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Adaptive Decomposition by Weighted Inner Functions: A Generalization of Fourier Series

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Abstract In recent study adaptive decomposition of functions into basic functions of analytic instantaneous frequencies has been sought. Fourier series is a particular case of such decomposition. Adaptivity addresses certain optimal property of the decomposition. The present paper presents a fast decomposition of functions in the $\mathcal{L}^2(\partial\mathbb{D})$ spaces into a series of inner and weighted inner functions of increasing frequencies.

Keywords Fourier series · Inner and outer functions · Hardy space · The Nevanlinna factorization theorem · Blaschke product · Analytic signal · Instantaneous frequency and amplitude · Mono-components · Adaptive decomposition of functions

Mathematics Subject Classification (2000) 42A50 · 32A30 · 32A35 · 46J15

1 Introduction

In signal analysis instantaneous frequency of a given real-valued signal (function) $s(t), t \in \mathbb{R}$, is defined to be the function $\theta'(t)$, when it is defined, where the function

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$\theta(t)$ is defined through the *analytic signal associated with* s, viz. $s(t) + iHs(t) = \rho(t)e^{i\theta(t)}$, where Hs is the *Hilbert transform* of s. From the physics view of point, if $\theta'(t)$ stands as a qualified instantaneous frequency function, then necessarily it should satisfy $\theta'(t) \geq 0$, a.e. (or, alternatively, $\theta'(t) \leq 0$) (see [5, 10]).

A real-valued signal $s(t)$ has many phase-amplitude representations of the form $s(t) = \rho(t)\cos\theta(t)$, $\rho \geq 0$. If a pair of ρ and θ in such a representation is obtained from the associated analytic function $s + iHs$, then we say that $s(t) = \rho(t)\cos\theta(t)$ is the *analytic phase-amplitude representation* of $s(t)$. A characteristic property of the pair (ρ, θ) from an analytic phase-amplitude representation is

$$H(\rho\cos\theta) = \rho\sin\theta, \tag{1.1}$$

or, equivalently (by using the relation $H^2 = -I$, where I denotes the Identity operator),

$$H(\rho e^{i\theta}) = -i\rho e^{i\theta}, \tag{1.2}$$

when Hs is defined. Here we have in mind two types of signals in, respectively, $\mathcal{L}^p(\mathbb{R})$ and $\mathcal{L}^p(\partial\mathbb{D})$, $1 \leq p \leq \infty$, where $\mathbb{D}$ denotes the unit disc in the complex plane $\mathbb{C}$, and $\partial\mathbb{D}$ its boundary. Note that if $s \in \mathcal{L}^\infty(\mathbb{R})$, then a normalization is needed in order to have the exact relation (1.1) and (1.2). Or, otherwise, they hold modulo constants [6]. As a basic property of Hilbert transform we know that if $s(t) = \rho(t)\cos\theta(t)$ is the analytic phase-amplitude representation of s, then the associated analytic signal $s + iHs = \rho e^{i\theta}$ is of only non-negative Fourier spectrum. Fourier spectrum and θ' have indirect relations. Examples show that signals with positive Fourier spectrum can allow $\theta'(t) < 0$ in a set of positive Lebesgue measure. In such a case we say that $s(t)$ does not have (analytic) instantaneous frequency. In the present work our stand is that not all analytic signals have well defined (analytic) instantaneous frequency. Instead, we seek a decomposition into the form

$$s(t) = \sum_{k=1}^{N} \rho_k(t)\cos\theta_k(t) + r_N(t), \tag{1.3}$$

where for each k, the basic signal $\rho_k \cos\theta_k$ has a well defined instantaneous frequency (function), that is $\theta_k'(t) \geq 0$, a.e., and thus a well defined instantaneous amplitude $\rho(t)$, too. One can formulate the counterpart notion on finite intervals that is the case equivalent to the unit circle $\partial\mathbb{D}$, where the Hilbert transformation on the line is replaced by the *circular Hilbert transform* on the circle, viz. the principal value singular integral

$$HF(e^{it}) = \frac{1}{2\pi} p.v. \int_0^{2\pi} \cot\left(\frac{t-u}{2}\right) F(e^{iu})\,du. \tag{1.4}$$

Note that Fourier series expansion is a particular case of such decomposition. We formulate what we want into the following definition [13].

Definition 1.1 (Mono-component) Let $s(t) = \rho(t)\cos\theta(t)$ (or $s(t) = \rho(t)e^{i\theta(t)}$) be the analytic phase-amplitude representation of $s(t)$, that is

$$H(\rho\cos\theta) = \rho\sin\theta \quad (\text{or } H(\rho e^{i\theta}) = -i\rho e^{i\theta}),$$

where $\rho \geq 0$. If, moreover, there holds $\theta' \geq 0$, then s is said to be a real (or a complex) mono-component on the line. Using the circular Hilbert transformation, still denoted H, to replace the Hilbert transformation one defines mono-components on the unit circle.

For functions in Hardy spaces the decomposition is of the form

$$s(t) + iHs(t) = \sum_{k=1}^{N} \rho_k(t)e^{i\theta_k(t)} + R_N(t) \tag{1.5}$$

where s is real-valued and $s + iHs$ is the boundary value of a function in the Hardy space, and for each k, $\rho_k(t)e^{i\theta_k(t)}$ is a complex mono-component (see Corollary 2.6).

