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Hardy-Sobolev Spaces Decomposition in Signal Analysis

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Abstract Some fundamental formulas and relations in signal analysis are based on the amplitude-phase representations $s(t) = A(t)e^{i\varphi(t)}$ and $\hat{s}(\omega) = B(\omega)e^{i\psi(\omega)}$, where the amplitude functions $A(t)$ and $B(\omega)$ and the phase functions $\varphi(t)$ and $\psi(\omega)$ are assumed to be differentiable. They include the amplitude-phase representations of the first and second order means of the Fourier frequency and time, and the equivalence between two forms of the covariance. A proof of the uncertainty principle is also based on the amplitude-phase representations. In general, however, signals of finite energy do not necessarily have differentiable amplitude-phase representations. The study presented in this paper extends the classical formulas and relations to general signals of finite energy. Under the formulation of the phase and amplitude derivatives based on the Hardy-Sobolev spaces decomposition the extended formulas reveal new features, and contribute to the foundations of time-frequency analysis. The established theory is based on the equivalent classes of the L^2 space but not on particular representations of the classes. We also give a proof of the uncertainty principle by using the amplitude-phase representations defined through the Hardy-Sobolev spaces decomposition.

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1 Introduction

The work presented in this paper was motivated by the recent study on the relations between the Fourier frequency and the analytic frequency of a signal [4–7, 11, 13, 16, 23], and especially on the relations between their respective positivities. For a signal s of finite energy, we say that its *Fourier frequency is of the positivity property* if $\hat{s}\chi_- = 0$, where $\hat{s}$ is the Fourier transform of s and χ_- is the characteristic (indicator) function of the set $(-\infty, 0)$. In the case s is also be said to *possess* or to *be of only positive Fourier frequencies*. In contrast, we say that its *analytic frequency is of the positivity property* if its *analytic phase derivative* φ' (*analytic frequency*) is a non-negative measurable function, where φ is defined (if it could be, see next section) through the *analytic signal associated with* s, viz. $s(t) + iHs(t)$, where H stands for Hilbert transformation, defined by (see also the basic result 4 stated in Sect. 2)

$$Hs(t) \triangleq \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|t-u|>\epsilon} \frac{s(u)}{t-u} du. \tag{1.1}$$

The terminology analytic phase derivative will be often abbreviated as *phase derivative*. In this paper, when one side of an equality is defined through the other side, we write "$\triangleq$" instead of "$=$".

The function $s + iHs$ is the boundary value of an analytic function in the upper-half complex plane, and thus possesses only positive Fourier frequencies (see the basic results 3 and 4 in Sect. 2). One thought that the positivity of the Fourier frequency would imply the positivity of the analytic frequency, but it is not true. While outer functions in the upper-half complex plane (signals of minimum phase and minimum energy delay) are of only positive Fourier frequencies, they have negative phase derivatives on sets of positive Lebesgue measure [19]. The wrong thought is caused by mixing up the two different kinds of frequencies. The result (1.2) given in Theorem 1.1, however, indicates that they indeed have some relations.

In the sequel we always assume the signal under study is of unit energy, that is

$$\|s\|_2^2 = \int_{-\infty}^{\infty} |s(t)|^2 dt = 1.$$

A precise version of a result proved in [4] is as follows.

Theorem 1.1 *Let $s(t) = A(t)e^{i\varphi(t)} \in L^2(\mathbb{R})$, $A(t) = |s(t)|$. Assume that the classical derivatives $\frac{dA}{dt}$, $\frac{d\varphi}{dt}$ and $\frac{ds}{dt}$ all exist and are Lebesgue measurable, and $\frac{ds}{dt}$ is in $L^2(\mathbb{R})$. Then there holds*

$$\langle\omega\rangle = \int_{-\infty}^{\infty} \frac{d\varphi(t)}{dt} |s(t)|^2 dt, \tag{1.2}$$

where $\langle\omega\rangle$ *is defined by*

$$\langle\omega\rangle \triangleq \int_{-\infty}^{\infty} \omega|\hat{s}(\omega)|^2 d\omega.$$

The formula (1.2) shows that if the mean of the Fourier frequency is positive then the mean of the analytic frequency $\frac{d\varphi(t)}{dt}$ is also positive, and vice versa. In particular, if s is taken to be an analytic signal, that is, $s = s_1 + iHs_1$ for some signal $s_1 \in L^2(\mathbb{R})$, then s is of only positive Fourier frequencies and the above formulas imply

$$0 < \langle\omega\rangle = \int_0^{\infty} \omega|\hat{s}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} \frac{d\varphi(t)}{dt}|s(t)|^2 dt. \tag{1.3}$$

In such a case, however, there does not necessarily hold $\frac{d\varphi(t)}{dt} \geq 0$, a.e.

