J Fourier Anal Appl (2011) 17:932–975
DOI 10.1007/s00041-010-9162-1

Spectral Multipliers for Multidimensional Bessel Operators

J.J. Betancor · A.J. Castro · J. Curbelo

Received: 16 April 2010 / Revised: 9 November 2010 / Published online: 10 December 2010

Abstract In this paper we prove L^p-boundedness properties of spectral multipliers associated with multidimensional Bessel operators. In order to do this we estimate the L^p-norm of the imaginary powers of Bessel operators. We also prove that the Hankel multipliers of Laplace transform type on $(0, \infty)^n$ are principal value integral operators of weak type $(1, 1)$.

Keywords Spectral multipliers · Bessel operators · Laplace transform

Mathematics Subject Classification 42A45 · 42A50

1 Introduction

Our objective is to establish L^p-boundedness properties of spectral multipliers associated with multidimensional Bessel operators.

Communicated by Hans G. Feichtinger.

J.J. Betancor was partially supported by MTM2007/65609. A.J. Castro was supported by a grant for Master studies of "la Caixa". J. Curbelo was supported by a grant JAE-Predoc of the CSIC (Spain).

J.J. Betancor (✉) · A.J. Castro
Departamento de Análisis Matemático, Universidad de la Laguna, Campus de Anchieta, Avda. Astrofísico Francisco Sánchez, s/n, 38271 La Laguna, Sta. Cruz de Tenerife, Spain
e-mail: jbetanco@ull.es

A.J. Castro
e-mail: ajcastro@ull.es

J. Curbelo
Instituto de Ciencias Matemáticas (CSIC, UAM, UC3M, UCM) Consejo Superior de Investigaciones Cientificas, Calle Nicolás Cabrera, 15, 28049 Madrid, Spain
e-mail: jezabel.curbelo@icmat.es

It is suitable to start by specifying some notation that we will use along the paper. This one will allow us to simplify the writing of our results and to make their reading easier. By $n \geq 1$ we denote the dimension. We consider $\lambda = (\lambda_1, \dots, \lambda_n) \in (-1/2, \infty)^n$, $\alpha \in (-1/2, \infty)$, and $x = (x_1, \dots, x_n) \in (0, \infty)^n$. Also we reserve y and z to denote elements of $(0, \infty)^n$ and u and v of in $(0, \infty)$. Suppose that $(u, v, \alpha) \in (0, \infty)^2 \times (-1/2, \infty) \to F(u, v, \alpha) \in \mathbb{R}$. We define, for every $x, y \in (0, \infty)^n$ and $\lambda \in (-1/2, \infty)^n$,

$$F(x, y, \lambda) = \prod_{j=1}^{n} F(x_j, y_j, \lambda_j).$$

According to this convention we define, for instance,

$$(x.y)^{-\lambda} = \prod_{j=1}^{n} (x_j y_j)^{-\lambda_j} \quad \text{and} \quad (x/y)^{-\lambda} = \prod_{j=1}^{n} (x_j/y_j)^{-\lambda_j},$$

for every $x, y \in (0, \infty)^n$. We also introduce the measures $d\mu_\alpha(u) = u^{2\alpha} du$, on $(0, \infty)$, and $d\mu_\lambda(x) = \prod_{j=1}^{n} d\mu_{\lambda_j}(x_j)$, on $(0, \infty)^n$.

We consider the Hankel transformation on $(0, \infty)^n$ defined by

$$H_\lambda(f)(x) = \int_{(0,\infty)^n} \Phi_\lambda(x, y) f(y) d\mu_\lambda(y), \quad x \in (0, \infty)^n,$$

where $\Phi_\alpha(u, v) = (uv)^{-\alpha+1/2} J_{\alpha-1/2}(uv)$, $u, v \in (0, \infty)$ and $\alpha > -1/2$. Here J_ν denotes the Bessel function of the first kind and order ν. H_λ maps $L^1((0, \infty)^n, d\mu_\lambda(x))$ into $L^\infty((0, \infty)^n, d\mu_\lambda(x))$ $(= L^\infty((0, \infty)^n, dx))$ and it can be extended as an isometry to $L^2((0, \infty)^n, d\mu_\lambda(x))$ being $H_\lambda^{-1} = H_\lambda$ on $L^2((0, \infty)^n, d\mu_\lambda(x))$ (see [2]).

If $m \in L^\infty((0, \infty)^n, dx)$ the multiplier operator T_λ^m associated with the Hankel transformation H_λ is defined by

$$T_\lambda^m(f) = H_\lambda\big(m H_\lambda(f)\big), \quad f \in L^2\big((0, \infty)^n, d\mu_\lambda(x)\big).$$

It is clear that T_λ^m is bounded from $L^2((0, \infty)^n, d\mu_\lambda(x))$ into itself. L^p-boundedness properties for Hankel multipliers have been studied in [1, 3, 8–12, 17] and [21], amongst others.

For every $\alpha > -1/2$ we denote by $\Delta_{\alpha,u}$ the Bessel operator

$$\Delta_{\alpha,u} = -u^{-2\alpha} \frac{d}{du}\left(u^{2\alpha} \frac{d}{du}\right) = -\left(\frac{d^2}{du^2} + \frac{2\alpha}{u} \frac{d}{du}\right),$$

and if $\lambda \in (-1/2, \infty)^n$ we write

$$\Delta_\lambda = \sum_{j=1}^{n} \Delta_{\lambda_j, x_j}.$$

Since

$$\Delta_{\alpha,u} \Phi_\alpha(u, v) = v^2 \Phi_\alpha(u, v), \quad u, v \in (0, \infty)$$

(see [13, (5.3.7), p. 103]), it follows that, for each $x, y \in (0,\infty)^n$,

$$\Delta_\lambda \Phi_\lambda(x,y) = |y|^2 \Phi_\lambda(x,y), \tag{1}$$

where $|\cdot|$ represents the Euclidean norm in $\mathbb{R}^n$. We denote by $C_c^\infty((0,\infty)^n)$ the space that consists of all those smooth functions in $(0,\infty)^n$ that have compact support in $(0,\infty)^n$. From (1) we deduce that, for every $g \in C_c^\infty((0,\infty)^n)$,

$$H_\lambda(\Delta_\lambda g)(y) = |y|^2 H_\lambda(g)(y), \quad y \in (0,\infty)^n. \tag{2}$$

According to (2) we define the operator Δ_λ as follows

$$\Delta_\lambda f = H_\lambda\big(|y|^2 H_\lambda(f)\big), \quad f \in D(\Delta_\lambda), \tag{3}$$

where

$$D(\Delta_\lambda) = \big\{ f \in L^2\big((0,\infty)^n, d\mu_\lambda(x)\big) : |y|^2 H_\lambda(f) \in L^2\big((0,\infty)^n, d\mu_\lambda(x)\big)\big\}.$$

This operator Δ_λ is closed and selfadjoint in $L^2((0,\infty)^n, d\mu_\lambda(x))$. The heat semigroup $\{W_t^\lambda\}_{t>0}$ generated by $-\Delta_\lambda$ is defined as follows

$$W_t^\lambda(f)(x) = \int_{(0,\infty)^n} W_t^\lambda(x,y) f(y) d\mu_\lambda(y), \quad x \in (0,\infty)^n \text{ and } t > 0,$$

where ([22, p. 395])

$$\begin{aligned} W_t^\alpha(u,v) &= \int_0^\infty e^{-tz^2}(zu)^{-\alpha+1/2} J_{\alpha-1/2}(zu)(zv)^{-\alpha+1/2} J_{\alpha-1/2}(zv) d\mu_\alpha(z) \\ &= \frac{(uv)^{-\alpha+1/2}}{2t} I_{\alpha-1/2}\left(\frac{uv}{2t}\right) e^{-(u^2+v^2)/4t}, \quad t,u,v \in (0,\infty),\ \alpha > -1/2, \end{aligned} \tag{4}$$

and I_ν denotes the modified Bessel function of the first kind and order ν ([22, p. 395]). This semigroup $\{W_t^\lambda\}_{t>0}$ is a symmetric diffusion semigroup in the sense of Stein ([19, p. 65]).

If $m \in L^\infty(0,\infty)$, the spectral multiplier associated with Δ_λ, $m(\Delta_\lambda)$, defined by m coincides with the Hankel multiplier $T_\lambda^{\tilde m}$, where $\tilde m(y) = m(|y|^2)$, $y \in (0,\infty)^n$.

We now establish our main result. Suppose that $m \in L^\infty(0,\infty)$. Given $N \in \mathbb{N}$, we denote by $m_N(u,v)$ the function

$$m_N(u,v) = (uv)^N e^{-uv/2} m(v), \quad u,v \in (0,\infty),$$

and by $M_N(u,w)$ the Mellin transform of $m_N(u,v)$ with respect to the variable v, i.e.,

$$M_N(u,w) = \int_0^\infty v^{-iw-1} m_N(u,v) dv, \quad u,w \in (0,\infty).$$

Theorem 1.1 *Let* $1 < p < \infty$. *Assume that* $m \in L^\infty(0,\infty)$ *and that for some* $N \in \mathbb{N}$, *the following condition holds*

$$\int_{\mathbb{R}} \sup_{u>0} |M_N(u,w)| e^{\pi |w| |\frac{1}{2}-\frac{1}{p}|} dw < \infty.$$

Then, $m(\Delta_\lambda)$ *extends to a bounded operator on* $L^p((0,\infty)^n, d\mu_\lambda(x))$.

According to [14, Theorem 1] our Theorem 1.1 will be proved when we see that, for every $\beta \in \mathbb{R}$, the imaginary power $\Delta_\lambda^{i\beta}$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself and that

$$\begin{aligned} &\|\Delta_\lambda^{i\beta} f\|_{L^p((0,\infty)^n, d\mu_\lambda(x))} \le C e^{\pi |\beta| |\frac{1}{2}-\frac{1}{p}|} \|f\|_{L^p((0,\infty)^n, d\mu_\lambda(x))}, \\ &\quad f \in L^p\big((0,\infty)^n, d\mu_\lambda(x)\big), \end{aligned} \tag{5}$$

where $C > 0$ does not depend on β. Note that the bound in (5) is better than the one obtained in [4, p. 270] and [5, p. 736].

As it is well known the imaginary powers of an operator are special cases of spectral multipliers of Laplace transform type for the operator (see [19, p. 58]). We will study L^p-boundedness properties of Hankel multipliers of Laplace transform type because, actually, the effort is the same than to treat with the particular case of the imaginary powers $\Delta_\lambda^{i\beta}$.

According to (1) we say that m is of Laplace transform type associated with $\phi \in L^\infty(0,\infty)$ when

$$m(y) = |y|^2 \int_0^\infty e^{-t|y|^2} \phi(t) dt, \quad y \in (0,\infty)^n$$

([19, p. 121]). By [19, Corollary 3, p. 121] the Hankel multiplier T_λ^m is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, for every $1 < p < \infty$, provided that m is of Laplace transform type. Note that if $\beta \in \mathbb{R}$ and $\phi_\beta(t) = \frac{t^{-i\beta}}{\Gamma(1-i\beta)}$, $t \in (0,\infty)$, we have that

$$m_\beta(y) = |y|^2 \int_0^\infty e^{-t|y|^2} \phi_\beta(t) dt = |y|^{2\beta i}, \quad y \in (0,\infty)^n,$$

and $T_\lambda^{m_\beta} = \Delta_\lambda^{i\beta}$ (see (3)).

Our purpose is to estimate the L^p-norm of the operator T_λ^m. In order to do this, the following pointwise representation of T_λ^m on $C_c^\infty((0,\infty)^n)$ is crucial.

Theorem 1.2 *Let* $\lambda \in (-1/2,\infty)^n$. *Assume that* m *is of Laplace transform type associated with* $\phi \in L^\infty(0,\infty)$ *and that* $f \in C_c^\infty((0,\infty)^n)$. *Then*

$$\begin{aligned} T_\lambda^m(f)(x) = -\lim_{\varepsilon\to 0^+} \left(\Lambda(\varepsilon) f(x) + \int_{(0,\infty)^n,\ |y-x|>\varepsilon} f(y) K_\lambda^\phi(x,y) d\mu_\lambda(y) \right), \\ \text{a.e.} \quad x \in (0,\infty)^n, \end{aligned} \tag{6}$$

where

$$K_\lambda^\phi(x,y)=\int_0^\infty \phi(t)\frac{\partial}{\partial t}W_t^\lambda(x,y)dt,\quad x,y\in(0,\infty)^n,\ x\neq y,$$

and $\|\Lambda\|_{L^\infty(0,\infty)}\le C\|\phi\|_{L^\infty(0,\infty)}$, *where* $C>0$ *does not depend on* ϕ. *Moreover, if there exists the limit* $\phi(0^+)=\lim_{t\to 0^+}\phi(t)$, *then*

$$T_\lambda^m(f)(x)=C\phi\big(0^+\big)f(x)-\lim_{\varepsilon\to 0^+}\left(\int_{(0,\infty)^n,\ |y-x|>\varepsilon} f(y)K_\lambda^\phi(x,y)d\mu_\lambda(y)\right),$$
$$\text{a.e.}\quad x\in(0,\infty)^n,$$

being C *a positive constant.*

The existence of the limit in (6) for every $f\in L^p((0,\infty)^n,d\mu_\lambda(x))$, $1\le p<\infty$, can be proved by using the following fundamental result where the L^p-boundedness properties for the maximal operator associated with the principal value in (6) are established. We consider the maximal Fourier multiplier $T^{m,*}$ defined by

$$T^{m,*}(f)(x)=\sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} f(y)H^\phi(x,y)dy\right|,\quad x\in\mathbb{R}^n,$$

where

$$H^\phi(x,y)=\int_0^\infty \phi(t)\frac{\partial}{\partial t}\mathbb{W}_t(x,y)dt,\quad x,y\in\mathbb{R}^n,\ x\neq y, \tag{7}$$

and $\mathbb{W}_t(x,y)$ denotes the classical heat kernel, that is,

$$\mathbb{W}_t(x,y)=\frac{e^{-|x-y|^2/4t}}{(4\pi t)^{n/2}},\quad x,y\in\mathbb{R}^n \text{ and } t>0.$$

$T^{m,*}$ is the maximal operator associated with the Fourier multiplier T^m defined by

$$T^m f=(m\hat{f})\check{},\quad f\in L^2\big(\mathbb{R}^n,dx\big),$$

where, as usual, by $\hat{f}$ we denote the Fourier transform of f and by $\check{g}$ the inverse Fourier transform of g. It is well known that the operator $T^{m,*}$ is bounded from $L^p(\mathbb{R}^n,dx)$ into itself, for every $1<p<\infty$, and from $L^1(\mathbb{R}^n,dx)$ into $L^{1,\infty}(\mathbb{R}^n,dx)$. Also, T^m can be extended from $L^2(\mathbb{R}^n,dx)\cap L^p(\mathbb{R}^n,dx)$ to $L^p(\mathbb{R}^n,dx)$ as a bounded operator from $L^p(\mathbb{R}^n,dx)$ into itself, for every $1<p<\infty$, and from $L^1(\mathbb{R}^n,dx)$ into $L^{1,\infty}(\mathbb{R}^n,dx)$.

Theorem 1.3 *Let* $\lambda\in(-1/2,\infty)^n$. *Suppose that* m *is of Laplace transform type associated with* $\phi\in L^\infty(0,\infty)$. *The maximal operator* $T_\lambda^{m,*}$ *defined by*

$$T_\lambda^{m,*}(f)(x)=\sup_{\varepsilon>0}\left|\int_{(0,\infty)^n,\ |x-y|>\varepsilon} f(y)K_\lambda^\phi(x,y)d\mu_\lambda(y)\right|,\quad x\in(0,\infty)^n,$$

being K_λ^ϕ as in Theorem 1.2, is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, for every $1 < p < \infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$. Moreover, we have, for every $1 < p < \infty$ and $f \in L^p((0,\infty)^n, d\mu_\lambda(x))$,

$$\begin{aligned}&\|T_\lambda^{m,*}(f)\|_{L^p((0,\infty)^n,d\mu_\lambda(x))}\\&\quad\le C\big(\|\phi\|_{L^\infty(0,\infty)} + \|T^{m,*}\|_{p\mapsto p}\big)\|f\|_{L^p((0,\infty)^n,d\mu_\lambda(x))},\end{aligned}$$

and, for every $L^1((0,\infty)^n, d\mu_\lambda(x))$,

$$\begin{aligned}&\|T_\lambda^{m,*}(f)\|_{L^{1,\infty}((0,\infty)^n,d\mu_\lambda(x))}\\&\quad\le C\big(\|\phi\|_{L^\infty(0,\infty)} + \|T^{m,*}\|_{1\mapsto(1,\infty)}\big)\|f\|_{L^1((0,\infty)^n,d\mu_\lambda(x))},\end{aligned}$$

where $\|T^{m,*}\|_{p\mapsto p}$, $1 < p < \infty$, and $\|T^{m,*}\|_{1\mapsto(1,\infty)}$ denotes the norm of the operator $T^{m,*}$ between $L^p(\mathbb{R}^n, dx)$ into itself, $1 < p < \infty$, and $L^1(\mathbb{R}^n, dx)$ into $L^{1,\infty}(\mathbb{R}^n, dx)$, respectively. Here $C > 0$ is a constant which does not depend on the function ϕ.

From Theorems 1.2 and 1.3 we can deduce the following result.

Theorem 1.4 *Let $\lambda \in (-1/2,\infty)^n$. Assume that m is of Laplace transform type associated with $\phi \in L^\infty(0,\infty)$. For every $f \in L^p((0,\infty)^n, d\mu_\lambda(x))$, $1 \le p < \infty$, the limit*

$$\lim_{\varepsilon\to 0^+}\left(\Lambda(\varepsilon)f(x) + \int_{(0,\infty)^n,\ |x-y|>\varepsilon} f(y)K_\lambda^\phi(x,y)d\mu_\lambda(y)\right),$$

exists, for almost all $x \in (0,\infty)^n$. Here K_λ^ϕ and Λ are defined as in Theorem 1.2. Moreover, the operator $\mathbb{T}_\lambda^m$ defined by

$$\mathbb{T}_\lambda^m(f)(x) = -\lim_{\varepsilon\to 0^+}\left(\Lambda(\varepsilon)f(x) + \int_{(0,\infty)^n,\ |x-y|>\varepsilon} f(y)K_\lambda^\phi(x,y)d\mu_\lambda(y)\right),\quad \text{a.e.}\quad x \in (0,\infty)^n,$$

is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, for every $1 < p < \infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$. Also, we have that, for every $1 < p < \infty$ and $f \in L^p((0,\infty)^n, d\mu_\lambda(x))$,

$$\|\mathbb{T}_\lambda^m(f)\|_{L^p((0,\infty)^n,d\mu_\lambda(x))} \le C\big(\|\phi\|_{L^\infty(0,\infty)} + \|T^m\|_{p\mapsto p}\big)\|f\|_{L^p((0,\infty)^n,d\mu_\lambda(x))}, \tag{8}$$

and, for every $L^1((0,\infty)^n, d\mu_\lambda(x))$,

$$\begin{aligned}&\|\mathbb{T}_\lambda^m(f)\|_{L^{1,\infty}((0,\infty)^n,d\mu_\lambda(x))}\\&\quad\le C\big(\|\phi\|_{L^\infty(0,\infty)} + \|T^m\|_{1\mapsto(1,\infty)}\big)\|f\|_{L^1((0,\infty)^n,d\mu_\lambda(x))},\end{aligned} \tag{9}$$

where $\|T^m\|_{p\mapsto p}$, $1<p<\infty$, *and* $\|T^m\|_{1\mapsto(1,\infty)}$ *denotes the norm of the operator* T^m *between* $L^p(\mathbb{R}^n,dx)$ *into itself,* $1<p<\infty$, *and* $L^1(\mathbb{R}^n,dx)$ *into* $L^{1,\infty}(\mathbb{R}^n,dx)$ (*defined in the usual way*), *respectively. Here* $C>0$ *is a constant which does not depend on the function* ϕ.

