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On Gibbs-Wilbraham Phenomenon and the Arclength of Fourier Series

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Abstract The arclength of the graphs $\Gamma(S_N(f))$ of the partial sums $S_N(f)$ of the Fourier series of a piecewise C^1 function f with jump discontinuities is equal asymptotically to (the sum of all jumps of f) $\times L_N$, where L_N is the Lebesgue constant. This is an improvement of R. Strichartz (J. Fourier Anal. Appl. 6, 533–536, 2000).

Keywords Fourier series · Gibbs-Wilbraham phenomenon · Arclength of curve · Piecewise C^1 function · Lebesgue constant

1 Introduction

Let f be a 2π periodic piecewise C^1 function with a finite number of jump discontinuities. Let the Fourier coefficient $\hat{f}(n)$ and the associated Fourier partial sum $S_N(f)(x)$ denote in the usual notation:

$$\hat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \qquad S_N(f)(x) := \sum_{n=-N}^{N} \hat{f}(n)e^{inx}.$$

If f is discontinuous at some points, the arclength of the graphs $\Gamma(S_N(f))$ of $S_N(f)$ don't converge to the arclength of the graphs $\Gamma(f)$ of f. This is suggested easily by the Gibbs-Wilbraham phenomenon.

R. Strichartz (2000, [2]) proved the following three propositions on the arclength of the graphs $\Gamma(S_N(f))$.

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Proposition 1.1 *Let f be a piecewise C^1 function on the circle with a finite number of jump discontinuities. Then*

$$\mathrm{length}(\Gamma(S_N(f)) = O(\log N) \quad \text{as } N \to \infty.$$

Proposition 1.2 *Let f be a continuous piecewise C^1 function on the circle. Then*

$$\lim_{N\to\infty} \mathrm{length}(\Gamma(S_N(f)) = \mathrm{length}(\Gamma(f)).$$

Proposition 1.3 *Suppose*

$$f_N = \varphi_N * f$$

for $\{\varphi_N\}$ a positive approximate identity, and f is of bounded variation. Then

$$\lim_{N\to\infty} \mathrm{length}(\Gamma(f_N)) = \mathrm{length}(\Gamma(f)) \tag{1.1}$$

and moreover

$$\mathrm{length}(\Gamma(f_N)) \le \mathrm{length}(\Gamma(f)) \quad \text{for every } N.$$

Remark 1.4 The equality (1.1) of Proposition 1.3 needs some condition, for example continuity of f. Indeed, the left hand side is independent of the value of f at a point, whereas the right hand side depends on the value at a point. For example, let the function f_a be the following: $f_a(x) := 1/2\,\mathrm{sign}(x)$ (if $-1/2 \le x < 1/2$ and $x \ne 0$) and $:= a$ (if $x = 0$). Then the left hand side is independent of a, whereas the right hand side depends on a. In fact $\mathrm{length}(\Gamma(f_a)) = 2$ (if $|a| \le 1/2$), and $= 2|a| + 1$ (if $|a| > 1/2$). If, however, f is both continuous and of bounded variation, then the proof of Strichartz applies to prove (1.1). Moreover, if the kernels are even and f is of bounded variation, then the limit in equality (1.1) exists and is the length of $\Gamma(\bar{f})$) where $\bar{f}$ is the normalized function $(f(x+0) + f(x-0))/2$ (e.g. Pinsky [2, p. 47]).

2 Main Theorem

We can prove the following theorem. This is an improvement of Proposition 1.2 of Strichartz.

Theorem 2.1 *Let f be a piecewise C^1 function on the circle. Then*

$$\lim_{N\to\infty} \frac{\mathrm{length}(\Gamma(S_N(f))}{\mathrm{L}_N} = \text{the sum of all jumps of } f,$$

where L_N is the Nth Lebesgue constant, i.e. $\mathrm{L}_N = \frac{4}{\pi^2}\log N + O(1)$.

Proof Let $\{a_j\}_{j=0}^k$ be a sequence with $-\pi \le a_0 < a_1 < a_2 < \cdots < a_k = a_0 + 2\pi$.

Let f be C^1 on each interval (a_{j-1}, a_j) $(j = 1, \ldots, k)$ and let $f(a_j + 0)$, $f(a_j - 0)$ and $f'(a_j + 0)$, $f'(a_j - 0)$ be the right hand limit of f, the left hand limit of f and the right hand limit of f', the left hand limit of f' at a_j.