For a real-valued signal s_1 and $s = s_1 + iHs_1 = Ae^{i\varphi}$, if the phase derivative $\frac{d\varphi}{dt} \geq 0$, a.e., then the (*analytic*) *instantaneous frequency of* s_1 is said to exist, and defined to be $\frac{d\varphi}{dt}$. We call a real-valued signal *mono-component* if and only if it has such defined instantaneous frequency, or, equivalently, the associated analytic signal possesses positive (non-negative) phase derivative [18]. Note that in this set of definitions the phase derivative $\frac{d\varphi}{dt}$ should be suitably defined and the definition is given in Sect. 3. The positivity of the analytic phase derivative and its related issues have been the subject of much controversy. The positivity requirement for the frequency is supported by physics science. We now provide more background information in relation to time-frequency analysis for the interest in positive phase derivative.

As in Theorem 1.1, let $s(t) = A(t)e^{i\varphi(t)} \in L^2(\mathbb{R})$, $A(t) = |s(t)|$, and $\hat{s}(\omega) = B(\omega)e^{i\psi(\omega)}$, $B(\omega) = |\hat{s}(\omega)|$. One usually requires that a time-frequency distribution $P(t,\omega)$ of s should satisfy the positivity condition

$$P(t,\omega) \geq 0 \tag{1.4}$$

and the usual edge distribution conditions

$$P_F(\omega) \triangleq \int_{-\infty}^{\infty} P(t,\omega)dt = |\hat{s}(\omega)|^2 \quad \text{and} \quad P_T(t) \triangleq \int_{-\infty}^{\infty} P(t,\omega)d\omega = |s(t)|^2. \tag{1.5}$$

It is also reasonable to require

$$\langle\omega\rangle_t \triangleq \frac{1}{P_T(t)} \int_{-\infty}^{\infty} \omega P(t,\omega)d\omega = \frac{d\varphi(t)}{dt}, \tag{1.6}$$

and

$$\langle t\rangle_\omega \triangleq \frac{1}{P_F(\omega)} \int_{-\infty}^{\infty} tP(t,\omega)dt = -\frac{d\psi(\omega)}{d\omega}. \tag{1.7}$$

The left-hand-side of (1.6) is the conditional mean of the Fourier frequency at the time moment t, while that of (1.7) is the conditional mean of the time at the frequency ω. The Wigner-Ville distribution, for instance, satisfies the conditions (1.5), (1.6) and (1.7), but not (1.4).

We also expect the *weak finite support* property (see [4]), that is

$$\hat{s}\chi_- = 0 \quad \text{implies} \quad P(t,\cdot)\chi_- = 0, \quad \forall t. \tag{1.8}$$

The Wigner-Ville distribution has this property. It is now interesting to observe that if the existence of a time-frequency distribution $P(t,\omega)$ that satisfies (1.8), (1.4) and (1.6) is assured, then the positivity of Fourier frequency implies the positivity of analytic frequency. Because of the already mentioned fact that outer functions have only positive Fourier frequencies but have negative analytic frequencies on sets of positive measures, such distributions possibly exist only under additional conditions.

The positivity of phase derivative is crucial for defining meaningful instantaneous frequency function. The observation on outer functions tells that one cannot expect positivity of analytic phase derivative in general. What one could do would be to decompose signals of only positive Fourier frequencies into sums of mono-components [18]. Fourier series is the classical example of such a decomposition. Expansions in Takenaka-Malmquist systems as generalizations of Fourier series belong to the same category [1, 3, 10]. Now comes the concept *adaptive mono-component decomposition* where adaptivity refers to fast convergence of the decomposition [20, 21]. In an adaptive decomposition *intrinsic mono-components* of the given signal are extracted. The present work will not pursue this direction. The interested reader is referred to [18–20].

To pursue the study of positive phase derivative, we should first define the notion phase derivative. The present paper is set to generalize, with appropriate formulations, the classical and fundamental formulas such as (1.2), (1.9) and (1.10), while the classical phase derivatives $\frac{d\varphi}{dt}$ and $\frac{d\psi}{d\omega}$ may not exist. For all purposes the right formulation of phase derivative should be first done. In general cases $e^{i\varphi(t)}$ can be defined by $s/|s|$. By doing so, however, the phase function $\varphi(t)$ is not uniquely defined: it has infinitely many representations. A general s may not be a smooth function, and when it is, Hs may not be smooth, and $\varphi(t)$ may not have a smooth representation.