Since $C_c^\infty((0,\infty)^n)$ is a dense subspace of $L^p((0,\infty)^n,d\mu_\lambda(x))$, $1\le p<\infty$, it follows that, for every $f\in L^2((0,\infty)^n,d\mu_\lambda(x))$,

$$T_\lambda^m(f)(x)=-\lim_{\varepsilon\to 0^+}\left(\Lambda(\varepsilon)f(x)+\int_{(0,\infty)^n,\ |x-y|>\varepsilon} f(y)K_\lambda^\phi(x,y)d\mu_\lambda(y)\right),$$
$$\text{a.e.}\quad x\in(0,\infty)^n, \tag{10}$$

where $\|\Lambda\|_{L^\infty(0,\infty)}\le C\|\phi\|_{L^\infty(0,\infty)}$. Moreover, the extension of T_λ^m from $L^2((0,\infty)^n,d\mu_\lambda(x))\cap L^p((0,\infty)^n,d\mu_\lambda(x))$ to $L^p((0,\infty)^n,d\mu_\lambda(x))$ as a bounded operator from $L^p((0,\infty)^n,d\mu_\lambda(x))$ into itself, for every $1<p<\infty$, is given by (10), and $\|T_\lambda^m\|_{p\mapsto p}\le C(\|\phi\|_{L^\infty(0,\infty)}+\|T^m\|_{p\mapsto p})$. Note that we also deduce that T_λ^m can be extended from $L^2((0,\infty)^n,d\mu_\lambda(x))\cap L^1((0,\infty)^n,d\mu_\lambda(x))$ to $L^1((0,\infty)^n,d\mu_\lambda(x))$ as a bounded operator from $L^1((0,\infty)^n,d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n,d\mu_\lambda(x))$, and $\|T_\lambda^m\|_{1\mapsto(1,\infty)}\le C(\|\phi\|_{L^\infty(0,\infty)}+\|T^m\|_{1\mapsto(1,\infty)})$. Thus, we complete the result in [19, Corollary 3, p. 121] for spectral multipliers of Laplace transform type for the Bessel operator Δ_λ.

We summarize these results on T_λ^m in the following.

Corollary 1.1 *Let* $\lambda\in(-1/2,\infty)^n$. *Assume that* m *is of Laplace transform type associated with* $\phi\in L^\infty(0,\infty)$. *Then, the Hankel multiplier* T_λ^m *can be extended from* $L^2((0,\infty)^n,d\mu_\lambda(x))\cap L^p((0,\infty)^n,d\mu_\lambda(x))$ *to* $L^p((0,\infty)^n,d\mu_\lambda(x))$ *as a bounded operator from* $L^p((0,\infty)^n,d\mu_\lambda(x))$ *into itself, for every* $1<p<\infty$, *and from* $L^2((0,\infty)^n,d\mu_\lambda(x))\cap L^1((0,\infty)^n,d\mu_\lambda(x))$ *to* $L^1((0,\infty)^n,d\mu_\lambda(x))$ *as a bounded operator from* $L^1((0,\infty)^n,d\mu_\lambda(x))$ *into* $L^{1,\infty}((0,\infty)^n,d\mu_\lambda(x))$. *Moreover, we have, for every* $1<p<\infty$ *and* $f\in L^p((0,\infty)^n,d\mu_\lambda(x))$,

$$\|T_\lambda^m(f)\|_{L^p((0,\infty)^n,d\mu_\lambda(x))}\le C\big(\|\phi\|_{L^\infty(0,\infty)}+\|T^m\|_{p\mapsto p}\big)\|f\|_{L^p((0,\infty)^n,d\mu_\lambda(x))},$$

and, for every $f\in L^1((0,\infty)^n,d\mu_\lambda(x))$,

$$\|T_\lambda^m(f)\|_{L^{1,\infty}((0,\infty)^n,d\mu_\lambda(x))}\le C\big(\|\phi\|_{L^\infty(0,\infty)}+\|T^m\|_{1\mapsto(1,\infty)}\big)\|f\|_{L^1((0,\infty)^n,d\mu_\lambda(x))},$$

where $\|T^m\|_{p\mapsto p}$, $1<p<\infty$, *and* $\|T^m\|_{1\mapsto(1,\infty)}$ *denotes the norm of the operator* T^m *between* $L^p(\mathbb{R}^n,dx)$ *into itself,* $1<p<\infty$, *and* $L^1(\mathbb{R}^n,dx)$ *into* $L^{1,\infty}(\mathbb{R}^n,dx)$, *respectively. Here* $C>0$ *is a constant which does not depend on the function* ϕ.

This corollary can be seen as an extension to higher dimension of [3, Theorem 1.2]. However, in order to show it we use a different procedure than the one employed in the proof of [3, Theorem 1.2]. In [3] Calderón-Zygmund theory for singular integral operators is applied. Here, we work in a completely different way. We split the region $(0,\infty)^n\times(0,\infty)^n$ in two parts. The set

$$\Omega=\left\{(x,y)\in(0,\infty)^n\times(0,\infty)^n : x_j/2<y_j<2x_j,\ j=1,\dots,n\right\}$$

is called the local region. In Ω the kernel $K_\lambda^\phi(x,y)$ that defines the Hankel multiplier T_λ^m differs from $(x.y)^{-\lambda}H^\phi(x,y)$, where H^ϕ is the kernel associated with the Fourier multiplier T^m, by a kernel defining a bounded operator in $L^p((0,\infty)^n, d\mu_\lambda(x))$ for every $1<p<\infty$.

On $((0,\infty)^n \times (0,\infty)^n)\backslash\Omega$, called the global region, the kernel $|K_\lambda^\phi|$ defines a positive bounded operator from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, when $1<p<\infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$.

From Corollary 1.1 we deduce the following result for the imaginary power of Δ_λ.

Corollary 1.2 *Let $\beta\in\mathbb{R}$. Then, the operator $\Delta_\lambda^{i\beta}$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, for every $1<p<\infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$. Moreover, for every $1<p<\infty$, we have that*

$$\|\Delta_\lambda^{i\beta}(f)\|_{L^p((0,\infty)^n,d\mu_\lambda(x))} \le Ce^{\pi|\beta||\frac{1}{2}-\frac{1}{p}|}\|f\|_{L^p((0,\infty)^n,d\mu_\lambda(x))},$$
$$f\in L^p\big((0,\infty)^n, d\mu_\lambda(x)\big).$$

As it was mentioned earlier Theorem 1.1 follows from Corollary 1.2 by using [14, Theorem 1].

This paper is organized as follows. In Sect. 2 we recall some properties of Bessel functions and we establish some estimates for the heat kernel associated with the Bessel operators that will be very useful in Sect. 3. There we prove the main results of this paper that have been presented in this introduction.

Throughout this paper we will always denote by C a suitable positive constant that can change from one line to the next. Also, we will use repeatedly without saying it that, for every $k\ge 0$, $\sup_{z>0} z^k e^{-z}<\infty$.

2 Some Estimates Involving Bessel Functions

In this section we establish some estimates involving Bessel functions that we will need in the following sections.

The following properties of the Bessel function J_ν, $\nu>-1$, can be encountered in [13, pp. 104, 122 and 123]:

$$z^{-\nu}J_\nu(z)\sim\frac{1}{2^\nu\Gamma(\nu+1)},\quad \text{as } z\to 0; \tag{11}$$

$$\sqrt{z}J_\nu(z)=\mathcal{O}(1),\quad \text{as } z\to\infty; \tag{12}$$

and

$$\frac{d}{dz}\big(z^{-\nu}J_\nu(z)\big)=-z^{-\nu}J_{\nu+1}(z),\quad z\in(0,\infty). \tag{13}$$

For the modified Bessel function I_ν, $\nu>-1$, we have that ([13, pp. 108, 123 and 110]):

$$z^{-\nu}I_\nu(z)\sim\frac{1}{2^\nu\Gamma(\nu+1)},\quad \text{as } z\to 0; \tag{14}$$

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}\left(1 - \frac{4\nu^2-1}{8z} + \mathcal{O}\left(\frac{1}{z^2}\right)\right), \quad \text{as } z \to \infty; \tag{15}$$

and

$$\frac{d}{dz}\big(z^{-\nu} I_\nu(z)\big) = z^{-\nu} I_{\nu+1}(z), \quad z \in (0,\infty). \tag{16}$$

In the sequel we assume that $\mathbb{K}$ is a compact subset of $(0,\infty)$.

Lemma 2.1 *Let $\alpha > -1/2$. Then*
(a)

$$W_t^\alpha(u,v) \le C \begin{cases} \frac{e^{-(u^2+v^2)/4t}}{t^{\alpha+1/2}}, & uv \le t \\ (uv)^{-\alpha}\frac{e^{-(u-v)^2/4t}}{\sqrt{t}}, & uv > t. \end{cases}$$

(b) $W_t^\alpha(u,v) \le Ct^{-\alpha-1/2}e^{-v^2/20t}$, $t > 0$ *and* $0 < 2u < v < \infty$.
(c) $|W_t^\alpha(u,v) - \frac{t^{-\alpha-1/2}}{2^{2\alpha}\Gamma(\alpha+1/2)}| \le Ct^{-\alpha-3/2}$, $u,v \in \mathbb{K}$ *and* $t > 1$.
(d) $|W_t^\alpha(u,v) - (uv)^{-\alpha}\mathbb{W}_t(u,v)| \le C\sqrt{t}(uv)^{-\alpha-1}e^{-(u-v)^2/4t}$, $uv > t > 0$.

Proof (a) and (d) follow immediately from (14) and (15). Suppose now that $0 < 2u < v < \infty$ and $t > 0$. If $uv \le t$, (a) implies (b). Also, if $uv > t$, from (a) we deduce that

$$\begin{aligned} W_t^\alpha(u,v) &\le C(uv)^{-\alpha-1/2}\left(\frac{uv}{t}\right)^{1/2} e^{-v^2/16t} \le C(uv)^{-\alpha-1/2}\frac{v}{\sqrt{t}}e^{-v^2/16t} \\ &\le \frac{C}{t^{\alpha+1/2}}e^{-v^2/20t}, \end{aligned}$$

and (b) is shown.

By using (14) and (16) and the mean value theorem we obtain

$$\begin{aligned} &\left|W_t^\alpha(u,v) - \frac{t^{-\alpha-1/2}}{2^{2\alpha}\Gamma(\alpha+1/2)}\right| \\ &\quad\le \frac{1}{(2t)^{\alpha+1/2}}\left|\left(\frac{uv}{2t}\right)^{-\alpha+1/2} I_{\alpha-1/2}\left(\frac{uv}{2t}\right) - \frac{1}{2^{\alpha-1/2}\Gamma(\alpha+1/2)}\right| e^{-\frac{u^2+v^2}{4t}} \\ &\qquad + \frac{t^{-\alpha-1/2}}{2^{2\alpha}\Gamma(\alpha+1/2)}|e^{-\frac{u^2+v^2}{4t}} - 1| \\ &\quad\le \frac{uv}{(2t)^{\alpha+3/2}}e^{-\frac{u^2+v^2}{4t}} \sup_{z\in(0,uv/2t)}\left|\frac{d}{dz}\big(z^{-\alpha+1/2}I_{\alpha-1/2}(z)\big)\right| + C\frac{u^2+v^2}{t^{\alpha+3/2}} \\ &\quad\le C\left(\frac{(uv)^2}{t^{\alpha+5/2}} + \frac{u^2+v^2}{t^{\alpha+3/2}}\right) \\ &\quad\le Ct^{-\alpha-3/2}, \quad u,v \in \mathbb{K} \text{ and } t > 1. \end{aligned}$$

Thus (c) is proved. □

Lemma 2.2 *Let* $\alpha > -1/2$. *Then*

(a)

$$\left|\frac{\partial}{\partial u} W_t^\alpha(u,v)\right| \le C \begin{cases} t^{-\alpha-3/2}, & u,v\in\mathbb{K},\ t\ge 1\\ \frac{e^{-(u-v)^2/8t}}{t}, & u,v\in\mathbb{K},\ 0<t<1.\end{cases}$$

(b) $|\frac{\partial^2}{\partial u^2} W_t^\alpha(u,v)| \le C t^{-\alpha-3/2}$, $u,v\in\mathbb{K}$ *and* $t\ge 1$.

(c) $|\frac{\partial^2}{\partial u^2} W_t^\alpha(u,v) - (uv)^{-\alpha}\frac{\partial^2}{\partial u^2}\mathbb{W}_t(u,v)| \le Ct^{-1}e^{-(u-v)^2/8t}$, $u,v\in\mathbb{K}$ *and* $0<t<1$.

Proof According to (16) we get, for every $u,v,t\in(0,\infty)$,

$$\frac{\partial}{\partial u} W_t^\alpha(u,v) = \frac{e^{-(u^2+v^2)/4t}}{(2t)^{\alpha+1/2}}\left[\frac{v}{2t}\left(\frac{uv}{2t}\right)^{-\alpha+1/2} I_{\alpha+1/2}\left(\frac{uv}{2t}\right) - \frac{u}{2t}\left(\frac{uv}{2t}\right)^{-\alpha+1/2} I_{\alpha-1/2}\left(\frac{uv}{2t}\right)\right], \tag{17}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial u^2} W_t^\alpha(u,v) &= \frac{\partial^2}{\partial u^2}\left[\frac{\sqrt{2\pi}}{(2t)^\alpha}\left(\frac{uv}{2t}\right)^{-\alpha+1/2} I_{\alpha-1/2}\left(\frac{uv}{2t}\right)e^{-uv/2t}\mathbb{W}_t(u,v)\right]\\ &= \frac{\sqrt{2\pi}}{(2t)^\alpha}e^{-uv/2t}\left(\frac{uv}{2t}\right)^{-\alpha}\Bigg\{\Bigg[\left(\frac{uv}{2t}\right)^{1/2} I_{\alpha+3/2}\left(\frac{uv}{2t}\right)\\ &\quad+\left(\frac{uv}{2t}\right)^{1/2} I_{\alpha+1/2}\left(\frac{uv}{2t}\right)\frac{2t}{uv} - 2\left(\frac{uv}{2t}\right)^{1/2} I_{\alpha+1/2}\left(\frac{uv}{2t}\right)\\ &\quad+\left(\frac{uv}{2t}\right)^{1/2} I_{\alpha-1/2}\left(\frac{uv}{2t}\right)\Bigg]\mathbb{W}_t(u,v)\left(\frac{v}{2t}\right)^2\\ &\quad+\frac{v}{t}\Bigg[\left(\frac{uv}{2t}\right)^{1/2} I_{\alpha+1/2}\left(\frac{uv}{2t}\right)\\ &\quad-\left(\frac{uv}{2t}\right)^{1/2} I_{\alpha-1/2}\left(\frac{uv}{2t}\right)\Bigg]\frac{\partial}{\partial u}\mathbb{W}_t(u,v)\\ &\quad+\left[\left(\frac{uv}{2t}\right)^{1/2} I_{\alpha-1/2}\left(\frac{uv}{2t}\right)\right]\frac{\partial^2}{\partial u^2}\mathbb{W}_t(u,v)\Bigg\}.\end{aligned} \tag{18}$$

(a) can be deduced from (17) by using (14) and (15). (b) follows from (14) and (18). In order to establish (c) we estimate three different parts on (18) using (15). We have that, for every $0 < t \le 1$ and $u, v \in \mathbb{K}$,

$$\bullet\ \frac{e^{-uv/2t}}{t^\alpha}\left(\frac{uv}{t}\right)^{-\alpha}\left(\frac{v}{t}\right)^2\left|\left(\frac{uv}{2t}\right)^{1/2} I_{\alpha+3/2}\left(\frac{uv}{2t}\right) + \left(\frac{uv}{2t}\right)^{1/2} I_{\alpha+1/2}\left(\frac{uv}{2t}\right)\frac{2t}{uv} - 2\left(\frac{uv}{2t}\right)^{1/2} I_{\alpha+1/2}\left(\frac{uv}{2t}\right) + \left(\frac{uv}{2t}\right)^{1/2} I_{\alpha-1/2}\left(\frac{uv}{2t}\right)\right| \le C;$$

$$\bullet\ \frac{ve^{-uv/2t}}{t^{\alpha+1}}\left(\frac{uv}{t}\right)^{-\alpha}\left|\left(\frac{uv}{2t}\right)^{1/2} I_{\alpha+1/2}\left(\frac{uv}{2t}\right) - \left(\frac{uv}{2t}\right)^{1/2} I_{\alpha-1/2}\left(\frac{uv}{2t}\right)\right| \le C;$$

and

$$\bullet\ (uv)^{-\alpha}\left|\left(\frac{uv}{2t}\right)^{1/2} I_{\alpha-1/2}\left(\frac{uv}{2t}\right)e^{-uv/2t} - \frac{1}{\sqrt{2\pi}}\right| \le Ct.$$

Then, these estimates and (18) allow us to conclude (c). □

Lemma 2.3 *Let $\alpha > -1/2$. Then*

(a)

$$\left|\frac{\partial}{\partial t} W_t^\alpha(u, v)\right| \le C \begin{cases} \frac{e^{-(u^2+v^2)/8t}}{t^{\alpha+3/2}}, & uv \le t \\ (uv)^{-\alpha}\frac{e^{-(u-v)^2/8t}}{t^{3/2}} & uv > t. \end{cases}$$

(b) $|\frac{\partial}{\partial t} W_t^\alpha(u, v)| \le C\frac{e^{-v^2/40t}}{t^{\alpha+3/2}}$, $t > 0$, *and* $0 < 2u < v < \infty$.

(c) $|\frac{\partial}{\partial t} W_t^\alpha(u, v) - (uv)^{-\alpha}\frac{\partial}{\partial t}\mathbb{W}_t(u, v)| \le C\frac{(uv)^{-\alpha-1}}{\sqrt{t}}e^{-(u-v)^2/16t}$, $uv > t$.

Proof According to (16) we get

$$\frac{\partial}{\partial t} W_t^\alpha(u, v) = \frac{e^{-(u^2+v^2)/4t}}{2^{\alpha+1/2}}\left\{-\frac{\alpha+1/2}{t^{\alpha+3/2}}\left(\frac{uv}{2t}\right)^{-\alpha+1/2} I_{\alpha-1/2}\left(\frac{uv}{2t}\right) - \frac{uv}{2t^{\alpha+5/2}}\left(\frac{uv}{2t}\right)^{-\alpha+1/2} I_{\alpha+1/2}\left(\frac{uv}{2t}\right) + \frac{u^2+v^2}{4t^{\alpha+5/2}}\left(\frac{uv}{2t}\right)^{-\alpha+1/2} I_{\alpha-1/2}\left(\frac{uv}{2t}\right)\right\}, \quad u, v, t \in (0, \infty). \tag{19}$$

Then, (a) for $uv \le t$ is deduced from (14). Moreover, from (15) it follows that

$$\left|\frac{\partial}{\partial t} W_t^\alpha(u, v)\right| \le C\frac{e^{-(u-v)^2/4t}}{t^{3/2}}(uv)^{-\alpha}\left(\frac{|u-v|^2}{t} + 1\right) \le C\frac{e^{-(u-v)^2/8t}}{t^{3/2}}(uv)^{-\alpha}, \quad uv > t.$$

Thus (a) is proved.

When $uv \le t$ (b) can be inferred immediately from (a). Also, if $uv > t$, from (a) we deduce

$$\begin{aligned}\left|\frac{\partial}{\partial t}W_t^\alpha(u,v)\right| &\le C\frac{e^{-v^2/32t}}{t}\left(\frac{uv}{t}\right)^{1/2}(uv)^{-\alpha-1/2}\\ &\le C\frac{e^{-v^2/32t}}{t^{\alpha+3/2}}\frac{v}{\sqrt{t}}\\ &\le C\frac{e^{-v^2/40t}}{t^{\alpha+3/2}},\quad 0<2u<v<\infty.\end{aligned}$$

From (19) by using (15) it follows that

$$\begin{aligned}&\frac{\partial}{\partial t}W_t^\alpha(u,v)\\ &=\left\{-\frac{\alpha+1/2}{t}\left(\frac{uv}{2t}\right)^{1/2}I_{\alpha-1/2}\left(\frac{uv}{2t}\right)-\frac{uv}{2t^2}\left(\frac{uv}{2t}\right)^{1/2}I_{\alpha+1/2}\left(\frac{uv}{2t}\right)\right.\\ &\quad\left.+\frac{u^2+v^2}{4t^2}\left(\frac{uv}{2t}\right)^{1/2}I_{\alpha-1/2}\left(\frac{uv}{2t}\right)\right\}\frac{(uv)^{-\alpha}}{\sqrt{2t}}e^{-(u^2+v^2)/4t}\\ &=\left\{-\frac{\alpha+1/2}{t}\left(1+\mathcal{O}\left(\frac{t}{uv}\right)\right)\right.\\ &\quad-\frac{uv}{2t^2}\left(1-\frac{4(\alpha+1/2)^2-1}{4}\frac{t}{uv}+\mathcal{O}\left(\left(\frac{t}{uv}\right)^2\right)\right)\\ &\quad\left.+\frac{u^2+v^2}{4t^2}\left(1-\frac{4(\alpha-1/2)^2-1}{4}\frac{t}{uv}+\mathcal{O}\left(\left(\frac{t}{uv}\right)^2\right)\right)\right\}\frac{(uv)^{-\alpha}}{\sqrt{4\pi t}}e^{-(u-v)^2/4t}\\ &=(uv)^{-\alpha}\frac{\partial}{\partial t}\mathbb{W}_t(u,v)+\mathcal{O}\left(\frac{(uv)^{-\alpha}}{t^{1/2}}e^{-(u-v)^2/8t}\left(\frac{1}{uv}+\frac{(u-v)^2}{(uv)^2}\right)\right),\quad uv>t.\end{aligned}$$

Since

$$\begin{aligned}&\frac{(uv)^{-\alpha}}{t^{1/2}}e^{-(u-v)^2/8t}\left(\frac{1}{uv}+\frac{(u-v)^2}{(uv)^2}\right)\\ &\quad\le C\frac{(uv)^{-\alpha-1}}{t^{1/2}}e^{-(u-v)^2/8t}\left(1+\frac{(u-v)^2}{t}\right)\\ &\quad\le C\frac{(uv)^{-\alpha-1}}{t^{1/2}}e^{-(u-v)^2/16t},\quad uv>t,\end{aligned}$$

(c) is established. □

3 Proofs of the Results

In the sequel we assume that

$$m(x)=|x|^2\int_0^\infty e^{-t|x|^2}\phi(t)dt,\quad x\in(0,\infty)^n,$$

where $\phi\in L^\infty(0,\infty)$ is not identically zero.

