delta_i^* e^{-\mathcal{M}_i^{1/2}l} f(x)\,\mathrm{d}l$$

$$= \int_t^\infty \delta_i^* \int_{\mathbb{R}_+^d} \int_0^1 \frac{l}{2\sqrt{\pi}} \frac{e^{\frac{l^2}{4\log r}}}{r(-\log r)^{3/2}} \mathcal{M}_i(x,y,r)\,\mathrm{d}r f(y)\,\mathrm{d}\mu_\alpha(y)\,\mathrm{d}l$$

$$= \int_t^\infty \int_{\mathbb{R}_+^d} \int_0^1 \frac{l}{2\sqrt{\pi}} \frac{e^{\frac{l^2}{4\log r}}}{r(-\log r)^{3/2}} \delta_i^* \mathcal{M}_i(x,y,r)\,\mathrm{d}r f(y)\,\mathrm{d}\mu_\alpha(y)\,\mathrm{d}l$$

$$= \int_{\mathbb{R}_+^d} \int_0^1 \frac{e^{\frac{t^2}{4\log r}}}{\sqrt{\pi} r(-\log r)^{1/2}} \delta_i^* \mathcal{M}_i(x,y,r)\,\mathrm{d}r f(y)\,\mathrm{d}\mu_\alpha(y)$$

$$= \int_{\mathbb{R}_+^d} \int_0^1 \frac{e^{\frac{t^2}{4\log r}}}{\sqrt{\pi}} w(r) \frac{\delta_i^* \mathcal{M}_i(x,y,r)}{\sqrt{r}\sqrt{1-r}} \frac{\mathrm{d}r}{\sqrt{r}} f(y)\,\mathrm{d}\mu_\alpha(y). \tag{7.3}$$

To justify the interchanging of δ_i^* and the double integral we should prove that

$$I = \int_{\mathbb{R}_+^d} \int_0^1 \frac{l}{2\sqrt{\pi}} \frac{e^{\frac{l^2}{4\log r}}}{r(-\log r)^{3/2}} \left| \frac{\partial \mathcal{M}_i}{\partial x_i}(x,y,r) \right| \mathrm{d}r |f(y)|\,\mathrm{d}\mu_\alpha(y)$$

is finite for every $x \in \mathbb{R}_+^d$. By applying Schwarz's inequality we get that

$$I \le \int_0^1 \frac{l}{2\sqrt{\pi}} \frac{e^{\frac{l^2}{4\log r}}}{r(-\log r)^{3/2}} \left(\int_{\mathbb{R}_+^d} \left| \frac{\partial \mathcal{M}_i}{\partial x_i}(x,y,r) \right|^2 \mathrm{d}\mu_\alpha(y) \right)^{1/2} \mathrm{d}r \|f\|_{L^2(\mathrm{d}\mu_\alpha)}.$$

By using estimates (7.1) and (7.2) one can prove that

$$\sum_{k:k_i>0} \frac{r^{|k|}}{k_i} \prod_{j=1}^d \frac{\Gamma(k_j+1)}{\Gamma(\alpha_j+k_j+1)} \left| \frac{\partial}{\partial x_i}(\delta_i L_k^\alpha)(x) \right| |\delta_i L_k^\alpha(y)|$$

is finite for every x and $r<1$, therefore

$$\frac{\partial \mathcal{M}_i}{\partial x_i}(x,y,r) = \sum_{k:k_i>0} \frac{r^{|k|}}{k_i} \prod_{j=1}^d \frac{\Gamma(k_j+1)}{\Gamma(\alpha_j+k_j+1)} \frac{\partial}{\partial x_i}(\delta_i L_k^\alpha)(x) \delta_i L_k^\alpha(y).$$

Then

$$\begin{aligned} \left\| \frac{\partial \mathcal{M}_i}{\partial x_i}(x,\cdot,r) \right\|_{L^2(\mathrm{d}\mu_\alpha)} &= \left(\sum_{k:k_i>0} \frac{r^{2|k|}}{k_i} \prod_{j=1}^d \frac{\Gamma(k_j+1)}{\Gamma(\alpha_j+k_j+1)} \left| \frac{\partial}{\partial x_i}(\delta_i L_k^\alpha)(x) \right|^2 \right)^{1/2} \\ &\le \sum_{k:k_i>0} \frac{r^{|k|}}{k_i^{1/2}} \left(\prod_{j=1}^d \frac{\Gamma(k_j+1)}{\Gamma(\alpha_j+k_j+1)} \right)^{1/2} \left| \frac{\partial}{\partial x_i}(\delta_i L_k^\alpha)(x) \right| \\ &\le C_\alpha(x) \sum_{k:k_i>0} |k|^{\frac{|\alpha|+d}{2}+1} r^{|k|}. \end{aligned}$$

This last bound is a consequence of the estimate (7.2). Therefore

$$\begin{aligned} I &\leq C_\alpha(x) \sum_{k:k_i>0} |k|^{\frac{|\alpha|+d}{2}+1} \int_0^1 \frac{l}{2\sqrt{\pi}} \frac{e^{\frac{l^2}{4\log r}}}{r(-\log r)^{3/2}} r^{|k|} dr \|f\|_{L^2(\mathrm{d}\mu_\alpha)} \\ &= C_\alpha(x) \sum_{k:k_i>0} |k|^{\frac{|\alpha|+d}{2}+1} e^{-|k|^{1/2} l} \|f\|_{L^2(\mathrm{d}\mu_\alpha)}. \end{aligned}$$

This last bound not only guarantees that indeed I is finite but also that we can apply Fubini's Theorem to justify equality (7.3).