Similar queries arise from the study of signals of minimum phase and minimum energy delay in relation to all-pass filters (see [14, 15]). In the complex analysis terminology, signals of minimum phase and minimum energy delay are outer functions and all-pass filters are inner functions in the corresponding domains. The related results in signal analysis are only interplay relations between the two types of analytic functions. To the authors' knowledge, before [19], no literature gave a valid proof for the fact that outer functions are of minimum phase, or, equivalently, inner functions are of positive "phase derivatives" on the boundary, and no literature gave a rigorous definition of the notion boundary phase derivative of the Z-transform of a general signal. The technicalities in the literature were only valid for proving the positivity of the phase derivatives of finite Blaschke products and of singular inner functions induced by a finite linear combination of shifted Dirac point measures. In [19], the existence and positivity of the well defined phase derivatives of the boundary values of inner functions are reduced to the classical Julia-Wolff-Carathéodory Theorem, and the existence and zero-mean property of the phase derivatives of the boundary values of a class of outer functions are proved.

In this paper, based on the definition of the phase and amplitude derivatives for functions in the Hardy-Sobolev spaces, we give generalizations of (1.2) and the following relations:

$$\sigma_\omega^2 = \int_{-\infty}^{\infty} \left(\frac{dA(t)}{dt}\right)^2 dt + \int_{-\infty}^{\infty} \left(\frac{d\varphi(t)}{dt} - \langle\omega\rangle\right)^2 A^2(t)dt, \tag{1.9}$$

and

$$\mathrm{Cov}_{t\omega} \triangleq \left\langle t\frac{d\varphi(t)}{dt}\right\rangle - \langle t\rangle\langle\omega\rangle = -\left\langle \omega\frac{d\psi(\omega)}{d\omega}\right\rangle - \langle t\rangle\langle\omega\rangle, \tag{1.10}$$

where

$$\left\langle t\frac{d\varphi(t)}{dt}\right\rangle \triangleq \int_{-\infty}^{\infty} t\frac{d\varphi(t)}{dt}A^2(t)dt, \quad \text{and} \quad \left\langle \omega\frac{d\psi(\omega)}{d\omega}\right\rangle \triangleq \int_{-\infty}^{\infty} \omega\frac{d\psi(\omega)}{d\omega}B^2(\omega)d\omega \tag{1.11}$$

Our contribution in this paper is two-fold: (i) We generalize the concept "phase derivative", $\frac{d\varphi}{dt}$, to non-smooth square-integrable signals in the Sobolev space; and (ii) As application of the new notion of amplitude and phase derivatives, we show that the formulas (1.2), (1.9), (1.10) can be extended to general signals with appropriate forms. We also give a proof of the classical uncertainty principle by using the generalized phase derivatives.

Throughout the paper we assume that s is complex-valued. In applications we usually assume s is real-valued, and the Hilbert transform Hs contributes the pure imaginary part to form the associated analytic signal $s + iHs$. The generalizations of the phase and amplitude derivatives are achieved via the Hardy-Sobolev spaces decomposition of functions in the Sobolev space.

Throughout the paper we denote by $\mathbb{R}$ the real axis, by $\mathbb{C}$ the complex plane, and by $\mathbb{C}^+$ and $\mathbb{C}^-$ the upper- and lower-half complex planes, respectively. We proceed to introduce the Hardy spaces in the upper- and the lower-half complex planes. For $1 \le p < \infty$ the totality of the analytic functions in the upper-half complex plane $\mathbb{C}^+$ under the norm

$$\|s\|_p \triangleq \sup_{y>0} \left(\int_{-\infty}^{\infty} |s(t+iy)|^p dt\right)^{1/p} < \infty \tag{1.12}$$

forms a Banach space. For $0 < p < 1$, the totality of the analytic functions in $\mathbb{C}^+$ satisfying (1.12) under the distance

$$d(s,u) \triangleq \sup_{y>0} \int_{-\infty}^{\infty} |s(t+iy) - u(t+iy)|^p dt < \infty$$

forms a complete metric space. For $p = \infty$ the totality of the analytic functions in $\mathbb{C}^+$ under the norm

$$\|s\|_\infty = \sup\{|s(z)| \,:\, z \in \mathbb{C}^+\} < \infty$$

forms a Banach space. In all the three cases we denote the space by $H^p(\mathbb{C}^+)$, and call it a Hardy space. Similarly one defines the Hardy spaces $H^p(\mathbb{C}^-)$ for the lower-half complex plane.