We fix some notation. If $x=(x_1,\dots,x_n)\in(0,\infty)^n$, for every $i=1,\dots,n$, $\overline{x}^i=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ (being understood in the obvious way when $i=0$ and $i=n$). Suppose that $(u,v,\alpha)\in(0,\infty)^2\times(-1/2,\infty)\to F(u,v,\alpha)\in\mathbb{R}$. We define, for every $x,y\in(0,\infty)^n$, $\lambda\in(-1/2,\infty)^n$, and, for every $i=1,\dots,n$,

$$F\big(\overline{x}^i,\overline{y}^i,\overline{\lambda}^i\big)=\prod_{j=1,\,j\neq i}^n F(x_j,y_j,\lambda_j),$$

and, also, we consider the measure $d\mu_{\overline{\lambda}^i}(\overline{x}^i)=\prod_{j=1,\,j\neq i}^n d\mu_{\lambda_j}(x_j)$ on $(0,\infty)^{n-1}$.

3.1 Poof of Theorem 1.2

In order to prove this result we need firstly to see the following.

Lemma 3.1 *Let $\lambda\in(-1/2,\infty)^n$. Assume that $f\in C_c^\infty((0,\infty)^n)$. Then,*

$$T_\lambda^m(f)(x)=\int_0^\infty\phi(t)H_\lambda\big(|y|^2e^{-t|y|^2}H_\lambda(f)(y)\big)(x)dt,\quad\text{a.e.}\quad x\in(0,\infty)^n.$$

Proof Let $g\in C_c^\infty((0,\infty)^n)$. By using Plancherel equality for Hankel transform we get

$$\begin{aligned}\int_{(0,\infty)^n}&T_\lambda^m(f)(x)g(x)d\mu_\lambda(x)\\&=\int_{(0,\infty)^n}|y|^2\int_0^\infty e^{-t|y|^2}\phi(t)dt\,H_\lambda(f)(y)H_\lambda(g)(y)d\mu_\lambda(y)\\&=\int_0^\infty\phi(t)\int_{(0,\infty)^n}|y|^2e^{-t|y|^2}H_\lambda(f)(y)H_\lambda(g)(y)d\mu_\lambda(y)dt.\end{aligned}$$

The interchange of the order of integration is justify by using Hölder inequality and that H_λ is an isometry in $L^2((0,\infty)^n,d\mu_\lambda(x))$. Indeed, we can write

$$\begin{aligned}\int_{(0,\infty)^n}&|y|^2\int_0^\infty\big|\phi(t)\big|e^{-t|y|^2}dt\big|H_\lambda(f)(y)\big|\big|H_\lambda(g)(y)\big|d\mu_\lambda(y)\\&\le C\|f\|_{L^2((0,\infty)^n,d\mu_\lambda(x))}\|g\|_{L^2((0,\infty)^n,d\mu_\lambda(x))}<\infty.\end{aligned}$$

Then, by using again Plancherel equality for H_λ, we obtain

$$\int_{(0,\infty)^n}T_\lambda^m(f)(x)g(x)d\mu_\lambda(x)$$

$$= \int_0^\infty \phi(t) \int_{(0,\infty)^n} H_\lambda\big(|y|^2 e^{-t|y|^2} H_\lambda(f)(y)\big)(x) g(x) d\mu_\lambda(x) dt.$$

According to (11) and (12), for each $0 \le r \le n$,

$$\int_0^{1/x_1} \cdots \int_0^{1/x_r} \int_{1/x_{r+1}}^\infty \cdots \int_{1/x_n}^\infty \big|\Phi_\lambda(x,y)\big| |y|^2 e^{-t|y|^2} \big|H_\lambda(f)(y)\big| d\mu_\lambda(y)$$

$$\le C \int_{(0,\infty)^n} |y|^2 e^{-t|y|^2} \big|H_\lambda(f)(y)\big| \prod_{j=r+1}^n (x_j y_j)^{-\lambda_j} d\mu_\lambda(y), \quad x \in (0,\infty)^n.$$

Then, since $|y|^l H_\lambda(f)$ is bounded on $(0,\infty)^n$, for every $l \in \mathbb{N}$, and $g \in C_c^\infty((0,\infty)^n)$, we get, for every $0 \le r \le n$,

$$\int_0^\infty |\phi(t)| \int_{(0,\infty)^n} \int_0^{1/x_1} \cdots \int_0^{1/x_r} \int_{1/x_{r+1}}^\infty \cdots \int_{1/x_n}^\infty \big|\Phi_\lambda(x,y)\big|$$

$$\times |y|^2 e^{-t|y|^2} \big|H_\lambda(f)(y)\big| d\mu_\lambda(y) \big|g(x)\big| d\mu_\lambda(x) dt$$

$$\le C \int_0^\infty \int_{(0,\infty)^n} \int_{(0,\infty)^n} |y|^2 e^{-t|y|^2} \big|H_\lambda(f)(y)\big| \big|g(x)\big|$$

$$\times \prod_{j=r+1}^n (x_j y_j)^{-\lambda_j} d\mu_\lambda(y) d\mu_\lambda(x) dt$$

$$\le C \int_{(0,\infty)^n} \big|g(x)\big| \prod_{j=r+1}^n x_j^{-\lambda_j} d\mu_\lambda(x)$$

$$\times \int_{(0,\infty)^n} \left(\int_0^\infty |y|^2 e^{-t|y|^2} dt \right) \big|H_\lambda(f)(y)\big| \prod_{j=r+1}^n y_j^{-\lambda_j} d\mu_\lambda(y)$$

$$\le C \left(\int_{(0,\infty)^n} \big|g(x)\big| \prod_{j=r+1}^n x_j^{-\lambda_j} d\mu_\lambda(x) \right)$$

$$\times \left(\int_{(0,\infty)^n} \big|H_\lambda(f)(y)\big| \prod_{j=r+1}^n y_j^{-\lambda_j} d\mu_\lambda(y) \right) < \infty.$$

We conclude that

$$\int_{(0,\infty)^n} T_\lambda^m(f)(x) g(x) d\mu_\lambda(x)$$

$$= \int_{(0,\infty)^n} \left\{ \int_0^\infty \phi(t) H_\lambda\big(|y|^2 e^{-t|y|^2} H_\lambda(f)(y)\big)(x) dt \right\} g(x) d\mu_\lambda(x).$$

Thus, the proof of this lemma finishes. □

We now prove Theorem 1.2. The procedure employed to show it is inspired in [7]. Assume that $n \geq 2$. When $n = 1$ we can proceed in a similar way. By Lemma 3.1 and (2) we can write, for a.e. $x \in (0,\infty)^n$,

$$T_\lambda^m(f)(x) = \int_0^\infty \phi(t) H_\lambda\big(e^{-t|y|^2} H_\lambda(\Delta_\lambda f)(y)\big)(x)\,dt. \tag{20}$$

Fix $x \in (0,\infty)^n$ such that (20) holds. According to (11) and (12), for each $0 \leq r, s \leq n$, it follows that,

$$\begin{aligned}
&\int_0^{1/x_1} \cdots \int_0^{1/x_r} \int_{1/x_{r+1}}^\infty \cdots \int_{1/x_n}^\infty \big|\Phi_\lambda(x,y)\big| e^{-t|y|^2} \\
&\quad \times \int_0^{1/y_1} \cdots \int_0^{1/y_s} \int_{1/y_{s+1}}^\infty \cdots \int_{1/y_n}^\infty \big|\Phi_\lambda(y,z)\big|\big|\Delta_\lambda f(z)\big| d\mu_\lambda(z) d\mu_\lambda(y) \\
&\leq C \prod_{j=r+1}^n x_j^{-\lambda_j} \int_{(0,\infty)^n} e^{-t|y|^2} \prod_{j=r+1}^n y_j^{-\lambda_j} \prod_{j=1}^s y_j^{2\lambda_j} \prod_{j=s+1}^n y_j^{\lambda_j} dy \\
&\quad \times \int_{(0,\infty)^n} \big|\Delta_\lambda f(z)\big| \prod_{j=1}^s z_j^{2\lambda_j} \prod_{j=s+1}^n z_j^{\lambda_j} dz < \infty, \quad t > 0.
\end{aligned}$$

Then, by interchanging the order of integration and by using (4), we get

$$\begin{aligned}
&H_\lambda\big(e^{-t|y|^2} H_\lambda(\Delta_\lambda f)(y)\big)(x) \\
&\quad = \int_{(0,\infty)^n} \Delta_\lambda f(z) \prod_{j=1}^n \int_0^\infty e^{-ty_j^2} (x_j y_j)^{-\lambda_j+1/2} J_{\lambda_j-1/2}(x_j y_j) \\
&\qquad \times (y_j z_j)^{-\lambda_j+1/2} J_{\lambda_j-1/2}(y_j z_j) d\mu_{\lambda_j}(y_j) d\mu_\lambda(z) \\
&\quad = \int_{(0,\infty)^n} \Delta_\lambda f(z) W_t^\lambda(x,z) d\mu_\lambda(z), \quad t > 0.
\end{aligned} \tag{21}$$

We choose $a > 1$ such that $\operatorname{supp} f \subset K^n$, where $K = [1/a, a]$. From Lemma 2.1(a) and (c), it follows that

$$\begin{aligned}
&\left| W_t^\lambda(x,z) - \frac{t^{-\sum_{j=1}^n(\lambda_j+1/2)}}{2^{2\lambda}\Gamma(\lambda+1/2)} \right| \\
&\quad \leq \sum_{i=1}^n \prod_{j=1}^{i-1} \frac{t^{-(\lambda_j+1/2)}}{2^{2\lambda_j}\Gamma(\lambda_j+1/2)} \left| W_t^{\lambda_i}(x_i,z_i) - \frac{t^{-(\lambda_i+1/2)}}{2^{2\lambda_i}\Gamma(\lambda_i+1/2)} \right| \prod_{j=i+1}^n W_t^{\lambda_j}(x_j,z_j) \\
&\quad \leq C \frac{1}{t^{\sum_{j=1}^n(\lambda_j+1/2)+1}}, \quad t \geq 1 \text{ and } z \in K^n.
\end{aligned} \tag{22}$$

On the other hand, by (2) and (11) we have

$$\begin{aligned}
&\int_{(0,\infty)^n} \Delta_\lambda f(z) d\mu_\lambda(z) \\
&\quad = \prod_{j=1}^n 2^{\lambda_j - 1/2}\Gamma(\lambda_j + 1/2) \lim_{y\to 0} \int_{(0,\infty)^n} \Delta_\lambda f(z)\Phi_\lambda(y,z) d\mu_\lambda(z) \\
&\quad = \prod_{j=1}^n 2^{\lambda_j - 1/2}\Gamma(\lambda_j + 1/2) \lim_{y\to 0} |y|^2 H_\lambda(f)(y) = 0. \qquad (23)
\end{aligned}$$

According to (20), (21), and (23) (suggested by (22)), we can write

$$\begin{aligned}
&T_\lambda^m(f)(x) \\
&= \int_0^\infty \phi(t) \int_{(0,\infty)^n} \Delta_\lambda f(z) \left(W_t^\lambda(x,z) - \frac{\chi_{(1,\infty)}(t) t^{-\sum_{j=1}^n (\lambda_j + 1/2)}}{2^{2\lambda}\Gamma(\lambda + 1/2)} \right) d\mu_\lambda(z) dt. \qquad (24)
\end{aligned}$$

Our next objective is to make that the Bessel operator Δ_λ pass from f to the second factor in the last integral. In order to do this we will apply partial integration carefully. The integral in (24) is absolutely convergent. Indeed, in order to do this we split the integral in the following way

$$\begin{aligned}
&\int_0^\infty |\phi(t)| \int_{(0,\infty)^n} |\Delta_\lambda f(z)| \left| W_t^\lambda(x,z) - \frac{\chi_{(1,\infty)}(t) t^{-\sum_{j=1}^n (\lambda_j + 1/2)}}{2^{2\lambda}\Gamma(\lambda + 1/2)} \right| d\mu_\lambda(z) dt \\
&\leq C \left(\int_0^1 + \int_1^\infty \right) \int_{(0,\infty)^n} |\Delta_\lambda f(z)| \\
&\quad \times \left| W_t^\lambda(x,z) - \frac{\chi_{(1,\infty)}(t) t^{-\sum_{j=1}^n} (\lambda_j + 1/2)}{2^{2\lambda}\Gamma(\lambda + 1/2)} \right| d\mu_\lambda(z) dt \\
&= I_1(x) + I_2(x).
\end{aligned}$$

Since by using the inversion formula for Hankel transform and (4) we get

$$\int_0^\infty W_t^\alpha(u,v) v^{2\alpha} dv = 1, \quad t, u \in (0,\infty),$$

when $\alpha > -1/2$, it follows that

$$I_1(x) \leq C \int_0^1 \int_{(0,\infty)^n} |\Delta_\lambda f(z)| W_t^\lambda(x,z) d\mu_\lambda(z) dt \leq C.$$

Also, by (22) we obtain

$$I_2(x) \leq C \int_1^\infty \frac{dt}{t^{\sum_{j=1}^n (\lambda_j + 1/2) + 1}} \int_{K^n} |\Delta_\lambda f(z)| d\mu_\lambda(z) \leq C.$$

By (24) we have

$$T_\lambda^m(f)(x) = \lim_{\varepsilon\to 0^+}\int_0^\infty \phi(t)\int_{(0,\infty)^n,|z-x|>\varepsilon}\Delta_\lambda f(z)\Big(W_t^\lambda(x,z) - \frac{\chi_{(1,\infty)}(t)t^{-\sum_{j=1}^n(\lambda_j+1/2)}}{2^{2\lambda}\Gamma(\lambda+1/2)}\Big)d\mu_\lambda(z)dt.$$

Assume that ε is small enough, for instance, $0<\varepsilon<x_i/2$, $i=1,\dots,n$. We now analyze the integral

$$I^\varepsilon(x,t) = \int_{\substack{K^n\\ |z-x|>\varepsilon}}\Delta_{\lambda_1,z_1}f(z)\Big(W_t^\lambda(x,z) - \frac{\chi_{(1,\infty)}(t)t^{-\sum_{j=1}^n(\lambda_j+1/2)}}{2^{2\lambda}\Gamma(\lambda+1/2)}\Big)d\mu_\lambda(z),$$

$$t>0.$$

The study of the integral involving Δ_{λ_j,z_j}, $j=2,\dots,n$, can be made in a similar way. We can write

$$\begin{aligned}I^\varepsilon(x,t) &= \int_{|\bar z^1-\bar x^1|>\varepsilon}\int_0^\infty \Delta_{\lambda_1,z_1}f(z)\Big(W_t^\lambda(x,z) - \frac{\chi_{(1,\infty)}(t)t^{-\sum_{j=1}^n(\lambda_j+1/2)}}{2^{2\lambda}\Gamma(\lambda+1/2)}\Big)d\mu_\lambda(z)\\ &\quad+\int_{|\bar z^1-\bar x^1|<\varepsilon}\Big(\int_0^{x_1-\sqrt{\varepsilon^2-|\bar z^1-\bar x^1|^2}}+\int_{x_1+\sqrt{\varepsilon^2-|\bar z^1-\bar x^1|^2}}^\infty\Big)\Delta_{\lambda_1,z_1}f(z)\\ &\quad\times\Big(W_t^\lambda(x,z) - \frac{\chi_{(1,\infty)}(t)t^{-\sum_{j=1}^n(\lambda_j+1/2)}}{\prod_{j=1}^n 2^{2\lambda_j}\Gamma(\lambda_j+1/2)}\Big)d\mu_\lambda(z)\\ &= I_1^\varepsilon(x,t)+I_2^\varepsilon(x,t),\quad t>0.\end{aligned}$$

By partial integration we obtain

$$\begin{aligned}&\int_0^\infty \Delta_{\lambda_1,z_1}f(z)\Big(W_t^\lambda(x,z) - \frac{\chi_{(1,\infty)}(t)t^{-\sum_{j=1}^n(\lambda_j+1/2)}}{2^{2\lambda}\Gamma(\lambda+1/2)}\Big)d\mu_{\lambda_1}(z_1)\\ &= -z_1^{2\lambda_1}\frac{\partial}{\partial z_1}f(z)\Big(W_t^\lambda(x,z) - \frac{\chi_{(1,\infty)}(t)t^{-\sum_{j=1}^n(\lambda_j+1/2)}}{2^{2\lambda}\Gamma(\lambda+1/2)}\Big)\Big]_0^\infty\\ &\quad+z_1^{2\lambda_1}f(z)\frac{\partial}{\partial z_1}W_t^\lambda(x,z)\\ &\quad+\int_0^\infty f(z)\Delta_{\lambda_1,z_1}W_t^{\lambda_1}(x_1,z_1)W_t^{\bar\lambda^1}(\bar x^1,\bar z^1)d\mu_{\lambda_1}(z_1)\\ &= \int_{1/a}^a f(z)\Delta_{\lambda_1,z_1}W_t^{\lambda_1}(x_1,z_1)W_t^{\bar\lambda^1}(\bar x^1,\bar z^1)d\mu_{\lambda_1}(z_1),\quad t>0 \text{ and } \bar z^1\in K^{n-1}.\end{aligned}$$

Differentiating in (4), by using (13) and [13, (5.3.6), p. 103], we get $-\Delta_{\alpha,u}W_t^\alpha(u,v) = \frac{\partial}{\partial t}W_t^\alpha(u,v)$, $u,v,t>0$, $\alpha>-1/2$, and it follows that

$$
\begin{aligned}
&I_1^\varepsilon(x,t)\\
&\quad=\int_{|\bar z^1-\bar x^1|>\varepsilon}\int_0^\infty \Delta_{\lambda_1,z_1}f(z)\left(W_t^\lambda(x,z)-\frac{\chi_{(1,\infty)}(t)t^{-\sum_{j=1}^n(\lambda_j+1/2)}}{2^{2\lambda}\Gamma(\lambda+1/2)}\right)d\mu_\lambda(z)\\
&\quad=-\int_{|\bar z^1-\bar x^1|>\varepsilon}\int_0^\infty f(z)\frac{\partial}{\partial t}W_t^{\lambda_1}(x_1,z_1)W_t^{\bar\lambda^1}(\bar x^1,\bar z^1)d\mu_\lambda(z),\quad t>0. \qquad (25)
\end{aligned}
$$

In a similar way we obtain

$$
\begin{aligned}
&\left(\int_0^{x_1-\sqrt{\varepsilon^2-|\bar z^1-\bar x^1|^2}}+\int_{x_1+\sqrt{\varepsilon^2-|\bar z^1-\bar x^1|^2}}^\infty\right)\Delta_{\lambda_1,z_1}f(z)\\
&\quad\times\left(W_t^\lambda(x,z)-\frac{\chi_{(1,\infty)}(t)t^{-\sum_{j=1}^n(\lambda_j+1/2)}}{2^{2\lambda}\Gamma(\lambda+1/2)}\right)d\mu_{\lambda_1}(z_1)\\
&=\left(\int_0^{x_1-\sqrt{\varepsilon^2-|\bar z^1-\bar x^1|^2}}+\int_{x_1+\sqrt{\varepsilon^2-|\bar z^1-\bar x^1|^2}}^\infty\right)f(z)\Delta_{\lambda_1,z_1}W_t^{\lambda_1}(x_1,z_1)\\
&\quad\times W_t^{\bar\lambda^1}(\bar x^1,\bar z^1)d\mu_{\lambda_1}(z_1)\\
&\quad-\left(H_1\left(x,x_1-\sqrt{\varepsilon^2-\left|\bar z^1-\bar x^1\right|^2},\bar z^1,t\right)-H_1\left(x,x_1+\sqrt{\varepsilon^2-\left|\bar z^1-\bar x^1\right|^2},\bar z^1,t\right)\right)\\
&\quad+H_2\left(x,x_1-\sqrt{\varepsilon^2-\left|\bar z^1-\bar x^1\right|^2},\bar z^1,t\right)-H_2\left(x,x_1+\sqrt{\varepsilon^2-\left|\bar z^1-\bar x^1\right|^2},\bar z^1,t\right),\\
&\quad|\bar x^1-\bar z^1|<\varepsilon \text{ and } t>0,
\end{aligned}
$$

where

$$
\begin{aligned}
H_1(x,z,t)&=z_1^{2\lambda_1}\frac{\partial}{\partial z_1}f(z)\left(W_t^\lambda(x,z)-\frac{\chi_{(1,\infty)}(t)t^{-\sum_{j=1}^n(\lambda_j+1/2)}}{2^{2\lambda}\Gamma(\lambda+1/2)}\right),\\
&t>0 \text{ and } z\in(0,\infty)^n,
\end{aligned}
$$

and

$$
H_2(x,z,t)=z_1^{2\lambda_1}f(z)\frac{\partial}{\partial z_1}W_t^\lambda(x,z),\quad t>0 \text{ and } z\in(0,\infty)^n.
$$

