

Nonuniform Sampling and Recovery of Multidimensional Bandlimited Functions by Gaussian Radial-Basis Functions

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Abstract Let $S \subset \mathbb{R}^d$ be a bounded subset with positive Lebesgue measure. The Paley-Wiener space associated to S , PW_S , is defined to be the set of all square-integrable functions on \mathbb{R}^d whose Fourier transforms vanish outside S . A sequence $(x_j : j \in \mathbb{N})$ in \mathbb{R}^d is said to be a Riesz-basis sequence for $L_2(S)$ (equivalently, a complete interpolating sequence for PW_S) if the sequence $(e^{-i\langle x_j, \cdot \rangle} : j \in \mathbb{N})$ of exponential functions forms a Riesz basis for $L_2(S)$. Let $(x_j : j \in \mathbb{N})$ be a Riesz-basis sequence for $L_2(S)$. Given $\lambda > 0$ and $f \in PW_S$, there is a unique sequence (a_j) in ℓ_2 such that the function

$$I_\lambda(f)(x) := \sum_{j \in \mathbb{N}} a_j e^{-\lambda \|x - x_j\|_2^2}, \quad x \in \mathbb{R}^d,$$

is continuous and square integrable on \mathbb{R}^d , and satisfies the condition $I_\lambda(f)(x_n) = f(x_n)$ for every $n \in \mathbb{N}$. This paper studies the convergence of the interpolant $I_\lambda(f)$ as λ tends to zero, *i.e.*, as the variance of the underlying Gaussian tends to infinity. The following result is obtained: Let $\delta \in (\sqrt{2/3}, 1]$ and $0 < \beta < \sqrt{3\delta^2 - 2}$. Suppose that $\delta B_2 \subset Z \subset B_2$, and let $(x_j : j \in \mathbb{N})$ be a Riesz basis sequence for $L_2(Z)$. If $f \in PW_{\beta B_2}$, then $f = \lim_{\lambda \rightarrow 0^+} I_\lambda(f)$ in $L_2(\mathbb{R}^d)$ and uniformly on \mathbb{R}^d . If $\delta = 1$, then one may take β to be 1 as well, and this reduces to a known theorem in the

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univariate case. However, if $d \geq 2$, it is not known whether $L_2(B_2)$ admits a Riesz-basis sequence. On the other hand, in the case when $\delta < 1$, there do exist bodies Z satisfying the hypotheses of the theorem (in any space dimension).

Keywords Scattered data · Gaussian interpolation · Multidimensional bandlimited functions

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1 Introduction

The theoretical study of interpolation has long played a prominent role in the development of the theory of approximation, whilst the computational aspects of the subject have found a natural outlet in numerical analysis. Given the inherent naturalness of the subject, and the variety of interesting questions it generates, it is not surprising that interpolation theory continues to be an active area of research.

Among the diverse issues associated to interpolation, the theory of ‘cardinal interpolation’ has been a well-studied theme over many years. The term refers to the interpolation of data on a regular, usually infinite, grid. When the interpolation at such a grid is done via spline functions, in particular, one encounters ‘cardinal spline interpolation’. Rooted in Isaac Schoenberg’s seminal work, this subject was further developed by a number of his successors.

The last century has witnessed enormous advances in the state of the art of electrical engineering and telecommunication theory. Included among the myriad branches of this discipline is the theory of ‘sampling’, which is an integral part of signal analysis. By their very nature, the theories of sampling and interpolation are intertwined; indeed, at a basic level both are in essence one and the same. Philosophical considerations apart, there are also solid and captivating mathematical connections between the subjects. Connections of this nature, in the context of cardinal spline interpolation, were brought out by Schoenberg himself; see, for instance [14], especially his remarks on p. 228 there. In this paper Schoenberg showed, among other things, how bandlimited functions can be recovered as limits of their cardinal-spline interpolants, as the degree of the underlying spline tends to infinity. Substantial extensions of this theme have since been carried out by many, both in the univariate and multivariate settings. More recently, it was also discovered that the theory of cardinal spline interpolation, and its connections to sampling, have a strong resonance in the theory of radial-basis functions, especially involving Gaussians. One consequence of these developments is the fact that bandlimited functions can also be recovered via their Gaussian cardinal interpolants, in a suitable limiting sense.

Moving away from the realm of gridded data, it is natural to look for comparable connections between interpolation at irregularly spaced points—often referred to as scattered-data interpolation—via splines and radial-basis functions, and the now classical theory of nonuniform sampling. In the case of splines, the analogue of Schoenberg’s theorem to which we alluded above was obtained in [8], and a counterpart of this result for Gaussians was given later in [13].

The focus in [13] is on the one-dimensional case, although it does include a relatively straightforward extension to higher dimensions in terms of tensor products. The purpose of the present article is to present a general multidimensional result. Here we overcome our earlier obstacles by extending the techniques of [8] and [13] in such a way that the radial symmetry of the multidimensional Gaussian can be exploited fully. As in [8] and [13], our setting for the interpolation problem also involves Riesz-basis sequences. However, we are forced to settle for less in the multivariate situation, because the existence of suitable Riesz-basis sequences in higher dimensions is a subtle matter depending on the geometry of the underlying domain. A more detailed discussion of this issue is the subject of the concluding remarks in Sect. 3.

The rest of the paper is organized as follows. In the next section we lay out some background material of a general nature. More specific preliminaries are given in the subsequent section, which concludes with the statement of the main result. The proof of the latter is detailed in the final section.