The Hardy spaces have a well developed theory connecting complex analysis and harmonic analysis. The latest developments of the Hardy spaces mainly concern the real-Hardy spaces with the index region $0 < p \leq 1$, and mainly via real analysis methods. For our purpose we concentrate on the complex Hardy spaces $H^2(\mathbb{C}^\pm)$.

We adopt the notation $L_n^2(\mathbb{R})$ for the Sobolev spaces [24], that is

$$L_n^2(\mathbb{R}) = \left\{ s(t) \in L^2(\mathbb{R}) : \left(\frac{d^*}{dt}\right)^n s(t) \in L^2(\mathbb{R}) \right\}$$

with the norm defined by

$$\sqrt{\|s\|_2^2 + \left\| \left(\frac{d^*}{dt}\right)^n s \right\|_2^2},$$

where $(\frac{d^*}{dt})^n s(t)$ denotes the n-th distributional derivative of s.

Throughout this paper we assume signals s under study satisfy

$$s \in L^2(\mathbb{R}), \quad \omega\hat{s}(\omega) \in L^2(\mathbb{R}). \tag{1.13}$$

Signals satisfying the condition (1.13) belong to $L_1^2(\mathbb{R})$ (see Lemma 2.5).

If $s \in L^2(\mathbb{R})$, then we usually perform the Hardy spaces decomposition $s = s^+ + s^-$, $s^\pm = (1/2)(s \pm iHs)$, and $s^\pm$ are, respectively, the boundary values of the analytic functions

$$s^\pm(z) = \frac{\pm 1}{2\pi i} \int_{-\infty}^{\infty} \frac{s(u)}{u - z} du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\omega} \chi_\pm(\omega) e^{-y\omega} \hat{s}(\omega) d\omega, \quad z = t + iy \in \mathbb{C}^\pm.$$

The last equal relation is a consequence of (2.4), where $\chi_\pm = \chi_{\mathbb{R}^\pm}$, $\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}^- = (-\infty, 0)$, and, in general, χ_E is the characteristic function of the Lebesgue measurable set E that takes value 1 on E and 0 otherwise. If s satisfies the conditions in (1.13), that is $s \in L_1^2(\mathbb{R})$, then

$$s^\pm \in H^\pm L_1^2(\mathbb{R}) = \left\{ s \in H^2(\mathbb{C}^\pm) : \frac{d^* s}{dt} \in H^2(\mathbb{C}^\pm) \right\},$$

where $H^\pm L_1^2(\mathbb{R})$ are called the Hardy-Sobolev spaces in the upper- and lower-half complex planes, respectively. Thus, we have the decomposition

$$L_1^2(\mathbb{R}) = H^+ L_1^2(\mathbb{R}) \oplus H^- L_1^2(\mathbb{R}).$$

The spaces $H^+ L_1^2(\mathbb{R})$ and $H^- L_1^2(\mathbb{R})$ are mutually orthogonal. We note that in the function space notations we mix up the analytic functions in the respective domains with their boundary values due to certain isometric isomorphism relations in the respective contexts.

It is for the functions in the Hardy-Sobolev spaces that we are able to define the phase and amplitude derivatives. They are defined to be the non-tangential boundary limits of the same quantities in the respective domains in which the functions are analytic. The details will be given in Sect. 3.

Both the Hardy and Sobolev spaces have been well studied, and have long histories with ample applications. The Hardy-Sobolev spaces in recent years have undergone a new phase of development [2, 12]. To the authors' knowledge, no applications in signal analysis have been noted.

In Sect. 2 we discuss the useful relations between some five types of derivatives, and in Sect. 3 we define the amplitude and phase derivatives for s^+ and s^-. In Sect. 4 we deal with the first and second order means of the Fourier frequency. In Sect. 5 we give the results on the mean of the time and the duration. In Sect. 6 we study the covariance, and in Sect. 7 we give a proof of the uncertainty principle that is all based on the generalized phase-amplitude representations. In Sect. 8 we draw the conclusions and give some remarks.

