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Optimal Embeddings of Spaces of Generalized Smoothness in the Critical Case

Susana D. Moura · Júlio S. Neves · Cornelia Schneider

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Abstract We study necessary and sufficient conditions for embeddings of Besov and Triebel-Lizorkin spaces of generalized smoothness $B_{p,q}^{(n/p,\Psi)}(\mathbb{R}^n)$ and $F_{p,q}^{(n/p,\Psi)}(\mathbb{R}^n)$, respectively, into generalized Hölder spaces $\Lambda_{\infty,r}^{\mu(\cdot)}(\mathbb{R}^n)$. In particular, we are able to characterize optimal embeddings for this class of spaces provided $q > 1$. These results improve the embedding assertions given by the continuity envelopes of $B_{p,q}^{(n/p,\Psi)}(\mathbb{R}^n)$ and $F_{p,q}^{(n/p,\Psi)}(\mathbb{R}^n)$, which were obtained recently solving an open problem of D.D. Haroske in the classical setting.

Keywords Function spaces of generalized smoothness · Optimal embeddings

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1 Introduction

The aim of this paper is to improve the results obtained in [24], where the authors computed the continuity envelopes (which are closely related to questions of sharp

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S.D. Moura (✉) · J.S. Neves
CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal
e-mail: smpsd@mat.uc.pt

J.S. Neves
e-mail: jsn@mat.uc.pt

C. Schneider
Department Mathematik, Universität Erlangen-Nürnberg, Haberstr. 2, 91058 Erlangen, Germany
e-mail: schneider@am.uni-erlangen.de

Birkhäuser

embeddings) for Besov and Triebel-Lizorkin spaces of generalized smoothness in a limiting case—the so-called critical case $s = \frac{n}{p}$. In this situation it was proven that the optimal index is ∞. This assertion is surprising in comparison with all the previously known results, where the optimal index was always q for the Besov spaces and p for the Triebel-Lizorkin spaces. In particular, in [24] the former result of Haroske in [20, Theorem 9.10] was improved and the open question posed by her in [20, Remark 9.11] was answered. However, it turns out now that in this critical situation the continuity envelopes of the spaces mentioned above do not yield optimal embedding results. A similar situation occurred in [16] and [17], but in the context of Bessel-potential-type spaces in the limiting case. However, the technics used there were completely different from the ones considered here.

In this paper we prove that

$$B^{(\frac{n}{p},\Psi)}_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n), \tag{1.1}$$

if, and only if,

$$\sup_{\varkappa\in(0,1)} \left(\int_{\varkappa}^{1} \mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} \left(\int_0^{\varkappa} \Psi(s)^{-q'}\frac{\mathrm{d}s}{s}\right)^{\frac{1}{q'}} < \infty$$

(with the usual modification if $r = \infty$ and/or $q' = \infty$), provided that $0 < p \leq \infty$, $0 < q \leq r \leq \infty$, Ψ is a slowly varying function such that $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}_0} \in \ell_{q'}$ and $\mu \in \mathcal{L}_r$ (see Theorem 3.2 and Remark 3.4 below and Sect. 2 for precise definitions).

In particular (cf. Corollary 3.3), when $q > 1$ and $r \in [q,\infty]$, the embedding (1.1) with $\mu = \lambda_{qr}$, where

$$\lambda_{qr}(t) := \Psi(t)^{\frac{q'}{r}} \left(\int_0^t \Psi(s)^{-q'}\frac{\mathrm{d}s}{s}\right)^{\frac{1}{q'}+\frac{1}{r}}, \quad t \in (0,1],$$

is sharp with respect to the parameter μ, that is, the target space $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ in (1.1) and the space $\Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\mathbb{R}^n)$ (i.e., the target space in (1.1) with $\mu = \lambda_{qr}$) satisfy $\Lambda^{\lambda_{qr}(\cdot)}_{\infty,r}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$. The embedding with $r = q$ and $\mu = \lambda_{qr} = \lambda_{qq}$ is optimal (*i.e.*, it is the best possible embedding among all the embeddings considered in (1.1)). An interesting case is the one with $p = \infty$, concerning the space $B^{(0,\Psi)}_{\infty,q}(\mathbb{R}^n)$.

For example, if $\Psi(t) \sim (1-\ln t)^{\alpha}$,

$$B^{(\frac{n}{p},(1-\ln t)^{\alpha})}_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{(1-\ln t)^{-\alpha+1}}_{\infty,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{(1-\ln t)^{-\alpha+\frac{1}{q'}+\frac{1}{r}}}_{\infty,r}(\mathbb{R}^n),$$

provided $1 < q \leq r \leq \infty$, $0 < p \leq \infty$ and $\alpha > \frac{1}{q'}$, where the first embedding is optimal.

If $0 < q \leq 1$, we have $\frac{1}{q'} = 0$, and in this case

$$B^{(\frac{n}{p},(1-\ln t)^{\alpha})}_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{(1-\ln t)^{-\alpha+\frac{1}{q}}}_{\infty,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{(1-\ln t)^{-\alpha+\frac{1}{r}}}_{\infty,r}(\mathbb{R}^n),$$

provided $0 < q \le r \le \infty$, $0 < p \le \infty$ and $\alpha > 0$, where the first embedding is optimal.

Note that when $\Psi(t) \sim (1 - \ln t)^\alpha$ with $\alpha < \frac{1}{q'}$, the space $B_{p,q}^{(\frac{n}{p},(1-\ln t)^\alpha)}(\mathbb{R}^n)$ is not embedded into $L_\infty(\mathbb{R}^n)$. Thus, it does not make sense to study embeddings into Hölder-type spaces but rather into Lorentz-Zygmund-type spaces. We refer to [8], where the authors dealt with this situation, and to [33] and [20] concerning the classical situation.

In terms of F-spaces we obtain similar results, with the usual replacement of q by p.

The paper is organized as follows. Section 2 contains notation, definitions, preliminary assertions and auxiliary results. In Sect. 3 we state our main results, providing necessary and sufficient conditions for the embeddings to hold, and derive optimal weights and sharp embedding assertions.

2 Preliminaries

2.1 General Notation

For a real number a, let $a_+ := \max(a, 0)$ and let $[a]$ denote its integer part. For $p \in (0, \infty]$, the number p' is defined by $1/p' := (1 - 1/p)_+$ with the convention that $1/\infty = 0$. By c, c_1, c_2, etc. we denote positive constants independent of appropriate quantities. For two non-negative expressions (i.e. functions or functionals) $\mathcal{A}$, $\mathcal{B}$, the symbol $\mathcal{A} \lesssim \mathcal{B}$ (or $\mathcal{A} \gtrsim \mathcal{B}$) means that $\mathcal{A} \le c\mathcal{B}$ (or $c\mathcal{A} \ge \mathcal{B}$). If $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{A} \gtrsim \mathcal{B}$, we write $\mathcal{A} \sim \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are equivalent. Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding is bounded. Furthermore, $L_p(\mathbb{R}^n)$, with $0 < p \le \infty$, is the usual Lebesgue space, with respect to the Lebesgue measure, endowed with the usual quasi-norm $\|\cdot \mid L_p(\mathbb{R}^n)\|$. The space of all scalar-valued (real or complex), bounded and continuous functions on $\mathbb{R}^n$ is denoted by $C_B(\mathbb{R}^n)$, which is equipped with the $L_\infty(\mathbb{R}^n)$-norm.

2.2 Slowly Varying Functions

Definition 2.1 A positive and measurable function Ψ defined on the interval (0, 1] is said to be *slowly varying* if

$$\lim_{t\to 0^+} \frac{\Psi(st)}{\Psi(t)} = 1, \qquad s \in (0, 1]. \tag{2.1}$$

Example 2.2 The following functions are examples of slowly varying functions:

(i) $\Psi(x) = (1 + |\log x|)^a (1 + \log(1 + |\log x|))^b$, $x \in (0, 1]$, $a, b \in \mathbb{R}$,
(ii) $\Psi(x) = \exp(|\log x|^c)$, $x \in (0, 1]$, $c \in (0, 1)$.

We remark that the function in Example 2.2(i) is also an admissible function in the sense of [12, 13], which means that Ψ is a positive monotone function defined on $(0,1]$ such that $\Psi(2^{-2j}) \sim \Psi(2^{-j})$, $j \in \mathbb{N}$. It can be proved that an admissible function is, up to equivalence, a slowly varying function.

The proposition below collects some properties of slowly varying functions which will be useful in what follows. We refer to the monograph [3] for details and further properties.