The totality of all the mono-components in each context, $\mathbb{R}$ or $\partial\mathbb{D}$, is denoted by $\mathcal{MC}$. The notion of mono-components in relation to the question of adaptive decomposition is proposed in [13]. There has been a series of work for finding various types of mono-components, including [10, 12, 13, 15, 16, 18, 20–22].

The set $\mathcal{MC}$ is not a basis, nor exists there orthogonality between its members. So far one finds that it is a rather large set including Blaschke products of finite and infinite zeros and singular inner functions [14], and their weighted forms [15] (called *weighted unimodular forms* below) and p-starlike functions [13], etc. The theory of mono-components has roots in complex Hardy spaces and conformal mappings. Below we, in particular, note a stream to find weight forms of unimodular mono-components that motivated recent study on the Bedrosian identity.

By unimodular mono-components we refer to the mono-components with $\rho \equiv 1$; or, equivalently, all the functions $s(t) = \cos\theta(t)$ for which

$$H\cos\theta = \sin\theta \quad \text{and} \quad \theta' \geq 0. \tag{1.6}$$

Earlier observations along this line are restricted to the finite Blaschke products case ([10] and [12]). Being aware of those basic unimodular mono-components, based on the engineers' experience that Fourier frequencies of the amplitude part should be lower than those of the phase signal part, one naturally seeks ways of constructing new and non-unimodular mono-components by using the idea of the Bedrosian identity [3]. In other words, having a function θ satisfying (1.6), one searches for $\rho \geq 0$ such that

$$H(\rho\cos\theta) = \rho H(\cos\theta). \tag{1.7}$$

Should such ρ exists, then we have (1.1), as well as $\theta'(t) \geq 0$ a.e. [15–17, 20–22] and [19]. A recent result on inner functions [14] asserts that the relation (1.6), as a matter of fact, implies $\theta'(t) \geq 0$ (see Theorem 2.1 below). So, the condition (1.1) alone characterizes the class of unimodular mono-components.

The proposed decomposition (1.3) is a generalization of the Fourier expansions in the series and the integral forms in, respectively, the straight line and the unit circle contexts. Adaptivity is application dependent. Thus the decomposition may not be unique. One branch of the stream deals with Takenaka-Malmquist systems (TM systems) involving weighted (finite) Blaschke products [15, 19]. A TM system consists of a sequence of mono-components. In the traditional studies, however, they are not adaptive. They are orthogonal bases determined by a sequence of points $\{a_k\}$ in the unit disc, where the choices of $a_1, a_2, \ldots$ obey the rule

$$\sum_{k=1}^{\infty}(1-|a_k|)=\infty \tag{1.8}$$

in order to make the system to be an orthogonal basis. The TM system and its variations are all based on a previously determined sequence of points $\{a_k\}$ irrelevant to the signal to be decomposed. For adaptive decomposition the sequence $\{a_k\}$ determining the system must be adaptively chosen in relation to the given signal, and thus it may not satisfy the condition (1.8), and the resulting system may not be a basis of the whole space. The present study is one of this adaptive scheme involving weighted inner functions. It offers high adaptivity and fast convergence.

To stress on the main idea we will work on the unit circle corresponding to periodic signals, and on the square-integrable case. Below C and C_p denote constants. The values of appearances of constant C and C_p may vary from one occurrence to another.

2 Adaptive Decomposition of Signals in $\mathcal{H}^2(\mathbb{D})$

Below we will briefly recall the necessary knowledge of Hardy spaces (see [6]). A function $F(z)$ holomorphic in the unit disk $\mathbb{D}$ is said to be in the *Hardy space* $\mathcal{H}^p(\mathbb{D})$ if it satisfies

$$\|F\|_{\mathcal{H}^p} := \sup_{0<r<1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|F(re^{i\theta})|^p\,d\theta\right\}^{1/p} < \infty, \quad 1\le p<\infty, \tag{2.9}$$

and is said to be in the *Hardy space* $\mathcal{H}^\infty(\mathbb{D})$ (the space of bounded holomorphic functions) if it satisfies

$$\|F\|_{\mathcal{H}^\infty} := \sup_{z\in\mathbb{D}}|F(z)| < \infty. \tag{2.10}$$

For $F(z)$ in $\mathcal{H}^p(\mathbb{D})$, $1\le p\le\infty$, there exists non-tangential boundary limit (or non-tangential boundary value), denoted $F(e^{it})$, where $F(e^{it})\in\mathcal{L}^p(\partial\mathbb{D})$. The functions $F(z)\in\mathcal{H}^\infty(\mathbb{D})$ with the property $|F(e^{it})|=1$ a.e. are called *inner functions*. The two particular types of inner functions are Blaschke products and singular inner functions.

A general Blaschke product is an inner function defined by an infinite product

$$B(z) = Cz^m\prod_{|z_n|\neq 0}\frac{-\overline{z}_n}{|z_n|}\frac{z-z_n}{1-\overline{z}_n z}, \tag{2.11}$$

where C is a unimodular constant, the zeros $z_1, z_2, \dots$ necessarily satisfy the condition

$$\sum_{n=1}^{\infty}(1-|z_n|)<\infty \tag{2.12}$$

in order to make the infinite product convergent. The role of the constant unimodular factors $\frac{-\bar{z}_n}{|z_n|}$ is to modify the arguments so to make the infinite product convergent. If there are only finitely many zeros, then those unimodular constant factors are unnecessary. Each factor is a Möbius transform.