Now calling $\widetilde{U}_i^\alpha(x,y,r) = \frac{\delta_i^* \mathcal{M}_i(x,y,r)}{\sqrt{r}\sqrt{1-r}} \prod_{j=1}^d e^{-y_j}$, and after a series of calculations we get kernel (3.2). □

Proof of Lemma 4.3 From (3.18), $u(l) + |x|^2 - |y|^2 \geq u_0 + |x|^2 - |y|^2 \geq 0$, then

$$e^{-\nu u(l)} \leq e^{-\nu u_0} \leq e^{-\nu(|y|^2 - |x|^2)}$$

and

$$\begin{aligned} (u(l) + |x|^2 - |y|^2)^m e^{-\nu u(l)} &= (u(l) + |x|^2 - |y|^2)^m e^{-\nu(u(l)+|x|^2-|y|^2)} e^{\nu(|x|^2-|y|^2)} \\ &\lesssim e^{\nu(|x|^2-|y|^2)}. \end{aligned}$$

And the proof of Lemma 4.3 follows. □

Proof of Lemma 4.4 Let us notice that conditions $|x-y| \geq q_-^{1/2}(x^2, y^2, s) > \frac{C}{1+|x|}$ and $|x| \geq |y|$ imply $|x| \geq c > 0$.

Now let us write

$$\int_0^1 \frac{e^{-\eta u(l)}}{l^{|\alpha|+d+1}} \frac{\mathrm{d}l}{\sqrt{1-l}} = \int_0^{\frac{1}{2|x|^2}} + \int_{\frac{1}{2|x|^2}}^{1/2} + \int_{1/2}^1 = I + II + III.$$

It is immediate the boundedness of III by a constant. On the other hand

$$II \leq \int_{\frac{1}{2|x|^2}}^{1/2} \frac{\mathrm{d}l}{l^{|\alpha|+d+1}} \lesssim |x|^{2|\alpha|+2d}.$$

Regarding I let us observe that by using Remark 3.8 we obtain the following inequality:

$$\begin{aligned} q_-^{1/2}((1-l)x^2, y^2, s) &= |\sqrt{1-l}X - Y| \\ &\geq |X-Y| - (1-\sqrt{1-l})|X| \\ &\geq |X-Y| - l|X| \\ &= q_-^{1/2}(x^2, y^2, s) - l|x|. \end{aligned} \tag{7.4}$$

Since $0 \le l \le 1/(2|x|^2)$, then $2l|x| \le \frac{1}{|x|} \le \frac{C}{1+|x|} < q_-^{1/2}(x^2, y^2, s)$ and therefore by applying this to (7.4) we have $q_-^{1/2}((1-l)x^2, y^2, s) \ge \frac{q_-^{1/2}(x^2,y^2,s)}{2}$. Thus, $u(l) \ge \frac{q_-(x^2,y^2,s)}{4l}$ and

$$I \lesssim \int_0^{\frac{1}{2|x|^2}} \frac{e^{-\frac{\eta q_-(x^2,y^2,s)}{4l}}}{l^{|\alpha|+d+1}} \mathrm{d}t \lesssim \frac{1}{q_-^{|\alpha|+d}(x^2, y^2, s)} \lesssim |x|^{2|\alpha|+2d}.$$

□

Proof of Lemma 4.5 From Lemma 4.3 with $m = 3/2$ and $\nu = \frac{1}{4(|\alpha|+d)}$, (3.16), (3.20), and (3.17) we have

$$\begin{aligned}
&\int_0^1 (1 \vee (u(l) + |x|^2 - |y|^2)^{3/2}) \frac{e^{-u(l)}}{l^{|\alpha|+d+1}} \frac{\mathrm{d}l}{\sqrt{1-l}} \\
&\quad \lesssim \int_0^1 e^{\nu|x|^2} e^{-\nu|y|^2} \frac{e^{-(1-\nu)u(l)}}{l^{|\alpha|+d+1}} \frac{\mathrm{d}l}{\sqrt{1-l}} \\
&\quad \lesssim \frac{e^{\nu|x|^2} e^{-\nu|y|^2}}{u_0^{1/2}} \int_0^1 (u(l))^{1/2} \frac{e^{-(1-\nu)u(l)}}{l^{|\alpha|+d+1}} \frac{\mathrm{d}l}{\sqrt{1-l}} \\
&\quad \lesssim e^{\nu|x|^2} e^{-\nu|y|^2} \int_0^1 \left(\frac{e^{-u(l)}}{l^{|\alpha|+d}} \right)^{1-2\nu} u^{1/2}(l) \frac{e^{-\nu u(l)}}{l^{3/2}} \frac{\mathrm{d}l}{\sqrt{1-l}} \\
&\quad \lesssim e^{\nu|x|^2} e^{-\nu|y|^2} \left(\frac{e^{-u_0}}{l_0^{|\alpha|+d}} \right)^{1-2\nu} \frac{e^{-\nu u_0}}{l_0^{1/2}} \\
&\quad = e^{\nu|x|^2} e^{-\nu|y|^2} \frac{e^{-(1-\nu)u_0}}{l_0^{|\alpha|+d}} \\
&\quad \lesssim e^{\nu|x|^2} e^{-\nu|y|^2} \left(\frac{q_+}{q_-} \right)^{(|\alpha|+d)/2} e^{-(1-\nu)u_0}.
\end{aligned}$$

□

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