2 Preliminaries

Let $d \in \mathbb{N}$. For $1 \leq p < \infty$ and a measurable set $S \subset \mathbb{R}^d$ with positive Lebesgue measure $m(S)$, we denote by $L_p(S)$ the space of complex-valued functions which are p -integrable on S (with respect to the Lebesgue measure). For $f \in L_p(S)$, we denote its standard L_p -norm by $\|f\|_{L_p(S)}$, or by $\|f\|_p$, when the context is clear. We also denote by $\|\cdot\|_p$, the ℓ_p -norm, $1 \leq p \leq \infty$, on the space of (finite and infinite) sequences. The space of continuous functions on \mathbb{R}^d is denoted by $C(\mathbb{R}^d)$, and

$$C_0(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : \lim_{\|x\|_2 \rightarrow \infty} f(x) = 0\}.$$

If $f \in L_1(\mathbb{R}^d)$, then the Fourier transform of f , \hat{f} , is defined as follows:

$$\hat{f}(x) := \int_{\mathbb{R}^d} f(u)e^{-i\langle x,u \rangle} du, \quad x \in \mathbb{R}^d. \tag{1}$$

The Fourier transform of $g \in L_2(\mathbb{R}^d)$ is denoted by $\mathcal{F}[g]$. The assignment $g \mapsto \mathcal{F}[g]$ is the unique extension of the map

$$\widehat{(\cdot)} : L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)$$

to a bounded linear operator on $L_2(\mathbb{R}^d)$, satisfying the following Plancherel-Parseval relation:

$$\|\mathcal{F}[g]\|_2^2 = (2\pi)^d \|g\|_2^2 \quad \text{for all } g \in L_2(\mathbb{R}^d). \tag{2}$$

If $g \in L_2(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and $\mathcal{F}[g] \in L_1(\mathbb{R}^d)$, then the following inverse formula holds:

$$g(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}[g](u)e^{i\langle u,x \rangle} du, \quad \text{for all } x \in \mathbb{R}^d. \tag{3}$$

For $\lambda > 0$ we define the *Gaussian function* $g_\lambda : \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$g_\lambda(x) = e^{-\lambda\|x\|_2^2}, \quad \text{for all } x \in \mathbb{R}^d,$$

and recall that

$$\hat{g}_\lambda(u) = \left(\frac{\pi}{\lambda}\right)^{d/2} e^{-\|u\|_2^2/(4\lambda)}, \quad \text{for all } u \in \mathbb{R}^d. \tag{4}$$

The functions we wish to interpolate are the so called *bandlimited* or *Paley-Wiener functions* on \mathbb{R}^d . Specifically, for a bounded and measurable $S \subset \mathbb{R}^d$ with $m(S) > 0$, we define

$$PW_S = \{g \in L_2(\mathbb{R}^d) : \mathcal{F}[g] = 0 \text{ almost everywhere outside } S\}.$$

If S is as above and $g \in PW_S$, then the Fourier inversion formula implies the relations

$$g(u) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}[g](x)e^{i(x,u)} dx = \frac{1}{(2\pi)^d} \int_A \mathcal{F}[g](x)e^{i(x,u)} dx, \tag{5}$$

for almost all $u \in \mathbb{R}^d$. As $L_2(S) \subset L_1(S)$ and $\mathcal{F}[g] = 0$ almost everywhere outside S , it follows that $\mathcal{F}[g] \in L_1(\mathbb{R}^d)$, so the Riemann-Lebesgue Lemma asserts that the last expression in (5), as a function of u , belongs to $C_0(\mathbb{R}^d)$. So we may assume that (5) holds for all $u \in \mathbb{R}^d$. Moreover, the Bunyakovskii–Cauchy–Schwarz (BCS) Inequality and (5) combine to show that

$$|g(u)| \leq \frac{m^{1/2}(S)}{(2\pi)^d} \|\mathcal{F}[g]\|_{L_2(S)} = \frac{m^{1/2}(S)}{(2\pi)^{d/2}} \|g\|_{L_2(\mathbb{R}^d)}, \quad u \in \mathbb{R}^d. \tag{6}$$

3 Gaussian Interpolants Associated to Riesz-Basis Sequences

We begin by assembling some basic facts about bases in Hilbert spaces (cf. [17]). Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_H)$ be a separable (complex) infinite dimensional Hilbert space. We say that $(h_j : j \in \mathbb{N})$ is a *Riesz basis* for \mathcal{H} if every element h in \mathcal{H} admits a unique representation of the form

$$h = \sum_{j \in \mathbb{N}} a_j h_j, \quad \text{with} \quad \sum_{j \in \mathbb{N}} |a_j|^2 < \infty. \tag{7}$$

One can then show that there exists a unique bounded sequence $(h_j^* : j \in \mathbb{N}) \subset \mathcal{H}$ so that $a_j = \langle h, h_j^* \rangle_H$, for all $j \in \mathbb{N}$. We call the h_j^* 's the *coordinate functionals* for (h_j) . The sequence (h_j^*) is also a Riesz basis for \mathcal{H} , and its coordinate functionals are the h_j 's. Moreover there exists a positive constant R_b so that

$$\frac{1}{R_b} \left(\sum_{j \in \mathbb{N}} |c_j| \right)^{1/2} \leq \left\| \sum_{j \in \mathbb{N}} c_j h_j \right\| \leq R_b \left(\sum_{j \in \mathbb{N}} |c_j| \right)^{1/2} \tag{8}$$

for every square-summable sequence $(c_j : j \in \mathbb{N})$.

Let S be a bounded subset of \mathbb{R}^d with positive Lebesgue measure. We say that a sequence $(x_j : j \in \mathbb{N}) \subset \mathbb{R}^d$ is a *Riesz-basis sequence* for $L_2(S)$ if the sequence

$(e^{-i(x_j, \cdot)} : j \in \mathbb{N})$ is a Riesz basis for $L_2(S)$. The following pair of observations will be useful.