Step 1. Let functions h_j be defined as follows:

$$h_j(x) := \big(M_j(x-a_{j-1}) + f(a_{j-1}+0)\big)\,\chi_{[a_{j-1},a_j)}(x),$$

where $M_j = \frac{f(a_j-0)-f(a_{j-1}+0)}{a_j-a_{j-1}}$ and $\chi_{[a_{j-1},a_j)}(x)$ is the characteristic function of $[a_{j-1},a_j)$. Then we have $h_j(a_{j-1}+0) = f(a_{j-1}+0)$ and $h_j(a_j-0) = f(a_j-0)$. Moreover we define functions g_j as follows:

$$g_j(x) := f(x)\chi_{[a_{j-1},a_j)}(x) - h_j(x) \quad (j = 1,\dots,k).$$

Then g_j is a continuous piecewise C^1 function on the circle. Therefore by Proposition 1.2 of Strichartz, the following holds.

$$\lim_{N\to\infty} \mathrm{length}(\Gamma(S_N(g_j)) = \mathrm{length}(\Gamma(g_j)) \;\text{ for each } j = 1,\dots,k.$$

Step 2. Suppose $h(x) = (M(x-a)+\alpha)\chi_{[a,b)}(x)$, then we have

$$\begin{aligned}\hat{h}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\left(\frac{-\beta e^{-inb}+\alpha e^{-ina}}{in} - M\frac{e^{-inb}-e^{-ina}}{(in)^2}\right) \quad (n\neq 0),\end{aligned}$$

where $\beta = h(b)$.

$$\begin{aligned}&S_N(h)(x)\\ &\quad= \hat{h}(0) + \frac{1}{2\pi}\left\{-\beta\sum_{0<|n|\le N}\frac{e^{in(x-b)}}{in} + \alpha\sum_{0<|n|\le N}\frac{e^{in(x-a)}}{in}\right.\\ &\qquad\left. - M\left(\sum_{0<|n|\le N}\frac{e^{in(x-b)}}{(in)^2} - \sum_{0<|n|\le N}\frac{e^{in(x-a)}}{(in)^2}\right)\right\},\\ &\frac{d}{dx}(S_N(h)(x))\\ &\quad= \frac{1}{2\pi}\left\{-\beta\sum_{0<|n|\le N} e^{in(x-b)} + \alpha\sum_{0<|n|\le N} e^{in(x-b)}\right.\\ &\qquad\left. - M\left(\sum_{0<|n|\le N}\frac{e^{in(x-b)}}{in} - \sum_{0<|n|\le N}\frac{e^{in(x-a)}}{in}\right)\right\}\\ &\quad= \frac{1}{2\pi}\{-\beta\,(D_N(x-b)-1) + \alpha\,(D_N(x-a)-1)\}\\ &\qquad - M(S_N(\varphi)(x-b) - S_N(\varphi)(x-a)),\end{aligned}$$

where D_N is the Dirichlet kernel and $\varphi(x)$ is the 2π-periodic sawtooth function.

Next, we obtain L^1-estimations of four terms.

$$\|S_N(\varphi)\|_1 \le \left(2\pi\int_{-\pi}^{\pi}|S_N(\varphi)(x)|^2dx\right)^{\frac{1}{2}} \le \left(2\sum_{n=1}^{N}\frac{1}{n^2}\right)^{\frac{1}{2}} \le \frac{\pi}{\sqrt{3}}.$$

On the other hand, suppose $-\pi \le a < b \le \pi$ and $b-a < 2\pi$. Let δ be a positive number which satisfies $2\delta < b-a \le 2\pi - 2\delta$. Then $a+\delta < b-\delta$, $b+\delta < a+2\pi-\delta$ and $-2\pi+\delta < a-b-\delta < a-b+\delta < -\delta$.

$$\begin{aligned}
&\int_{-\pi}^{\pi}|-\beta D_N(x-b)+\alpha D_N(x-a)|dx \\
&\quad = \int_{[a-\delta,a+\delta]} + \int_{(a+\delta,b-\delta)} + \int_{[b-\delta,b+\delta]} + \int_{(b+\delta,a+2\pi-\delta)} = I_1+I_2+I_3+I_4. \\
&I_1 = |\alpha|\int_{a-\delta}^{a+\delta}|D_N(x-a)|dx + O\left(\int_{a-\delta}^{a+\delta}|D_N(x-b)|dx\right) \\
&\quad = |\alpha|\int_{-\delta}^{\delta}|D_N(x)|dx + O\left(\int_{a-b-\delta}^{a-b+\delta}|D_N(x)|dx\right) \\
&\quad = |\alpha|\int_{-\pi}^{\pi}|D_N(x)|dx + O\left(\int_{\delta<|x|\le\pi}|D_N(x)|dx\right),
\end{aligned}$$

where L_N is the Nth Lebesgue constant, i.e.