Proposition 2.3 *Let Ψ be a slowly varying function.*

(i) *For any $\delta > 0$ there exists $c = c(\delta) > 1$ such that*

$$\frac{1}{c} s^{\delta} \le \frac{\Psi(st)}{\Psi(t)} \le c s^{-\delta}, \quad t, s \in (0,1].$$

(ii) *For each $\alpha > 0$ there is a decreasing function ϕ and an increasing function φ such that*

$$t^{-\alpha}\Psi(t) \sim \phi(t) \quad \text{and} \quad t^{\alpha}\Psi(t) \sim \varphi(t).$$

(iii) *If $\int_0^1 \Psi(s)\frac{ds}{s} < \infty$, then $\tilde{\Psi}$ defined by $\tilde{\Psi}(t) = \int_0^t \Psi(s)\frac{ds}{s}$, $t \in (0,1]$, is a slowly varying function such that*

$$\lim_{t\to 0^+} \frac{\tilde{\Psi}(t)}{\Psi(t)} = \infty.$$

(iv) *Ψ^r, $r \in \mathbb{R}$, is a slowly varying function.*

(v) *If Φ is a slowly varying function as well, so is $\Psi\Phi$.*

Remark 2.4 It follows easily from the last proposition that

$$\Psi(t) \sim \Psi\big(2^{-j}\big) \sim \Psi\big(2^{-(j+1)}\big), \quad t \in \big[2^{-(j+1)}, 2^{-j}\big],\ j \in \mathbb{N}_0.$$

The next proposition provides a very useful discretization method, which coincides partially with [24, Proposition 2.5].

Proposition 2.5 *Let Ψ be a slowly varying function.*

(i) *Then*

$$\int_t^1 \Psi(s)\frac{\mathrm{d}s}{s} \sim \sum_{j=0}^{[|\log t|]} \Psi\big(2^{-j}\big), \quad t \in (0, 2^{-1}].$$

(ii) *Moreover, if $\int_0^1 \Psi(s)\frac{\mathrm{d}s}{s} < \infty$, then*

$$\int_0^t \Psi(s)\frac{\mathrm{d}s}{s} \sim \sum_{j=[|\log t|]}^{\infty} \Psi\big(2^{-j}\big), \quad t \in (0,1].$$

A corresponding assertion holds if we replace the integral and the sum by suprema.

We complement the previous proposition by a discrete version of [29, (3.2)], also cf. [24, Lemma 2.6].

Lemma 2.6 *Let* $0 < u \le \infty$ *and* Ψ *be a slowly varying function.*

(i) *Then*

$$\left(\sum_{j=k}^{\infty} 2^{-ju}\Psi\left(2^{-j}\right)^{u}\right)^{1/u} \sim 2^{-k}\Psi\left(2^{-k}\right), \quad k \in \mathbb{N} \tag{2.2}$$

(*with the usual modification if* $u = \infty$).

(ii) *Furthermore, we have*

$$\left(\sum_{j=0}^{k} 2^{ju}\Psi\left(2^{-j}\right)^{-u}\right)^{1/u} \sim 2^{k}\Psi\left(2^{-k}\right)^{-1}, \quad k \in \mathbb{N} \tag{2.3}$$

(*with the usual modification if* $u = \infty$).

Proof Suppose that $0 < u < \infty$. In order to prove (i) let $\varepsilon \in (0, u)$. Using the fact that $t^{\varepsilon}\Psi(t)^{u}$ is equivalent to an increasing function, cf. Proposition 2.3(ii), we obtain for $k \in \mathbb{N}$,

$$\begin{aligned}\sum_{j=k}^{\infty} 2^{-ju}\Psi\left(2^{-j}\right)^{u} &= \sum_{j=k}^{\infty} 2^{j(\varepsilon-u)}\left(2^{-j}\right)^{\varepsilon}\Psi\left(2^{-j}\right)^{u}\\ &\lesssim \left(2^{-k}\right)^{\varepsilon}\Psi\left(2^{-k}\right)^{u}\sum_{j=k}^{\infty} 2^{j(\varepsilon-u)}\\ &= 2^{-k\varepsilon}\Psi\left(2^{-k}\right)^{u} 2^{k(\varepsilon-u)}\sum_{j=0}^{\infty} 2^{j(\varepsilon-u)}\\ &\lesssim 2^{-uk}\Psi\left(2^{-k}\right)^{u}.\end{aligned}$$

This completes the proof since the reverse inequality is clear. A corresponding proof for (ii) can be found in [24, Lemma 2.6]. The proof in the case $u = \infty$ is analogous. □

2.3 Function Spaces of Generalized Smoothness

In the sequel, let $\mathcal{S}(\mathbb{R}^n)$ stand for the Schwartz space of all complex-valued rapidly decreasing C^{∞} functions on $\mathbb{R}^n$ and we denote by $\mathcal{S}'(\mathbb{R}^n)$ its topological dual, the space of all tempered distributions. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ be a function such that

$$\varphi_0(x) = 1 \quad \text{for } |x| \le 1 \quad \text{and} \quad \operatorname{supp}\varphi_0 \subset \left\{x \in \mathbb{R}^n : |x| \le 2\right\}. \tag{2.4}$$

For each $j \in \mathbb{N}$, we define

$$\varphi_j(x) := \varphi_0\left(2^{-j}x\right) - \varphi_0\left(2^{-j+1}x\right), \quad x \in \mathbb{R}^n. \tag{2.5}$$

Then, since $\sum_{j=0}^{\infty}\varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$, the sequence $(\varphi_j)_{j\in\mathbb{N}_0}$ is a dyadic resolution of unity. Given any $f \in \mathcal{S}'$, we denote by $\widehat{f}$ and $f^{\vee}$ its Fourier transform and its inverse Fourier transform, respectively.

Definition 2.7 Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and let Ψ be a slowly varying function according to Definition 2.1.

(i) Then $B^{(s,\Psi)}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\| f \mid B^{(s,\Psi)}_{p,q}\big(\mathbb{R}^n\big)\big\| := \left(\sum_{j=0}^{\infty} 2^{jsq}\Psi\big(2^{-j}\big)^q \big\|(\varphi_j \widehat{f})^\vee \mid L_p\big(\mathbb{R}^n\big)\big\|^q\right)^{1/q} \tag{2.6}$$

(with the usual modification if $q = \infty$) is finite.

(ii) Let $0 < p < \infty$. Then $F^{(s,\Psi)}_{p,q}(\mathbb{R}^n)$ is defined to be the set of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\| f \mid F^{(s,\Psi)}_{p,q}\big(\mathbb{R}^n\big)\big\| := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq}\Psi\big(2^{-j}\big)^q |(\varphi_j \widehat{f})^\vee(\cdot)|^q\right)^{1/q} \,\Big|\, L_p\big(\mathbb{R}^n\big)\right\| \tag{2.7}$$

(with the usual modification if $q = \infty$) is finite.

Remark 2.8 The above spaces were introduced by Edmunds and Triebel in [12, 13] and also considered by Moura in [25, 26] when Ψ is an admissible function. For basic properties of the spaces above, like the independence of these spaces from the resolution of unity $(\varphi_j)_{j\in\mathbb{N}_0}$, according to (2.4) and (2.5), in the sense of equivalent quasi-norms, we refer to [14] in a more general setting. Taking $\Psi \equiv 1$ bring us back to the classical Besov and Triebel-Lizorkin spaces denoted by $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$, respectively. If $\Psi(t) = (1 + |\log t|)^b$, $b \in \mathbb{R}$, we obtain the spaces considered by Leopold in [22] and [23]. Denoting by A either B or F, we have for all $\varepsilon > 0$ the following elementary embeddings between classical spaces and spaces of generalized smoothness

$$A^{s+\varepsilon}_{p,q}\big(\mathbb{R}^n\big) \hookrightarrow A^{(s,\Psi)}_{p,q}\big(\mathbb{R}^n\big) \hookrightarrow A^{s-\varepsilon}_{p,q}\big(\mathbb{R}^n\big).$$

The next assertion on embeddings between Besov and Triebel-Lizorkin spaces of generalized smoothness will enable us to handle embedding assertions involving Triebel-Lizorkin spaces of generalized smoothness in a very simple way by using the results for B-spaces. We refer to [9, Proposition 3.4, Example 3.5] for a proof in the case of Ψ being an admissible function and to [7, Lemma 1] for a more general situation.