We have, by (25), that

$$
\begin{aligned}
I_1^\varepsilon(x,t)&+\int_{|\bar z^1-\bar x^1|<\varepsilon}\left(\int_0^{x_1-\sqrt{\varepsilon^2-|\bar z^1-\bar x^1|^2}}+\int_{x_1+\sqrt{\varepsilon^2-|\bar z^1-\bar x^1|^2}}^\infty\right)\\
&\times f(z)\Delta_{\lambda_1,z_1}W_t^{\lambda_1}(x_1,z_1)W_t^{\bar\lambda^1}(\bar x^1,\bar z^1)d\mu_\lambda(z)\\
&=-\int_{|z-x|>\varepsilon}f(z)\frac{\partial}{\partial t}W_t^{\lambda_1}(x_1,z_1)W_t^{\bar\lambda^1}(\bar x^1,\bar z^1)d\mu_\lambda(z),\quad t>0.
\end{aligned}
$$

Since $f \in C_c^\infty((0,\infty)^n)$, by using mean value Theorem, (22), Lemmas 2.1(a) and 2.2(a), we get

$$\begin{aligned}
&\left|H_1\left(x, x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}, \bar{z}^1, t\right)-H_1\left(x, x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}, \bar{z}^1, t\right)\right| \\
&\quad\leq \Bigg|\left(x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right)^{2\lambda_1}\left(\frac{\partial}{\partial z_1} f\right)\left(x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}, \bar{z}^1\right) \\
&\qquad-\left(x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right)^{2\lambda_1}\left(\frac{\partial}{\partial z_1} f\right)\left(x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}, \bar{z}^1\right)\Bigg| \\
&\qquad\times\left|W_t^{\lambda_1}\left(x_1, x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right) W_t^{\bar{\lambda}^1}\left(\bar{x}^1, \bar{z}^1\right)-\frac{t^{-\sum_{j=1}^n(\lambda_j+1/2)}}{2^{2\lambda} \Gamma(\lambda+1/2)}\right| \\
&\qquad+\left|\left(x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right)^{2\lambda_1}\left(\frac{\partial}{\partial z_1} f\right)\left(x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}, \bar{z}^1\right)\right| \\
&\qquad\times\Bigg|W_t^{\lambda_1}\left(x_1, x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right) \\
&\qquad-W_t^{\lambda_1}\left(x_1, x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right)\Bigg| W_t^{\bar{\lambda}^1}\left(\bar{x}^1, \bar{z}^1\right) \\
&\quad\leq C \frac{\varepsilon}{t^{\sum_{j=1}^n(\lambda_j+1/2)+1}}, \quad t \geq 1,\ \bar{z}^1 \in K^{n-1} \text{ and } \left|\bar{z}^1-\bar{x}^1\right|<\varepsilon.
\end{aligned} \tag{26}$$

Also, from Lemmas 2.1(a), and 2.2(a), we deduce

$$\begin{aligned}
&\left|H_1\left(x, x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}, \bar{z}^1, t\right)-H_1\left(x, x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}, \bar{z}^1, t\right)\right| \\
&\quad\leq C\Bigg\{\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2} \frac{(x_1(x_1+\sqrt{\varepsilon^2-|\bar{z}^1-\bar{x}^1|^2}))^{-\lambda_1}}{\sqrt{t}} \\
&\qquad+\left|\left(x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right)^{2\lambda_1}\left(\frac{\partial}{\partial z_1} f\right)\left(x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}, \bar{z}^1\right)\right| \\
&\qquad\times \int_{x_1-\sqrt{\varepsilon^2-|\bar{z}^1-\bar{x}^1|^2}}^{x_1+\sqrt{\varepsilon^2-|\bar{z}^1-\bar{x}^1|^2}}\left|\frac{\partial}{\partial z_1} W_t^{\lambda_1}(x_1, z_1)\right| d z_1\Bigg\} W_t^{\bar{\lambda}^1}\left(\bar{x}^1, \bar{z}^1\right) \\
&\quad\leq C\left\{\frac{\varepsilon}{\sqrt{t}}+\int_{x_1-\sqrt{\varepsilon^2-|\bar{z}^1-\bar{x}^1|^2}}^{x_1+\sqrt{\varepsilon^2-|\bar{z}^1-\bar{x}^1|^2}} \frac{e^{-|x_1-z_1|^2/8t}}{t} d z_1\right\} \frac{e^{-|\bar{x}^1-\bar{z}^1|^2/4t}}{t^{(n-1)/2}}, \\
&\quad 0<t<1,\ \bar{z}^1 \in K^{n-1} \text{ and } \left|\bar{z}^1-\bar{x}^1\right|<\varepsilon.
\end{aligned} \tag{27}$$

By (26), we can write

$$\begin{aligned}
&\int_1^\infty|\phi(t)| \int_{\bar{z}^1 \in K^{n-1},|\bar{z}^1-\bar{x}^1|<\varepsilon}\Bigg| H_1\left(x, x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}, \bar{z}^1, t\right) \\
&\qquad-H_1\left(x, x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}, \bar{z}^1, t\right)\Bigg| d \mu_{\bar{\lambda}^1}(\bar{z}^1) d t
\end{aligned}$$

$$\leq C\varepsilon \int_1^\infty \int_{|\bar{z}^1-\bar{x}^1|<\varepsilon} \frac{1}{t^{\sum_{j=1}^n(\lambda_j+1/2)+1}} d\mu_{\bar{\lambda}^1}(\bar{z}^1)dt \longrightarrow 0, \quad \text{as } \varepsilon \to 0^+.$$

By (27) and by using [20, Lemma 1.1], when $n > 2$ it follows

$$\begin{aligned}
&\int_0^1 |\phi(t)| \int_{\bar{z}^1\in K^{n-1},|\bar{z}^1-\bar{x}^1|<\varepsilon} \Big|H_1\Big(x, x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}, \bar{z}^1, t\Big) \\
&\quad - H_1\Big(x, x_1 + \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}, \bar{z}^1, t\Big)\Big| d\mu_{\bar{\lambda}^1}(\bar{z}^1)dt \\
&\leq C\Bigg(\varepsilon \int_0^1 \int_{\bar{z}^1\in K^{n-1},|\bar{z}^1-\bar{x}^1|<\varepsilon} \frac{e^{-|\bar{z}^1-\bar{x}^1|^2/4t}}{t^{n/2}} d\bar{z}^1 dt \\
&\quad + \int_0^1 \int_{\bar{z}^1\in K^{n-1},|\bar{z}^1-\bar{x}^1|<\varepsilon} \int_{x_1-\sqrt{\varepsilon^2-|\bar{z}^1-\bar{x}^1|^2}}^{x_1+\sqrt{\varepsilon^2-|\bar{z}^1-\bar{x}^1|^2}} \frac{e^{-|z-x|^2/8t}}{t^{(n+1)/2}} dz dt\Bigg) \\
&\leq C\Bigg(\varepsilon \int_{|\bar{z}^1-\bar{x}^1|<\varepsilon} \frac{d\bar{z}^1}{|\bar{z}^1-\bar{x}^1|^{n-2}} + \int_{|z-x|<\varepsilon} \frac{dz}{|z-x|^{n-1}}\Bigg) \longrightarrow 0, \quad \text{as } \varepsilon \to 0^+.
\end{aligned}$$

For $n = 2$ we can proceed analogously.

We now write, for each $t > 0$, $\bar{z}^1 \in K^{n-1}$ and $|\bar{z}^1 - \bar{x}^1| < \varepsilon$,

$$\begin{aligned}
&\Big|H_2\Big(x, x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}, \bar{z}^1, t\Big) - H_2\Big(x, x_1 + \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}, \bar{z}^1, t\Big)\Big| \\
&\quad \leq \Big|\Big(x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}\Big)^{2\lambda_1} f\Big(x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}, \bar{z}\Big) \\
&\qquad - \Big(x_1 + \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}\Big)^{2\lambda_1} f\Big(x_1 + \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}, \bar{z}^1\Big)\Big| \\
&\qquad \times \left|\frac{\partial}{\partial z_1} W_t^{\lambda_1}\Big(x_1, x_1 + \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}\Big)\right| W_t^{\bar{\lambda}^1}(\bar{x}^1, \bar{z}^1) \\
&\qquad + \Big|\Big(x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}\Big)^{2\lambda_1} f\Big(x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}, \bar{z}^1\Big)\Big| \\
&\qquad \times \left|\frac{\partial}{\partial z_1} W_t^{\lambda_1}\Big(x_1, x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}\Big)\right. \\
&\qquad \left. - \frac{\partial}{\partial z_1} W_t^{\lambda_1}\Big(x_1, x_1 + \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}\Big)\right| W_t^{\bar{\lambda}^1}(\bar{x}^1, \bar{z}^1).
\end{aligned}$$

By Lemmas 2.1(a), and 2.2(a), mean value Theorem lead to

$$\begin{aligned}
&\Big|\Big(x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}\Big)^{2\lambda_1} f\Big(x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}, \bar{z}^1\Big) \\
&\quad - \Big(x_1 + \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}\Big)^{2\lambda_1} f\Big(x_1 + \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}, \bar{z}^1\Big)\Big| \\
&\quad \times \left|\frac{\partial}{\partial z_1} W_t^{\lambda_1}\Big(x_1, x_1 + \sqrt{\varepsilon^2 - |\bar{z}^1-\bar{x}^1|^2}\Big)\right| W_t^{\bar{\lambda}^1}(\bar{x}^1, \bar{z}^1)
\end{aligned}$$

$$\leq C\frac{\varepsilon}{t^{\lambda_1+3/2}}\prod_{j=2}^{n}\frac{1}{t^{\lambda_j+1/2}}, \quad t\geq 1,\ \bar{z}^1\in K^{n-1} \text{ and } \left|\bar{z}^1-\bar{x}^1\right|<\varepsilon. \tag{28}$$

On the other hand, Lemmas 2.1(a), and 2.2(b), lead to

$$\begin{aligned}&\left|\left(x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right)^{2\lambda_1}f\left(x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2},\bar{z}^1\right)\right|\\&\quad\times\left|\frac{\partial}{\partial z_1}W_t^{\lambda_1}\left(x_1,x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right)\right.\\&\quad\left.-\frac{\partial}{\partial z_1}W_t^{\lambda_1}\left(x_1,x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right)\right|W_t^{\bar{\lambda}^1}\left(\bar{x}^1,\bar{z}^1\right)\\&\leq C\frac{\varepsilon}{t^{\lambda_1+3/2}}\prod_{j=2}^{n}\frac{1}{t^{\lambda_j+1/2}}, \quad t\geq 1,\ \bar{z}^1\in K^{n-1} \text{ and } \left|\bar{z}^1-\bar{x}^1\right|<\varepsilon.\end{aligned} \tag{29}$$

From (28) and (29) we deduce that

$$\begin{aligned}&\int_1^\infty|\phi(t)|\int_{\bar{z}^1\in K^{n-1},\,|\bar{z}^1-\bar{x}^1|<\varepsilon}\left|H_2\left(x,x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2},\bar{z},t\right)\right.\\&\quad\left.-H_2\left(x,x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2},\bar{z},t\right)\right|d\mu_{\bar{\lambda}^1}\left(\bar{z}^1\right)dt\\&\leq C\varepsilon\int_1^\infty\int_{\bar{z}^1\in K^{n-1},|\bar{z}^1-\bar{x}^1|<\varepsilon}\frac{1}{t^{\sum_{j=1}^n(\lambda_j+1/2)+1}}d\mu_{\bar{\lambda}^1}\left(\bar{z}^1\right)dt\longrightarrow 0,\quad \text{as } \varepsilon\to 0^+.\end{aligned}$$

By mean value Theorem and Lemmas 2.1(a), and 2.2(a), it has,

$$\begin{aligned}&\left|\left(x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right)^{2\lambda_1}f\left(x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2},\bar{z}^1\right)\right.\\&\quad\left.-\left(x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right)^{2\lambda_1}f\left(x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2},\bar{z}^1\right)\right|\\&\quad\times\left|\frac{\partial}{\partial z_1}W_t^{\lambda_1}\left(x_1,x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right)\right|W_t^{\bar{\lambda}^1}\left(\bar{x}^1,\bar{z}^1\right)\\&\leq C\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\frac{1}{t^{(n+1)/2}}\left(\bar{x}^1.\bar{z}^1\right)^{-\bar{\lambda}^1}e^{-\frac{|\bar{x}^1-\bar{z}^1|^2}{4t}-\frac{\varepsilon^2-|\bar{x}^1-\bar{z}^1|^2}{8t}}\\&\leq C\varepsilon\frac{1}{t^{(n+1)/2}}e^{-\frac{\varepsilon^2}{8t}},\quad 0<t<1,\ \bar{z}^1\in K^{n-1} \text{ and } \left|\bar{z}^1-\bar{x}^1\right|<\varepsilon.\end{aligned}$$

Then [20, Lemma 1.1] allows us to get

$$\begin{aligned}&\int_0^1|\phi(t)|\int_{|\bar{x}^1-\bar{z}^1|<\varepsilon}\left|\left(x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right)^{2\lambda_1}f\left(x_1-\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2},\bar{z}^1\right)\right.\\&\quad\left.-\left(x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right)^{2\lambda_1}f\left(x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2},\bar{z}^1\right)\right|\\&\quad\times\left|\frac{\partial}{\partial z_1}W_t^{\lambda_1}\left(x_1,x_1+\sqrt{\varepsilon^2-\left|\bar{z}^1-\bar{x}^1\right|^2}\right)\right|W_t^{\bar{\lambda}^1}\left(\bar{x}^1,\bar{z}^1\right)d\mu_{\bar{\lambda}^1}\left(\bar{z}^1\right)dt\end{aligned}$$

$$\leq C\varepsilon \int_{|\bar{x}^1-\bar{z}^1|<\varepsilon} \int_0^1 \frac{e^{-\varepsilon^2/8t}}{t^{(n+1)/2}} dt d\bar{z}^1$$

$$\leq C\varepsilon \longrightarrow 0, \quad \text{as } \varepsilon \to 0^+.$$

Also, we write, for each $0 < t < 1$, $\bar{z}^1 \in K^{n-1}$ and $|\bar{z}^1 - \bar{x}^1| < \varepsilon$,

$$\begin{aligned}
&\Big(x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1 - \bar{x}|^2}\Big)^{2\lambda_1} f\Big(x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1 - \bar{x}^1|^2}, \bar{z}\Big) \\
&\quad \times \bigg(\frac{\partial}{\partial z_1} W_t^{\lambda_1}\Big(x_1, x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1 - \bar{x}^1|^2}\Big) \\
&\quad - \frac{\partial}{\partial z_1} W_t^{\lambda_1}\Big(x_1, x_1 + \sqrt{\varepsilon^2 - |\bar{z}^1 - \bar{x}^1|^2}\Big)\bigg) W_t^{\bar{\lambda}^1}(\bar{x}^1, \bar{z}^1) \\
&= \Big(\Big(x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1 - \bar{x}^1|^2}\Big)^{2\lambda_1} f\Big(x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1 - \bar{x}^1|^2}, \bar{z}^1\Big) - x_1^{2\lambda_1} f(x)\Big) \\
&\quad \times \bigg(\frac{\partial}{\partial z_1} W_t^{\lambda_1}\Big(x_1, x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1 - \bar{x}^1|^2}\Big) \\
&\quad - \frac{\partial}{\partial z_1} W_t^{\lambda_1}\Big(x_1, x_1 + \sqrt{\varepsilon^2 - |\bar{z}^1 - \bar{x}^1|^2}\Big)\bigg) W_t^{\bar{\lambda}^1}(\bar{x}^1, \bar{z}^1) \\
&\quad + x_1^{2\lambda_1} f(x)\bigg(\frac{\partial}{\partial z_1} W_t^{\lambda_1}\Big(x_1, x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1 - \bar{x}^1|^2}\Big) \\
&\quad - \frac{\partial}{\partial z_1} W_t^{\lambda_1}\Big(x_1, x_1 + \sqrt{\varepsilon^2 - |\bar{z}^1 - \bar{x}^1|^2}\Big)\bigg) W_t^{\bar{\lambda}^1}(\bar{x}^1, \bar{z}^1).
\end{aligned}$$

By using Lemmas 2.1(a), and 2.2(a), and proceeding as above we obtain,

$$\begin{aligned}
&\int_0^1 |\phi(t)| \int_{\bar{z}^1 \in K^{n-1}, |\bar{x}^1 - \bar{z}^1| < \varepsilon} \Big|\Big(x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1 - \bar{x}^1|^2}\Big)^{2\lambda_1} \\
&\qquad \times f\Big(x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1 - \bar{x}^1|^2}, \bar{z}^1\Big) - x_1^{2\lambda_1} f(x)\Big| \\
&\qquad \times \bigg|\frac{\partial}{\partial z_1} W_t^{\lambda_1}\Big(x_1, x_1 - \sqrt{\varepsilon^2 - |\bar{z}^1 - \bar{x}^1|^2}\Big) \\
&\qquad \times - \frac{\partial}{\partial z_1} W_t^{\lambda_1}\Big(x_1, x_1 + \sqrt{\varepsilon^2 - |\bar{z}^1 - \bar{x}^1|^2}\Big)\bigg| W_t^{\bar{\lambda}^1}(\bar{x}^1, \bar{z}^1) d\mu_{\bar{\lambda}^1}(\bar{z}^1) \\
&\quad \leq C\varepsilon \int_{\bar{z}^1 \in K^{n-1}, |\bar{x}^1 - \bar{z}^1| < \varepsilon} \int_0^1 \frac{e^{-\varepsilon^2/8t}}{t^{(n+1)/2}} dt d\bar{z}^1 \\
&\quad \leq C\varepsilon \longrightarrow 0, \quad \text{as } \varepsilon \to 0^+.
\end{aligned}$$

From the above estimates we conclude that

$$\int_0^\infty \phi(t)\int_{(0,\infty)^n}\Delta_{\lambda_1,z_1}f(z)\left(W_t^\lambda(x,z)-\frac{\chi_{(1,\infty)}(t)t^{-\sum_{j=1}^n(\lambda_j+1/2)}}{2^{2\lambda}\Gamma(\lambda+1/2)}\right)d\mu_\lambda(z)dt$$

$$= -\lim_{\varepsilon\to 0^+}\Bigg[\int_0^\infty \phi(t)\int_{|x-z|>\varepsilon} f(z)\frac{\partial}{\partial t}W_t^{\lambda_1}(x_1,z_1)W_t^{\bar\lambda^1}(\bar x^1,\bar z^1)d\mu_\lambda(z)dt$$

$$+ f(x)x_1^{2\lambda_1}\int_0^1\phi(t)\int_{|\bar x^1-\bar z^1|<\varepsilon}\int_{x_1-\sqrt{\varepsilon^2-|\bar x^1-\bar z^1|^2}}^{x_1+\sqrt{\varepsilon^2-|\bar x^1-\bar z^1|^2}}\frac{\partial^2}{\partial z_1^2}W_t^{\lambda_1}(x_1,z_1)$$

$$\times W_t^{\bar\lambda^1}(\bar x^1,\bar z^1)dz_1 d\mu_{\bar\lambda^1}(\bar z^1)dt\Bigg]$$

$$= -\lim_{\varepsilon\to 0^+}\Bigg[\int_0^\infty \phi(t)\int_{|x-z|>\varepsilon} f(z)\frac{\partial}{\partial t}W_t^{\lambda_1}(x_1,z_1)W_t^{\bar\lambda^1}(\bar x^1,\bar z^1)d\mu_\lambda(z)dt$$

$$+ f(x)x_1^{2\lambda_1}\int_0^1\phi(t)\int_{|x-z|<\varepsilon}\frac{\partial^2}{\partial z_1^2}W_t^{\lambda_1}(x_1,z_1)W_t^{\bar\lambda^1}(\bar x^1,\bar z^1)d\mu_{\bar\lambda^1}(\bar z^1)dz_1dt\Bigg].$$

Then, by invoking Lemmas 2.1(a), and 2.2(c), and [20, Lemma 1.1] we have that

$$\int_0^1|\phi(t)|\int_{|x-z|<\varepsilon}\left|\frac{\partial^2}{\partial z_1^2}W_t^{\lambda_1}(x_1,z_1)-(x_1z_1)^{-\lambda_1}\frac{\partial^2}{\partial z_1^2}\mathbb{W}_t(x_1,z_1)\right|$$

$$\times W_t^{\bar\lambda^1}(\bar x^1,\bar z^1)d\mu_{\bar\lambda^1}(\bar z^1)dz_1dt$$

$$\le C\int_{|x-z|<\varepsilon}\int_0^1\frac{e^{-|x-z|^2/8t}}{t^{(n+1)/2}}dt\,d\mu_{\bar\lambda^1}(\bar z^1)dz_1$$

$$\le C\int_{|x-z|<\varepsilon}\frac{dz}{|x-z|^{n-1}}\le C\varepsilon\longrightarrow 0,\quad \text{as } \varepsilon\to 0^+.$$

Also, Lemma 2.1(a) and (d), leads to, for every $0<t<1$, and $z\in K^n$,

$$\Bigg|\prod_{j=1}^{i-1}(x_jz_j)^{-\lambda_j}\frac{\partial^2}{\partial z_1^2}\mathbb{W}_t(x_1,z_1)\prod_{j=2}^{i-1}\mathbb{W}_t(x_j,z_j)\big(W_t^{\lambda_i}(x_i,z_i)-(x_iz_i)^{-\lambda_i}\mathbb{W}_t(x_i,z_i)\big)$$

$$\times\prod_{j=i+1}^n W_t^{\lambda_j}(x_j,z_j)\Bigg|\le C\frac{e^{-|x-z|^2/8t}}{t^{n/2}},\quad i=2,\dots,n.$$

We have that for every $i=2,\dots,n$,

$$\int_0^1 |\phi(t)| \int_{|x-z|<\varepsilon} \left| \prod_{j=1}^{i-1} (x_j z_j)^{-\lambda_j} \frac{\partial^2}{\partial z_1^2} \mathbb{W}_t(x_1, z_1) \right.$$

$$\times \prod_{j=2}^{i-1} \mathbb{W}_t(x_j, z_j) \big(W_t^{\lambda_i}(x_i, z_i) - (x_i z_i)^{-\lambda_i} W_t(x_i, z_i) \big)$$

$$\left. \times \prod_{j=i+1}^{n} W_t^{\lambda_j}(x_j, z_j) \right| \prod_{j=2}^{n} z_j^{2\lambda_j} dz dt$$

$$\leq C \int_{|x-z|<\varepsilon} \int_0^1 \frac{e^{-|x-z|^2/8t}}{t^{n/2}} dt dz$$

$$\leq C \int_{|x-z|<\varepsilon} \frac{dz}{|x-z|^{n-2}} \longrightarrow 0, \quad \text{as } \varepsilon \to 0^+,$$

provided that $n > 2$. When $n = 2$ we proceed in a similar way.