A general singular inner function is an inner function with the form

$$S(z)=\exp\left(-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\right), \tag{2.13}$$

where $d\mu$ is a finite positive Borel measure that is singular with respect to the Lebesgue measure dt. A typical example is $d\mu=\delta(t-t_0)\,dt$, the Dirac point mass measure at t_0. A singular inner function has no zero points in the unit disc, with non-tangential boundary values of module 1 almost everywhere on $\partial\mathbb{D}$, and takes infinite many times of any value $\zeta\in\mathbb{D}$ in any neighborhood of any point at which the singular measure concentrates.

Another type of $\mathcal{H}^p(\partial\mathbb{D})$ functions having no zeros in $\mathbb{D}$ is *outer functions*. They are of the form

$$O(z)=Ce^{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log h(t)\,dt}, \tag{2.14}$$

where C is a unimodular constant, $h\geq 0$ and $\log h\in\mathcal{L}^1(\partial\mathbb{D})$. The function $h\in\mathcal{L}^p(\partial\mathbb{D})$ if and only if $O\in\mathcal{H}^p(\mathbb{D})$. In fact, the module of the boundary value $|O|=h$ a.e.

Nevanlinna's Factorization Theorem for $\mathcal{H}^p(\mathbb{D})$ functions asserts that $F\in\mathcal{H}^p(\mathbb{D})$ if and only if

$$F=OBS, \tag{2.15}$$

where O is an outer function with $|O|=|F|$ on the boundary, B is a Blaschke product formed by all the zeros of F, that automatically meet the requirement (2.12), S is a singular inner function defined by $d\mu$, where $d\mu$ has the characteristic property that

$$\log|F(e^{it})|\,dt-d\mu$$

gives rise to the least harmonic majorant of $\log|F(z)|$. The product BS is called the inner function part of F. We note that for a given function F in some Hardy space one can first determine its outer function part by

$$O(z)=e^{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log|F(e^{it})|\,dt}$$

up to a unimodular constant. The inner function part $I=BS$ is then determined by the Factorization Theorem.

Now we turn to the connection with mono-components. It is well known that $s(t)=\rho(t)e^{i\theta(t)}$ is the boundary value of a holomorphic function in $\mathcal{H}^p(\mathbb{D})$ if and only if $H(\rho e^{i\theta})=-i\rho e^{i\theta}$, module a constant [2], or equivalently, $H(\rho\cos\theta)=\rho\sin\theta$, module a constant, where H stands for the circular Hilbert transformation. Thus, boundary values of the $\mathcal{H}^p(\mathbb{D})$-functions automatically satisfy the Hilbert transform condition in Definition 1.1. The condition $\theta'(t)\geq 0$ a.e. is not automatically satisfied, though. For instance, conformal mappings are sense preserving. But mono-components require more than that: They require that the phase $\theta(t)$ is an increasing function. It is a known fact that the phase of a Möbius transform is a harmonic measure, and its derivative is the Poisson kernel, and thus positive [6]. This positivity can be easily generalized to the finite Blaschke product case. For singular inner functions given by a point Dirac mass a direct computation shows $\theta'(t)\geq 0$ [14]. For general inner functions, including infinite Blaschke products and singular inner functions given by arbitrary positive Borel measure perpendicular to the Lebesgue measure, [14] asserts that there always holds $\theta'(t)\geq 0$, a.e. The meaning of the derivative, however, should be suitably interpreted. The non-tangential boundary value of an inner function I satisfies $|I(e^{it})|=1$ a.e. When we write this as $I(e^{it})=e^{i\theta(t)}$, the phase function $\theta(t)$ is not uniquely defined. Or, when we uniquely define it by restricting its values to, for instance, $[0,2\pi)$, then the function may not be continuous. In particular, the boundary value is defined only almost everywhere through non-tangential boundary limit. It would then be difficult to talk about derivatives of the phase function on the boundary. A notion of phase derivative should be suitably defined.

With an abuse of notation (as $\theta(t)$ is not defined) the phase derivative $\theta'(t)$ is defined to be the limit of $\theta_r'(t)$ as $r\to 1-$, where $F(re^{it})=\rho_r(t)e^{i\theta_r(t)}$. It is shown in [14] that for inner functions the limits exist for almost all points on the unit circle. Moreover,

$$\lim_{r\to 1-}\theta_r'(t)=\lim_{r\to 1-}\operatorname{Re}\frac{re^{it}F'(re^{it})}{F(re^{it})}\geq 0,\quad \text{a.e.} \tag{2.16}$$

Essentially, the existence of the limits and the positivity together are nothing more than the content of the classical Julia-Wolff-Carathéodory Theorem [14]. If the inner function has analytic continuation cross an interval on the boundary, then the above defined phase derivative coincides with the traditional derivative $\theta'(t)$ on the interval. With this generalization of phase derivative we have

Theorem 2.1 *Assume that $\theta(t)$ is a measurable function. Then*

$$H(\cos\theta)=\sin\theta \tag{2.17}$$

if and only if $e^{i\theta(t)}$ is the non-tangential boundary limit of an inner function. In the case, there holds $\theta'(t)\geq 0$ a.e., where the derivative is defined through the limit given in (2.16).