Proposition 3.1 *Let S be as above and let (x_k) be a Riesz-basis sequence for $L_2(S)$.*

- (a) *There is a $q > 0$ so that $\|x_k - x_\ell\| \geq q$ for $k \neq \ell$.*
- (b) *There is a constant $C > 0$ so that*

$$\left(\sum_{k \in \mathbb{N}} |f(x_k)|^2 \right)^{1/2} \leq C \|f\|_{L_2(\mathbb{R}^d)}, \quad \text{for every } f \in PW_S.$$

Proof If (a) were not true, there would be two subsequences (k_j) and (ℓ_j) such that $\lim_{j \rightarrow \infty} \|x_{k_j} - x_{\ell_j}\| = 0$, whence the Dominated Convergence Theorem implies that $\lim_{j \rightarrow \infty} \|e^{-i(x_{k_j}, \cdot)} - e^{-i(x_{\ell_j}, \cdot)}\|_{L_2(S)} = 0$. Let $(e_j^* : j \in \mathbb{N})$ be the coordinate functionals for $(e^{-i(x_j, \cdot)} : j \in \mathbb{N})$. Since $\langle e^{-i(x_n, \cdot)}, e_m^* \rangle = \delta_{mn}$, for $m, n \in \mathbb{N}$, we have (with $\langle \cdot, \cdot \rangle_S = \langle \cdot, \cdot \rangle_{L_2(S)}$) $\langle e^{-i(x_{k_j}, \cdot)} - e^{-i(x_{\ell_j}, \cdot)}, e_{\ell_j}^* \rangle_S = 1$, for $j \in \mathbb{N}$. But this contradicts the boundedness of the sequence $(\|e_j^*\| : j \in \mathbb{N})$.

(b) Let $f \in PW_S$. Considering $\mathcal{F}[f]$ as a function in $L_2(S)$ and recalling that (e_j^*) is also a Riesz basis for $L_2(S)$, we may write

$$\mathcal{F}[f] = \sum_{j \in \mathbb{N}} \langle \mathcal{F}[f], e^{-i(x_j, \cdot)} \rangle_S e_j^* = (2\pi)^d \sum_{j \in \mathbb{N}} f(x_j) e_j^*,$$

where the second equality stems from (5). The asserted result follows from (7) and (8) applied to the Riesz basis (e_j^*) . □

We now begin our discussion of Gaussian interpolants associated to Riesz-basis sequences. As the first step we state the following result which can be deduced from [11, Lemma 2.1] and the M. Riesz Convexity Theorem.

Proposition 3.2 *Let $q > 0$ and suppose that (x_j) is a q -separated sequence in \mathbb{R}^d , i.e.*

$$\inf_{k \neq \ell} \|x_k - x_\ell\| \geq q.$$

If $\lambda > 0$, and if (a_j) is a bounded sequence of complex numbers, then the function $\mathbb{R}^d \ni x \mapsto \sum a_j g_\lambda(x - x_j)$ is continuous and bounded. Moreover, the infinite matrix $(g_\lambda(x_j - x_k))_{j,k \in \mathbb{N}}$ acts as a bounded operator on ℓ_p , for all $1 \leq p \leq \infty$.

This next theorem is an important discovery in the quantitative theory of radial-basis functions.

Theorem 3.3 [10, Theorem 2.3] *Let λ and q be fixed positive numbers. There exists a number θ , depending only on d, λ , and q , such that the following holds: if (x_j) is any q -separated sequence in \mathbb{R}^d , then $\sum_{j,k} \xi_j \bar{\xi}_k g_\lambda(\|x_j - x_k\|_2) \geq \theta \sum_j |\xi_j|^2$, for every sequence of complex numbers (ξ_j) .*

Corollary 3.4 *Suppose that λ is a fixed positive number. Let $(x_j : j \in \mathbb{N}) \subset \mathbb{R}^d$ be q -separated for some $q > 0$. Then the matrix $(g_\lambda(x_k - x_j))_{k, j \in \mathbb{N}}$ is boundedly invertible on ℓ_2 . In particular, given a square-summable sequence $(d_k : k \in \mathbb{N})$, there exists a unique square-summable sequence $(a_j^{(\lambda)} : j \in \mathbb{N})$ such that*

$$\sum_{j \in \mathbb{N}} a_j^{(\lambda)} g_\lambda(x_k - x_j) = d_k, \quad k \in \mathbb{N}.$$

The interpolation operators, whose study will occupy the rest of the paper, are introduced in the following theorem, which is a necessary prelude to the main result; its proof will be given in the next section.

Proposition 3.5 *Let $d \in \mathbb{N}$ and let $Z \subset \mathbb{R}^d$ be convex, symmetric about the origin and bounded with $m(Z) > 0$. Let λ be a fixed positive number, and let $(x_j : j \in \mathbb{N}) \subset \mathbb{R}^d$ be a Riesz-basis sequence for $L_2(Z)$.*

For any $f \in PW_Z$, there exists a unique square-summable sequence $(a_j^{(\lambda)} : j \in \mathbb{N})$ such that

$$\sum_{j \in \mathbb{N}} a_j^{(\lambda)} g_\lambda(x_k - x_j) = f(x_k), \quad k \in \mathbb{N}. \quad (9)$$

The Gaussian Interpolation Operator $I_\lambda : PW_Z \rightarrow L_2(\mathbb{R}^d)$, defined by

$$I_\lambda(f)(\cdot) = \sum_{j \in \mathbb{N}} a_j^{(\lambda)} g_\lambda((\cdot) - x_j), \quad (10)$$

where $(a_j^{(\lambda)} : j \in \mathbb{N})$ satisfies (9), is a well-defined, bounded linear operator from PW_Z to $L_2(\mathbb{R}^d)$. Moreover, $I_\lambda(f) \in C_0(\mathbb{R}^d)$.

We now state the main result of the paper.

Theorem 3.6 *Let $\delta \in (\sqrt{2/3}, 1]$ and $0 < \beta < \sqrt{3\delta^2 - 2}$. Assume that $Z \subset \mathbb{R}^d$ is convex and symmetric about the origin, such that $\delta B_2 \subset Z \subset B_2$, and let $(x_j : j \in \mathbb{N})$ be a Riesz-basis sequence for $L_2(Z)$. Let I_λ be the associated Gaussian Interpolation Operator. Then for every $f \in PW_{\beta B_2}$ we have $f = \lim_{\lambda \rightarrow 0^+} I_\lambda(f)$ in $L_2(\mathbb{R}^d)$ and uniformly on \mathbb{R}^d .*

Remarks The statement of Theorem 3.6 includes the case $\delta = 1$, i.e., $Z = B_2$. Moreover, in this case, the proof allows one to take $\beta = 1$. However, unless $d = 1$ (in which case one obtains Theorems 4.3 and 4.4 in [13]), this result may well be vacuous, for it is not known if $L_2(B_2)$ admits a Riesz-basis sequence for $d \geq 2$. In fact, Kristian Seip and Joaquim Ortega-Cerdà have informed us that the prevailing belief is that, when $d \geq 2$, there is no Riesz basis for $L_2(B_2)$ consisting of exponentials; the latter has also demonstrated this to be the case for certain allied spaces [12]. This interesting problem is closely related to interpolatory properties of the associated Paley-Wiener space. It is also connected to Fuglede's work [3] and his conjecture, and to the recent studies reported in [5, 6], and [16].