2 Technical Preparations

For any $\alpha > 0$, define the α-cone $\Gamma_\alpha^+(t)$ at $t \in \mathbb{R}$ by

$$\Gamma_\alpha^+(t) \triangleq \{(x, y) : |x - t| < \alpha y, 0 < y < \infty\}.$$

For an analytic function s in $\mathbb{C}^+$ define its α-non-tangential maximal function by

$$M_\alpha^+ s(t) \triangleq \sup_{z \in \Gamma_\alpha^+(t)} |s(z)|.$$

Note that $M_\alpha^+ s$ is a function defined on $\mathbb{R}$. It is a fundamental result that $M_\alpha^+ s \in L^2(\mathbb{R})$ if and only if $s \in H^2(\mathbb{C}^+)$ [9]. This result presents the equivalence between the two statements of which one is dependent of α while the other is not. This shows that the condition $M_\alpha^+ s \in L^2(\mathbb{R})$ is independent of α. In other words, if for one $\alpha_0 \in (0, \infty)$ there holds $M_{\alpha_0}^+ s \in L^2(\mathbb{R})$, then $M_\alpha^+ s \in L^2(\mathbb{R})$ holds for all $\alpha \in (0, \infty)$. One defines the α-non-tangential maximal function for analytic functions in the lower-half complex plane $\mathbb{C}^-$ in the same way through the α-cone $\Gamma_\alpha^-(t)$ symmetric to $\Gamma_\alpha^+(t)$ with respect to the real-axis. The theory for the lower-half complex plane is parallel to that for the upper-half complex plane. We write $\Gamma(t)$ for either $\Gamma_\alpha^+(t)$ or $\Gamma_\alpha^-(t)$ for some $\alpha \in (0, \infty)$ depending on the context. To indicate a non-tangential limit σ being independent of $\alpha > 0$, we adopt the notation

$$\lim_{\Gamma:\, z \to t} s(z) \triangleq \sigma.$$

This, in fact, holds for all $s \in H^2(\mathbb{C}^\pm)$. Here we allow the limit σ to be ∞, including $\pm\infty$. In the context this value σ is denoted by $s(t)$. The correspondence between $s(z) \in H^2(\mathbb{C}^\pm)$ and its boundary value $s(t) \in L^2(\mathbb{R})$ is one to one with equal norms in their respective spaces. For meromorphic functions in the respective domains, we use the same limit notation but for *truncated α-cones*. Recall that a function is *meromorphic* in a domain if it is analytic throughout the domain except at its poles. This is exactly what is needed in defining the phase and amplitude derivatives (see Theorem 3.1).

Now we list five types of derivatives. Assume that $s \in L^2(\mathbb{R})$.

(i) The *distributional* or *weak derivative* $\frac{d^*s}{dt}$. Based on the distribution theory this type of derivative always exists.
(ii) The *classical* or *strong derivative* $\frac{ds}{dt}$. It may or may not exist.
(iii) The *Fourier transform derivative*. It is defined to be the inverse Fourier transform of $i\omega\hat{s}(\omega)$ in $L^2(\mathbb{R})$. It exists if $s \in L_1^2(\mathbb{R})$.
(iv) The *analytic derivatives*. Further assuming $s \in L_1^2(\mathbb{R})$, and decomposing s into the sum $s = s^+ + s^-$, where $s^\pm$ are the non-tangential boundary values of the associated analytic functions $s^\pm(z)$ in the Hardy spaces $H^2(\mathbb{C}^\pm)$, one can show that $s^{\pm\prime}(z)$ are also in $H^2(\mathbb{C}^\pm)$, respectively, and the non-tangential limits

$$\lim_{\Gamma: z\to t} s^{\pm\prime}(z) \triangleq s^{\pm\prime}(t), \quad z \in \mathbb{C}^\pm,$$

exist, and are called analytic derivatives.
(v) The *boundary derivative*. They are defined through the non-tangential boundary limits

$$\lim_{\Gamma: z\to t} \frac{s^\pm(z) - s^\pm(t)}{z - t} \triangleq s^{\pm\prime}(t),$$

provided that the non-tangential boundary values $s^\pm(t)$ exist, and the non-tangential limits on the left-hand-side exist.

In order to define the phase and amplitude derivatives for $s \in L_1^2(\mathbb{R})$ a close study on the relations between the five types of derivatives is now necessary.

The Fourier transform of $s \in L^1(\mathbb{R})$ is defined by

$$\hat{s}(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\omega} s(t)dt. \tag{2.1}$$

If $\hat{s}$ is also in $L^1(\mathbb{R})$, then the inversion formula holds, that is

$$s(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\omega} \hat{s}(\omega)d\omega, \quad \text{a.e.} \tag{2.2}$$

There holds the Plancherel Theorem

$$\|\hat{s}\|_2^2 = \|s\|_2^2, \quad s \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Through a density argument, both the Fourier transformation and its inverse can be extended to $L^2(\mathbb{R})$ in which the Plancherel Theorem still holds. When we use the formulas (2.1) and (2.2) for $L^2(\mathbb{R})$ functions, we keep in mind that the convergence of the integrals is in the L^2 sense.