Proposition 2.9 *Let Ψ be a slowly varying function. Let $0 < p_0 < p < p_1 \le \infty$, $0 < q \le \infty$ and let $s, s_0, s_1 \in \mathbb{R}$ be such that $s_0 - n/p_0 = s - n/p = s_1 - n/p_1$. Then*

$$B^{(s_0,\Psi)}_{p_0,u}\big(\mathbb{R}^n\big) \hookrightarrow F^{(s,\Psi)}_{p,q}\big(\mathbb{R}^n\big) \hookrightarrow B^{(s_1,\Psi)}_{p_1,v}\big(\mathbb{R}^n\big) \quad \textit{if, and only if,} \quad 0 < u \le p \le v \le \infty.$$

The following result gives a characterization of the Besov spaces of generalized smoothness by means of Peetre's maximal function. The proof runs in the same way as that of [25, Theorem 1.7(i)] for Ψ being an admissible function.

Theorem 2.10 *Let $(\varphi_j)_{j\in\mathbb{N}_0}$ be a smooth dyadic resolution of unity as above. Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and let Ψ be a slowly varying function. Let $a > n/p$, then*

$$\| f \mid B^{(s,\Psi)}_{p,q}(\mathbb{R}^n)\|^* := \left(\sum_{j=0}^{\infty} 2^{jsq}\Psi(2^{-j})^q \|(\varphi_j^* f)_a \mid L_p(\mathbb{R}^n)\|^q\right)^{1/q}$$

(*with the usual modification if $q=\infty$*) *is an equivalent quasi-norm in $B^{(s,\Psi)}_{p,q}(\mathbb{R}^n)$, where the Peetre's maximal function $(\varphi_j^* f)_a$ is defined by*

$$(\varphi_j^* f)_a(x) := \sup_{z\in\mathbb{R}^n} \frac{|(\varphi_j \widehat{f})^\vee(x-z)|}{(1+2^j|z|)^a} \quad \textit{for } x \in \mathbb{R}^n.$$

An important tool in our later considerations is the characterization of the spaces of generalized smoothness by means of atomic decompositions. We state this here for the B-spaces only.

We need some preparation. As for $\mathbb{Z}^n$, it stands for the lattice of all points in $\mathbb{R}^n$ with integer components, $Q_{\nu m}$ denotes a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-\nu}m = (2^{-\nu}m_1, \dots, 2^{-\nu}m_n)$, and with side length $2^{-\nu}$, where $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q.

Definition 2.11 Let $s \in \mathbb{R}$, $0 < p \le \infty$, $K \in \mathbb{N}_0$, $L+1 \in \mathbb{N}_0$ and $d > 1$. The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an $(s, p, \Psi)_{K,L}$-*atom* if for some $\nu \in \mathbb{N}_0$ the following assumptions are satisfied

(i) $\operatorname{supp} a \subset dQ_{\nu m}$ for some $m \in \mathbb{Z}^n$,
(ii) $|\mathrm{D}^\alpha a(x)| \le 2^{-\nu(s-\frac{n}{p})+|\alpha|\nu}\Psi(2^{-\nu})^{-1}$ for $|\alpha| \le K$, $x \in \mathbb{R}^n$,
(iii) $\int_{\mathbb{R}^n} x^\beta a(x)\,\mathrm{d}x = 0$ for $|\beta| \le L$.

If conditions (i) and (ii) are satisfied for $\nu = 0$, then a is called an 1_K-*atom*.

Remark 2.12 In the sequel, we will write $a_{\nu m}$ instead of a, to indicate the localization and size of an $(s, p, \Psi)_{K,L}$-atom a, i.e. if $\operatorname{supp} a \subset dQ_{\nu m}$. If $L = -1$, then (iii) simply means that no moment conditions are required.

We define the relevant sequence spaces.

Definition 2.13 Let $0 < p, q \le \infty$ and $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$. Then

$$b_{p,q} = \left\{\lambda : \|\lambda | b_{p,q}\| = \left(\sum_{\nu=0}^{\infty}\left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p\right)^{q/p}\right)^{1/q} < \infty\right\}$$

(with the usual modification if $p = \infty$ and/or $q = \infty$).

The following theorem provides an atomic characterization.

Theorem 2.14 *Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and Ψ be a slowly varying function. Let $d > 1$, $K \in \mathbb{N}_0$ and $L + 1 \in \mathbb{N}_0$ with*

$$K \ge \big(1 + [s]\big)_+ \quad \textit{and} \quad L \ge \max\big(-1, [\sigma_p - s]\big)$$

be fixed, where $\sigma_p = n(\frac{1}{p} - 1)_+$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{(s,\Psi)}_{p,q}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}, \quad \textit{convergence being in } \mathcal{S}'\big(\mathbb{R}^n\big), \tag{2.8}$$

where $a_{\nu m}$ are 1_K-atoms ($\nu = 0$) or $(s, p, \Psi)_{K,L}$-atoms ($\nu \in \mathbb{N}$), according to Definition 2.11, and $\lambda \in b_{p,q}$. Furthermore

$$\inf \|\lambda | b_{p,q}\|, \tag{2.9}$$

where the infimum is taken over all admissible representations (2.8), is an equivalent quasi-norm in $B^{(s,\Psi)}_{p,q}(\mathbb{R}^n)$.

The previous theorem coincides with [25, Theorem 1.18(ii)] in case of Ψ being an admissible function. The general case is covered by [14, Theorem 4.4.3] and [4, Theorem 2.3.7(i)]. We refer as well to [15] and [32] for the classical situation.

The next result characterizes the embedding of $B^{(n/p,\Psi)}_{p,q}(\mathbb{R}^n)$ and $F^{(n/p,\Psi)}_{p,q}(\mathbb{R}^n)$ into $C_B(\mathbb{R}^n)$. The case of Ψ being an admissible function is covered by [8, Proposition 3.13]. We refer to [5, Corollary 3.10 & Remark 3.11] and to [7, Proposition 4.4] for a more general situation.

Theorem 2.15 *Let $0 < p, q \le \infty$ and Ψ be a slowly varying function.*

(i) *Then*

$$B^{(n/p,\Psi)}_{p,q}\big(\mathbb{R}^n\big) \hookrightarrow C_B\big(\mathbb{R}^n\big) \quad \textit{if, and only if,} \quad \big(\Psi\big(2^{-j}\big)^{-1}\big)_{j \in \mathbb{N}_0} \in \ell_{q'}.$$

(ii) *Assume $0 < p < \infty$. Then*

$$F^{(n/p,\Psi)}_{p,q}\big(\mathbb{R}^n\big) \hookrightarrow C_B\big(\mathbb{R}^n\big) \quad \textit{if, and only if,} \quad \big(\Psi\big(2^{-j}\big)^{-1}\big)_{j \in \mathbb{N}_0} \in \ell_{p'}.$$

For each $f \in C_B(\mathbb{R}^n)$, $\omega(f, \cdot)$ stands for the modulus of continuity of f and it is defined by

$$\omega(f, t) := \sup_{|h| \le t} \sup_{x \in \mathbb{R}^n} |\Delta_h f(x)| = \sup_{|h| \le t} \big\| \Delta_h f | L_\infty\big(\mathbb{R}^n\big)\big\|, \quad t > 0,$$

with $\Delta_h f(x) := f(x + h) - f(x)$, $x, h \in \mathbb{R}^n$.

Let $r \in (0, \infty]$ and let $\mathcal{L}_r$ be the class of all continuous functions $\lambda : (0, 1] \to (0, \infty)$ such that

$$\left(\int_0^1 \frac{1}{\lambda(t)^r} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} = \infty \tag{2.10}$$

and

$$\left(\int_0^1 \frac{t^r}{\lambda(t)^r}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} < \infty \tag{2.11}$$

(with the usual modification if $r = \infty$).

Definition 2.16 Let $0 < r \le \infty$ and $\mu \in \mathcal{L}_r$. The generalized Hölder space $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ consists of all functions $f \in C_B(\mathbb{R}^n)$ for which the quasi-norm

$$\|f|\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)\| := \|f|L_\infty(\mathbb{R}^n)\| + \left(\int_0^1 \left[\frac{\omega(f,t)}{\mu(t)}\right]^r \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}$$

is finite (with the usual modification if $r = \infty$).

Standard arguments show that the space $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ is complete, cf. [28, Theorem 3.1.4]. Conditions (2.10) and (2.11) are natural. In fact, if (2.10) does not hold, then $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ coindices with $C_B(\mathbb{R}^n)$. If (2.11) does not hold, then the space $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ contains only constant functions.

If $r = \infty$, we can assume without loss of generality in the definition of $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ that all the elements μ of $\mathcal{L}_r$ are continuous increasing functions on the interval (0, 1] such that $\lim_{t\to 0^+} \mu(t) = 0$ (cf. [17]).