On the other hand, there do exist bodies Z satisfying the hypotheses of Theorem 3.6 (in any space dimension); specifically these are *zonotopes*. Firstly, given any $\delta \in (0, 1)$, there exist zonotopes Z such that $\delta B_2 \subset Z \subset B_2$; this fact, well known to convex geometers, may be deduced, for instance, as a consequence of [4, Theorem 4.1.10]. As to the existence of Riesz-basis sequences for $L_2(Z)$, this is proved in [9] for $d = 2$; the higher dimensional version of this theorem—to which [9] alludes—was communicated to us by Yuri Lyubarskii (private correspondence).

It is also known that Riesz-basis sequences exist for $L_2(T^d)$, where T^d is a symmetric cube centred at the origin. Furthermore, in this case, one can provide sufficient conditions under which a set of distinct points in \mathbb{R}^d forms a Riesz-basis sequence for $L_2(T^d)$; see, for example, [15] and [1]. These conditions lead to multivariate generalizations of Kadec’s famous “1/4-theorem” [7]. However, cubes do not quite serve our purpose here; our argument makes essential use of the additional flexibility offered by zonotopes.

In view of the scarcity of suitable Riesz-basis sequences in higher dimensions, we think it is of considerable interest to discover an analogue of Theorem 3.6—even at the expense of forgoing interpolation, if need be—for *frame sequences*, namely, sequences $(x_j : j \in \mathbb{N})$ for which the corresponding exponentials constitute a frame for $L_2(B_2)$. Such sequences are easier to come by, thanks to Beurling’s penetrating insights and masterly work on balayage [2]. We shall indicate later why the method employed in the current paper does not lend itself to the case of frame sequences.

4 Proof of the Main Result

For $m \in \mathbb{N}$ we define a linear bounded operator A_m on $L_2(Z)$ as follows: Let $(e_k^*) \subset L_2(Z)$ be the coordinate functionals for $(e^{-i\langle x_k, \cdot \rangle} : k \in \mathbb{N})$, i.e., for every $h \in L_2(Z)$,

$$h = \sum_{k \in \mathbb{N}} \langle h, e_k^* \rangle_Z e^{-i\langle \cdot, x_k \rangle} = \sum_{k \in \mathbb{N}} \int_Z h(\xi) \overline{e_k^*(\xi)} d\xi e^{-i\langle \cdot, x_k \rangle}. \tag{11}$$

Note that for $a = (a_1, a_2, \dots, a_d) \in \mathbb{R}^d$ we have

$$\begin{aligned} \left\| \sum_{k \in \mathbb{N}} \langle h, e_k^* \rangle_Z e^{-i\langle \cdot, x_k \rangle} \right\|_{L_2(a+Z)} &= \left\| \sum_{k \in \mathbb{N}} \langle h, e_k^* \rangle_Z e^{-i\langle \cdot + a, x_k \rangle} \right\|_{L_2(Z)} \\ &= \left\| \sum_{k \in \mathbb{N}} e^{-i\langle a, x_k \rangle} \langle h, e_k^* \rangle_Z e^{-i\langle \cdot, x_k \rangle} \right\|_{L_2(Z)} \leq R_b^2 \|h\|_{L_2(Z)}, \end{aligned} \tag{12}$$

where R_b is the constant satisfying (8). Thus the following extension $E(h)$ of h is locally square integrable, hence defined almost everywhere on \mathbb{R}^d .

$$E(h)(x) = \sum_{k \in \mathbb{N}} \langle h, e_k^* \rangle_Z e^{-i\langle x, x_k \rangle}, \quad x \in \mathbb{R}^d. \tag{13}$$

Let $m \in \mathbb{N}$, and define $A_m : L_2(Z) \rightarrow L_2(Z)$ by

$$A_m(h)(\xi) = E(h)(2^m(\xi))\chi_{Z \setminus (1/2)Z}(\xi) \tag{14}$$

For $h \in L_2(Z)$ it follows from (12) that

$$\begin{aligned} \|A_m(h)\|_{L_2(Z)}^2 &= \int_{Z \setminus (1/2)Z} |E(h)(2^m u)|^2 du \\ &= 2^{-dm} \int_{2^m Z \setminus 2^{m-1}Z} |E(h)(v)|^2 dv \leq 2^{-dm} C^m R_b^d \|h\|_{L_2(Z)}^2, \end{aligned} \tag{15}$$

where C is the number of translates of Z which are needed to cover $2Z$. The constant C can be bounded by a number which only depends on d , and an induction argument shows that at most C^m translates of Z are needed to cover $2^m Z$.

Proof of Proposition 3.5 Let $\lambda > 0$. By Proposition 3.1 and Corollary 3.4, there is a positive constant κ so that, for each $f \in PW_Z$, there is a sequence $(a_j^{(\lambda)}) \in \ell_2$ satisfying (9) and the estimate

$$\|(a_j^{(\lambda)})\|_2 \leq \kappa \|f\|_2. \tag{16}$$

Proposition 3.2 ensures that the function $I_\lambda(f)$, as defined in (9), is continuous and bounded whenever $f \in PW_Z$.