Hence, we conclude that

$$\int_0^\infty \phi(t) \int_{(0,\infty)^n} \Delta_{\lambda_1, z_1} f(z) \left(W_t^\lambda(x, z) - \frac{\chi_{(1,\infty)}(t) t^{-\sum_{j=1}^n (\lambda_j + 1/2)}}{2^{2\lambda} \Gamma(\lambda + 1/2)} \right) d\mu_\lambda(z) dt$$

$$= - \lim_{\varepsilon \to 0^+} \left[\int_0^\infty \phi(t) \int_{|x-z|>\varepsilon} f(z) \frac{\partial}{\partial t} W_t^{\lambda_1}(x_1, z_1) W_t^{\bar{\lambda}^1}(\bar{x}^1, \bar{z}^1) d\mu_\lambda(z) dt \right.$$

$$\left. + f(x) x_1^{2\lambda_1} \int_0^1 \phi(t) \int_{|x-z|<\varepsilon} (x.z)^{-\lambda} \frac{\partial^2}{\partial z_1^2} \mathbb{W}_t(x_1, z_1) \mathbb{W}_t(\bar{x}^1, \bar{z}^1) z_1^{-2\lambda_1} d\mu_\lambda(z) dt \right].$$

On the other hand, the mean value Theorem allows us to write, for every $\eta \in \mathbb{R}$, and $j = 1, \dots, n$,

$$|z_j^\eta - x_j^\eta| \left| \frac{\partial^2}{\partial z_1^2} \mathbb{W}_t(x_1, z_1) \mathbb{W}_t(\bar{x}^1, \bar{z}^1) \right| \leq C |z_j - x_j| \frac{e^{-|x-z|^2/8t}}{t^{n/2+1}} \leq C \frac{e^{-|x-z|^2/16t}}{t^{(n+1)/2}},$$

$t > 0$ and $z \in K^n$.

Then, by proceeding as above we obtain

$$\int_0^\infty \phi(t) \int_{(0,\infty)^n} \Delta_{\lambda_1, z_1} f(z) \left(W_t^\lambda(x, z) - \frac{\chi_{(1,\infty)}(t) t^{-\sum_{j=1}^n (\lambda_j + 1/2)}}{\prod_{j=1}^n 2^{2\lambda_j} \Gamma(\lambda_j + 1/2)} \right) d\mu_\lambda(z) dt$$

$$= - \lim_{\varepsilon \to 0^+} \left[\int_0^\infty \phi(t) \int_{|x-z|>\varepsilon} f(z) \frac{\partial}{\partial t} W_t^{\lambda_1}(x_1, z_1) W_t^{\bar{\lambda}^1}(\bar{x}^1, \bar{z}^1) d\mu_\lambda(z) dt \right.$$

$$\left. + f(x) \int_0^1 \phi(t) \int_{|x-z|<\varepsilon} \frac{\partial^2}{\partial z_1^2} \frac{e^{-|x-z|^2/4t}}{(2\sqrt{\pi t})^n} dz dt \right]$$

$$= - \lim_{\varepsilon\to 0^+}\left[\int_0^\infty \phi(t)\int_{|x-z|>\varepsilon} f(z)\frac{\partial}{\partial t}W_t^{\lambda_1}(x_1,z_1)W_t^{\bar{\lambda}^1}\left(\bar{x}^1,\bar{z}^1\right)d\mu_\lambda(z)dt\right.$$

$$\left.+ f(x)\int_0^1 \phi(t)\int_{|y|<\varepsilon}\frac{\partial^2}{\partial y_1^2}\frac{e^{-|y|^2/4t}}{(2\sqrt{\pi t})^n}dydt\right]. \tag{30}$$

Also, we have that, for every $i=2,\dots,n$,

$$\int_0^\infty \phi(t)\int_{(0,\infty)^n}\Delta_{\lambda_i,z_i}f(z)\left(W_t^\lambda(x,z)-\frac{\chi_{(1,\infty)}(t)t^{-\sum_{j=1}^n(\lambda_j+1/2)}}{2^{2\lambda}\Gamma(\lambda+1/2)}\right)d\mu_\lambda(z)dt$$

$$= - \lim_{\varepsilon\to 0^+}\left[\int_0^\infty \phi(t)\int_{|x-z|>\varepsilon} f(z)\frac{\partial}{\partial t}W_t^{\lambda_i}(x_i,z_i)W_t^{\bar{\lambda}^i}\left(\bar{x}^i,\bar{z}^i\right)d\mu_\lambda(z)dt\right.$$

$$\left.+ f(x)\int_0^1 \phi(t)\int_{|y|<\varepsilon}\frac{\partial^2}{\partial y_i^2}\frac{e^{-|y|^2/4t}}{(2\sqrt{\pi t})^n}dydt\right]. \tag{31}$$

Then, by (30) and (31), it follows that

$$T_\lambda^m f(x) = - \lim_{\varepsilon\to 0^+}\left[\int_0^\infty \phi(t)\int_{|x-z|>\varepsilon} f(z)\frac{\partial}{\partial t}W_t^{\lambda}(x,z)d\mu_\lambda(z)dt\right.$$

$$\left.+ nf(x)\int_0^1 \phi(t)\int_{|y|<\varepsilon}\frac{\partial^2}{\partial y_1^2}\frac{e^{-|y|^2/4t}}{(2\sqrt{\pi t})^n}dydt\right]. \tag{32}$$

Next the second term in the last sum is analyzed. We define

$$\Lambda(\varepsilon) = n\int_0^1 \phi(t)\int_{|y|<\varepsilon}\frac{\partial^2}{\partial y_1^2}\frac{e^{-|y|^2/4t}}{(2\sqrt{\pi t})^n}dydt, \quad \varepsilon>0.$$

We can write

$$\Lambda(\varepsilon) = n\int_0^1 \phi(t)\int_{|\bar{y}^1|<\varepsilon}\int_{-\sqrt{\varepsilon^2-|\bar{y}^1|^2}}^{\sqrt{\varepsilon^2-|\bar{y}^1|^2}}\frac{\partial^2}{\partial y_1^2}\frac{e^{-|y|^2/4t}}{(2\sqrt{\pi t})^n}dy_1d\bar{y}^1dt$$

$$= n\int_0^1 \phi(t)\int_{|\bar{y}^1|<\varepsilon}\left[\frac{\partial}{\partial y_1}\left(\frac{e^{-(y_1^2+|\bar{y}^1|^2)/4t}}{(2\sqrt{\pi t})^n}\right)_{|y_1=\sqrt{\varepsilon^2-|\bar{y}^1|^2}}\right.$$

$$\left.-\frac{\partial}{\partial y_1}\left(\frac{e^{-(y_1^2+|\bar{y}^1|^2)/4t}}{(2\sqrt{\pi t})^n}\right)_{|y_1=-\sqrt{\varepsilon^2-|\bar{y}^1|^2}}\right]d\bar{y}^1dt, \quad \varepsilon>0.$$

Hence, [20, Lemma 1.1] leads to

$$\left|\Lambda(\varepsilon)\right| \le C\|\phi\|_{L^\infty(0,\infty)}\int_0^1\int_{|\bar{y}^1|<\varepsilon}\frac{e^{-\varepsilon^2/8t}}{t^{(n+1)/2}}d\bar{y}^1dt$$

$$\leq C\|\phi\|_{L^\infty(0,\infty)} \int_{|\bar{y}^1|<\varepsilon} \int_0^1 \frac{e^{-\varepsilon^2/8t}}{t^{(n+1)/2}} dt d\bar{y}^1$$

$$\leq \frac{C}{\varepsilon^{n-1}} \|\phi\|_{L^\infty(0,\infty)} \int_{|\bar{y}^1|<\varepsilon} d\bar{y}^1$$

$$\leq C\|\phi\|_{L^\infty(0,\infty)}, \quad \varepsilon > 0.$$

Suppose now that there exists $\phi(0^+) = \lim_{t\to 0^+} \phi(t)$. Then, we have that

$$\lim_{\varepsilon\to 0^+} \int_0^1 \phi(t) \int_{|y|<\varepsilon} \frac{\partial^2}{\partial y_1^2} \frac{e^{-|y|^2/4t}}{(2\sqrt{\pi t})^n} dy dt = -M\phi\big(0^+\big), \tag{33}$$

for a certain $M > 0$. Indeed, by making changes of variables we obtain

$$\begin{aligned}
&\int_0^1 \phi(t) \int_{|y|<\varepsilon} \frac{\partial^2}{\partial y_1^2} \frac{e^{-|y|^2/4t}}{(2\sqrt{\pi t})^n} dy dt \\
&= \int_0^{1/\varepsilon^2} \phi\big(s\varepsilon^2\big) \int_{|z|<1} \frac{\partial^2}{\partial z_1^2} \frac{e^{-|z|^2/4s}}{(2\sqrt{\pi s})^n} dz ds \\
&= \int_0^{1/\varepsilon^2} \phi\big(s\varepsilon^2\big) \int_{|\bar{z}^1|<1} \frac{\partial}{\partial z_1} \frac{e^{-|z|^2/4s}}{(2\sqrt{\pi s})^n} \Big]_{z_1=-\sqrt{1-|\bar{z}^1|^2}}^{z_1=\sqrt{1-|\bar{z}^1|^2}} d\bar{z}^1 ds \\
&= -\int_0^{1/\varepsilon^2} \phi\big(s\varepsilon^2\big) \int_{|\bar{z}^1|<1} \frac{z_1}{2s} \frac{e^{-|z|^2/4s}}{(2\sqrt{\pi s})^n} \Big]_{z_1=-\sqrt{1-|\bar{z}^1|^2}}^{z_1=\sqrt{1-|\bar{z}^1|^2}} d\bar{z}^1 ds \\
&= -\int_0^{1/\varepsilon^2} \phi\big(s\varepsilon^2\big) \int_{|\bar{z}^1|<1} \frac{\sqrt{1-|\bar{z}^1|^2}}{s} \frac{e^{-1/4s}}{(2\sqrt{\pi s})^n} d\bar{z}^1 ds \\
&= -M \int_0^{1/\varepsilon^2} \phi\big(s\varepsilon^2\big) \frac{e^{-1/4s}}{s^{n/2+1}} ds, \quad \varepsilon > 0,
\end{aligned}$$

where

$$M = \frac{1}{(2\sqrt{\pi})^n} \int_{|\bar{z}^1|<1} \sqrt{1-\big|\bar{z}^1\big|^2} d\bar{z}^1.$$

Moreover, denoting by $\chi_{[0,1]}$ the characteristic function of [0, 1], we have that

$$\int_0^{1/\varepsilon^2} \phi\big(s\varepsilon^2\big) \frac{e^{-1/4s}}{s^{n/2+1}} ds = \int_0^\infty \phi\big(s\varepsilon^2\big) \chi_{[0,1]}\big(s\varepsilon^2\big) \frac{e^{-1/4s}}{s^{n/2+1}} ds, \quad \varepsilon > 0.$$

Then, by using the dominated convergence theorem, it follows that

$$\lim_{\varepsilon\to 0^+} \int_0^{1/\varepsilon^2} \phi\big(s\varepsilon^2\big) \frac{e^{-1/4s}}{s^{n/2+1}} ds = \phi\big(0^+\big) \int_0^\infty \frac{e^{-1/4s}}{s^{n/2+1}} ds.$$

Hence, there exists $M > 0$ for which (33) holds.

By using Lemmas 2.1 (a), and 2.3 (a), we get, for every $z \in K^n$,

$$\left|\frac{\partial}{\partial t} W_t^\lambda(x,z)\right| \leq \sum_{i=1}^n \left|\frac{\partial}{\partial t} W_t^{\lambda_i}(x_i,z_i)\right| W_t^{\bar{\lambda}^i}\left(\bar{x}^i,\bar{z}^i\right)$$
$$\leq C \begin{cases} t^{-\sum_{j=1}^n(\lambda_j+1/2)+1}, & t>1 \\ e^{-|x-z|^2/8t} t^{-(n+2)/2}, & 0<t\leq 1. \end{cases}$$

Hence, since $f \in C_c^\infty((0,\infty)^n)$, it follows that, for every $\varepsilon > 0$

$$\int_0^\infty |\phi(t)| \int_{(0,\infty)^n,\, |x-z|>\varepsilon} |f(z)| \left|\frac{\partial}{\partial t} W_t^\lambda(x,z)\right| d\mu_\lambda(z) dt < \infty.$$

Then, we can interchange the order of integration on the integrals in (32) and by using (33) we conclude that

$$T_\lambda^m f(x) = -\lim_{\varepsilon\to 0^+} \int_{\substack{(0,\infty)^n \\ |z-x|>\varepsilon}} f(z) \int_0^\infty \phi(t) \frac{\partial}{\partial t} W_t^\lambda(x,z) dt\, d\mu_\lambda(z) + C\phi\left(0^+\right) f(x),$$

for a certain $C > 0$.

3.2 Proof of Theorem 1.3.

In order to established the L^p-boundedness properties for the maximal operator $T_\lambda^{m,*}$ we consider the operator

$$T_{loc,\lambda}^{m,*}(f)(x) = \sup_{\varepsilon>0} \left| \int_{L(x),|x-y|>\varepsilon} f(y)(x.y)^{-\lambda} H^\phi(x,y) d\mu_\lambda(y) \right|, \quad x \in (0,\infty)^n,$$

where, for every $x \in (0,\infty)^n$,

$$L(x) = \left\{ y \in (0,\infty)^n \;:\; x_j/2 < y_j < 2x_j,\ j=1,\ldots,n \right\},$$

and H^ϕ is defined in (7).

We have that

$$\begin{aligned} T_\lambda^{m,*}(f)(x) &\leq \left|T_\lambda^{m,*}(f)(x) - T_{loc,\lambda}^{m,*}(f)(x)\right| + T_{loc,\lambda}^{m,*}(f)(x) \\ &\leq \sup_{\varepsilon>0}\left| \int_{(0,\infty)^n\setminus L(x),|x-y|>\varepsilon} f(y) K_\lambda^\phi(x,y) d\mu_\lambda(y) \right| \\ &\quad + \sup_{\varepsilon>0} \left| \int_{L(x),|x-y|>\varepsilon} f(y)\left(K_\lambda^\phi(x,y) - (x.y)^{-\lambda} H^\phi(x,y)\right) d\mu_\lambda(y) \right| \\ &\quad + T_{loc,\lambda}^{m,*}(f)(x) \\ &\leq \int_{(0,\infty)^n\setminus L(x)} |f(y)||K_\lambda^\phi(x,y)| d\mu_\lambda(y) \\ &\quad + \int_{L(x)} |f(y)|\left|K_\lambda^\phi(x,y) - (x.y)^{-\lambda} H^\phi(x,y)\right| d\mu_\lambda(y) + T_{loc,\lambda}^{m,*}(f)(x) \end{aligned}$$

$$= \mathcal{G}_\lambda(|f|)(x) + \mathcal{L}_\lambda(|f|)(x) + T^{m,*}_{loc,\lambda}(f)(x), \quad x \in (0,\infty)^n. \tag{34}$$

We are going to show the L^p-boundedness properties for the operators $\mathcal{G}^\phi_\lambda$, $\mathcal{L}_\lambda$ and $T^{m,*}_{loc,\lambda}$. Throughout the remainder of this proof the constant $C > 0$ does not depend on ϕ.

3.2.1 The Operator $T^{m,*}_{loc,\lambda}$

For every $j = (j_1, \dots, j_n) \in \mathbb{Z}^n$, the dyadic cube Q_j is defined by

$$Q_j = \{y \in (0,\infty)^n : 2^{j_i} \le y_i < 2^{j_i+1},\ i = 1, \dots, n\},$$

and the cube $\tilde{Q}_j$ is given by

$$\tilde{Q}_j = \{y \in (0,\infty)^n : 2^{j_i-1} \le y_i < 2^{j_i+2},\ i = 1, \dots, n\}.$$

It is clear that if $j \in \mathbb{Z}^n$, $x \in Q_j$ and $y \in L(x)$, then $y \in \tilde{Q}_j$. We can write

$$\begin{aligned}
&\int_{L(x),|x-y|>\varepsilon} f(y)(x.y)^{-\lambda} H^\phi(x,y)\,d\mu_\lambda(y) \\
&\quad = \int_{\tilde{Q}_j,|x-y|>\varepsilon} f(y)\left(\frac{y}{x}\right)^\lambda H^\phi(x,y)\,dy \\
&\qquad - \int_{\tilde{Q}_j\setminus L(x),|x-y|>\varepsilon} f(y)\left(\frac{y}{x}\right)^\lambda H^\phi(x,y)\,dy, \quad x \in Q_j,\ j \in \mathbb{Z}^n \text{ and } \varepsilon > 0.
\end{aligned}$$

Let $j = (j_1, \dots, j_n) \in \mathbb{Z}^n$. It has $\tilde{Q}_j \setminus L(x) = \bigcup_{i=1}^n (\tilde{Q}^+_{j,i} \cup \tilde{Q}^-_{j,i})$ where

$$\tilde{Q}^+_{j,i} = \{y \in (0,\infty)^n : 2^{j_\ell-1} \le y_\ell < 2^{j_\ell+2}, \ell = 1, \dots, n,\ \ell \ne i;\ 2x_i < y_i < 2^{j_i+2}\}$$

and

$$\tilde{Q}^-_{j,i} = \{y \in (0,\infty)^n : 2^{j_\ell-1} \le y_\ell < 2^{j_\ell+2}, \ell = 1, \dots, n,\ \ell \ne i;\ 2^{j_i-1} < y_i < x_i/2\}$$

for $i = 1, \dots, n$.