The space $\Lambda^{\mu(\cdot)}_{\infty,\infty}(\mathbb{R}^n)$, cf. [27, Proposition 3.5], coincides with the space $C^{0,\mu(\cdot)}(\mathbb{R}^n)$ defined by

$$\|f|C^{0,\mu(\cdot)}(\mathbb{R}^n)\| := \sup_{x\in\mathbb{R}^n} |f(x)| + \sup_{x,y\in\mathbb{R}^n,\ 0<|x-y|\le 1} \frac{|f(x)-f(y)|}{\mu(|x-y|)} < \infty.$$

If $\mu(t) = t$, $t \in (0, 1]$, then $\Lambda^{\mu(\cdot)}_{\infty,\infty}(\mathbb{R}^n)$ coincides with the space $\mathrm{Lip}(\mathbb{R}^n)$ of the Lipschitz functions. If $\mu(t) = t^\alpha$, $\alpha \in (0, 1]$, then the space $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ coincides with the space $C^{0,\alpha,r}(\mathbb{R}^n)$ introduced in [1]. Furthermore, if $\mu(t) = t|\log t|^\beta$, $\beta > \frac{1}{r}$ (with $\beta \ge 0$ if $r = \infty$), the space $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ coincides with the space $\mathrm{Lip}^{(1,-\beta)}_{\infty,r}(\mathbb{R}^n)$ of generalized Lipschitz functions presented and studied in [10, 11, 19].

2.4 Hardy Inequalities

In the sequel, discrete weighted Hardy inequalities will be indispensable for our proofs. There is a vast amount of literature concerning this topic. We merely rely on results as can be found in [18, pp. 17–20], adapted to our situation. In this context we refer as well to [2, Theorem 1.5] and [30].

Let $0 < q, r \le \infty$ and $(b_n)_{n\in\mathbb{N}_0}$, $(d_n)_{n\in\mathbb{N}_0}$ be non-negative sequences. Consider the inequalities

$$\left(\sum_{j=0}^\infty \left(\sum_{k=0}^j a_k d_k\right)^r {b_j}^r\right)^{\frac{1}{r}} \lesssim \left(\sum_{n=0}^\infty a_n^q\right)^{\frac{1}{q}}$$

$$\text{for all non-negative sequences } (a_n)_{n\in\mathbb{N}_0} \tag{2.12}$$

and

$$\left(\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}a_k d_k\right)^r {b_j}^r\right)^{\frac{1}{r}} \lesssim \left(\sum_{n=0}^{\infty}a_n^q\right)^{\frac{1}{q}} \quad \text{for all non-negative sequences } (a_n)_{n\in\mathbb{N}_0} \tag{2.13}$$

(with the usual modification if $r=\infty$ or $q=\infty$).

Theorem 2.17

(i) *Let* $0<q\le r\le\infty$. *Then*, (2.12) *is satisfied if*, *and only if*,

$$\sup_{N\ge 0}\left(\sum_{j=N}^{\infty}{b_j}^r\right)^{\frac{1}{r}}\left(\sum_{k=0}^{N}{d_k}^{q'}\right)^{\frac{1}{q'}}<\infty \tag{2.14}$$

and, *furthermore*, (2.13) *is satisfied if*, *and only if*,

$$\sup_{N\ge 0}\left(\sum_{j=0}^{N}{b_j}^r\right)^{\frac{1}{r}}\left(\sum_{k=N}^{\infty}{d_k}^{q'}\right)^{\frac{1}{q'}}<\infty \tag{2.15}$$

(*with the usual modification if* $r=\infty$ *or* $q'=\infty$).

(ii) *Let* $0<r<q\le\infty$. *Then*, (2.12) *is satisfied if*, *and only if*,

$$\left\{\sum_{N=0}^{\infty}\left(\sum_{j=N}^{\infty}{b_j}^r\right)^{\frac{u}{q}} b_N^r\left(\sum_{k=0}^{N}{d_k}^{q'}\right)^{\frac{u}{q'}}\right\}^{\frac{1}{u}}<\infty \tag{2.16}$$

and, *furthermore*, (2.13) *is satisfied if*, *and only if*,

$$\left\{\sum_{N=0}^{\infty}\left(\sum_{j=0}^{N}{b_j}^r\right)^{\frac{u}{q}} b_N^r\left(\sum_{k=N}^{\infty}{d_k}^{q'}\right)^{\frac{u}{q'}}\right\}^{\frac{1}{u}}<\infty \tag{2.17}$$

(*with the usual modification if* $q'=\infty$), *where* $\frac{1}{u}:=\frac{1}{r}-\frac{1}{q}$.

3 Main Results

We start by providing extremal functions, which will play a key role for proving necessity in the main theorem below. For related assertions, but different, see [33, pp. 220–221], [6, Proposition 2.4] and [24, Proposition 3.1].

Proposition 3.1 *Let* $0<p,q\le\infty$ *and let* Ψ *be a slowly varying function. Furthermore*, *let* h *be a compactly supported* C^∞ *function on* $\mathbb{R}$ *defined by* $h(y)=e^{-\frac{1}{1-y^2}}$

for $|y|<1$ *with* $\int_{\mathbb{R}} h(y)dy=0$ *and* $h(y)\le 0$ *for* $|y|\ge 1$. *For each* $b=(b_j)_{j\in\mathbb{N}_0}\in \ell_q$, *let* f_b *be defined by*

$$f_b(x) := \sum_{j=0}^{\infty} b_j \Psi\big(2^{-j}\big)^{-1} \prod_{k=1}^{n} h\big(2^j x_k\big), \quad x=(x_k)_{k=1}^n \in \mathbb{R}^n. \tag{3.1}$$

(i) *Then* $f_b \in B^{(n/p,\Psi)}_{p,q}(\mathbb{R}^n)$ *and*

$$\big\| f_b \mid B^{(n/p,\Psi)}_{p,q}\big(\mathbb{R}^n\big)\big\| \le c_1 \|b|\ell_q\| \tag{3.2}$$

for some $c_1>0$ *independent of* b.

(ii) *If* $b_j\ge 0$, $j\in\mathbb{N}_0$, *then*

$$\frac{\omega(f_b,2^{-k})}{2^{-k}} \ge c_2 2^k \sum_{j=k}^{\infty} b_j \Psi\big(2^{-j}\big)^{-1}, \quad k\in\mathbb{N}_0, \tag{3.3}$$

and

$$\frac{\omega(f_b,2^{-k})}{2^{-k}} \ge c_3 \sum_{j=0}^{k} b_j 2^j \Psi\big(2^{-j}\big)^{-1}, \quad k\in\mathbb{N}_0, \tag{3.4}$$

for some $c_2, c_3>0$ *depending only on the function* h.

Proof Since the functions

$$a_j(x) := \Psi\big(2^{-j}\big)^{-1} \prod_{k=1}^{n} h\big(2^j x_k\big), \quad x=(x_k)_{k=1}^n\in\mathbb{R}^n,\ j\in\mathbb{N}_0,$$

are (up to constants, independently of j) 1_K-atoms ($j=0$) or $(n/p,p,\Psi)_{K,0}$-atoms ($j\in\mathbb{N}$), for some fixed $K\in\mathbb{N}$ with $K>n/p$, and $b\in\ell_q$, then (3.2) is an immediate consequence of the atomic decomposition theorem, cf. Theorem 2.14.

Let us now prove (ii). Let $k\in\mathbb{N}_0$ and let $\eta\in(0,1)$ be fixed. Then, putting temporarily $c=\prod_{k=2}^n h(0)$, we obtain

$$\begin{aligned}\frac{\omega(f_b,2^{-k})}{2^{-k}} &\ge 2^k\big(f_b(0)-f_b\big(-\eta 2^{-k},0,\ldots,0\big)\big)\\ &= 2^k \sum_{j=0}^{\infty} b_j\Psi\big(2^{-j}\big)^{-1}\big(h(0)-h\big(-\eta 2^{j-k}\big)\big)\cdot c\\ &\ge 2^k \sum_{j=k}^{\infty} b_j\Psi\big(2^{-j}\big)^{-1}\big(h(0)-h\big(-\eta 2^{j-k}\big)\big)\cdot c\\ &\ge c_2 2^k \sum_{j=k}^{\infty} b_j \Psi\big(2^{-j}\big)^{-1}. \end{aligned} \tag{3.5}$$

The second last estimate above holds true, since $h(0) - h(-\eta 2^{j-k}) > 0$ for $j < k$. The last inequality above follows from the fact that $h(0) - h(-\eta 2^{j-k}) \geq h(0) - h(-\eta) > 0$ for all $j \geq k$. This shows the estimate (3.3).