Now we summarize the results that are often recalled and regarded as *basic results* throughout the paper (see also [9, 22]).

1. s belongs to $H^2(\mathbb{C}^\pm)$ if and only if $M^\pm_\alpha s$ belongs to $L^2(\mathbb{R})$, $\alpha > 0$. Such s and $M^\pm_\alpha s$ are of equivalent norms in their respective spaces.
2. If $s \in H^2(\mathbb{C}^\pm)$, then the *non-tangential boundary limit* or *non-tangential boundary value*

$$\lim_{\Gamma: z \to t} s(z) = s(t)$$

exists for a.e. $t \in \mathbb{R}$. We use the notation s for both $s(z)$ and $s(t)$. They are of equal norms in their respective spaces.
3. If $s \in L^2(\mathbb{R})$, then $s = s^+ + s^-$, $\hat{s^\pm} \triangleq \chi_\pm \hat{s}$, $s^\pm \in H^2(\mathbb{C}^\pm)$, where

$$s^\pm(t) = \frac{\pm 1}{\sqrt{2\pi}} \int_0^{\pm\infty} e^{it\omega} \hat{s}(\omega) d\omega \quad \text{and} \quad s^\pm(z) = \frac{\pm 1}{2\pi i} \int_{-\infty}^{\infty} \frac{s(u)}{u - z} du, \quad z \in \mathbb{C}^\pm. \tag{2.3}$$

The decomposition is orthogonal and unique.
4. For $s \in L^2(\mathbb{R})$ we have the Plemelj Theorem

$$\lim_{y \to \pm 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{s(u)}{u - (t + iy)} du = \frac{1}{2} s(t) \pm \frac{i}{2} Hs(t) = s^\pm(t), \quad \text{a.e.},$$

where Hs is the *Hilbert transform* of s,

$$Hs(t) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{s(u)}{t - u} du = \frac{-i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\omega} \text{sgn}(\omega) \hat{s}(\omega) d\omega,$$

where p.v. stands for the *principal value* of the integral defined through (1.1) and $\text{sgn}(\omega)$ is the signum function that takes value 1 for $\omega > 0$ and value -1 for $\omega < 0$.
5.

$$H^2 = -I \ (I \text{ being the identity operator}), \qquad \|Hs\|_{L^2(\mathbb{R})} = \|s\|_{L^2(\mathbb{R})},$$

and the adjoint operator of H is $-H$.
6. The following inverse Fourier transforms play important roles: For $z = x + iy$, $\pm y > 0$,

$$\left[\frac{1}{(\cdot) - z} \right]^\vee (\omega) = \pm \sqrt{2\pi} i \chi_\pm(\omega) e^{ix\omega} e^{-y\omega}. \tag{2.4}$$

The following result shows that, in the Hardy spaces, the existence of the analytic derivative implies the existence of the Fourier transform derivatives of the same and lower orders, and vice versa.

Lemma 2.1 *Let $s \in H^2(\mathbb{C}^\pm)$ and n a positive integer. Then*

(i) *$s^{(n)} \in H^2(\mathbb{C}^\pm)$ if and only if $\omega^n \hat{s} \in L^2(\mathbb{R})$.*
(ii) *$s^{(n)} \in H^2(\mathbb{C}^\pm)$ implies $s^{(k)} \in H^2(\mathbb{C}^\pm)$, $k = 1, \ldots, n - 1$.*

Proof of Lemma We only prove the lemma for $H^2(\mathbb{C}^+)$. For $H^2(\mathbb{C}^-)$ the proof is similar.

(i) For $s \in H^2(\mathbb{C}^+)$, denote $s_y(t) = s(t+iy)$. By invoking the basic results 3 and 6, and the Plancherel Theorem, we have

$$\begin{aligned} s_y(t) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{s(u)}{u-(t+iy)}du \\ &= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{it\omega}e^{-y\omega}\hat{s}(\omega)d\omega. \end{aligned}$$

Therefore,

$$(s_y)^\wedge(\omega) = \chi_+(\omega)e^{-y\omega}\hat{s}(\omega).$$

As a consequence,

$$(s_y^{(n)})^\wedge(\omega) = (i\omega)^n\chi_+(\omega)e^{-y\omega}\hat{s}(\omega).$$