Next we show that I_λ is a bounded operator on $L_2(\mathbb{R}^d)$. Let $f \in PW_Z$ and let $(a_j^{(\lambda)}) \in \ell_2$ be the sequence given above. By (8), the function $Q := \sum_{k \in \mathbb{N}} a_k^{(\lambda)} e^{-i\langle \cdot, x_k \rangle}$ is square integrable on Z , so (12) ensures that $\|Q\|_{L_2(a+Z)} \leq R_b^2 \|Q\|_{L_2(Z)}$ whenever $a \in \mathbb{R}^d$. In particular, Q is locally square integrable, hence locally integrable, on \mathbb{R}^d . Combining these facts with the exponential decay of \hat{g}_λ , we find, via a standard periodization argument, that the function

$$w : \mathbb{R}^d \ni x \mapsto \left(\frac{\pi}{\lambda}\right)^{d/2} e^{-\|x\|^2/(4\lambda)} \sum_{k \in \mathbb{N}} a_k^{(\lambda)} e^{-i\langle x, x_k \rangle},$$

belongs to $L_2(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$. Moreover, using (8), (12), and (16), we arrive at the estimate

$$\|w\|_{L_2(\mathbb{R}^d)} \leq C' \|f\|_{L_2(\mathbb{R}^d)}, \tag{17}$$

where C' depends only on λ and R_b . As w is in $L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ and $I_\lambda(f)$ is continuous, it follows from general principles that w is the Fourier transform of $I_\lambda(f)$. Thus $I_\lambda(f) \in C_0(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ and $\|I_\lambda(f)\|_{L_2(\mathbb{R}^d)} \leq C'(2\pi)^{-d/2} \|f\|_{L_2(\mathbb{R}^d)}$, by (17) and (2). □

Proof of Theorem 3.6 Now fix $f \in PW_Z$ and write $I_\lambda(f)$ as

$$I_\lambda(f)(\cdot) = \sum_{j \in \mathbb{N}} a_j^{(\lambda)} g_\lambda(\cdot - x_j).$$

Recall from the preceding paragraph that the Fourier transform of $I_\lambda(f)$ is given by

$$\mathcal{F}[I_\lambda(f)](u) = \left(\frac{\pi}{\lambda}\right)^{d/2} e^{-\|u\|_2^2/(4\lambda)} \sum_{j \in \mathbb{N}} a_j^{(\lambda)} e^{-i\langle x_j, u \rangle}, \quad u \in \mathbb{R}^d. \tag{18}$$

The proof of Theorem 3.6 proceeds in three steps.

Step 1. We claim that there is a constant $D_1 < \infty$ and $\lambda_0 > 0$, only depending on (x_j) , so that

$$\|\mathcal{F}[I_\lambda(f)]\|_2 \leq D_1 e^{(1-\delta^2)/(4\lambda)} \|\mathcal{F}(f)\|_2, \quad \lambda \in (0, \lambda_0].$$

We start by defining

$$H_\lambda(u) = \left(\frac{\pi}{\lambda}\right)^{d/2} \sum_{j \in \mathbb{N}} a_j^{(\lambda)} e^{-i\langle x_j, u \rangle} = e^{\|u\|_2^2/(4\lambda)} \mathcal{F}[I_\lambda(f)](u), \quad u \in \mathbb{R}^d,$$

and let $h_\lambda = H_\lambda|_Z \in L_2(Z)$ (thus $H_\lambda = E(h_\lambda)$).

Suppose that $k \in \mathbb{N}$. Equation (3) implies that

$$(2\pi)^d f(x_k) = \int_Z \mathcal{F}[f](u) e^{i\langle x_k, u \rangle} du = \langle \mathcal{F}[f], e^{-i\langle x_k, \cdot \rangle} \rangle_Z. \tag{19}$$

On the other hand, (9) and (10) assert that

$$\begin{aligned} (2\pi)^d f(x_k) &= (2\pi)^d I_\lambda(f)(x_k) \\ &= \int_{\mathbb{R}^d} \mathcal{F}[I_\lambda(f)](u) e^{i\langle x_k, u \rangle} du \quad (\text{by (3)}) \\ &= \int_{\mathbb{R}^d} e^{-\|u\|_2^2/(4\lambda)} H_\lambda(u) e^{i\langle x_k, u \rangle} du \\ &= \int_Z e^{-\|u\|_2^2/(4\lambda)} H_\lambda(u) e^{i\langle x_k, u \rangle} du \\ &\quad + \sum_{m=1}^\infty \int_{2^m Z \setminus 2^{m-1} Z} e^{-\|u\|_2^2/(4\lambda)} H_\lambda(u) e^{i\langle x_k, u \rangle} du \\ &= \int_Z e^{-\|u\|_2^2/(4\lambda)} H_\lambda(u) e^{i\langle x_k, u \rangle} du \\ &\quad + \sum_{m=1}^\infty 2^{dm} \int_{Z \setminus 2^{-1} Z} e^{-\|2^m v\|_2^2/(4\lambda)} H_\lambda(2^m v) e^{i\langle x_k, 2^m v \rangle} dv \\ &= \int_Z e^{-\|u\|_2^2/(4\lambda)} h_\lambda(u) e^{i\langle x_k, u \rangle} du \\ &\quad + \sum_{m=1}^\infty 2^{dm} \int_{Z \setminus 2^{-1} Z} e^{-\|2^m v\|_2^2/(4\lambda)} A_m(h_\lambda)(v) \overline{A_m(e^{-i\langle x_k, \cdot \rangle})(v)} dv \end{aligned}$$

$$\begin{aligned}
&= \left\langle e^{-\|\cdot\|_2^2/(4\lambda)} h_\lambda, e^{-i(x_k, \cdot)} \right\rangle_Z \\
&\quad + \sum_{m=1}^{\infty} 2^{dm} \left\langle e^{-\|2^m(\cdot)\|_2^2/(4\lambda)} A_m(h_\lambda), A_m(e^{-i(x_k, \cdot)}) \right\rangle_Z \\
&= \left\langle \mathcal{F}[I_\lambda(f)], e^{-i(x_k, \cdot)} \right\rangle_Z \\
&\quad + \sum_{m=1}^{\infty} \left\langle 2^{dm} A_m^*(e^{-\|2^m(\cdot)\|_2^2/(4\lambda)} A_m(h_\lambda)), e^{-i(x_k, \cdot)} \right\rangle_Z \\
&= \left\langle \mathcal{F}[I_\lambda(f)] + \sum_{m=1}^{\infty} 2^{dm} A_m^*(e^{-\|2^m(\cdot)\|_2^2/(4\lambda)} A_m(h_\lambda)), e^{-i(x_k, \cdot)} \right\rangle_Z. \quad (20)
\end{aligned}$$