The proof of (3.4) is similar. We estimate

$$\begin{aligned}\frac{\omega(f_b, 2^{-k})}{2^{-k}} &\geq 2^k \sum_{j=0}^{\infty} b_j \Psi\big(2^{-j}\big)^{-1}\big(h(0) - h\big(-\eta 2^{j-k}\big)\big) \cdot c \\ &\geq 2^k \sum_{j=0}^{k} b_j \Psi\big(2^{-j}\big)^{-1}\big(h(0) - h\big(-\eta 2^{j-k}\big)\big) \cdot c \\ &= 2^k \sum_{j=0}^{k} b_j \Psi\big(2^{-j}\big)^{-1} \cdot \eta 2^{j-k} \cdot h'(\xi_{jk}) \cdot c \geq c_3 \sum_{j=0}^{k} b_j 2^j \Psi\big(2^{-j}\big)^{-1},\end{aligned} \tag{3.6}$$

for some $\xi_{jk} \in (-\eta 2^{j-k}, 0)$, observing that for $j \leq k$, $\xi_{jk} \in (-\eta, 0)$ and hence $h'(\xi_{jk}) \geq c_1 > 0$ for some c_1 which is independent of j and k. □

The following theorem characterizes optimal embeddings of Besov spaces with generalized smoothness into generalized Hölder spaces in the limiting case when $s = \frac{n}{p}$. In this context we also refer to [16, Theorem 4] and [17, Theorem 1.6, Corollary 1.7], where the authors obtained similar embedding results for Bessel-potential-type spaces in the limiting case. There, the technics were completely different from the ones considered here.

Theorem 3.2 *Let $0 < p \leq \infty$, $0 < q, r \leq \infty$, $\mu \in \mathcal{L}_r$, and let Ψ be a slowly varying function with*

$$\big(\Psi\big(2^{-j}\big)^{-1}\big)_{j \in \mathbb{N}_0} \in \ell_{q'}.$$

(i) *If $0 < q \leq r \leq \infty$, then*

$$B^{(\frac{n}{p}, \Psi)}_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n), \tag{3.7}$$

if, and only if,

$$\sup_{N \geq 0} \left(\sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} \left(\sum_{k=N}^{\infty} \Psi\big(2^{-k}\big)^{-q'}\right)^{\frac{1}{q'}} < \infty \tag{3.8}$$

(*with the usual modification if $r = \infty$ and/or $q' = \infty$*).

(ii) *If $0 < r < q \leq \infty$, then*

$$B^{(\frac{n}{p}, \Psi)}_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n), \tag{3.9}$$

if, and only if,

$$\left\{\sum_{N=0}^{\infty}\left(\sum_{j=0}^{N}\int_{2^{-(j+1)}}^{2^{-j}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)^{\frac{u}{q}}\left(\int_{2^{-(N+1)}}^{2^{-N}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)\right.$$
$$\left.\times\left(\sum_{k=N}^{\infty}\Psi(2^{-k})^{-q'}\right)^{\frac{u}{q'}}\right\}^{\frac{1}{u}}<\infty \tag{3.10}$$

and

$$\left\{\sum_{N=0}^{\infty}\left(\sum_{j=N}^{\infty}2^{-jr}\int_{2^{-(j+1)}}^{2^{-j}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)^{\frac{u}{q}}\cdot 2^{-Nr}\left(\int_{2^{-(N+1)}}^{2^{-N}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)\right.$$
$$\left.\times\left(\sum_{k=0}^{N}2^{kq'}\Psi(2^{-k})^{-q'}\right)^{\frac{u}{q'}}\right\}^{\frac{1}{u}}<\infty \tag{3.11}$$

(*with the usual modification if* $q'=\infty$), *where* $\frac{1}{u}:=\frac{1}{r}-\frac{1}{q}$.

Proof In the sequel we shall always assume that q and r are finite, since the limiting situations ($q=\infty$ and/or $r=\infty$) are proven in the same way with the obvious modifications.

Step 1: In order to prove sufficiency in (i), assume that (3.8) holds.

Let $f\in B^{(0,\Psi)}_{\infty,q}(\mathbb{R}^n)$ and let $a>0$. Then, by Theorem 2.15(i), we can make use of the following estimate which can be found in [31, 2.5.12 formulas (8), (9)], stating that for $|h|\le 2^{-j}$,

$$\|\Delta_h f|L_\infty(\mathbb{R}^n)\|\lesssim\sum_{k=0}^{j}2^{k-j}\left\|\left(\varphi_k^* f\right)_a|L_\infty(\mathbb{R}^n)\right\|$$
$$+\sum_{k=j+1}^{\infty}\left\|\left(\varphi_k^* f\right)_a|L_\infty(\mathbb{R}^n)\right\| \tag{3.12}$$

(the constant involved is independent of f). Using the fact that $\omega(f,\cdot)$ is monotonically increasing, together with (3.12), we have

$$\left(\int_0^1\left[\frac{\omega(f,t)}{\mu(t)}\right]^r\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}\sim\left(\sum_{j=0}^{\infty}\int_{2^{-(j+1)}}^{2^{-j}}\omega(f,t)^r\mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}$$
$$\lesssim\left(\sum_{j=0}^{\infty}\omega\left(f,2^{-j}\right)^r\underbrace{\int_{2^{-(j+1)}}^{2^{-j}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}}_{=:m_j}\right)^{\frac{1}{r}}$$

$$\lesssim \left(\sum_{j=0}^{\infty} m_j \left[\sum_{k=0}^{j} 2^{k-j} \left\| \left(\varphi_k^* f\right)_a | L_\infty \right\| + \sum_{k=j+1}^{\infty} \left\| \left(\varphi_k^* f\right)_a | L_\infty \right\| \right]^r \right)^{\frac{1}{r}}$$

$$\lesssim \underbrace{\left(\sum_{j=0}^{\infty} 2^{-jr} m_j \left[\sum_{k=0}^{j} 2^k \left\| \left(\varphi_k^* f\right)_a | L_\infty \right\| \right]^r \right)^{\frac{1}{r}}}_{=(I)} + \underbrace{\left(\sum_{j=0}^{\infty} m_j \left[\sum_{k=j}^{\infty} \left\| \left(\varphi_k^* f\right)_a | L_\infty \right\| \right]^r \right)^{\frac{1}{r}}}_{=(II)}. \tag{3.13}$$

Setting

$$b_j := 2^{-j} m_j^{\frac{1}{r}}, \quad a_k := \Psi\left(2^{-k}\right) \left\| \left(\varphi_k^* f\right)_a | L_\infty \right\|, \quad \text{and} \quad d_k := 2^k \Psi\left(2^{-k}\right)^{-1}, \tag{3.14}$$

an application of Theorem 2.17(i) to the first term of (3.13), yields

$$(I) \lesssim \left(\sum_{n=0}^{\infty} \Psi\left(2^{-n}\right)^q \left\| \left(\varphi_n^* f\right)_a | L_\infty \right\|^q \right)^{\frac{1}{q}} \sim \left\| f | B_{\infty,q}^{(0,\Psi)} \right\| \quad \text{for all } f \in B_{\infty,q}^{(0,\Psi)}(\mathbb{R}^n). \tag{3.15}$$

This can be seen as follows. Condition (3.8) gives

$$\int_{2^{-(N+1)}}^{2^{-N}} \mu(t)^{-r} \frac{\mathrm{d}t}{t} \lesssim \Psi\left(2^{-N}\right)^r \quad \text{for all } N,$$

which together with (2.3) and (2.2) yields

$$\sup_{N \ge 0} \left(\sum_{j=N}^{\infty} 2^{-jr} m_j \right)^{\frac{1}{r}} \left(\sum_{k=0}^{N} 2^{kq'} \Psi\left(2^{-k}\right)^{-q'} \right)^{\frac{1}{q'}}$$

$$\sim \sup_{N \ge 0} \left(\sum_{j=N}^{\infty} 2^{-jr} m_j \right)^{\frac{1}{r}} 2^N \Psi\left(2^{-N}\right)^{-1}$$

$$\lesssim \sup_{N \ge 0} \left(\sum_{j=N}^{\infty} 2^{-jr} \Psi\left(2^{-j}\right)^r \right)^{\frac{1}{r}} 2^N \Psi\left(2^{-N}\right)^{-1}$$

$$\lesssim 2^{-N} \Psi\left(2^{-N}\right) 2^N \Psi\left(2^{-N}\right)^{-1} \lesssim 1 < \infty \tag{3.16}$$

and (2.14) is satisfied. For the second term of (3.13), we put

$$b_j := m_j^{\frac{1}{r}}, \quad a_k := \Psi\big(2^{-k}\big)\big\|\big(\varphi_k^* f\big)_a|L_\infty\big\|, \quad \text{and} \quad d_k := \Psi\big(2^{-k}\big)^{-1}. \tag{3.17}$$