We will be working with the inner functions

$$N_n = I_n(z)z^{n-1}\prod_{j=1}^{n-1}\left[\left(\frac{z-a_j}{1-\overline{a_j}z}\right)^{d_j} I_j(z)\right], \quad n=1,2,\ldots, \tag{2.18}$$

and the weighted inner functions

$$M_n = \frac{z}{1-\overline{a_n}z}N_n = \frac{1}{1-\overline{a_n}z}I_n(z)z^n\prod_{j=1}^{n-1}\left[\left(\frac{z-a_j}{1-\overline{a_j}z}\right)^{d_j} I_j(z)\right], \quad n=1,2,\ldots, \tag{2.19}$$

where d_j are non-negative integers, and a_n are complex numbers in the unit disc $\mathbb{D}$, and I_n are inner functions [1]. The sequence $\{a_n\}$ will be consecutively chosen according to some optimal principle specified later. We show that the functions in the sequence $N_1, M_1, N_2, M_2, \ldots$ are mono-components and they form an orthogonal system.

Starlike functions in one complex variable are closely related to mono-components. We recall the following definitions [8, 11].

Definition 2.1 A domain Ω is said to be starlike about the origin if $0 \in \Omega$, and $tz \in \Omega$, $0 < t < 1$, whenever $z \in \Omega$. A univalent and holomorphic function $f : \mathbb{D} \to f(\mathbb{D})$ is said to be *starlike* about the origin if $f(0) = 0$ and $f(\mathbb{D})$ is starlike about the origin.

Definition 2.2 A domain Ω is said to be convex, if $tz_1 + (1-t)z_2 \in \Omega$, whenever $z_1, z_2 \in \Omega$ and $0 < t < 1$. A univalent and holomorphic function $f : \mathbb{D} \to f(\mathbb{D})$ is said to be *convex*, if $f(\mathbb{D})$ is convex.

It is easy to verify that if f is convex and $f(0) = 0$, then f is starlike about the origin. Let f be a conformal mapping with $f(0) = 0$ that maps $\mathbb{D}$ to a bounded domain Ω. By a famous theorem of Carathéodory the mapping f has a one to one continuous extension from $\overline{\mathbb{D}} \to \overline{\Omega}$. Then f is a starlike mapping if and only if $f'(0) \neq 0$, and

$$\operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}$$

(see, for instance, [11]). Assuming that f is smoothly extended to its boundary $f(e^{it}) = \rho(t)e^{i\theta(t)}$, by taking derivative respect to t, we obtain

$$\theta'(t) = \operatorname{Re}\left(z\frac{f'(z)}{f(z)}\bigg|_{z=e^{it}}\right) > 0.$$

On the other hand, such a mapping is a function in the Hardy $H^\infty(\mathbb{D})$ space, we therefore obtain that $f(e^{it})$ is a mono-component [13]. This result is generalized to the so called *p-starlike functions* [7]. The observations show the close connection of mono-components with the well studied mappings in one complex variable.

Lemma 2.2 *Functions in* $\{N_n\}$ *and* $\{M_n\}$ *are mono-components.*

Proof This is in fact a consequence of Theorem 2.1. Functions in $\{N_n\}$ are inner functions and thus are mono-components. To show that M_n are mono-components we note that the class $\mathcal{MC}$ is closed under certain multiplication. In fact, if F, G are mono-components and are boundary values of functions in, respectively, $\mathcal{H}^{p_1}(\mathbb{D})$ and $\mathcal{H}^{p_2}(\mathbb{D})$, and their product FG is the boundary value of some $\mathcal{H}^p(\mathbb{D})$, where $1 \le p, p_1, p_2 \le \infty$, then FG is also a mono-component. For a fixed $a_n \in \mathbb{D}$ to show that the bounded holomorphic function $M_n = \frac{z}{1-\overline{a_n}z}N_n$ is a mono-component, it suffices to show that the boundary value of the bounded holomorphic function

$$G_n(z) = \frac{z}{1-\overline{a_n}z}$$

has an increasing phase function. While this may be explicitly verified through computation [15], we give a geometrical proof here. For $a_n \in \mathbb{D}$ the fractional linear transform G_n maps the closed unit disc centered at the origin to a closed unit disc containing the origin. Since G_n is convex with $G(0) = 0$, G_n is starlike. A starlike function has an increasing phase. Thus G_n is a mono-component. The proof is complete. □

Lemma 2.3 *The two collections* $\{N_n\}$ *and* $\{M_n\}$ *together form an orthogonal system in* $\mathcal{H}^2(\mathbb{D})$.

The proof of Lemma 2.3 uses the fact that if $F(e^{it})$ is the non-tangential boundary value of a function $F \in \mathcal{H}^1(\mathbb{D})$, then

$$\int_{\partial\mathbb{D}} F(z)\,dz = 0.$$

This turns to be true for all $\mathcal{H}^p(\mathbb{D})$, $1 \le p \le \infty$, as we have $\mathcal{H}^p(\mathbb{D}) \subset \mathcal{H}^1(\mathbb{D})$. In the sequel we regard this as Cauchy's Theorem for $\mathcal{H}^p(\mathbb{D})$ functions, or Cauchy's Theorem in short.