As $(e^{-i(x_k, \cdot)} : k \in \mathbb{N})$ is a Riesz basis for $L_2(Z)$ (in particular a complete system), equations (19) and (20) lead to the identity

$$\mathcal{F}[f] = \mathcal{F}[I_\lambda(f)] + \sum_{m=1}^{\infty} 2^{dm} A_m^*(e^{-\|2^m(\cdot)\|_2^2/(4\lambda)} A_m(h_\lambda)) \quad \text{a.e. on } Z. \quad (21)$$

Suppose now that $h \in L_2(Z)$ and $m \in \mathbb{N}$. We deduce from (15) that

$$\begin{aligned}
&\|2^{md} A_m^*(e^{-\|2^m(\cdot)\|_2^2/(4\lambda)} A_m(h))\|_{L_2(Z)}^2 \\
&\leq C^m R_b^4 2^{md} \|e^{-\|2^m(\cdot)\|_2^2/(4\lambda)} A_m(h)\|_{L_2(Z)}^2 \\
&\leq C^m R_b^4 2^{md} \|e^{-2^{2m-2}\delta^2/(4\lambda)} A_m(h)\|_{L_2(Z)}^2 \\
&\quad \left(\text{since } \text{supp } A_m(h) \subset Z \setminus \frac{1}{2}Z \subset Z \setminus \frac{\delta}{2}B_2 \right) \\
&\leq (C^m R_b^4)^2 e^{-2^{2m-2}\delta^2/(2\lambda)} \|h\|_{L_2(Z)}^2,
\end{aligned}$$

whence

$$\|2^{md} A_m^*(e^{-\|2^m(\cdot)\|_2^2/(4\lambda)} A_m(h))\|_{L_2(Z)} \leq C^m R_b^4 e^{-2^{2m-2}\delta^2/(4\lambda)}.$$

Therefore the linear operator

$$\tau_\lambda : L_2(Z) \rightarrow L_2(Z), \quad h \mapsto \sum_{m \in \mathbb{N}} 2^{md} A_m^*(e^{-\|2^m(\cdot)\|_2^2/(4\lambda)} A_m(h))$$

is bounded. In fact, as there are numbers $\lambda_0 > 0$ and D , which depend only on C (which only depends on d), such that

$$\sum_{m \in \mathbb{N}} C^m e^{-2^{2m-2}\delta^2/(4\lambda)} \leq D e^{-\delta^2/(4\lambda)}, \quad \lambda \in (0, \lambda_0], \quad (22)$$

the operator norm of τ_λ obeys the following estimate:

$$\|\tau_\lambda\| \leq R_b^4 D e^{-\delta^2/(4\lambda)} \quad \text{whenever } \lambda < \lambda_0. \quad (23)$$

As the operator τ_λ is positive, (21) yields

$$\|\mathcal{F}[f]\|_2 \|h_\lambda\|_2 \geq \langle \mathcal{F}[f], h_\lambda \rangle_Z \geq (e^{-\| \cdot \|_2^2/(4\lambda)}) h_\lambda, h_\lambda \rangle_Z \geq e^{-1/(4\lambda)} \|h_\lambda\|_2^2.$$

Consequently,

$$\|h_\lambda\|_2 \leq e^{1/(4\lambda)} \|\mathcal{F}[f]\|_2. \tag{24}$$

Thus, from (21) and (23) we get

$$\|\mathcal{F}[I_\lambda(f)]\|_Z \|2\| \leq \|\mathcal{F}[f]\|_2 + \|\tau_\lambda(h_\lambda)\|_2 \leq (1 + R_b^4 D e^{(1-\delta^2)/(4\lambda)}) \|\mathcal{F}[f]\|_2. \tag{25}$$

Our next step is to estimate $\|\mathcal{F}[I_\lambda(f)]\|_{\mathbb{R}^d \setminus Z} \|2\|$. Equation (18) implies that

$$\begin{aligned} & \|\mathcal{F}[I_\lambda(f)]\|_{\mathbb{R}^d \setminus Z} \|2\|^2 \\ &= \int_{\mathbb{R}^d \setminus Z} e^{-\|u\|_2^2/(2\lambda)} |H_\lambda(u)|^2 du \\ &= \sum_{m=1}^\infty \int_{2^m Z \setminus 2^{m-1} Z} e^{-\|u\|_2^2/(2\lambda)} |H_\lambda(u)|^2 du \\ &= \sum_{m=1}^\infty 2^{dm} \int_{Z \setminus 2^{-1} Z} e^{-2^{2m} \|v\|_2^2/(2\lambda)} |A_m(h_\lambda)(v)|^2 dv \\ &\leq \sum_{m=1}^\infty 2^{dm} e^{-2^{2m} \delta^2/(8\lambda)} \|A_m(h_\lambda)\|_2^2 \quad (\text{as } \text{supp } A_m(h) \subset Z \setminus 2^{-1} Z) \\ &\leq R_b^4 \|h_\lambda\|_2^2 \sum_{m=1}^\infty C^m e^{-2^{2m} \delta^2/(8\lambda)} \quad (\text{by (15)}) \\ &\leq e^{1/(2\lambda)} R_b^4 \|\mathcal{F}[f]\|_2^2 \sum_{m=1}^\infty e^{-2^{2m} \delta^2/(8\lambda)} C^m \quad (\text{by (24)}). \end{aligned} \tag{26}$$

By changing λ_0 and D , if need be, one obtains, as in (22),

$$\sum_{m=1}^\infty e^{-2^{2m} \delta^2/(8\lambda)} C^m \leq D e^{-\delta^2/(2\lambda)}, \quad \lambda \in (0, \lambda_0]. \tag{27}$$

Combining (25) and (26) proves our claim.