An application of Theorem 2.17(i) gives

$$(II) \lesssim \left(\sum_{n=0}^{\infty} \Psi\big(2^{-n}\big)^q \big\|\big(\varphi_n^* f\big)_a|L_\infty\big\|^q\right)^{\frac{1}{q}} \sim \big\|f|B_{\infty,q}^{(0,\Psi)}\big\| \quad \text{for all } f \in B_{\infty,q}^{(0,\Psi)}(\mathbb{R}^n), \tag{3.18}$$

since, by (3.8),

$$\begin{aligned} &\sup_{N\geq 0}\left(\sum_{j=0}^{N} m_j\right)^{\frac{1}{r}} \left(\sum_{k=N}^{\infty} \Psi\big(2^{-k}\big)^{-q'}\right)^{\frac{1}{q'}} \\ &\quad = \sup_{N\geq 0}\left(\sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r}\,\frac{dt}{t}\right)^{\frac{1}{r}} \left(\sum_{k=N}^{\infty} \Psi\big(2^{-k}\big)^{-q'}\right)^{\frac{1}{q'}} < \infty \end{aligned} \tag{3.19}$$

and (2.15) is satisfied. Now, (3.13), together with (3.15), (3.18) and Theorem 2.15(i), yields

$$B_{\infty,q}^{(0,\Psi)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}(\mathbb{R}^n).$$

Since, by Proposition 2.9,

$$B_{p,q}^{(n/p,\Psi)}(\mathbb{R}^n) \hookrightarrow B_{\infty,q}^{(0,\Psi)}(\mathbb{R}^n),$$

we have the desired embedding

$$B_{p,q}^{(n/p,\Psi)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}(\mathbb{R}^n).$$

Step 2: Concerning sufficiency in (ii) again we have (3.13). Let $\frac{1}{u} := \frac{1}{r} - \frac{1}{q}$. Applying (2.16), using (3.14) we obtain for the first integral (I) the estimate (3.15), since

$$\begin{aligned} &\left\{\sum_{N=0}^{\infty}\left(\sum_{j=N}^{\infty} 2^{-jr} \int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r}\,\frac{dt}{t}\right)^{\frac{u}{q}} 2^{-Nr}\left(\int_{2^{-(N+1)}}^{2^{-N}} \mu(t)^{-r}\,\frac{dt}{t}\right)\right. \\ &\quad \left.\times\left(\sum_{k=0}^{N} 2^{kq'}\Psi(2^{-k})^{-q'}\right)^{\frac{u}{q'}}\right\}^{\frac{1}{u}} < \infty \end{aligned}$$

is bounded by (3.11). For the second integral (II) in (3.13), an application of (2.17) yields (3.18), since inserting (3.17) we obtain

$$\Bigg\{\sum_{N=0}^{\infty}\bigg(\sum_{j=0}^{N}\int_{2^{-(j+1)}}^{2^{-j}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\bigg)^{\frac{u}{q}}\bigg(\int_{2^{-(N+1)}}^{2^{-N}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\bigg)$$
$$\times\bigg(\sum_{k=N}^{\infty}\Psi\big(2^{-k}\big)^{-q'}\bigg)^{\frac{u}{q'}}\Bigg\}^{\frac{1}{u}}<\infty,$$

which is bounded by (3.10).

Step 3: Concerning necessity in (i) and (ii), assume we have the embedding

$$B^{(\frac{n}{p},\Psi)}_{p,q}(\mathbb{R}^n)\hookrightarrow\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n),\quad 0<q,r\le\infty,$$

which means that

$$\bigg(\int_0^1\bigg(\frac{\omega(f,t)}{\mu(t)}\bigg)^r\frac{\mathrm{d}t}{t}\bigg)^{\frac{1}{r}}\lesssim\|f|B^{(\frac{n}{p},\Psi)}_{p,q}\|\quad\text{for all } f\in B^{(\frac{n}{p},\Psi)}_{p,q}(\mathbb{R}^n).$$

In particular, for each non-negative sequence $(a_n)_{n\in\mathbb{N}_0}$, using the function f_a constructed in (3.1), Propostion 3.1, we have

$$\begin{aligned}
\|a|\ell_q\|&\gtrsim\bigg(\int_0^1\bigg(\frac{\omega(f_a,t)}{\mu(t)}\bigg)^r\frac{\mathrm{d}t}{t}\bigg)^{\frac{1}{r}}\\
&\sim\bigg(\sum_{k=0}^{\infty}\int_{2^{-(k+1)}}^{2^{-k}}\bigg(\frac{\omega(f_a,t)}{t}\bigg)^r\frac{t^{r-1}\mathrm{d}t}{\mu(t)^r}\bigg)^{\frac{1}{r}}\\
&\gtrsim\bigg(\sum_{k=0}^{\infty}\bigg(\frac{\omega(f_a,2^{-k})}{2^{-k}}\bigg)^r\int_{2^{-(k+1)}}^{2^{-k}}\mu(t)^{-r}t^{r-1}\mathrm{d}t\bigg)^{\frac{1}{r}}\\
&\gtrsim\bigg(\sum_{k=0}^{\infty}\bigg(2^k\sum_{j=k}^{\infty}a_j\Psi\big(2^{-j}\big)^{-1}\bigg)^r\int_{2^{-(k+1)}}^{2^{-k}}\mu(t)^{-r}t^{r-1}\mathrm{d}t\bigg)^{\frac{1}{r}}\\
&\sim\bigg(\sum_{k=0}^{\infty}\bigg(\sum_{j=k}^{\infty}a_j\Psi\big(2^{-j}\big)^{-1}\bigg)^r\int_{2^{-(k+1)}}^{2^{-k}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\bigg)^{\frac{1}{r}},
\end{aligned}\tag{3.20}$$

where we used the fact that $\frac{\omega(f,t)}{t}$ is equivalent to a monotonically decreasing function and the estimate (3.3). Putting

$$d_j=\Psi\big(2^{-j}\big)^{-1}\quad\text{and}\quad b_k=\bigg(\int_{2^{-(k+1)}}^{2^{-k}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\bigg)^{\frac{1}{r}},\tag{3.21}$$

from (3.20) we obtain

$$\|a|\ell_q\| \gtrsim \left(\sum_{k=0}^{\infty}\left(\sum_{j=k}^{\infty} a_j d_j\right)^r {b_k}^r\right)^{\frac{1}{r}} \quad \text{for all non-negative sequences } (a_n)_{n\in\mathbb{N}_0}, \tag{3.22}$$

which is the Hardy-type inequality (2.13). Now the necessary conditions (3.8) and (3.10) follow from Theorem 2.17. If we apply the estimate (3.4) instead of (3.3) in (3.20), we obtain

$$\|a|\ell_q\| \gtrsim \left(\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j \Psi\left(2^{-j}\right)^{-1} 2^j\right)^r 2^{-kr}\int_{2^{-(k+1)}}^{2^{-k}} \mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} \tag{3.23}$$

for all non-negative sequences $(a_n)_{n\in\mathbb{N}_0}$. Now setting

$$d_j = \Psi\left(2^{-j}\right)^{-1} 2^j \quad \text{and} \quad b_k = 2^{-k}\left(\int_{2^{-(k+1)}}^{2^{-k}} \mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}, \tag{3.24}$$

we obtain

$$\|a|\ell_q\| \gtrsim \left(\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j d_j\right)^r {b_k}^r\right)^{\frac{1}{r}}$$

$$\text{for all non-negative sequences } (a_n)_{n\in\mathbb{N}_0}, \tag{3.25}$$

which is the Hardy-type inequality (2.12). Theorem 2.17 now yields (3.11). This finally completes the proof. □

In terms of optimal weights we have the following result.

Corollary 3.3 *Let* $1 < q \le \infty$, $0 < p, r \le \infty$, $\mu \in \mathcal{L}_r$, *and let* Ψ *be a slowly varying function with*

$$\left(\Psi\left(2^{-j}\right)^{-1}\right)_{j\in\mathbb{N}_0} \in \ell_{q'}.$$

Furthermore, let $\lambda_{qr} \in \mathcal{L}_r$ *be defined by*

$$\lambda_{qr}(t) := \Psi(t)^{\frac{q'}{r}}\left(\int_0^t \Psi(s)^{-q'}\frac{\mathrm{d}s}{s}\right)^{\frac{1}{q'}+\frac{1}{r}}, \quad t \in (0,1]. \tag{3.26}$$

We consider the embedding

$$B_{p,q}^{(\frac{n}{p},\Psi)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}(\mathbb{R}^n). \tag{3.27}$$

(i) *If* $1 < q \le r \le \infty$, *then* (3.27) *holds if, and only if,*

$$\sup_{N\ge 0} \frac{(\sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r} \frac{dt}{t})^{\frac{1}{r}}}{(\sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \lambda_{qr}(t)^{-r} \frac{dt}{t})^{\frac{1}{r}}} < \infty \tag{3.28}$$

(*with the usual modification if* $r = \infty$).