We have that

$$\begin{aligned}
|H^\phi(x,y)| &\le \int_0^\infty |\phi(t)| \left| \frac{\partial}{\partial t} \frac{e^{-|x-y|^2/4t}}{t^{n/2}} \right| dt \le C\|\phi\|_{L^\infty(0,\infty)} \int_0^\infty \frac{e^{-|x-y|^2/8t}}{t^{n/2+1}} dt \\
&\le C\|\phi\|_{L^\infty(0,\infty)} \frac{1}{|x-y|^n} \int_0^\infty \frac{e^{-1/u}}{u^{n/2+1}} du \\
&\le C\|\phi\|_{L^\infty(0,\infty)} \frac{1}{|x-y|^n}, \quad x, y \in (0,\infty)^n.
\end{aligned} \tag{35}$$

By (35), for every $\varepsilon > 0$, we get

$$\left|\int_{\tilde{Q}_j\setminus L(x),|x-y|>\varepsilon} f(y)\left(\frac{y}{x}\right)^{\lambda} H^{\phi}(x,y)dy\right|$$

$$\leq \int_{\tilde{Q}_j\setminus L(x)} |f(y)|\left(\frac{y}{x}\right)^{\lambda} |H^{\phi}(x,y)|dy$$

$$\leq \sum_{i=1}^{n}\left(\int_{\tilde{Q}_{j,i}^{+}\cup\tilde{Q}_{j,i}^{-}} |f(y)|\left(\frac{y}{x}\right)^{\lambda}|H^{\phi}(x,y)|dy + |f(y)|\left(\frac{y}{x}\right)^{\lambda}|H^{\phi}(x,y)|dy\right)$$

$$\leq C\|\phi\|_{L^\infty(0,\infty)}\sum_{i=1}^{n}\int_{\tilde{Q}_{j,i}^{+}\cup\tilde{Q}_{j,i}^{-}} \frac{|f(y)|}{(x_i^2+|\bar{x}^i-\bar{y}^i|^2)^{n/2}}dy$$

$$\leq C\|\phi\|_{L^\infty(0,\infty)}\sum_{i=1}^{n}\int_{\tilde{Q}_{j,i}^{+}\cup\tilde{Q}_{j,i}^{-}} \frac{|f(y)|}{(2^{2j_i}+|\bar{x}^i-\bar{y}^i|^2)^{n/2}}dy, \quad x\in Q_j.$$

Then, for each $x \in Q_j$,

$$\sup_{\varepsilon>0}\left|\int_{\tilde{Q}_j\setminus L(x),|x-y|>\varepsilon} f(y)\left(\frac{y}{x}\right)^{\lambda} H^{\phi}(x,y)dy\right| \leq C\|\phi\|_{L^\infty(0,\infty)}\sum_{i=1}^{n}\int_{\tilde{Q}_{j,i}^{+}\cup\tilde{Q}_{j,i}^{-}} \frac{|f(y)|}{(2^{2j_i}+|\bar{x}^i-\bar{y}^i|^2)^{n/2}}dy. \tag{36}$$

For every $i = 1,\dots,n$, we define

$$f_{j,i}(\bar{y}^i) = \int_{2^{j_i-1}}^{2^{j_i+2}} |f(y)|dy_i\,\chi_{\prod_{\ell=1,\ell\neq i}^{n}[2^{j_\ell-1},2^{j_\ell+2}]}(\bar{y}^i), \quad \bar{y}^i\in(0,\infty)^{n-1}.$$

From (36) it follows that

$$\sup_{\varepsilon>0}\left|\int_{\tilde{Q}_j\setminus L(x),|x-y|>\varepsilon} f(y)\left(\frac{y}{x}\right)^{\lambda} H^{\phi}(x,y)dy\right|$$

$$\leq C\|\phi\|_{L^\infty(0,\infty)}\sum_{i=1}^{n}\int_{(0,\infty)^{n-1}} \frac{f_{j,i}(\bar{y}^i)}{(2^{2j_i}+|\bar{x}^i-\bar{y}^i|^2)^{n/2}}d\bar{y}^i$$

$$\leq C\|\phi\|_{L^\infty(0,\infty)}\sum_{i=1}^{n}\left(\sum_{k=0}^{\infty}\int_{\substack{(0,\infty)^{n-1}\\ 2^{k+j_i}<|\bar{y}^i-\bar{x}^i|<2^{k+j_i+1}}} \frac{f_{j,i}(\bar{y}^i)}{(2^{2j_i}+|\bar{x}^i-\bar{y}^i|^2)^{n/2}}d\bar{y}^i\right.$$

$$\left. + \int_{(0,\infty)^{n-1},|\bar{y}^i-\bar{x}^i|<2^{j_i}} \frac{f_{j,i}(\bar{y}^i)}{(2^{2j_i}+|\bar{x}^i-\bar{y}^i|^2)^{n/2}}d\bar{y}^i\right)$$

$$\leq C\|\phi\|_{L^\infty(0,\infty)} \sum_{i=1}^{n} \Bigg(\sum_{k=0}^{\infty} \frac{1}{2^{nj_i}(1+2^{2k})^{n/2}} \int_{|\bar{y}^i - \bar{x}^i| < 2^{k+j_i+1}} f_{j,i}\big(\bar{y}^i\big) d\bar{y}^i$$

$$+ \frac{1}{2^{nj_i}} \int_{|\bar{y}^i - \bar{x}^i| < 2^{j_i}} f_{j,i}\big(\bar{y}^i\big) d\bar{y}^i \Bigg)$$

$$\leq C\|\phi\|_{L^\infty(0,\infty)} \sum_{i=1}^{n} \frac{1}{2^{j_i}} M_{n-1}(f_{j,i})\big(\bar{x}^i\big), \quad x \in Q_j.$$

Here, M_{n-1} represents the Hardy-Littlewood maximal function on $\mathbb{R}^{n-1}$. We recall that M_{n-1} is a bounded operator from $L^p(\mathbb{R}^{n-1}, dx)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^{n-1}, dx)$ into $L^{1,\infty}(\mathbb{R}^{n-1}, dx)$.

On the other hand, it is known that the maximal operator $T^{m,*}$ defined by

$$T^{m,*}(f)(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} f(y) H^\phi(x,y) dy \right|, \quad x \in \mathbb{R}^n,$$

is bounded from $L^p(\mathbb{R}^n, dx)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^n, dx)$ into $L^{1,\infty}(\mathbb{R}^n, dx)$. We denote by $\|T^{m,*}\|_{p\mapsto p}$, $1 < p < \infty$, and $\|T^{m,*}\|_{1\mapsto(1,\infty)}$ the norm of those operators.

We write, as usual, $\mu_\lambda(E) = \int_E d\mu_\lambda(x)$, and, for every $i = 1, \dots, n$, $\mu_{\bar{\lambda}^i}(E) = \int_E d\mu_{\bar{\lambda}^i}(\bar{x}^i)$, for each Lebesgue measurable set E in $(0,\infty)^n$ and in $(0,\infty)^{n-1}$, respectively.

Let $\gamma > 0$. We have that,

$$\mu_\lambda\big(\big\{x \in (0,\infty)^n : T^{m,*}_{loc,\lambda}(f)(x) > \gamma\big\}\big)$$

$$= \sum_{j\in\mathbb{Z}^n} \mu_\lambda\big(\big\{x \in Q_j : T^{m,*}_{loc,\lambda}(f)(x) > \gamma\big\}\big)$$

$$\leq \sum_{j\in\mathbb{Z}^n} \Bigg(\mu_\lambda\big(\big\{x \in Q_j : T^{m,*}\big((y/x)^\lambda f\chi_{\tilde{Q}_j}\big)(x) > C\gamma\big\}\big)$$

$$+ \mu_\lambda\left(\left\{x \in Q_j : \sup_{\varepsilon>0} \left| \int_{\tilde{Q}_j \setminus L(x)} f(y) \left(\frac{y}{x}\right)^\lambda H^\phi(x,y) dy \right| > C\gamma \right\}\right)\Bigg)$$

$$\leq \sum_{j\in\mathbb{Z}^n} \Bigg(\frac{C}{\gamma} \|T^{m,*}\|_{1\mapsto(1,\infty)} 2^{2j\cdot\lambda} \|f\chi_{\tilde{Q}_j}\|_{L^1(\mathbb{R}^n)}$$

$$+ \sum_{i=1}^{n} \mu_\lambda\big(\big\{x \in \tilde{Q}_j : M_{n-1}(f_{j,i})(\bar{x}^i) > 2^{j_i} C\gamma / \|\phi\|_{L^\infty(0,\infty)}\big\}\big) \Bigg)$$

$$
\begin{aligned}
&\le \sum_{j\in\mathbb{Z}^n}\Bigg(\frac{C}{\gamma}\|T^{m,*}\|_{1\mapsto(1,\infty)}\|f\chi_{\tilde{Q}_j}\|_{L^1((0,\infty)^n,d\mu_\lambda(x))}\\
&\quad+\sum_{i=1}^n 2^{(2\lambda_i+1)j_i}\mu_{\bar{\lambda}^i}\Bigg(\Bigg\{\bar{x}^i\in\prod_{\ell=1,\,\ell\neq i}^n[2^{j_\ell},2^{j_\ell+1}):M_{n-1}(f_{j,i})\big(\bar{x}^i\big)\\
&\qquad\qquad > 2^{j_i}C\gamma/\|\phi\|_{L^\infty(0,\infty)}\Bigg\}\Bigg)\Bigg)\\
&\le \frac{C}{\gamma}\sum_{j\in\mathbb{Z}^n}\Bigg(\|T^{m,*}\|_{1\mapsto(1,\infty)}\|f\chi_{\tilde{Q}_j}\|_{L^1((0,\infty)^n,d\mu_\lambda(x))}\\
&\quad+\|\phi\|_{L^\infty(0,\infty)}\sum_{i=1}^n\|f_{j,i}\|_{L^1(\mathbb{R}^{n-1})}2^{2j\cdot\lambda}\Bigg)\\
&\le \frac{C}{\gamma}\big(\|T^{m,*}\|_{1\mapsto(1,\infty)}+\|\phi\|_{L^\infty(0,\infty)}\big)\sum_{j\in\mathbb{Z}^n}\|f\chi_{\tilde{Q}_j}\|_{L^1((0,\infty)^n,d\mu_\lambda(x))}\\
&\le \frac{C}{\gamma}\big(\|T^{m,*}\|_{1\mapsto(1,\infty)}+\|\phi\|_{L^\infty(0,\infty)}\big)\|f\|_{L^1((0,\infty)^n,d\mu_\lambda(x))},\\
&f\in L^1\big((0,\infty)^n,d\mu_\lambda(x)\big).
\end{aligned}
$$

Hence, $T^{m,*}_{loc,\lambda}$ is bounded from $L^1((0,\infty)^n,d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n,d\mu_\lambda(x))$, and $\|T^{m,*}_{loc,\lambda}\|_{1\mapsto(1,\infty)}\le C(\|\phi\|_{L^\infty(0,\infty)}+\|T^{m,*}\|_{1\mapsto(1,\infty)})$. Here, $\|T^{m,*}_{loc,\lambda}\|_{1\mapsto(1,\infty)}$ denotes the norm of $T^{m,*}_{loc,\lambda}$ as a bounded operator from $L^1((0,\infty)^n,d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n,d\mu_\lambda(x))$. Also, if $1<p<\infty$, we have

$$
\begin{aligned}
&\|T^{m,*}_{loc,\lambda}(f)\|^p_{L^p((0,\infty)^n,d\mu_\lambda(x))}\\
&\quad\le\sum_{j\in\mathbb{Z}^n}\|T^{m,*}_{loc,\lambda}(f)\chi_{Q_j}\|^p_{L^p(\mathbb{R}^n,dx)}2^{2j\cdot\lambda}\\
&\quad\le C\sum_{j\in\mathbb{Z}^n}\Bigg(\int_{Q_j}|T^{m,*}\big((y/x)^\lambda f\chi_{\tilde{Q}_j}\big)(x)|^p dx\\
&\qquad+\|\phi\|^p_{L^\infty(0,\infty)}\sum_{i=1}^n\frac{1}{2^{j_ip}}\int_{Q_j}|M_{n-1}(f_{j,i})(\bar{x}^i)|^p d\bar{x}^i dx_i\Bigg)2^{2j\cdot\lambda}\\
&\quad\le C\sum_{j\in\mathbb{Z}^n}\Bigg(\|T^{m,*}\|^p_{p\mapsto p}\int_{\tilde{Q}_j}|f(x)|^p dx\\
&\qquad+\|\phi\|^p_{L^\infty(0,\infty)}\sum_{i=1}^n\frac{1}{2^{j_ip}}\int_{2^{j_i}}^{2^{j_i+1}}\|f_{j,i}\|^p_{L^p(\mathbb{R}^{n-1})}dx_i\Bigg)2^{2j\cdot\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\big(\|T^{m,*}\|_{p\mapsto p} + \|\phi\|_{L^\infty(0,\infty)}\big)^p \sum_{j\in\mathbb{Z}^n} \int_{\tilde{Q}_j} |f(x)|^p dx 2^{2j\cdot\lambda} \\
&\leq C\big(\|T^{m,*}\|_{p\mapsto p} + \|\phi\|_{L^\infty(0,\infty)}\big)^p \|f\|^p_{L^p((0,\infty)^n, d\mu_\lambda(x))}, \\
&f \in L^p\big((0,\infty)^n, d\mu_\lambda(x)\big).
\end{aligned}
$$

It is proved that $T^{m,*}_{loc,\lambda}$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, and the norm $\|T^{m,*}_{loc,\lambda}\|_{p\mapsto p}$ of $T^{m,*}_{loc,\lambda}$ satisfies that $\|T^{m,*}_{loc,\lambda}\|_{p\mapsto p} \leq C(\|T^{m,*}\|_{p\mapsto p} + \|\phi\|_{L^\infty(0,\infty)})$, for every $1 < p < \infty$.

3.2.2 The Operator $\mathcal{L}_\lambda$

We have that

$$
\begin{aligned}
\mathcal{L}_\lambda(|f|)(x) &\leq \int_{L(x)} |f(y)||K^\phi_\lambda(x,y) - (x.y)^{-\lambda} H^\phi(x,y)| d\mu_\lambda(y) \\
&\leq \int_{L(x)} |f(y)| \int_0^\infty |\phi(t)| \bigg| \frac{\partial}{\partial t} W^\lambda_t(x,y) \\
&\quad - (x.y)^{-\lambda} \frac{\partial}{\partial t} \mathbb{W}_t(x,y) \bigg| dt d\mu_\lambda(y) \\
&\leq C\|\phi\|_{L^\infty(0,\infty)} \sum_{i=1}^n \int_{L(x)} |f(y)| \int_0^\infty \bigg| W^{\bar{\lambda}^i}_t\big(\bar{x}^i, \bar{y}^i\big) \frac{\partial}{\partial t} W^{\lambda_i}_t(x_i, y_i) \\
&\quad - (x.y)^{-\lambda} \mathbb{W}_t\big(\overline{x}^i, \overline{y}^i\big) \frac{\partial}{\partial t} \mathbb{W}_t(x_i, y_i) \bigg| dt d\mu_\lambda(y), \quad x \in (0,\infty)^n.
\end{aligned}
\tag{37}
$$

We analyze the operator defined by the first summand. The other ones can be studied similarly. It follows that, for each $x \in (0,\infty)^n$,

$$
\begin{aligned}
\mathcal{L}^1_\lambda(f)(x) &= \int_{L(x)} |f(y)| \int_0^\infty \bigg| W^{\bar{\lambda}^1}_t\big(\bar{x}^1, \bar{y}^1\big) \frac{\partial}{\partial t} W^{\lambda_1}_t(x_1, y_1) \\
&\quad - (x.y)^{-\lambda} \mathbb{W}_t\big(\overline{x}^1, \overline{y}^1\big) \frac{\partial}{\partial t} \mathbb{W}_t(x_1, y_1) \bigg| dt d\mu_\lambda(y) \\
&\leq \int_{L(x)} |f(y)| \int_0^\infty \bigg| \frac{\partial}{\partial t} W^{\lambda_1}_t(x_1, y_1) \\
&\quad - (x_1 y_1)^{-\lambda_1} \frac{\partial}{\partial t} \mathbb{W}_t(x_1, y_1) \bigg| W^{\bar{\lambda}^1}_t\big(\bar{x}^1, \bar{y}^1\big) dt d\mu_\lambda(y) \\
&\quad + \sum_{i=2}^n \int_{L(x)} |f(y)| \int_0^\infty (x_1 y_1)^{-\lambda_1} \bigg| \frac{\partial}{\partial t} \mathbb{W}_t(x_1, y_1) \bigg|
\end{aligned}
$$

$$\times \prod_{j=2}^{i-1} \mathbb{W}_t(x_j, y_j)(x_j y_j)^{-\lambda_j} \left| W_t^{\lambda_i}(x_i, y_i) - (x_i y_i)^{-\lambda_i} \mathbb{W}_t(x_i, y_i) \right|$$

$$\times \prod_{j=i+1}^{n} W_t^{\lambda_j}(x_j, y_j) dt d\mu_\lambda(y)$$

$$= \sum_{i=1}^{n} \mathcal{L}_\lambda^{1,i}(f)(x).$$

We now use Lemmas 2.1(a), and 2.3(a) and (c), to obtain the following

$$\begin{aligned}
\mathcal{L}_\lambda^{1,1}(f)(x) &= \int_{L(x)} |f(y)| \int_0^\infty \left| \frac{\partial}{\partial t} W_t^{\lambda_1}(x_1, y_1) \right. \\
&\quad \left. - (x_1 y_1)^{-\lambda_1} \frac{\partial}{\partial t} \mathbb{W}_t(x_1, y_1) \right| W_t^{\bar{\lambda}^1}(\bar{x}^1, \bar{y}^1) dt d\mu_\lambda(y) \\
&\leq \int_{L(x)} |f(y)| \left\{ \int_0^{x_1 y_1} \frac{(x_1 y_1)^{-\lambda_1 - 1}}{t^{1/2}} e^{-\frac{(x_1 - y_1)^2}{16t}} \right. \\
&\quad \times \prod_{j=2}^{n} \left(\frac{1}{x_j^{2\lambda_j + 1}} + (x_j y_j)^{-\lambda_j} \frac{e^{-\frac{(x_j - y_j)^2}{4t}}}{\sqrt{t}} \right) dt \\
&\quad + \int_{x_1 y_1}^\infty \left(\frac{1}{t^{\lambda_1 + \frac{3}{2}}} + (x_1 y_1)^{-\lambda_1} \frac{1}{t^{\frac{3}{2}}} \right) \\
&\quad \left. \times \prod_{j=2}^{n} \left(\frac{1}{x_j^{2\lambda_j + 1}} + (x_j y_j)^{-\lambda_j} \frac{e^{-\frac{(x_j - y_j)^2}{4t}}}{\sqrt{t}} \right) dt \right\} d\mu_\lambda(y), \qquad (38)
\end{aligned}$$

for every $x \in (0, \infty)^n$.

Then, the operator $\mathcal{L}_\lambda^{1,1}$ is controlled by operators of the following type:

$$\begin{aligned}
\Lambda_\lambda^\ell(g)(x) &= \sup_{t>0} \left| \int_{\frac{x_{\ell+1}}{2}}^{2x_{\ell+1}} \cdots \int_{\frac{x_n}{2}}^{2x_n} \prod_{j=\ell+1}^{n} x_j^{-2\lambda_j - 1} \right. \\
&\quad \left. \times \int_{\frac{x_1}{2}}^{2x_1} \cdots \int_{\frac{x_\ell}{2}}^{2x_\ell} \prod_{j=1}^{\ell} (x_j y_j)^{-\lambda_j} \frac{e^{-\frac{(x_j - y_j)^2}{16t}}}{\sqrt{t}} g(y) d\mu_\lambda(y) \right|,
\end{aligned}$$

where $0 \leq \ell \leq n$, $\ell \in \mathbb{N}$. For each $0 \leq \ell \leq n$, $\ell \in \mathbb{N}$, Λ_λ^ℓ is bounded operator from $L^p((0, \infty)^n, d\mu_\lambda(x))$ into itself, for every $1 < p < \infty$, and from $L^1((0, \infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0, \infty)^n, d\mu_\lambda(x))$. Indeed, let $k \in \mathbb{N}$. We consider the maximal operators

$$\Omega_\beta(g)(x)=\sup_{t>0}\left|\int_{\frac{x_1}{2}}^{2x_1}\cdots\int_{\frac{x_k}{2}}^{2x_k}\prod_{j=1}^{k}(x_jy_j)^{-\beta_j}\frac{e^{-\frac{(x_j-y_j)^2}{16t}}}{\sqrt{t}}g(y)d\mu_\beta^k(y)\right|,\quad x\in(0,\infty)^k,$$

where $\beta=(\beta_1,\dots,\beta_k)\in\mathbb{R}^k$, $d\mu_\beta^k(y)=\prod_{j=1}^k y_j^{2\beta_j}dy$. We denote by μ_β^k the measure defined by $\mu_\beta^k(E)=\int_E d\mu_\beta^k(y)$, for every Lebesgue measurable set $E\subset(0,\infty)^k$. We also consider the maximal operator $\mathcal{M}_k$ given by

$$\mathcal{M}_k(g)(x)=\sup_{t>0}\left|\int_{(0,\infty)^k}\frac{e^{-|x-y|^2/16t}}{t^{k/2}}g(y)dy\right|,\quad x\in(0,\infty)^k.$$

For every $j=(j_1,\dots,j_k)\in\mathbb{Z}^k$ we define

$$Q_j^k=\{y=(y_1,\dots,y_k)\in(0,\infty)^k:\ 2^{j_i}\le y_i<2^{j_i+1},\ i=1,\dots,k\},$$

and

$$\tilde{Q}_j^k=\{y=(y_1,\dots,y_k)\in(0,\infty)^k:\ 2^{j_i-1}\le y_i<2^{j_i+2},\ i=1,\dots,k\}.$$

Assume that $\gamma>0$. Since, as it well known, the operator $\mathcal{M}_k$ is bounded from $L^1((0,\infty)^k,dx)$ into $L^{1,\infty}((0,\infty)^k,dx)$, we have

$$\begin{aligned}
&\mu_\beta^k(\{x\in(0,\infty)^k:\Omega_\beta(g)(x)>\gamma\})\\
&\quad=\sum_{j\in\mathbb{Z}^k}\mu_\beta^k(\{x\in Q_j^k:\Omega_\beta(g)(x)>\gamma\})\\
&\quad\le\sum_{j\in\mathbb{Z}^k}2^{2\beta\cdot j}\mu_0^k(\{x\in(0,\infty)^k:\mathcal{M}_k(|g|\chi_{\tilde{Q}_j^k})(x)>C\gamma\})\\
&\quad\le\frac{C}{\gamma}\sum_{j\in\mathbb{Z}^k}2^{2\beta\cdot j}\int_{\tilde{Q}_j^k}|g(y)|dy\\
&\quad\le\frac{C}{\gamma}\int_{(0,\infty)^k}|g(y)|d\mu_\beta^k(y),\quad g\in L^1((0,\infty)^k,d\mu_\beta^k(x)).
\end{aligned}$$

Also, $\mathcal{M}_k$ is bounded from $L^p((0,\infty)^k,dx)$ into itself, for every $1<p<\infty$, and it has

$$\begin{aligned}
&\int_{(0,\infty)^k}|\Omega_\beta(g)(x)|^p d\mu_\beta^k(x)\\
&\quad\le C2^{2\beta\cdot j}\sum_{j\in\mathbb{Z}^k}\int_{Q_j^k}|\Omega_\beta(g)(x)|^p dx\\
&\quad\le C\sum_{j\in\mathbb{Z}^k}2^{2\beta\cdot j}\int_{(0,\infty)^k}|\mathcal{M}_k(|g|\chi_{\tilde{Q}_j^k})(x)|^p dx
\end{aligned}$$

$$\leq C \sum_{j\in\mathbb{Z}^k} 2^{2\beta\cdot j} \int_{\tilde{Q}_j^k} |g(y)|^p dy$$
$$\leq C \int_{(0,\infty)^k} |g(y)|^p d\mu_\beta^k(y), \quad g \in L^p\big((0,\infty)^k, d\mu_\beta^k(x)\big),$$

and Ω_β is bounded from $L^p((0,\infty)^k, d\mu_\beta^k(x))$ into itself, for every $1 < p < \infty$.