Remark The opening argument in the foregoing step makes critical use of the assumption that $(x_j : j \in \mathbb{N})$ is a Riesz-basis sequence, for it relies on the fact that every member of $L_2(Z)$ admits a *unique* expansion in terms of the exponentials $(e^{-i(x_j, \cdot)} : j \in \mathbb{N})$. This guarantees the relation $(\pi/\lambda)^{d/2} a_j^{(\lambda)} = \langle h_\lambda, e_j^* \rangle_Z$ for every positive integer j , which, in turn, validates the assertion $H_\lambda = E(h_\lambda)$. Sadly, this approach falls short in the case of frame sequences, because frame representations are no longer unique.

Step 2. Let $f \in PW_{\beta B_2}$. There is a positive constant D_2 such that

$$\|f - I_\lambda(f)\|_2 \leq D_2 e^{(\beta^2 - 3\delta^2 + 2)/(4\lambda)} \|f\|_2, \quad (28)$$

for all $0 < \lambda < \lambda_0$.

Remark Note that (28) implies that $\lim_{\lambda \rightarrow 0^+} I_\lambda(f) = f$ in $L_2(\mathbb{R}^d)$.

To prove (28) we define $\tilde{\tau}_\lambda = e^{1/(4\lambda)} \tau_\lambda$,

$$M_\lambda : L_2(Z) \rightarrow L_2(Z), \quad h \mapsto e^{-(1 - \|\cdot\|_2^2)/(4\lambda)} h, \quad \text{and}$$

$$L_\lambda : L_2(Z) \rightarrow L_2(Z), \quad h \mapsto R \circ \mathcal{F} \circ I_\lambda \circ \mathcal{F}^{-1}(h),$$

where $R : L_2(\mathbb{R}^d) \rightarrow L_2(Z)$ is the restriction map.

Proposition 4.1 *The map $\text{Id} + \tilde{\tau}_\lambda \circ M_\lambda$ is an invertible operator on $L_2(Z)$, and $(\text{Id} + \tilde{\tau}_\lambda \circ M_\lambda)^{-1} = L_\lambda$.*

Proof Let $h \in PW_Z$. From (21) we obtain (a.e. on Z)

$$\begin{aligned} \mathcal{F}[h] &= \mathcal{F}[I_\lambda(h)] + \tau_\lambda(e^{\|\cdot\|^2/(4\lambda)} \mathcal{F}[I_\lambda(h)]) \\ &= \mathcal{F}[I_\lambda(h)] + \tilde{\tau}_\lambda \circ M_\lambda(\mathcal{F}[I_\lambda(h)]) = (\text{Id} + \tilde{\tau}_\lambda \circ M_\lambda)L_\lambda(\mathcal{F}[h]). \end{aligned}$$

This implies that $\text{Id} + \tilde{\tau}_\lambda \circ M_\lambda$ is surjective and is a left inverse of the bounded operator L_λ . Next we show that $\text{Id} + \tilde{\tau}_\lambda \circ M_\lambda$ is also injective. To that end, let $(\text{Id} + \tilde{\tau}_\lambda \circ M_\lambda)(h) = 0$ for some $h \in L_2(Z)$. Then

$$\begin{aligned} 0 &= \langle (\text{Id} + \tilde{\tau}_\lambda \circ M_\lambda)(h), M_\lambda(h) \rangle_Z = \langle h, M_\lambda(h) \rangle_Z + \langle \tilde{\tau}_\lambda(M_\lambda(h)), M_\lambda(h) \rangle_Z \\ &\geq \langle h, M_\lambda(h) \rangle_Z \geq 0, \end{aligned}$$

the first inequality above being a consequence of the positivity of $\tilde{\tau}_\lambda$. Hence $\langle h, M_\lambda(h) \rangle_Z = 0$, which implies that $h = 0$, because M_λ is a strictly positive operator. The injectivity of $\text{Id} + \tilde{\tau}_\lambda \circ M_\lambda$ follows. Thus $\text{Id} + \tilde{\tau}_\lambda \circ M_\lambda$ is invertible, and its inverse is L_λ . \square

Proposition 4.1 provides the following identity on Z :

$$\begin{aligned} \mathcal{F}[f] - \mathcal{F}[I_\lambda(f)] &= [\text{Id} - (\text{Id} + \tilde{\tau}_\lambda \circ M_\lambda)^{-1}](\mathcal{F}[f]) \\ &= (\text{Id} + \tilde{\tau}_\lambda \circ M_\lambda)^{-1} \circ \tilde{\tau}_\lambda \circ M_\lambda(\mathcal{F}[f]). \end{aligned}$$

If $f \in PW_{\beta B_2}$, then (23) and Step 1 provide

$$\begin{aligned} \|\mathcal{F}[f] - \mathcal{F}[I_\lambda(f)]\|_2 &\leq \|(\text{Id} + \tilde{\tau}_\lambda \circ M_\lambda)^{-1}\| \|\tilde{\tau}_\lambda\| \|M_\lambda(\mathcal{F}[f])\|_2 \\ &\leq D_1 e^{(1 - \delta^2)/(4\lambda)} e^{1/(4\lambda)} R_b^2 D e^{(-\delta^2)/(4\lambda)} \|M_\lambda(\mathcal{F}[f])\|_2 \\ &=: D' e^{(1 - 2\delta^2)/(4\lambda)} \|e^{\|\cdot\|^2/(4\lambda)}(\mathcal{F}[f])\|_2 \\ &\leq D' e^{(\beta^2 + 1 - 2\delta^2)/(4\lambda)} \|\mathcal{F}[f]\|. \end{aligned} \quad (29)$$