By the Plancherel Theorem,

$$\int_{-\infty}^{\infty} |s^{(n)}(t+iy)|^2 dt = \int_0^{\infty} \omega^{2n}e^{-2y\omega}|\hat{s}(\omega)|^2 d\omega. \tag{2.5}$$

Therefore, $s^{(n)} \in H^2(\mathbb{C}^+)$ if and only if $\omega^n\hat{s}(\omega) \in L^2(\mathbb{R})$. The assertion (ii) is well known in the Sobolev space theory whose proof is omitted. □

Definition 2.2 We say that s has the *non-tangential boundary derivative* or, in brief, *boundary derivative* $s'(t)$ at $t \in \mathbb{R}$, if a finite limit $s(t) = \lim_{\Gamma:z\to t} s(z)$ exists, and

$$\lim_{\Gamma:z\to t}\frac{s(z)-s(t)}{z-t} \triangleq s'(t). \tag{2.6}$$

The limit allows the infinite values. Replacing s with $s^{(k)}$ in the above definition, we can inductively define the $(k+1)$-order *non-tangential boundary derivative* $s^{(k+1)}$.

So far the notation $s'(t)$ has two meanings: one is the boundary derivative defined by (2.6), and the other is the limit

$$\lim_{\Gamma:z\to t} s'(z) \tag{2.7}$$

(the fourth type derivative, or analytic derivative). Thanks to the following lemma the notation does not cause confusion.

Lemma 2.3 *Let* $s \in L_1^2(\mathbb{R})$. *Then*

(i) *Analytic derivative* s' *defined by* (2.7) *exists.*
(ii) *Boundary derivative* s' *defined by* (2.6) *exists as a function in* $L^2(\mathbb{R})$ *that coincides with the analytic derivative.*

Proof of Lemma By invoking (i) of Lemma 2.1, the assertion (i) is a consequence of the basic result 2. To prove (ii) we use the same method as in the proof for the counterpart result in the disc (see [17, p. 79]). □

Combining (ii) of Lemma 2.1 with Lemma 2.3, we have

Corollary 2.4 *Let s and $s^{(n)}$ belong to $H^2(\mathbb{C}^{\pm})$, $n \geq 1$. Then*

(i) *All the boundary derivatives $s', \ldots, s^{(n)}$ defined through (2.6) exist.*
(ii) *The boundary derivatives coincide with the corresponding analytic derivatives defined through (2.7), and they all belong to $L^2(\mathbb{R})$.*

Let $s \in L^2(\mathbb{R})$. As a temperate distribution, s has a distributional derivative $\frac{d^*s}{dt}$. What is interesting is the case when the temperate distribution $\frac{d^*s}{dt}$ belongs to $L^2(\mathbb{R})$. The latter means that there exists a function $h \in L^2(\mathbb{R})$ such that for all $\phi \in \mathcal{S}$, the Schwarz class of rapidly decreasing functions, there holds

$$\langle h, \phi \rangle = -\langle s, \phi' \rangle,$$

and $\frac{d^*s}{dt} = h(t)$. We have the following result.

Lemma 2.5 *Let $s \in L^2(\mathbb{R})$ and n a positive integer. Then the following conditions are equivalent.*

(i) $\omega^n \hat{s} \in L^2(\mathbb{R})$.
(ii) $(\frac{d^*}{dt})^n s$ *exists in* $L^2(\mathbb{R})$.

When they hold, we have $((\frac{d^*}{dt})^n s)\hat{}(\omega) = (i\omega)^n \hat{s}(\omega)$.

This result is well known and proved by standard techniques [25]. Below we provide a proof by using the Hardy spaces decomposition.

Proof of Lemma We first show (i) implies (ii). Assume $\omega^n \hat{s} \in L^2(\mathbb{R})$. Then both $\omega^n \hat{s^+}$ and $\omega^n \hat{s^-}$ are in $L^2(\mathbb{R})$, where $s^{\pm} \in H^2(\mathbb{C}^{\pm})$. Due to (2.5), we have, for all $y > 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} |s^{+(n)}(t+iy)|^2 dt &= \int_0^{\infty} \omega^{2n} e^{-2y\omega} |\hat{s}(\omega)|^2 d\omega \\ &\leq \int_0^{\infty} \omega^{2n} |\hat{s}(\omega)|^2 d\omega \\ &< \infty.\end{aligned}$$

This shows that $s^{+(n)} \in H^2(\mathbb{C}^+)$, and therefore $(s^{+(n)})_y$ has a $L^2(\mathbb{R})$ limit, h^+, as $y \to 0+$ [9, p. 57]. By the definition of distributional derivative, for every $y > 0$ and $\phi \in \mathcal{S}$,

$$\langle (s^{+(n)})_y, \phi \rangle = \langle (s^+)_y, (-1)^n \phi^{(n)} \rangle.$$