(ii) *If* $0 < r < q \le \infty$ *and* $q > 1$, *then* (3.27) *holds if, and only if,*

$$\left\{ \sum_{N=0}^{\infty} \frac{(\sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r} \frac{dt}{t})^{u/q}}{(\sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \lambda_{qr}(t)^{-r} \frac{dt}{t})^{u/r}} \int_{2^{-(N+1)}}^{2^{-N}} \mu(s)^{-r} \frac{ds}{s} \right\}^{\frac{1}{u}} < \infty \tag{3.29}$$

(*with the usual modification if* $q = \infty$), *where* $\frac{1}{u} := \frac{1}{r} - \frac{1}{q}$.

(iii) *Let* $r \in [q, \infty]$. *Among the embeddings in* (3.27), *that one with* $\mu = \lambda_{qr}$, *is sharp with respect to the parameter* μ.

(iv) *Among the embeddings in* (3.27), *that one with* $\mu = \lambda_{qq}$ *and* $r = q$, *i.e.*,

$$B_{p,q}^{(\frac{n}{p},\Psi)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,q}^{\lambda_{qq}(\cdot)}(\mathbb{R}^n), \tag{3.30}$$

is optimal.

Proof Concerning (i) Theorem 3.2 shows that (3.27) holds if, and only if,

$$\sup_{N\ge 0} \left(\sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r} \frac{dt}{t} \right)^{\frac{1}{r}} \left(\sum_{k=N}^{\infty} \Psi\left(2^{-k}\right)^{-q'} \right)^{\frac{1}{q'}} < \infty,$$

which is equivalent to

$$\sup_{\varkappa\in(0,1/2)} \left(\int_{\varkappa}^{1} \mu(t)^{-r} \frac{dt}{t} \right)^{\frac{1}{r}} \left(\int_{0}^{2\varkappa} \Psi(s)^{-q'} \frac{ds}{s} \right)^{\frac{1}{q'}} < \infty. \tag{3.31}$$

Since

$$\begin{aligned} \left(\int_{\varkappa}^{1} \lambda_{qr}(t)^{-r} \frac{dt}{t} \right)^{-\frac{1}{r}} &= \left(\int_{\varkappa}^{1} \Psi(t)^{-q'} \left(\int_{0}^{t} \Psi(s)^{-q'} \frac{ds}{s} \right)^{-\frac{r}{q'}-1} \frac{dt}{t} \right)^{-\frac{1}{r}} \\ &\sim \left(\int_{0}^{\varkappa} \Psi(t)^{-q'} \frac{dt}{t} \right)^{\frac{1}{q'}} \sim \left(\int_{0}^{2\varkappa} \Psi(t)^{-q'} \frac{dt}{t} \right)^{\frac{1}{q'}} \\ \text{for all } \varkappa \in \left(0, \frac{1}{2}\right], \end{aligned} \tag{3.32}$$

and as singularities of functions in question are only at 0, this means that (3.31) is equivalent to (3.28).

Turning towards (ii) the same argument used above shows that (3.10) is equivalent to (3.29). Now, necessity follows from Theorem 3.2(ii). Let $\frac{1}{u} := \frac{1}{r} - \frac{1}{q}$. As for sufficiency, we observe that

$$\begin{aligned} A_1 &:= \left\{ \int_0^{1/2} \left(\int_\varkappa^1 \frac{\mu(t)^{-r}}{t} \mathrm{d}t \right)^{\frac{u}{q}} \cdot \frac{\mu(\varkappa)^{-r}}{\varkappa} \cdot \left(\int_\varkappa^1 \frac{\lambda_{qq}(t)^{-q}}{t} \mathrm{d}t \right)^{-\frac{u}{q}} \mathrm{d}\varkappa \right\}^{\frac{1}{u}} \\ &\lesssim \left\{ \int_0^1 \left(\int_\varkappa^1 \frac{\mu(t)^{-r}}{t} \mathrm{d}t \right)^{\frac{u}{q}} \cdot \frac{\mu(\varkappa)^{-r}}{\varkappa} \cdot \left(\int_0^\varkappa \Psi(t)^{-q'} \frac{\mathrm{d}t}{t} \right)^{\frac{u}{q'}} \mathrm{d}\varkappa \right\}^{\frac{1}{u}} \\ &\lesssim \left\{ \sum_{N=0}^\infty \left(\int_{2^{-(N+1)}}^1 \mu(t)^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{u}{q}} \left(\int_0^{2^{-N}} \Psi(t)^{-q'} \frac{\mathrm{d}t}{t} \right)^{\frac{u}{q'}} \right. \\ &\qquad \left. \times \int_{2^{-(N+1)}}^{2^{-N}} \mu(\varkappa)^{-r} \frac{\mathrm{d}\varkappa}{\varkappa} \right\}^{\frac{1}{u}} \end{aligned} \tag{3.33}$$

is bounded by (3.10). But now, since $\omega(f, \cdot)$ is increasing, [21, Proposition 2.1(ii)] implies

$$\Lambda_{\infty,q}^{\lambda_{qq}(\cdot)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}(\mathbb{R}^n).$$

This and (3.27) (with $\mu = \lambda_{qq}$ and $r = q$, which follows from part (i)), yield

$$B_{p,q}^{(\frac{n}{p},\Psi)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,q}^{\lambda_{qq}(\cdot)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}(\mathbb{R}^n), \quad 0 < r < q \le \infty, \ q > 1. \tag{3.34}$$

This completes the proof of (ii).

Let us now prove (iii). We need to show that the target space $\Lambda_{\infty,r}^{\mu(\cdot)}(\mathbb{R}^n)$ in (3.27) and the space $\Lambda_{\infty,r}^{\lambda_{qr}(\cdot)}(\mathbb{R}^n)$ (that is, the target space in (3.27) with $\mu = \lambda_{qr}$) satisfy

$$\Lambda_{\infty,r}^{\lambda_{qr}(\cdot)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}(\mathbb{R}^n). \tag{3.35}$$

Indeed, since $\omega(f, \cdot)$ is increasing, this last embedding holds if

$$\sup_{\varkappa \in (0,\frac{1}{2})} \frac{(\int_\varkappa^1 \mu(t)^{-r} \frac{\mathrm{d}t}{t})^{\frac{1}{r}}}{(\int_\varkappa^1 \lambda_{qr}(t)^{-r} \frac{\mathrm{d}t}{t})^{\frac{1}{r}}} < \infty \tag{3.36}$$

(cf. [21, Proposition 2.1(i)], see also [17, Theorem 3.6(i)]), which is equivalent to (3.28). The proof of (iii) is complete.

We turn our attention towards (iv). We need to show that the target space $\Lambda_{\infty,r}^{\mu(\cdot)}(\mathbb{R}^n)$ in (3.27) and the space $\Lambda_{\infty,q}^{\lambda_{qq}(\cdot)}(\mathbb{R}^n)$ (that is, the target space in (3.27) with $\mu = \lambda_{qq}$ and $r = q$) satisfy

$$\Lambda_{\infty,q}^{\lambda_{qq}(\cdot)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}(\mathbb{R}^n). \tag{3.37}$$

Since $\omega(f,\cdot)$ is increasing, this last embedding holds, for $q \le r$, if

$$\sup_{\varkappa\in(0,\frac{1}{2})} \frac{(\int_\varkappa^1 \mu(t)^{-r}\frac{dt}{t})^{\frac{1}{r}}}{(\int_\varkappa^1 \lambda_{qq}(t)^{-q}\frac{dt}{t})^{\frac{1}{q}}} \approx \sup_{\varkappa\in(0,\frac{1}{2})} \frac{(\int_\varkappa^1 \mu(t)^{-r}\frac{dt}{t})^{\frac{1}{r}}}{(\int_\varkappa^1 \lambda_{qr}(t)^{-r}\frac{dt}{t})^{\frac{1}{r}}} < \infty \tag{3.38}$$

(cf. [21, Proposition 2.1(i)], see also [17, Theorem 3.6(i)]), which is equivalent to (3.28). In the case $r < q$ we obtained (3.34) when proving (ii), which gives the desired embedding. □

Remark 3.4

(i) Theorem 3.2 could be improved as follows. If we had

$$\sum_{j=N}^{\infty} 2^{(N-j)r} \int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r}\frac{dt}{t} \lesssim \sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r}\frac{dt}{t}, \tag{3.39}$$

for all $\mu \in \mathcal{L}_r$, $N \in \mathbb{N}$, then (3.10) implies (3.11) and therefore (3.11) could be omitted. In particular, (3.39) seems to be natural since it holds true for functions

$$\mu(t) = t^\alpha, \quad \alpha > 0 \quad \text{and} \quad \mu(t) = \lambda_{qr}(t),$$

defined in (3.26) with $1 < q \le \infty$ and $0 < r \le \infty$.