Now the first inequality in (26) yields

$$\begin{aligned}
 & \| \mathcal{F}[I_\lambda(f)] \|_{\mathbb{R}^d \setminus Z} \|_2^2 \\
 & \leq \sum_{m=1}^\infty 2^{dm} e^{-2^{2m} \delta^2 / (8\lambda)} \| A_m(h_\lambda) \|_2^2 \\
 & \leq R_b^4 \sum_{m=1}^\infty C^m e^{-2^{2m} \delta^2 / (8\lambda)} \| e^{\|\cdot\|_2^2 / (4\lambda)} \mathcal{F}[I_\lambda(f)] \|_Z \|_2^2 \quad \text{by (15)} \\
 & \leq R_b^4 D e^{-\delta^2 / (2\lambda)} \| e^{\|\cdot\|_2^2 / (4\lambda)} \mathcal{F}[I_\lambda(f)] \|_Z \|_2^2 \quad \text{by (27)} \\
 & \leq R_b^4 D [\| e^{(\|\cdot\|_2^2 - \delta^2) / (4\lambda)} (\mathcal{F}[I_\lambda(f)] \|_Z - \mathcal{F}(f)) \|_2 + \| e^{(\|\cdot\|_2^2 - \delta^2) / (4\lambda)} \mathcal{F}(f) \|_2]^2 \\
 & \leq R_b^4 D [e^{(1-\delta^2) / (4\lambda)} \| (\mathcal{F}[I_\lambda(f)] \|_Z - \mathcal{F}(f)) \|_2 + \| e^{(\|\cdot\|_2^2 - \delta^2) / (4\lambda)} \mathcal{F}(f) \|_2]^2.
 \end{aligned}
 \tag{30}$$

If we restrict to $f \in PW_{\beta B_2}$, then, by (29),

$$\begin{aligned}
 & \| \mathcal{F}[I_\lambda(f)] \|_{\mathbb{R}^d \setminus Z} \|_2^2 \\
 & \leq R_b^4 D [D' e^{(\beta^2 + 1 - 2\delta^2) / (4\lambda)} e^{(1-\delta^2) / (4\lambda)} + e^{(\beta^2 - \delta^2) / (4\lambda)}]^2 \| \mathcal{F}(f) \|_2^2.
 \end{aligned}$$

Combining this with (2), and using (29) once again, we obtain the following estimate for some constant D_2 :

$$\| f - I_\lambda f \|_2 \leq D_2 e^{(\beta^2 - 3\delta^2 + 2) / (4\lambda)} \| f \|_2.$$

Step 3. Suppose that $f \in PW_{\beta B_2}$. There exist constants $\lambda_1 \in (0, \lambda_0]$ and D_3 so that

$$| I_\lambda(f)(x) - f(x) | \leq D_3 e^{(\beta^2 - 3\delta^2 + 2) / (4\lambda)} \| f \|_2,$$

for all $0 < \lambda \leq \lambda_1$ and $x \in \mathbb{R}^d$. In particular $\lim_{\lambda \rightarrow 0^+} I_\lambda(f) = f$ uniformly on \mathbb{R}^d .

We first observe that, we can find, as before, numbers $\lambda_1 \in (0, \lambda_0]$ and $D'' > 0$, such that

$$m^{1/2}(Z) R_b^2 \sum_{m=1}^\infty C^{m/2} 2^{dm/2} e^{(4-2^{2m}) / (16\lambda)} \leq D'' \quad \text{whenever } 0 < \lambda \leq \lambda_1. \tag{31}$$

Let $x \in \mathbb{R}^d$ and $f \in PW_{\beta B_2}$. We use (3) to write

$$\begin{aligned}
 & | I_\lambda(f)(x) - f(x) | \\
 & = \frac{1}{(2\pi)^d} \left| \int_Z [\mathcal{F}[I_\lambda(f)](u) - \mathcal{F}[f](u)] e^{ixu} du + \int_{\mathbb{R}^d \setminus Z} \mathcal{F}[I_\lambda(f)](u) e^{ixu} du \right| \\
 & \leq \frac{1}{(2\pi)^d} [\| \mathcal{F}[I_\lambda(f)] \|_Z - \mathcal{F}[f] \|_1 + \| \mathcal{F}[I_\lambda(f)] \|_{\mathbb{R}^d \setminus Z} \|_1].
 \end{aligned}
 \tag{32}$$

From the BCS inequality and (29) we deduce that $\lim_{\lambda \rightarrow 0^+} \|\mathcal{F}[I_\lambda(f)]\|_Z - \|\mathcal{F}[f]\|_1 = 0$, and an argument similar to that in (26) yields

$$\begin{aligned}
 & \|\mathcal{F}[I_\lambda(f)]\|_{\mathbb{R}^d \setminus Z} \|_1 \\
 &= \sum_{m=1}^{\infty} \int_{2^m Z \setminus 2^{m-1} Z} e^{-\|u\|_2^2/(4\lambda)} |H_\lambda(u)| du \\
 &= \sum_{m=1}^{\infty} 2^{dm} \int_{Z \setminus 2^{-1} Z} e^{-2^{2m} \|v\|_2^2/(4\lambda)} |A_m(h_\lambda)(v)| dv \\
 &\leq m^{1/2} (Z) \sum_{m=1}^{\infty} 2^{dm} \|e^{-2^{2m} \|\cdot\|_2^2/(4\lambda)} A_m(h_\lambda)\|_2 \quad (\text{by the BCS inequality}) \\
 &\leq m^{1/2} (Z) R_b^2 \sum_{m=1}^{\infty} C^{m/2} 2^{dm/2} e^{-2^{2m} \delta^2/(16\lambda)} \|h_\lambda\|_2 \\
 &\quad \left(\text{by (15) and since } \text{supp}(A_m(h)) \subset Z \setminus \frac{1}{2}Z \right) \\
 &= D'' \|e^{(\|\cdot\|_2^2 - \delta^2)/(4\lambda)} \mathcal{F}[I_\lambda(f)]\|_Z \|_2 \quad (\text{by (32)}) \\
 &\leq D'' e^{(1-\delta^2)/(4\lambda)} [\|\mathcal{F}[I_\lambda(f)]\|_Z - \|\mathcal{F}[f]\|_2 + \|\mathcal{F}[f]\|_2] \\
 &\leq D' D'' e^{(\beta^2 + 2 - 3\delta^2)/(4\lambda)} \|\mathcal{F}[f]\| \quad (\text{by (29)})
 \end{aligned}$$

This concludes the proof. \square

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