On the other hand, it is not hard to see that, for every $k \in \mathbb{N}$ and $\beta = (\beta_1, \dots, \beta_k) \in \mathbb{R}^k$, the operator Z_β defined by

$$Z_\beta(g)(x) = \prod_{j=1}^k x_j^{-2\beta_j - 1} \int_{x_1/2}^{2x_1} \cdots \int_{x_k/2}^{2x_k} g(y) d\mu_\beta^k(y), \quad x \in (0,\infty)^k,$$

is bounded from $L^p((0,\infty)^k, d\mu_\beta^k(x))$ into itself, for every $1 \leq p < \infty$.

Then, the L^p-boundedness properties of operators Ω_β and Z_β, $\beta = (\beta_1, \dots, \beta_k) \in (-1/2, \infty)^k$, $k \in \mathbb{N}$, allow us, by using [6, Proposition 1], to conclude that, for every $\ell \in \mathbb{N}$, $0 \leq \ell \leq n$, the operator Λ_λ^ℓ is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, for every $1 < p < \infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$. Hence, the operator $\mathcal{L}_\lambda^{1,1}$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, for every $1 < p < \infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$.

For every $i = 2, \dots, n$, by using Lemma 2.1(a) and (d), we have

$$\mathcal{L}_\lambda^{1,i}(f)(x) = \int_{L(x)} |f(y)| \int_0^\infty (x_1 y_1)^{-\lambda_1} \left| \frac{\partial}{\partial t} \mathbb{W}_t(x_1, y_1) \right|$$
$$\times \prod_{j=2}^{i-1} \mathbb{W}_t(x_j, y_j)(x_j y_j)^{-\lambda_j} \left| W_t^{\lambda_i}(x_i, y_i) - (x_i y_i)^{-\lambda_i} \mathbb{W}_t(x_i, y_i) \right|$$
$$\times \prod_{j=i+1}^n W_t^{\lambda_j}(x_j, y_j) dt d\mu_\lambda(y)$$
$$\leq C \int_{L(x)} |f(y)| \left\{ \int_0^{x_i y_i} \frac{e^{-\sum_{j=1}^{i-1} \frac{(x_j - y_j)^2}{4t}}}{t^{\frac{i+1}{2}}} \right.$$
$$\times \prod_{j=1}^{i-1} (x_j y_j)^{-\lambda_j} (x_i y_i)^{-\lambda_i - 1} \sqrt{t} e^{-\frac{(x_i - y_i)^2}{4t}}$$
$$\times \prod_{j=i+1}^n \left(\frac{1}{x_j^{2\lambda_j + 1}} + (x_j y_j)^{-\lambda_j} \frac{e^{-\frac{(x_j - y_j)^2}{4t}}}{\sqrt{t}} \right) dt$$

$$+\int_{x_iy_i}^{\infty}\prod_{j=1}^{i-1}(x_jy_j)^{-\lambda_j}\frac{e^{-\sum_{j=1}^{i-1}\frac{(x_j-y_j)^2}{4t}}}{t^{\frac{i+1}{2}}}\left(\frac{1}{t^{\lambda_i+\frac{1}{2}}}+(x_iy_i)^{-\lambda_i}\frac{e^{-\frac{(x_i-y_i)^2}{4t}}}{\sqrt{t}}\right)$$

$$\times\prod_{j=i+1}^{n}\left(\frac{1}{x_j^{2\lambda_j+1}}+(x_jy_j)^{-\lambda_j}\frac{e^{-\frac{(x_j-y_j)^2}{4t}}}{\sqrt{t}}\right)dt\Bigg\}d\mu_\lambda(y),\quad x\in(0,\infty)^n. \tag{39}$$

Then, the operator $\mathcal{L}_\lambda^{1,i}$, $i=1,\dots,n$, can be controlled by operator of type Λ_λ^ℓ, $0\le\ell\le n$. Thus, we conclude that $\mathcal{L}_\lambda^{1,i}$, $i=1,\dots,n$, is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, for every $1<p<\infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$.

From the assertions proved in (38) and (39), $i=2,\dots,n$, we deduce that the operator $\mathcal{L}_\lambda^1$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, for every $1<p<\infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$.

Thus, from (37), we conclude that the operator $\mathcal{L}_\lambda$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, and $\|\mathcal{L}_\lambda\|_{p\mapsto p}\le C\|\phi\|_{L^\infty(0,\infty)}$, for every $1<p<\infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$, and $\|\mathcal{L}_\lambda\|_{1\mapsto(1,\infty)}\le C\|\phi\|_{L^\infty(0,\infty)}$.

3.2.3 The Operator $\mathcal{G}_\lambda$

It is clear that $\mathcal{G}_\lambda$ can be decomposed in a sum of operators like the following one

$$\begin{aligned}
&S_\lambda^{k,\ell}(g)(x)\\
&=\int_0^{\frac{x_1}{2}}\cdots\int_0^{\frac{x_k}{2}}\int_{\frac{x_{k+1}}{2}}^{2x_{k+1}}\cdots\int_{\frac{x_\ell}{2}}^{2x_\ell}\int_{2x_{\ell+1}}^{\infty}\cdots\int_{2x_n}^{\infty}|g(y)||K_\lambda^\phi(x,y)|d\mu_\lambda(y)\\
&=\sum_{i=1}^n\int_0^{\frac{x_1}{2}}\cdots\int_0^{\frac{x_k}{2}}\int_{\frac{x_{k+1}}{2}}^{2x_{k+1}}\cdots\int_{\frac{x_\ell}{2}}^{2x_\ell}\int_{2x_{\ell+1}}^{\infty}\cdots\int_{2x_n}^{\infty}|g(y)||K_\lambda^{\phi,i}(x,y)|d\mu_\lambda(y)\\
&=\sum_{i=1}^n S_\lambda^{k,\ell,i}(g)(x),\quad x\in(0,\infty)^n,
\end{aligned}\tag{40}$$

where $0\le k\le\ell\le n$, $k\ne0$ or $\ell\ne n$, and

$$K_\lambda^{\phi,i}(x,y)=\int_0^\infty W_t^{\bar\lambda^i}(\bar{x}^i,\bar{y}^i)\frac{\partial}{\partial t}W_t^{\lambda_i}(x_i,y_i)\phi(t)dt,\quad x,y\in(0,\infty)^n,$$

for every $i=1,\dots,n$.

We now study the operators $S_\lambda^{k,\ell,i}$. According to Lemmas 2.1(b), and 2.3(b), we infer

$$\left|S_\lambda^{0,0,i}(g)(x)\right|\le C\|\phi\|_{L^\infty(0,\infty)}\int_{2x_1}^\infty\cdots\int_{2x_n}^\infty|g(y)|\int_0^\infty\frac{e^{-|y|^2/40t}}{t^{\sum_{j=1}^n(\lambda_j+1/2)+1}}dtd\mu_\lambda(y)$$

$$\leq C\|\phi\|_{L^\infty(0,\infty)} \int_{2x_1}^\infty \cdots \int_{2x_n}^\infty |g(y)| \frac{1}{(\sum_{j=1}^n y_j^2)^{\sum_{j=1}^n(\lambda_j+1/2)}} d\mu_\lambda(y)$$

$$\leq C\|\phi\|_{L^\infty(0,\infty)} \int_{2x_1}^\infty \cdots \int_{2x_n}^\infty |g(y)| \frac{1}{y_1 \dots y_n} dy,$$

$x \in (0,\infty)^n$ and $i = 1, \dots, n$.

It is not hard to see that, for every $k \in \mathbb{N}$ the operator

$$\mathcal{S}_k(g)(x) = \int_{2x_1}^\infty \cdots \int_{2x_k}^\infty |g(y)| \frac{1}{y_1 \dots y_k} dy, \quad x \in (0,\infty)^k,$$

is bounded from $L^p((0,\infty)^k, d\mu_\beta^k(x))$ into itself, for every $1 \leq p < \infty$ and $\beta = (\beta_1, \dots, \beta_k) \in (-1/2, \infty)^k$.

Hence the operator $S_\lambda^{0,0,i}$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$, and $\|S_\lambda^{0,0,i}\|_{p\mapsto p} \leq C\|\phi\|_{L^\infty(0,\infty)}$, for every $1 \leq p < \infty$ and $i = 1, \dots, n$.

Let $\ell, k \in \mathbb{N}$, $1 \leq \ell \leq k$, and $\beta = (\beta_1, \dots, \beta_k) \in (-1/2, \infty)^k$. We define the operator $\mathcal{H}_\beta^\ell$ by

$$\mathcal{H}_\beta^\ell(g)(x) = \int_0^{\frac{x_1}{2}} \cdots \int_0^{\frac{x_k}{2}} |g(y)| \int_0^\infty \prod_{j=1}^{\ell-1} W_t^{\beta_j}(x_j, y_j) \left| \frac{\partial}{\partial t} W_t^{\beta_\ell}(x_\ell, y_\ell) \right|$$

$$\times \prod_{j=\ell+1}^{k} W_t^{\beta_j}(x_j, y_j) dt d\mu_\beta^k(y), \quad x \in (0,\infty)^k.$$

By taking into account symmetries, Lemmas 2.1(b), and 2.3(b), we get that

$$|\mathcal{H}_\beta^\ell(g)(x)| \leq C \int_0^{\frac{x_1}{2}} \cdots \int_0^{\frac{x_k}{2}} |g(y)| \int_0^\infty \frac{e^{-|x|^2/40t}}{t^{\sum_{j=1}^k(\beta_j+1/2)+1}} dt d\mu_\beta^k(y)$$

$$\leq \frac{C}{(\sum_{j=1}^k x_j^2)^{\sum_{j=1}^k(\beta_j+1/2)}} \int_0^{\frac{x_1}{2}} \cdots \int_0^{\frac{x_k}{2}} |g(y)| d\mu_\beta^k(y),$$

$$x \in (0,\infty)^k. \tag{41}$$

The operator $\mathbb{H}_\beta$ given by

$$\mathbb{H}_\beta(g)(x) = \frac{1}{(\sum_{j=1}^k x_j^2)^{\sum_{j=1}^k(\beta_j+1/2)}} \int_0^{\frac{x_1}{2}} \cdots \int_0^{\frac{x_k}{2}} g(y) d\mu_\beta^k(y), \quad x \in (0,\infty)^k$$

is bounded from $L^p((0,\infty)^k, d\mu_\beta^k(x))$ into itself, for every $1 < p < \infty$, and from $L^1((0,\infty)^k, d\mu_\beta^k(x))$ into $L^{1,\infty}((0,\infty)^k, d\mu_\beta^k(x))$. Indeed, assume that $\gamma > 0$. We have

$$\begin{aligned}
&\mu_\beta^k\big(\{x\in(0,\infty)^k:|\mathbb{H}_\beta(g)(x)|>\gamma\}\big)\\
&\quad\le \mu_\beta^k\left(\left\{x\in(0,\infty)^k:\sum_{j=1}^k x_j^2\le\left(\frac{1}{\gamma}\|g\|_{L^1((0,\infty)^k,d\mu_\beta^k(x))}\right)^{\frac{1}{\sum_{j=1}^k(\beta_j+1/2)}}\right\}\right)\\
&\quad\le \mu_\beta^k\left(\left\{x\in(0,\infty)^k:0\le x_j\le\left(\frac{1}{\gamma}\|g\|_{L^1((0,\infty)^k,d\mu_\beta^k(x))}\right)^{\frac{1}{\sum_{j=1}^k(2\beta_j+1)}},\right.\right.\\
&\qquad\qquad \left.\left. j=1,\dots,k\right\}\right)\\
&\quad\le \frac{C}{\gamma}\|g\|_{L^1((0,\infty)^k,d\mu_\beta^k(x))},\quad g\in L^1\big((0,\infty)^k,d\mu_\beta^k(x)\big).
\end{aligned}$$

Hence, $\mathbb{H}_\beta$ is bounded from $L^1((0,\infty)^k,d\mu_\beta^k(x))$ into $L^{1,\infty}((0,\infty)^k,d\mu_\beta^k(x))$.

On the other hand, we can write

$$\mathbb{H}_\beta(g)(x)\le\frac{1}{x_1^{2\beta_1+1}}\int_0^{x_1}\frac{1}{x_2^{2\beta_2+1}}\int_0^{x_2}\cdots\frac{1}{x_k^{2\beta_k+1}}\int_0^{x_k}|g(y)|d\mu_\beta^k(y),\quad x\in(0,\infty)^k.$$

Since, as it is well known, the Hardy type operator $\mathbb{H}_\alpha$ given by, for every $\alpha>-1/2$,

$$\mathbb{H}_\alpha(g)(u)=\frac{1}{u^{2\alpha+1}}\int_0^u g(v)v^{2\alpha}dv,\quad u\in(0,\infty),$$

is bounded from $L^p((0,\infty),u^{2\alpha}du)$ into itself, for every $1<p<\infty$ (see [15]), we deduce that $\mathbb{H}_\beta$ is a bounded operator from $L^p((0,\infty)^k,d\mu_\beta^k(x))$ into itself, for every $1<p<\infty$.

Then, from (41) we infer that the operator $\mathcal{H}_\beta^\ell$ is bounded from $L^p((0,\infty)^k, d\mu_\beta^k(x))$ into itself, for every $1<p<\infty$, and from $L^1((0,\infty)^k, d\mu_\beta^k(x))$ into $L^{1,\infty}((0,\infty)^k, d\mu_\beta^k(x))$. Since $|S_\lambda^{n,n,i}(g)|\le\|\phi\|_{L^\infty(0,\infty)}\mathcal{H}^i_{\lambda_1,\dots,\lambda_n}(|g|)$, it follows that $S_\lambda^{n,n,i}$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself and $\|S_\lambda^{n,n,i}\|_{p\mapsto p}\le C\|\phi\|_{L^\infty(0,\infty)}$, for every $1<p<\infty$, and from $L^1((0,\infty)^n,d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$, and $\|S_\lambda^{n,n,i}\|_{1\mapsto(1,\infty)}\le C\|\phi\|_{L^\infty(0,\infty)}$, for every $i=1,\dots,n$.

Assume that $1\le i\le k<n$. By using Lemmas 2.1(b), and 2.3(b), we have that

$$\begin{aligned}
\big|S_\lambda^{k,k,i}(g)(x)\big|&\le C\|\phi\|_{L^\infty(0,\infty)}\int_0^{\frac{x_1}{2}}\cdots\int_0^{\frac{x_k}{2}}\int_{2x_{k+1}}^\infty\cdots\int_{2x_n}^\infty\\
&\qquad\times\int_0^\infty\frac{e^{-(\sum_{j=1}^k x_j^2+\sum_{j=k+1}^n y_j^2)/40t}}{t^{\sum_{j=1}^n(\lambda_j+1/2)+1}}dt|g(y)|d\mu_\lambda(y)\\
&\le C\|\phi\|_{L^\infty(0,\infty)}\int_0^{\frac{x_1}{2}}\cdots\int_0^{\frac{x_k}{2}}\\
&\qquad\times\int_0^\infty\frac{e^{-\sum_{j=1}^k x_j^2/40t}}{t^{\sum_{j=1}^k(\lambda_j+1/2)+1}}dt\int_{2x_{k+1}}^\infty\cdots\int_{2x_n}^\infty\frac{|g(y)|}{\prod_{j=k+1}^n y_j^{2\lambda_j+1}}d\mu_\lambda(y)
\end{aligned}$$

$$\leq C\|\phi\|_{L^\infty(0,\infty)} \frac{1}{(\sum_{j=1}^k x_j^2)^{\sum_{j=1}^k(\lambda_j+1/2)}} \times \int_0^{\frac{x_1}{2}} \cdots \int_0^{\frac{x_k}{2}} \left(\int_{2x_{k+1}}^\infty \cdots \int_{2x_n}^\infty \frac{|g(y)|}{y_{k+1}\dots y_n} dy_n \dots dy_{k+1} \right) d\mu_\lambda^k(y),$$

for every $x \in (0,\infty)^n$.

According to [6, Proposition 1] and by taken into account the L^p-boundedness properties of the operator $\mathbb{H}_{\lambda_1,\dots,\lambda_k}$ and $\mathcal{S}_{n-k}$ we conclude that the operator $S_\lambda^{k,k,i}$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, and $\|S_\lambda^{k,k,i}\|_{p\mapsto p} \leq C\|\phi\|_{L^\infty(0,\infty)}$, for every $1<p<\infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$, and $\|S_\lambda^{k,k,i}\|_{1\mapsto(1,\infty)} \leq C\|\phi\|_{L^\infty(0,\infty)}$. In a similar way we can see that if $0<k<i\leq n$ the operator $S_\lambda^{k,k,i}$ satisfies the same L^p-boundedness properties.