(ii) Note that condition (3.8) is equivalent to the following integral version,

$$\sup_{\varkappa\in(0,1)} \left(\int_\varkappa^1 \mu(t)^{-r}\frac{dt}{t}\right)^{\frac{1}{r}} \left(\int_0^\varkappa \Psi(s)^{-q'}\frac{ds}{s}\right)^{\frac{1}{q'}} < \infty \tag{3.40}$$

(with the usual modification if $r = \infty$ and/or $q' = \infty$).

(iii) If $1 < q \le \infty$, using the terminology of [24], the authors obtained in [24, Theorem 3.4] that $(\frac{\lambda_{q\infty}(t)}{t}, \infty)$ is the continuity envelope of $B_{p,q}^{(\frac{n}{p},\Psi)}(\mathbb{R}^n)$, which means in this situation that

$$B_{p,q}^{(\frac{n}{p},\Psi)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{\lambda_{q\infty}(\cdot)}(\mathbb{R}^n)$$

only holds when $r = \infty$. Note that this also follows from Corollary 3.3, because condition (3.28) is not satisfied when $\mu = \lambda_{q\infty}$ and $r < \infty$ (this follows by applying l'Hôpital rule to the quotient in (3.38) and by Proposition 2.3(iii)), but it is satisfied with $\mu = \lambda_{q\infty}$ and $r = \infty$. Moreover, from Corollary 3.3(iv), (3.37) and (3.38),

$$B_{p,q}^{(\frac{n}{p},\Psi)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,q}^{\lambda_{qq}(\cdot)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,\infty}^{\lambda_{q\infty}(\cdot)}(\mathbb{R}^n).$$

Therefore, in this limiting case, we have an instance of the phenomenon where the continuity envelope does not yield the optimal embedding, since Theorem 3.2 provides an even better result. A similar situation occurs for Bessel-potential-type spaces in the limiting case, cf. [16, Theorem 4, Remark 5] and [17, Theorem 1.6, Corollary 1.7].

(iv) If $0 < q \le 1$, which is not considered in Corollary 3.3, similar results can be obtained from Theorem 3.2 provided we impose, for instance, that the derivative of Ψ is negative on the open interval (0, 1) and $\lim_{t\to 0^+} \Psi(t) = \infty$. We refer to the end of the introduction for an example. Under this circuntances, the continuous envelope of $B^{(\frac{n}{p},\Psi)}_{p,q}(\mathbb{R}^n)$ was obtained in [24, Theorem 3.5]. Furthermore, if $\Psi \equiv 1$, condition (3.8) implies the violation of condition (2.10). Note that if (2.10) is not satisfied, then $\Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$.

In terms of the Triebel-Lizorkin spaces our results read as follows.

Corollary 3.5 *Let $0 < p < \infty$, $0 < q, r \le \infty$, $\mu \in \mathcal{L}_r$, and let Ψ be a slowly varying function with*

$$\left(\Psi(2^{-j})^{-1}\right)_{j\in\mathbb{N}_0} \in \ell_{p'}.$$

(i) *If $0 < p \le r < \infty$ and $p < r$ if $r = \infty$, then*

$$F^{(\frac{n}{p},\Psi)}_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n), \tag{3.41}$$

if, and only if,

$$\sup_{N\ge 0}\left(\sum_{j=0}^{N}\int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}\left(\sum_{k=N}^{\infty}\Psi\left(2^{-k}\right)^{-p'}\right)^{\frac{1}{p'}} < \infty \tag{3.42}$$

(with the usual modification if $r = \infty$ and/or $p' = \infty$).

(ii) *If $0 < r < p < \infty$, then*

$$F^{(\frac{n}{p},\Psi)}_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n), \tag{3.43}$$

if, and only if,

$$\left\{\sum_{N=0}^{\infty}\left(\sum_{j=0}^{N}\int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)^{\frac{u}{p}}\left(\int_{2^{-(N+1)}}^{2^{-N}} \mu(t)^{-r}\frac{\mathrm{d}t}{t}\right) \times\left(\sum_{k=N}^{\infty}\Psi(2^{-k})^{-p'}\right)^{\frac{u}{p'}}\right\}^{\frac{1}{u}} < \infty \tag{3.44}$$

and

$$\left\{\sum_{N=0}^{\infty}\left(\sum_{j=N}^{\infty}2^{-jr}\int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)^{\frac{u}{p}} 2^{-Nr}\left(\int_{2^{-(N+1)}}^{2^{-N}} \mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)\right.$$

$$\times \left(\sum_{k=0}^{N} 2^{kp'} \Psi(2^{-k})^{-p'} \right)^{\frac{u}{p'}} \Bigg\}^{\frac{1}{u}} < \infty \tag{3.45}$$

(*with the usual modification if* $p' = \infty$), *where* $\frac{1}{u} := \frac{1}{r} - \frac{1}{p}$.

Proof Using Proposition 2.9 together with Theorem 3.2 we have

$$B_{p_2,p}^{(\frac{n}{p_2},\Psi)}(\mathbb{R}^n) \hookrightarrow F_{p,q}^{(\frac{n}{p},\Psi)}(\mathbb{R}^n) \hookrightarrow B_{p_1,p}^{(\frac{n}{p_1},\Psi)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}(\mathbb{R}^n),$$

yielding the desired result. □

Remark 3.6 In particular, it turns out that for the F-spaces our results are independent of the parameter q.

Corollary 3.3 can now be reformulated as follows.

Corollary 3.7 *Let* $1 < p < \infty$, $0 < r, q \le \infty$, $\mu \in \mathcal{L}_r$, *and let* Ψ *be a slowly varying function with*

$$\left(\Psi(2^{-j})^{-1}\right)_{j\in\mathbb{N}_0} \in \ell_{p'}.$$

Furthermore, let $\lambda_{pr} \in \mathcal{L}_r$ *be defined by*

$$\lambda_{pr}(t) := \Psi(t)^{\frac{p'}{r}} \left(\int_0^t \Psi(s)^{-p'} \frac{\mathrm{d}s}{s} \right)^{\frac{1}{p'}+\frac{1}{r}}, \quad t \in (0,1]. \tag{3.46}$$

We consider the embedding

$$F_{p,q}^{(\frac{n}{p},\Psi)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}(\mathbb{R}^n). \tag{3.47}$$

(i) *If* $1 < p \le r < \infty$ *and* $1 < p < r$ *if* $r = \infty$, *then* (3.47) *holds if, and only if,*

$$\sup_{N\ge 0} \frac{(\sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r} \frac{\mathrm{d}t}{t})^{\frac{1}{r}}}{(\sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \lambda_{pr}(t)^{-r} \frac{\mathrm{d}t}{t})^{\frac{1}{r}}} < \infty \tag{3.48}$$

(*with the usual modification if* $r = \infty$).

(ii) *If* $0 < r < p < \infty$ *and* $p > 1$, *then* (3.47) *holds if, and only if,*

$$\left\{ \sum_{N=0}^{\infty} \frac{(\sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r} \frac{\mathrm{d}t}{t})^{u/p}}{(\sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \lambda_{pr}(t)^{-r} \frac{\mathrm{d}t}{t})^{u/r}} \int_{2^{-(N+1)}}^{2^{-N}} \mu(s)^{-r} \frac{\mathrm{d}s}{s} \right\}^{\frac{1}{u}} < \infty, \tag{3.49}$$

where $\frac{1}{u} := \frac{1}{r} - \frac{1}{p}$.

(iii) *Let* $r \in [p, \infty]$. *Among the embeddings in* (3.47), *that one with* $\mu = \lambda_{pr}$, *is sharp with respect to the parameter* μ.

(iv) *Among the embeddings in* (3.47), *that one with* $\mu = \lambda_{pp}$ *and* $r = p$, *i.e.*,

$$F_{p,q}^{(\frac{n}{p},\Psi)}(\mathbb{R}^n) \hookrightarrow \Lambda_{\infty,p}^{\lambda_{pp}(\cdot)}(\mathbb{R}^n), \tag{3.50}$$

is optimal.

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