Proof First, within the collection $\{N_n\}$ any two different functions are orthogonal. In fact, for $l \ge 1$, and $z \in \partial\mathbb{D}$ there holds

$$N_{n+l}(z)\overline{N}_n(z) = z^l I(z),$$

where $I(z)$ is an inner function. Applying Cauchy's theorem for $\mathcal{H}^p(\mathbb{D})$ functions, we have

$$\int_0^{2\pi} N_{n+l}(e^{it})\overline{N}_n(e^{it})\,dt = -i\int_{\partial\mathbb{D}} z^{l-1} I(z)\,dz = 0.$$

Similarly, any two different functions in the collection $\{M_n\}$ are orthogonal. For $z \in \partial\mathbb{D}$,

$$M_{n+l}(z)\overline{M}_n(z) = \frac{z}{1-\overline{a}_{n+l}z}N_{n+l}\frac{\overline{z}}{1-a_n\overline{z}}\overline{N}_n = \frac{z}{1-\overline{a}_{n+l}z}\frac{1}{z-a_n}z^l I(z),$$

where the inner function $I(z)$ has a_n as a zero. Thus the last expression is of the form $z^{1+l}F(z)$, where $F(z)$ is the boundary value of a function in $\mathcal{H}^\infty$. By invoking Cauchy's Theorem again, we have

$$\int_0^{2\pi} M_{n+l}(e^{it})\overline{M}_n(e^{it})\,dt = 0.$$

To exam the orthogonality between M_n and N_k we first have

$$M_n\overline{N}_n = \frac{z}{1-\overline{a}_n z},$$

and obviously M_n and N_n are orthogonal. For M_{n+l} and N_n we have

$$M_{n+l}(z)\overline{N}_n(z) = \frac{z}{1-\overline{a}_{n+l}z}N_{n+l}(z)\overline{N}_n(z).$$

The same reasoning for the orthogonality between N_{n+l} and N_n then implies the orthogonality between M_{n+l} and N_n. Finally we check M_n and N_{n+l}. We have

$$N_{n+l}(z)\overline{M}_n(z) = \frac{1-\overline{a}_{n+l}z}{z}M_{n+l}(z)\overline{M}_n(z).$$

From the proof of the orthogonality between $M_{n+l}(z)$ and $M_n(z)$ the right-hand-side of the last equality is of the form $z^l F(z)$, where $F(z)$ is the boundary value of a function in $\mathcal{H}^\infty(\mathbb{D})$ and again we have the orthogonality. The proof is complete. □

The mono-components N_n and M_n result from the following process of decomposition of functions in $\mathcal{H}^p(\mathbb{D})$. Any function $F \in \mathcal{H}^p(\mathbb{D})$, $1 \le p \le \infty$, can be decomposed by recursively employing Nevanlinna's Factorization Theorem. In each of the recurrence steps below we subtract a linear function from an outer function so that at least two factors z and $z - a_i$, and hopefully more inner function factors too, can be factorized out, namely,

$$\begin{aligned}
F(z) &= O_1(z)I_1(z)\\
&= \left[O_1(z) - \frac{B_1 z}{1-\overline{a_1}z} - A_1\right]I_1(z) + \left[\frac{B_1 z}{1-\overline{a_1}z} + A_1\right]I_1(z)\\
&= O_2(z)z\left(\frac{z-a_1}{1-\overline{a_1}z}\right)^{d_1} I_1(z)I_2(z) + \left[\frac{B_1 z}{1-\overline{a_1}z} + A_1\right]I_1(z)\\
&= \left[O_2(z) - \frac{B_2 z}{1-\overline{a_2}z} - A_2\right]z\left(\frac{z-a_1}{1-\overline{a_1}z}\right)^{d_1} I_1(z)I_2(z)\\
&\quad + \left[\frac{B_2 z}{1-\overline{a_2}z} + A_2\right]z\left(\frac{z-a_1}{1-\overline{a_1}z}\right)^{d_1} I_1(z)I_2(z) + \left[\frac{B_1 z}{1-\overline{a_1}z} + A_1\right]I_1(z)\\
&= O_3(z)I_3(z)z^2\prod_{j=1}^{2}\left[\left(\frac{z-a_j}{1-\overline{a_j}z}\right)^{d_j} I_j(z)\right]
\end{aligned}$$

$$+\left[A_2+\frac{B_2z}{1-\overline{a_2}z}\right]z\left(\frac{z-a_1}{1-\overline{a_1}z}\right)^{d_1}I_1(z)I_2(z)+\left[\frac{B_1z}{1-\overline{a_1}z}+A_1\right]I_1(z)$$

$$\vdots$$

$$= O_{n+1}(z)I_{n+1}(z)z^n\prod_{j=1}^{n}\left[\left(\frac{z-a_j}{1-\overline{a_j}z}\right)^{d_j}I_j(z)\right]$$

$$+\left[A_n+\frac{B_nz}{1-\overline{a_n}z}\right]I_n(z)z^{n-1}\prod_{j=1}^{n-1}\left[\left(\frac{z-a_j}{1-\overline{a_j}z}\right)^{d_j}I_j(z)\right]$$

$$+\cdots$$

$$+\left[A_2+\frac{B_2z}{1-\overline{a_2}z}\right]I_2(z)z\left(\frac{z-a_1}{1-\overline{a_1}z}\right)^{d_1}I_1(z)+\left[\frac{B_1z}{1-\overline{a_1}z}+A_1\right]I_1(z)$$