$$\mathrm{L}_N = \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_N(x)|\,dx = \frac{4}{\pi^2}\log N + O(1)$$

(e.g. Finch [1]). Therefore we have

$$\begin{aligned}
&I_1 = 2\pi|\alpha|\mathrm{L}_N + O(1), \quad \text{and also} \quad I_3 = 2\pi|\beta|\mathrm{L}_N + O(1). \\
&I_2 = O\left(\int_{a+\delta}^{b-\delta}|D_N(x-a)|dx\right) + O\left(\int_{a+\delta}^{b-\delta}|D_N(x-b)|dx\right) \\
&\quad = O\left(\int_{\delta}^{b-a-\delta}|D_N(x)|dx\right) + O\left(\int_{a-b+\delta}^{-\delta}|D_N(x)|dx\right) \\
&\quad = O\left(\int_{\delta<|x|\le\pi}|D_N(x)|dx\right) = O(1).
\end{aligned}$$

And also $I_4 = O(1)$. Consequently we have

$$\|S_N(h)'\|_1 = (|\alpha|+|\beta|)\mathrm{L}_N + O(1).$$

Step 3. By

$$f=\sum_{j=1}^{k} g_j+\sum_{j=1}^{k} h_j, \qquad S_N(f)'=\sum_{j=1}^{k} S_N(g_j)'+\sum_{j=1}^{k} S_N(h_j)'.$$

$$\|S_N(f)'\|_1=\left\|\sum_{j=1}^{k} S_N(h_j)'\right\|_1+O\left(\sum_{j=1}^{k}\|S_N(g_j)'\|_1\right)=\left\|\sum_{j=1}^{k} S_N(h_j)'\right\|_1+O(1)$$

(by Step 1). Now

$$\begin{aligned}
\sum_{j=1}^{k} S_N(h_j)'(x) &= \frac{1}{2\pi}\sum_{j=1}^{k}\big\{-f(a_j-0)(D_N(x-a_j)-1)\\
&\quad + f(a_{j-1}+0)(D_N(x-a_{j-1})-1)\big\}\\
&\quad -\sum_{j=1}^{k} M_j S_n(\varphi)(x-a_j)+\sum_{j=1}^{k} M_j S_n(\varphi)(x-a_{j-1})\\
&= \frac{1}{2\pi}\left\{\sum_{j=0}^{k-1}\{f(a_j+0)-f(a_j-0)\}(D_N(x-a_j)-1)\right\}\\
&\quad -\sum_{j=1}^{k} M_j S_N(\varphi)(x-a_j)+\sum_{j=1}^{k} M_j S_N(\varphi)(x-a_{j-1})\\
&= \frac{1}{2\pi}\left\{\sum_{j=0}^{k-1}\{f(a_j+0)-f(a_j-0)\}D_N(x-a_j)\right\}\\
&\quad -\sum_{j=1}^{k} M_j S_N(\varphi)(x-a_j)+\sum_{j=1}^{k} M_j S_N(\varphi)(x-a_{j-1})\\
&\quad -\frac{1}{2\pi}\sum_{j=0}^{k-1}(f(a_j+0)-f(a_j-0)).
\end{aligned}$$

By the same method with Step 2, we have the following equalities.

$$\left\|\sum_{j=1}^{k} S_N(h_j)'\right\|_1=\left(\sum_{j=0}^{k-1}|f(a_j+0)-f(a_j-0)|\right)\mathrm{L}_N+O(1),$$

$$\|S_N(f)'\|_1=\left(\sum_{j=0}^{k-1}|f(a_j+0)-f(a_j-0)|\right)\mathrm{L}_N+O(1).$$

Step 4. Conclusion.

$$\begin{aligned}\text{length}(\Gamma(S_N(f))) &= \int_{-\pi}^{\pi}\sqrt{1+|S_N(f))'(x)|^2}dx = \int_{-\pi}^{\pi}|S_N(f)'(x)|dx + O(1)\\ &= \left(\sum_{j=0}^{k-1}|f(a_j+0)-f(a_j-0)|\right)\mathrm{L}_N + O(1).\end{aligned}$$

□

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References

1. Finch, S.R.: Mathematical Constants. Encyclopedia of Mathematics and Its Applications, vol. 94, Cambridge (2003)
2. Pinsky, M.A.: Introduction to Fourier Analysis and Wavelets. Graduate Studies in Mathematics, vol. 102. Am. Math. Soc., Providence (2009)
3. Strichartz, R.S.: Gibbs' phenomenon and arclength. J. Fourier Anal. Appl. **6**, 533–536 (2000)