Let $1\leq i\leq k<n$. By using Lemmas 2.1(a) and (b), and 2.3(b), we get

$$\left|S_\lambda^{k,n,i}(g)(x)\right| \leq C\|\phi\|_{L^\infty(0,\infty)} \int_0^{\frac{x_1}{2}} \cdots \int_0^{\frac{x_k}{2}} \int_{\frac{x_{k+1}}{2}}^{2x_{k+1}} \cdots \int_{\frac{x_n}{2}}^{2x_n} \int_0^\infty \frac{e^{-\sum_{j=1}^k x_j^2/40t}}{t^{\sum_{j=1}^k(\lambda_j+1/2)+1}} \times \prod_{j=k+1}^n \left(\frac{1}{x_j^{2\lambda_j+1}} + (x_j y_j)^{-\lambda_j} \frac{e^{-(x_j-y_j)^2/4t}}{\sqrt{t}} \right) dt\,|g(y)|\,d\mu_\lambda(y),$$
$$x \in (0,\infty)^n. \tag{42}$$

Suppose that $k<\ell\leq n$, $\ell\in\mathbb{N}$, and define

$$\mathbb{S}_\lambda^{k,\ell}(g)(x) = \int_0^{\frac{x_1}{2}} \cdots \int_0^{\frac{x_k}{2}} \int_{\frac{x_{k+1}}{2}}^{2x_{k+1}} \cdots \int_{\frac{x_n}{2}}^{2x_n} \int_0^\infty \frac{e^{-(\sum_{j=1}^k x_j^2 + \sum_{j=k+1}^\ell (x_j-y_j)^2)/40t}}{t^{\sum_{j=1}^k(\lambda_j+1/2)+1+(\ell-k)/2}} dt \times \prod_{j=k+1}^\ell (x_j y_j)^{-\lambda_j} \prod_{j=\ell+1}^n \frac{1}{x_j^{2\lambda_j+1}} g(y)\,d\mu_\lambda(y), \quad x\in(0,\infty)^n.$$

Note that the operator in the right hand side of (42) is controlled by a linear combination of operators like $\mathbb{S}_\lambda^{k,\ell}$. We have that, for each $x\in(0,\infty)^n$,

$$|\mathbb{S}_\lambda^{k,\ell}(g)(x)| \leq C \int_0^{\frac{x_1}{2}} \cdots \int_0^{\frac{x_k}{2}} \int_{\frac{x_{k+1}}{2}}^{2x_{k+1}} \cdots \int_{\frac{x_\ell}{2}}^{2x_\ell} \times \frac{\prod_{j=k+1}^\ell (x_j y_j)^{-\lambda_j}}{(\sum_{j=1}^k x_j^2 + \sum_{j=k+1}^\ell (x_j-y_j)^2)^{\sum_{j=1}^k(\lambda_j+1/2)+(\ell-k)/2}} \times \left(\frac{1}{\prod_{j=\ell+1}^n x_j^{2\lambda_j+1}} \int_{\frac{x_{\ell+1}}{2}}^{2x_{\ell+1}} \cdots \int_{\frac{x_n}{2}}^{2x_n} |g(y)| \prod_{j=\ell+1}^n y_j^{2\lambda_j} dy_n \dots dy_{\ell+1} \right) \times \prod_{j=1}^\ell y_j^{2\lambda_j} dy_\ell \dots dy_1. \tag{43}$$

Assume that $r, s \in \mathbb{N}$, $0 < s < r$, and $\beta = (\beta_1, \dots, \beta_r) \in (-1/2, \infty)^r$. We consider the operator

$$Y_\beta^s(g)(x) = \int_0^{\frac{x_1}{2}} \cdots \int_0^{\frac{x_s}{2}} \int_{\frac{x_{s+1}}{2}}^{2x_{s+1}} \cdots \int_{\frac{x_r}{2}}^{2x_r} \times \frac{\prod_{j=s+1}^r (x_j y_j)^{-\beta_j}}{(\sum_{j=1}^s x_j^2 + \sum_{j=s+1}^r (x_j - y_j)^2)^{\sum_{j=1}^s (\beta_j + 1/2) + (r-s)/2}} \times g(y) d\mu_\beta^r(y), \quad x \in (0, \infty)^r.$$

By proceeding as in the proof of Case 3 in [16] we can see that the operator $Y^s_{\beta_1,\dots,\beta_r}$ is bounded from $L^1((0,\infty)^r, d\mu_\beta^r(x))$ into $L^{1,\infty}((0,\infty)^r, d\mu_\beta^r(x))$. Since the operator $Z_{\lambda_{\ell+1},\dots,\lambda_n}$ is bounded from $L^1((0,\infty)^{n-\ell}, \prod_{j=\ell+1}^n x_j^{2\lambda_j} dx)$ into itself, when $n > \ell$, by [6, Proposition 1], we deduce from (43) that the operator $\mathbb{S}_\lambda^{k,\ell}$ is bounded from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$.

On the other hand we also have that

$$\begin{aligned}
&|\mathbb{S}_\lambda^{k,\ell}(g)(x)| \\
&\leq C \int_0^{\frac{x_1}{2}} \cdots \int_0^{\frac{x_k}{2}} \int_0^\infty \frac{e^{-\sum_{j=1}^k x_j^2/40t}}{t^{\sum_{j=1}^k (\lambda_j + 1/2) + 1}} dt \\
&\quad \times \left\{ \frac{1}{\prod_{j=\ell+1}^n x_j^{2\lambda_j + 1}} \int_{\frac{x_{\ell+1}}{2}}^{2x_{\ell+1}} \cdots \int_{\frac{x_n}{2}}^{2x_n} \right. \\
&\quad \times \left(\sup_{t>0} \int_{\frac{x_{k+1}}{2}}^{2x_{k+1}} \cdots \int_{\frac{x_\ell}{2}}^{2x_l} \prod_{j=k+1}^{\ell} (x_j y_j)^{-\lambda_j} \frac{e^{-(x_j - y_j)^2/4t}}{\sqrt{t}} |g(y)| \right. \\
&\quad \times \left. \left. \prod_{j=k+1}^{\ell} y_j^{2\lambda_j} dy_\ell \dots dy_{k+1} \right) \prod_{j=\ell+1}^n y_j^{2\lambda_j} dy_n \dots dy_{\ell+1} \right\} \prod_{j=1}^k y_j^{2\lambda_j} dy_1 \dots dy_k \\
&\leq C \frac{1}{\prod_{j=1}^k x_j^{2\lambda_j + 1}} \int_0^{\frac{x_1}{2}} \cdots \int_0^{\frac{x_k}{2}} \left\{ \frac{1}{\prod_{j=\ell+1}^n x_j^{2\lambda_j + 1}} \int_{\frac{x_{\ell+1}}{2}}^{2x_{\ell+1}} \cdots \int_{\frac{x_n}{2}}^{2x_n} \right. \\
&\quad \times \left(\sup_{t>0} \int_{\frac{x_{k+1}}{2}}^{2x_{k+1}} \cdots \int_{\frac{x_\ell}{2}}^{2x_\ell} \prod_{j=k+1}^{n} (x_j y_j)^{-\lambda_j} \frac{e^{-(x_j - y_j)^2/4t}}{\sqrt{t}} |g(y)| \right. \\
&\quad \times \left. \left. \prod_{j=k+1}^{\ell} y_j^{2\lambda_j} dy_\ell \dots dy_{k+1} \right) \prod_{j=\ell+1}^n y_j^{2\lambda_j} dy_n \dots dy_{\ell+1} \right\} \\
&\quad \times \prod_{j=1}^k y_j^{2\lambda_j} dy_1 \dots dy_k, \quad x \in (0,\infty)^n.
\end{aligned}$$

According to the L^p-boundedness properties of the operators Ω_β, Z_β, and $\mathbb{H}_\beta$, $\beta = (\beta_1, \dots, \beta_r) \in (-1/2, \infty)^r$, $r \in \mathbb{N}$, by using [6, Proposition 1], we obtain that $\mathbb{S}_\lambda^{k,\ell}$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, for every $1 < p < \infty$.

We conclude that $S_\lambda^{k,n,i}$ is bounded $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, and $\|S_\lambda^{k,n,i}\|_{p\mapsto p} \le C\|\phi\|_{L^\infty(0,\infty)}$, for every $1 < p < \infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$, and $\|S_\lambda^{k,n,i}\|_{1\mapsto(1,\infty)} \le C\|\phi\|_{L^\infty(0,\infty)}$, provided that $1 \le i \le k < n$.

Suppose now that $1 \le k < i \le n$. By using Lemmas 2.1(b), and 2.3(a), we obtain

$$\begin{aligned}
\left|S_\lambda^{k,n,i}(g)(x)\right| &\le C\|\phi\|_{L^\infty(0,\infty)} \int_0^{\frac{x_1}{2}} \cdots \int_0^{\frac{x_k}{2}} \int_{\frac{x_{k+1}}{2}}^{2x_{k+1}} \cdots \int_{\frac{x_n}{2}}^{2x_n} \int_0^\infty \frac{e^{-\sum_{j=1}^k x_j^2/20t}}{t^{\sum_{j=1}^k(\lambda_j+1/2)}} \\
&\quad \times \prod_{j=k+1, j\neq i}^n \left(\frac{1}{x_j^{2\lambda_j+1}} + (x_j y_j)^{-\lambda_j} \frac{e^{(x_j-y_j)^2/10t}}{\sqrt{t}}\right) \\
&\quad \times \left((x_i y_i)^{-\lambda_i} \frac{e^{-(x_i-y_i)^2/10t}}{t^{3/2}} + \frac{e^{-(x_i^2+y_i^2)/4t}}{t^{\lambda_i+3/2}}\right) dt|g(y)|d\mu_\lambda(y) \\
&\le C\|\phi\|_{L^\infty(0,\infty)} \int_0^{\frac{x_1}{2}} \cdots \int_0^{\frac{x_k}{2}} \int_{\frac{x_{k+1}}{2}}^{2x_{k+1}} \cdots \int_{\frac{x_n}{2}}^{2x_n} \int_0^\infty \frac{e^{-\sum_{j=1}^k x_j^2/20t}}{t^{\sum_{j=1}^k(\lambda_j+1/2)+1}} \\
&\quad \times \prod_{j=k+1}^n \left(\frac{1}{x_j^{2\lambda_j+1}} + (x_j y_j)^{-\lambda_j} \frac{e^{(x_j-y_j)^2/10t}}{\sqrt{t}}\right) dt|g(y)|d\mu_\lambda(y),
\end{aligned}$$

$x \in (0,\infty)^n$.

Then, as above, $S_\lambda^{k,n,i}$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, for every $1 < p < \infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$, when $1 \le k < i \le n$.

By proceeding in a similar way we can see that the operator $S_\lambda^{k,\ell,i}$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, for every $1 < p < \infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$, when $0 \le k < \ell < n$, and $i = 1, \dots, n$.

By (40) we conclude that $S_\lambda^{k,\ell}$, $0 \le k \le \ell \le n$, $k \neq 0$ or $\ell \neq n$, is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, for every $1 < p < \infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$. Then, the operator $\mathcal{G}_\lambda$ has the same L^p-boundedness properties.

In all the above cases the norm of operators is controlled by $C\|\phi\|_{L^\infty(0,\infty)}$.

From (34) it follows that the maximal operator $T_\lambda^{m,*}$ is bounded from $L^p((0,\infty)^n, d\mu_\lambda(x))$ into itself, and $\|T_\lambda^{m,*}\|_{p\mapsto p} \le C(\|\phi\|_{L^\infty(0,\infty)} + \|T^{m,*}\|_{p\mapsto p})$, for every $1 < p < \infty$, and from $L^1((0,\infty)^n, d\mu_\lambda(x))$ into $L^{1,\infty}((0,\infty)^n, d\mu_\lambda(x))$, and $\|T_\lambda^{m,*}\|_{1\mapsto(1,\infty)} \le C(\|\phi\|_{L^\infty(0,\infty)} + \|T^{m,*}\|_{1\mapsto(1,\infty)})$.

Thus the proof of this theorem is completed.

3.3 Proof of Theorem 1.4

The existence of the limit that defines the operator $\mathbb{T}_\lambda^m$ and the L^p-boundedness properties of $\mathbb{T}_\lambda^m$ can be proved by taking into account Theorems 1.2 and 1.3 and using standard procedures.

Now we are going to show (8) and (9). We know that the Fourier multiplier T^m takes the pointwise representation

$$T^m(f)(x) = -\lim_{\varepsilon\to 0^+}\left(\tilde{\Lambda}(\varepsilon)f(x) + \int_{|x-y|>\varepsilon} f(y)H^\phi(x,y)dy\right), \quad \text{a.e.} \quad x\in\mathbb{R}^n,$$

for every $f\in L^p(\mathbb{R}^n)$, $1\le p<\infty$. Here H^ϕ is the function given by (7) and $\tilde{\Lambda}$ represents a measurable bounded function on $(0,\infty)^n$ such that $\|\tilde{\Lambda}\|_{L^\infty(0,\infty)}\le C\|\phi\|_{L^\infty(0,\infty)}$, where $C>0$ does not depend on ϕ.

For every $x\in(0,\infty)^n$, we define the local region $L(x)$ as in the proof of Theorem 1.3 and the operator

$$T^m_{loc,\lambda}(f)(x) = -\lim_{\varepsilon\to 0^+}\left(\tilde{\Lambda}(\varepsilon)f(x) + \int_{L(x),|x-y|>\varepsilon}(x.y)^{-\lambda}H^\phi(x,y)f(y)d\mu_\lambda(y)\right),$$
$$f\in C_c^\infty\big((0,\infty)^n\big).$$

We can write

$$\mathbb{T}_\lambda^m(f) = \big(\mathbb{T}_\lambda^m(f) - T^m_{loc,\lambda}(f)\big) + T^m_{loc,\lambda}(f),$$
$$f\in C_c^\infty\big((0,\infty)^n\big).$$

If $\mathcal{G}_\lambda$ and $\mathcal{L}_\lambda$ denote the operators defined in (34) we have that

$$\begin{aligned}|\mathbb{T}_\lambda^m(f)(x)| &\le (\|\Lambda\|_{L^\infty(0,\infty)} + \|\tilde{\Lambda}\|_{L^\infty(0,\infty)})|f(x)| \\ &\quad + \mathcal{G}_\lambda(|f|)(x) + \mathcal{L}_\lambda(|f|)(x) + |T^m_{loc,\lambda}(f)(x)|, \end{aligned}\tag{44}$$

for every $x\in(0,\infty)^n$. We consider cubes Q_j and $\tilde{Q}_j$, $j\in\mathbb{Z}^n$, as in Sect. 3.2.1. It follows that

$$\begin{aligned}T^m_{loc,\lambda}(f)(x) = -\lim_{\varepsilon\to 0^+}\Bigg(&\tilde{\Lambda}(\varepsilon)f(x) + \int_{L(x),|x-y|>\varepsilon}\left(\frac{y}{x}\right)^\lambda H^\phi(x,y)f(y)\chi_{\tilde{Q}_j}(y)dy \\ &- \int_{\tilde{Q}_j\setminus L(x),|x-y|>\varepsilon}\left(\frac{y}{x}\right)^\lambda H^\phi(x,y)f(y)dy\Bigg),\end{aligned}$$
$$\text{a.e.} \quad x\in Q_j,\ j\in\mathbb{Z}^n.$$

Then, for every $j\in\mathbb{Z}^n$,

$$\begin{aligned}|T^m_{loc,\lambda}(f)(x)| \le &\left|T^m\left(\left(\frac{y}{x}\right)^\lambda f\chi_{\tilde{Q}_j}\right)(x)\right| \\ &+ \sup_{\varepsilon>0}\left|\int_{\tilde{Q}_j\setminus L(x),|x-y|>\varepsilon}\left(\frac{y}{x}\right)^\lambda H^\phi(x,y)f(y)dy\right|, \quad \text{a.e.} \quad x\in Q_j.\end{aligned}$$

By proceeding as in Sect. 3.2.1 we conclude that

$$\|T^m_{loc,\lambda}\|_{p\to p} \le C\big(\|\phi\|_{L^\infty(0,\infty)} + \|T^m\|_{p\to p}\big), \quad 1<p<\infty,$$

and

$$\|T^m_{loc,\lambda}\|_{1\to(1,\infty)} \le C\big(\|\phi\|_{L^\infty(0,\infty)} + \|T^m\|_{1\to(1,\infty)}\big).$$

Moreover, according to the properties established in Sects. 3.2.2 and 3.2.3 we have that

$$\|\mathcal{N}\|_{p\to p} \le C\|\phi\|_{L^\infty(0,\infty)}, \quad 1<p<\infty,$$

and

$$\|\mathcal{N}\|_{1\to(1,\infty)} \le C\|\phi\|_{L^\infty(0,\infty)},$$

where $\mathcal{N}$ represents the operators $\mathcal{G}_\lambda$ and $\mathcal{L}_\lambda$.

Hence, by combining the above estimates with (44) we obtain (8) and (9). Thus the proof is completed.

3.4 Proof of Corollary 1.2

Let $\beta \in \mathbb{R}$. We define $\phi_\beta(t) = t^{-i\beta}/\Gamma(1-i\beta)$, $t\in(0,\infty)$, and $m_\beta(y) = |y|^2\int_0^\infty e^{-t|y|^2}\phi_\beta(t)dt$, $y\in(0,\infty)^n$. As it was mentioned $\Delta_\lambda^{i\beta} = T_\lambda^{m_\beta}$. Also, $\Delta^{i\beta} = T^{m_\beta}$. From Corollary 1.1 we deduce the L^p-boundedness properties for $\Delta_\lambda^{i\beta}$. Moreover, we have that

$$\|\Delta_\lambda^{i\beta}\|_{1\mapsto(1,\infty)} \le C\big(\|\phi_\beta\|_{L^\infty(0,\infty)} + \|\Delta^{i\beta}\|_{1\mapsto(1,\infty)}\big).$$

According to [18], $\|\Delta^{i\beta}\|_{1\mapsto(1,\infty)} \le C(1+|\beta|)^{n/2}$. Moreover, by [13, p. 14], we get

$$\|\phi_\beta\|_{L^\infty(0,\infty)} \le Ce^{|\beta|\pi/2}.$$

Then, we deduce that

$$\|\Delta_\lambda^{i\beta}\|_{1\mapsto(1,\infty)} \le Ce^{|\beta|\pi/2}.$$

On the other hand, since $\|m_\beta\|_{L^\infty(0,\infty)} = 1$ and H_λ is an isometry in $L^2((0,\infty)^n, d\mu_\lambda(x))$, we have that $\|T_\lambda^{m_\beta}\|_{2\mapsto 2} = 1$. By using now the classical Marcinkiewicz interpolation theorem and standard duality arguments we conclude that, for every $1<p<\infty$,

$$\|\Delta_\lambda^{i\beta}\|_{p\mapsto p} \le Ce^{|\beta|\pi|\frac{1}{2}-\frac{1}{p}|}.$$

Here, $C>0$ is always a constant which does not depend on β.

Thus we complete the proof of this corollary.

Acknowledgements We are very grateful to the referees for their helpful comments which have improved this paper. One of them pointed out us the connection between the L^p-boundedness properties for imaginary powers of an operator with the L^p-boundedness properties of general spectral multipliers associated with that operator. He suggested us to use the results in [5] and [14] to establish Theorem 1.1. The other referee suggested us to complete the paper with Sect. 3.3.

References

1. Betancor, J.J., Rodríguez-Mesa, L.: Weighted inequalities for Hankel convolution operators. Ill. J. Math. **44**(2), 230–245 (2000)
2. Betancor, J.J., Stempak, K.: Relating multipliers and transplantation for Fourier-Bessel expansions and Hankel transform. Tohoku Math. J. **53**(1), 109–129 (2001)
3. Betancor, J.J., Martínez, T., Rodríguez-Mesa, L.: Laplace transform type multipliers for Hankel transforms. Can. Math. Bull. **51**, 487–496 (2008)
4. Cowling, M.: Harmonic analysis on semigroups. Ann. Math. **117**, 267–283 (1983)
5. Cowling, M., Meda, S.: Harmonic analysis and ultracontractivity. Trans. Am. Math. Soc. **340**(2), 733–752 (1993)
6. Dinger, U.: Weak type $(1,1)$ estimates of the maximal function for the Laguerre semigroup in finite dimensions. Rev. Mat. Iberoam. **8**(1), 93–118 (1992)
7. García-Cuerva, J., Mauceri, G., Sjögren, P., Torrea, J.L.: Spectral multipliers for the Ornstein-Uhlenbeck semigroup. J. Anal. Math. **78**, 281–305 (1999)
8. Garrigós, G., Seeger, A.: Characterizations of Hankel multipliers. Math. Ann. **342**, 31–68 (2008)
9. Gasper, G., Trebels, W.: Multiplier criteria of Hormander type for Fourier series and applications to Jacobi series and Hankel transforms. Math. Ann. **242**(3), 225–240 (1979)
10. Gosselin, J., Stempak, K.: A weak type estimate for Fourier-Bessel multipliers. Proc. Am. Math. Soc. **106**(3), 655–662 (1989)
11. Guy, D.L.: Hankel transformations and weighted p-norms. Trans. Am. Math. Soc. **95**, 137–189 (1960)
12. Kapelko, R.: A multiplier theorem for Hankel transforms. Rev. Mat. Complut. **11**(2), 281–288 (1998)
13. Lebedev, N.N.: Special Functions and Their Applications. Dover, New York (1972)
14. Meda, S.: A general multiplier theorem. Proc. Am. Math. Soc. **110**(3), 639–647 (1990)
15. Muckenhoupt, B.: Hardy's inequality with weights. Stud. Math. **44**, 31–38 (1972)
16. Nowak, A., Sjögren, P.: Weak type $(1,1)$ estimates for maximal operators associated with various multi-dimensional systems of Laguerre functions. Indiana Univ. Math. J. **56**, 417–436 (2007)
17. Nowak, A., Stempak, K.: Weighted estimates for the Hankel transform transplantation operator. Tohoku Math. J. **58**, 277–301 (2006)
18. Sikora, A., Wright, J.: Imaginary powers of Laplace operators. Proc. Am. Math. Soc. **129**, 1745–1754 (2001)
19. Stein, E.M.: Topics in harmonic analysis related to the Littlewood-Paley theory. Ann. of Math. Studies, vol. 63. Princeton Univ. Press, Princeton (1970)
20. Stempak, K., Torrea, J.L.: Poisson integrals and Riesz transform for Hermite function expansions with weights. J. Funct. Anal. **202**, 443–472 (2003)
21. Stempak, K., Trebels, W.: Hankel multipliers and transplantation operators. Stud. Math. **126**(1), 51–66 (1997)
22. Watson, G.N.: A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge (1966)