$$= R_n(z)+(A_nN_n+B_nM_n)+\cdots+(A_1N_1+B_1M_1)$$

$$= R_n(z)+S_n(z), \tag{2.20}$$

where $R_n(z)=O_{n+1}N_{n+1}$, O_i are outer functions, I_i are inner functions, and N_i and M_i are, respectively, the types of inner and weighted inner functions formed from I_i and a_i defined in (2.18) and (2.19), S_n stands for the n-th partial sum, $A_i=O_i(0)$, B_i is chosen so that $O_i(z)-A_i-\frac{B_iz}{1-\overline{a}_iz}$ has a zero at $z=a_i$, that is

$$B_i=\begin{cases} O_i'(0), & \text{if } a_i=0;\\ a_i^{-1}(1-|a_i|^2)[O_i(a_i)-O_i(0)], & \text{if } a_i\neq 0,\end{cases} \tag{2.21}$$

and $d_i\geq 1$. The above process is valid for all p. We, in this study, restrict ourselves to the case $p=2$. For $p=2$ there is an optimal selection criterion for a_i. In fact, one can first show that there exists a point $a_i\in\mathbb{D}$ such that

$$\int_{-\pi}^{\pi}\left|O_i(e^{it})-A_i-\frac{B_ie^{it}}{1-\overline{a_i}e^{it}}\right|^2dt=\min_{a\in\mathbb{D}}\int_{-\pi}^{\pi}\left|O_i(e^{it})-A_i-\frac{B_ae^{it}}{1-\overline{a}e^{it}}\right|^2dt, \tag{2.22}$$

where the relation between B_a and a is the same as that between B_i and a_i given in (2.21). Below we will write both the $\mathcal{H}^p(\mathbb{D})$ norm and the $\mathcal{L}^p(\partial\mathbb{D})$ norm by $\|\ \|_p$ that causes no confusion.

Lemma 2.4 *For any function $F(z)\in\mathcal{H}^2(\mathbb{D})$ with $F(0)=0$, the value*

$$\min_{a\in\mathbb{D}}\|F-B_aE_a\|_2 \tag{2.23}$$

can be attained at a point in $\mathbb{D}$, where

$$B_0=F'(0),\qquad B_a=(1-|a|^2)\frac{F(a)}{a},\quad a\neq 0,$$

and

$$E_a(e^{it}) = \frac{e^{it}}{1 - \overline{a}e^{it}}.$$

Proof We first assume $F(z) \in \mathcal{H}^p(\mathbb{D}), 2 < p \le \infty$, and $F(0) = 0$. Then

$$\langle F - B_a E_a, F - B_a E_a\rangle = \langle F, F\rangle - B_a\langle E_a, F\rangle - \overline{B}_a\langle F, E_a\rangle + |B_a|^2\langle E_a, E_a\rangle,$$

where, since $F(z)/z \in \mathcal{H}^p(\mathbb{D})$, by the Cauchy Formula,

$$\langle F, E_a\rangle = \int_0^{2\pi} F(e^{it})\frac{e^{-it}}{1 - ae^{-it}}\,dt = \frac{1}{i}\int_{\partial\mathbb{D}} \frac{F(z)}{z}\frac{1}{z-a}\,dz = 2\pi\frac{F(a)}{a}.$$

Since the integral value of the Poisson kernel is identical to 1, we have

$$\langle E_a, E_a\rangle = \int_0^{2\pi} \frac{dt}{|e^{it} - a|^2} = \frac{2\pi}{1 - |a|^2}.$$

Therefore,

$$\|F - B_a E_a\|_2^2 = \|F\|_2^2 - 2\pi(1 - |a|^2)\frac{|F(a)|^2}{|a|^2}. \tag{2.24}$$

For any $F \in \mathcal{H}^p(\mathbb{D}), 0 < p \le \infty$, there holds

$$|F(z)| \le \left(\frac{1 + |z|}{1 - |z|}\right)^{1/p}\|F\|_p \le \frac{C_p}{(1 - |z|)^{1/p}}\|F\|_p, \quad |z| < 1, \tag{2.25}$$

where C_p is a constant. The result is referred to [6], p. 89 (proved similarly as for the upper-half plane case on p. 18). Using the above inequality for $2 < p \le \infty$, we have

$$(1 - |a|^2)\frac{|F(a)|^2}{|a|^2} \le C_p(1 - |a|)^{1-2/p}\|F\|_p^2 \to 0, \quad \text{as } |a| \to 1 - 0.$$

This last estimate together with (2.24) show that the continuous function $\|F - B_a E_a\|_2^2$ attains its minimum at a point in $\mathbb{D}$.

Now consider the case $F(z) \in \mathcal{H}^2(\mathbb{D})$ and $F(0) = 0$. In that case for any $\epsilon > 0$ there exists $F^{(1)} \in \mathcal{H}^\infty(\mathbb{D})$ such that

$$\|F - F^{(1)}\|_2 \le \epsilon, \qquad F^{(1)}(0) = 0.$$

This is always possible as we can take $F^{(1)}$ to be the n-th partial sum of the power series expansion of F with a sufficient large n. Then,

$$\begin{aligned}\|F - B_a E_a\|_2 &\le \|F - F^{(1)}\|_2 + \left\|F^{(1)} - B_a^{(1)}E_a\right\|_2 + \left\|B_a^{(1)}E_a - B_a E_a\right\|_2 \\ &\le \epsilon + \left\|F^{(1)} - B_a^{(1)}E_a\right\|_2 + \left\|B_a^{(1)}E_a - B_a E_a\right\|_2,\end{aligned} \tag{2.26}$$

where

$$B_0^{(1)} = F^{(1)\prime}(0), \qquad B_a^{(1)} = (1-|a|^2)\frac{F^{(1)}(a)}{a}, \quad a \neq 0.$$

Since $F^{(1)} \in \mathcal{H}^\infty(\mathbb{D})$, in view of (2.24), for any $a \in \mathbb{D}$,

$$\left\| F^{(1)} - B_a^{(1)} E_a \right\|_2 \le \|F^{(1)}\|_2 \le \|F\|_2 + \epsilon. \tag{2.27}$$

The inequalities (2.26) and (2.27), together with the estimate

$$\begin{aligned}\left\| B_a^{(1)} E_a - B_a E_a \right\|_2 &= \left| B_a^{(1)} - B_a \right| \|E_a\|_2 \\ &= (1-|a|^2)\left|\frac{F^{(1)}(a) - F(a)}{a}\right| \frac{\sqrt{2\pi}}{\sqrt{1-|a|^2}} \\ &\le C\sqrt{1-|a|^2}\left|\frac{F^{(1)}(a) - F(a)}{a}\right| \\ &\le C\|F^{(1)} - F\|_2 \\ &\le C\epsilon,\end{aligned}$$

where we used the estimate (2.25) for $p = 2$, give

$$\|F - B_a E_a\|_2 \le \|F\|_2 + C\epsilon. \tag{2.28}$$

Note that this inequality is valid for all $a \in \mathbb{D}$.

Exchanging the roles of F and $F^{(1)}$ and recalling that $\|F^{(1)} - B_a^{(1)} E_a\|_2$ tends to $\|F^{(1)}\|_2$ as $|a|$ tends to 1 (the proceeding case proved for $2 < p \le \infty$), we have

$$\begin{aligned}\|F - B_a E_a\|_2 &\ge \left\| F^{(1)} - B_a^{(1)} E_a \right\|_2 - \|F^{(1)} - F\|_2 - \left\| B_a^{(1)} E_a - B_a E_a \right\|_2 \\ &\ge \|F\|_2 - C\epsilon.\end{aligned} \tag{2.29}$$

From (2.28) and (2.29) we conclude

$$\lim_{|a|\to 1-} \|F - B_a E_a\|_2 = \|F\|_2.$$

As consequence, the minimum in (2.23) may be attained at a point of $\mathbb{D}$. The proof is complete. □

Remark 2.1 From the above proof we can set the selection criterion for a: choose $a \in \mathbb{D}$ so that the value

$$2\pi(1-|a|^2)\frac{|F(a)|^2}{|a|^2}$$

in (2.24) attains its maximum.

In this regards, the decomposition is highly adaptive, being dependent of the given function F through the choices of $\{a_i\}_{i=0}^\infty$ based on the minimization in (2.23).

We have the following convergence result.

Theorem 2.5 *In the notation of* (2.20), *for any choice of the sequence* $\{a_k\}$ *in* $\mathbb{D}$, *we have*

$$\lim_{n\to\infty} S_n = F$$

in the $\mathcal{H}^2$*-convergence sense.*

Proof In the notation of (2.20) we have $F = R_n + S_n$. We show that R_n is orthogonal with all the inner and weighted inner functions N_i, M_i, $i = 1, 2, \ldots, n$. To this end we note that $R_n = O_{n+1}N_{n+1}$. As in the proof of Lemma 2.3, the products $\frac{R_n(z)}{z}\overline{N_i}(z)$ and $\frac{R_n(z)}{z}\overline{M_i}(z)$, $i = 1, \ldots, n$, are all in $\mathcal{H}^1(\mathbb{D})$. Then Cauchy's theorem for $\mathcal{H}^p(\partial\mathbb{D})$ implies the orthogonality. As consequence, $R_n(z)$ and $S_n(z)$ are orthogonal.

Write F into its power series expansion

$$F(z) = \sum_{k=0}^{\infty} c_k z^k.$$

One notices that all the terms $c_0, c_1z, \ldots, c_{n-1}z^{n-1}$ in the expansion are only contained in $S_n(z)$ but not in $R_n(z)$. Denote

$$S_n(z) = \sum_{k=0}^{n-1} c_k z^k + L_n(z),$$

where $L_n(z)$ collects all the constant multiples of $z^l, l \geq n$, in $S_n(z)$. Obviously, $\sum_{k=0}^{n-1} c_k z^k$ and $L_n(z)$ are orthogonal. Note that the H^2-norm of a function in $H^2(\mathbb{D})$ coincides with the L^2-norm of its boundary function. It follows that

$$\|F\|_2^2 = \|R_n\|_2^2 + \|S_n\|_2^2 = \|R_n\|_2^2 + \|L_n\|_2^2 + 2\pi \sum_{k=0}^{n-1} |c_k|^2.$$

Therefore,

$$\|F - S_n\|_2^2 = 2\pi \sum_{k=n}^{\infty} |c_k|^2 - \|L_n\|_2^2 \leq 2\pi \sum_{k=n}^{\infty} |c_k|^2. \tag{2.30}$$

This shows that S_n converges to F at a rate faster than that of the Fourier $(n-1)$-th partial sum. The proof is complete. □

Let s be a real-valued function in $\mathcal{L}^2(\partial\mathbb{D})$. It has a Fourier series expansion in the $\mathcal{L}^2$-convergence sense

$$s(e^{it}) = \sum_{n=-\infty}^{\infty} c_n e^{int}.$$

We have the decomposition $s = s^+ + s^-$, where

$$s^+(e^{it}) = \sum_{n=0}^{\infty} c_n e^{int}, \quad s^-(e^{it}) = \sum_{n=-\infty}^{-1} c_n e^{int},$$

and s^+ and s^- are, respectively, the non-tangential boundary values of the functions in the Hardy spaces $\mathcal{H}^2(\mathbb{D})$ and $\mathcal{H}^2(\overline{\mathbb{D}}^c)$, viz.

$$s^+(z) = \sum_{n=0}^{\infty} c_n z^n, \quad |z| < 1,$$

and

$$s^-(z) = \sum_{n=-\infty}^{-1} c_n z^n, \quad |z| > 1.$$

Because the coefficients satisfy the relation

$$\overline{c}_n = c_{-n}, \quad n \neq 0,$$

on the boundary there holds

$$s = 2\,\mathrm{Re}\, s^+ - c_0.$$

We accordingly have

Corollary 2.6 *In the notation* (2.20) *for* $F = s^+ \in \mathcal{H}^2(\mathbb{D})$, *by denoting*

$$\rho_k^{(1)}(t)\cos\theta_k^{(1)}(t) = \mathrm{Re}[A_k N_k(e^{it})], \qquad \rho_k^{(2)}(t)\cos\theta_k^{(2)}(t) = \mathrm{Re}[B_k M_k(e^{it})]$$

and

$$s_n(e^{it}) = \sum_{k=1}^{n} \rho_k^{(1)}(t)\cos\theta_k^{(1)}(t) + \rho_k^{(2)}(t)\cos\theta_k^{(2)}(t),$$

we have

$$\lim_{n\to\infty} s_n = s$$

in $\mathcal{L}^2(\partial\mathbb{D})$.

Remark 2.2 We wish to stress on the adaptivity of the decomposition. There are two points to make. First, the parameters A_i, a_i and B_i are adaptively chosen. This makes the quantity $\|L_n\|^2$ be as large as possible. Therefore, smaller errors for the partial sums may result. The second is that the "un-wending" process by factorizing out the inner function part at each step may also result in smaller errors for the partial sums. Although the effect of un-wending in speeding up the convergence is hard to measure, the philosophy can be well explained by the energy-delay result in digital signal processing [4, 9]. We provide both the statement and the proof of the result for easy reference.

Let $s(z) = \sum_{n=0}^{\infty} c_n z^n$ be a signal in $\mathcal{H}^2(\mathbb{D})$, and g an inner function. Let $h(z) = s(z)g(z)$ be with the Taylor (or Fourier, if on the boundary) series expansion

$$h(z) = \sum_{n=o}^{\infty} d_n z^n.$$

Because inner functions are unimodule on the unit circle, the total energy does not change after the multiplication transform, viz.

$$\|s\|^2 = \|h\|^2.$$

The energy-delay effect of being multiplied by an inner function addresses the following fact: For any positive integer N,

$$\sum_{n=0}^{N} |c_n|^2 \geq \sum_{n=0}^{N} |d_n|^2, \tag{2.31}$$

and thus the energy is delayed at every partial sum after being multiplied by an inner function.

To show (2.31), set $s_N(z) = \sum_{n=0}^{N} c_n z^n$. Denote the corresponding sequences by, respectively,

$$\{s_N\} = (c_0, c_1, \ldots, c_N, 0, \ldots, 0, \ldots), \qquad \{g\} = (b_0, b_1, \ldots, b_n, \ldots).$$

Setting $h_N(z) = s_N(z)g(z)$, we have

$$(\{h_N\})_n = (\{s_N\} * \{g\})_n = \sum_{k=0}^{N} c_k b_{n-k}.$$

Then by the Plancherel theorem and the unimodular property of inner functions,

$$2\pi \sum_{n=0}^{N} |c_n|^2 = \|s_N\|^2 = \|h_N\|^2 = 2\pi \sum_{n=0}^{\infty} |(\{h_N\})_n|^2 \geq 2\pi \sum_{n=0}^{N} |(\{h_N\})_n|^2.$$

On the other hand, if $n \leq N$,

$$d_n = (\{h\})_n = \sum_{k=0}^{n} c_k b_{n-k} = \sum_{k=0}^{N} c_k b_{n-k} = (\{h_N\})_n.$$

So, we obtain

$$2\pi \sum_{n=0}^{N} |c_n|^2 \geq 2\pi \sum_{n=0}^{N} |d_n|^2,$$

as desired.

Remark 2.3 Mathematically the proposed decomposition may be constructively obtained. In practice it is up to the approximation error. For $F \in \mathcal{H}^2$, we can first obtain its outer function factor O_1 by (2.14) with $h(e^{it}) = |F(e^{it})|$. Once we have determined O_1 we can obtain the corresponding inner function factor $I_1 = F/O_1$. To find an optimal a_1 is to find local extrema of a smooth function of two real variables x and y in the open box $(-1, 1) \times (-1, 1)$ with the constraint condition $x^2 + y^2 = r^2$, $0 \le r < 1$. For functions in the $\mathcal{L}^2$ space of the boundary we seek for the Hardy space decomposition and then use Corollary 2.6.

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