Passing to the L^2-limits of $(s^{+(n)})_y$ and $(s^+)_y$, we obtain

$$\langle h^+, \phi \rangle = \langle s^+, (-1)^n \phi^{(n)} \rangle.$$

This shows that $(\frac{d^*}{dt})^n s^+ = h^+ \in L^2(\mathbb{R})$. Similarly we can show that there exists $h^- \in L^2(\mathbb{R})$ such that $(\frac{d^*}{dt})^n s^- = h^- \in L^2(\mathbb{R})$. Therefore, we have $(\frac{d^*}{dt})^n s = h^+ + h^- \triangleq h \in L^2(\mathbb{R})$. Although the proof that (ii) implies (i) is direct, we prefer to include it for the completeness. We assume that $(\frac{d^*}{dt})^n s$ exists in the $L^2(\mathbb{R})$. There holds

$$\left\langle \left(\frac{d^*}{dt}\right)^n s, \phi \right\rangle = \langle s, (-1)^n \phi^{(n)} \rangle. \tag{2.8}$$

Using Parseval's identity, we have

$$\left\langle \left(\left(\frac{d^*}{dt}\right)^n s\right)^{\hat{}}, \hat{\phi} \right\rangle = \langle \hat{s}, (-i(\cdot))^n \hat{\phi} \rangle = \langle (i(\cdot))^n \hat{s}, \hat{\phi} \rangle. \tag{2.9}$$

Since $\hat{\phi}$ runs over all $\mathcal{S}$, the boundedness of the left-hand-side shows $(i(\cdot))^n \hat{s} \in L^2(\mathbb{R})$. This proves that (ii) implies (i), and $((\frac{d^*}{dt})^n s)^{\hat{}}(\omega) = (i\omega)^n \hat{s}(\omega)$. □

Lemma 2.6 *Let $s \in L^2(\mathbb{R})$ and n a positive integer. Then the following assertions hold.*

(i) $\omega^n \hat{s}(\omega) \in L^2(\mathbb{R})$ *if and only if* $\omega^n \widehat{s^+}(\omega) \in L^2(\mathbb{R})$ *and* $\omega^n \widehat{s^-}(\omega) \in L^2(\mathbb{R})$.
(ii) $(\frac{d^*}{dt})^n s \in L^2(\mathbb{R})$ *if and only if* $(\frac{d^*}{dt})^n s^+ \in L^2(\mathbb{R})$ *and* $(\frac{d^*}{dt})^n s^- \in L^2(\mathbb{R})$.
(iii) $t^n s(t) \in L^2(\mathbb{R})$ *if and only if* $t^n s^+(t) \in L^2(\mathbb{R})$ *and* $t^n s^-(t) \in L^2(\mathbb{R})$.

Proof of Lemma (i) is a consequence of $\widehat{s^{\pm}} = \chi_{\pm}\hat{s}$. (ii) is a consequence of (i) and Lemma 2.5. Now we prove (iii). If $t^n s(t) \in L^2(\mathbb{R})$, then by Lemma 2.5, $(\frac{d^*}{d\omega})^n \hat{s}(\omega) \in L^2(\mathbb{R})$, and hence $\chi_+(\omega)(\frac{d^*}{d\omega})^n \hat{s}(\omega)$ and $\chi_-(\omega)(\frac{d^*}{d\omega})^n \hat{s}(\omega) \in L^2(\mathbb{R})$. We show that we can exchange the order of multiplying by $\chi_{\pm}$ and taking the distributional derivative $(\frac{d^*}{d\omega})^n$. Let $h \in L^2(\mathbb{R})$ be such that

$$\langle h, \varphi \rangle = \langle (-1)^n \hat{s}, \varphi^{(n)} \rangle, \quad \varphi \in \mathcal{S}. \tag{2.10}$$

It is easy to show

$$\langle \chi_{\pm} h, \varphi \rangle = \langle h, \chi_{\pm}\varphi \rangle$$

and

$$\langle (-1)^n \hat{s}, (\chi_{\pm}\varphi)^{(n)} \rangle = \langle (-1)^n \hat{s}, \chi_{\pm}\varphi^{(n)} \rangle = \langle (-1)^n \chi_{\pm}\hat{s}, \varphi^{(n)} \rangle.$$

In order to show

$$\langle \chi_{\pm} h, \varphi \rangle = \langle (-1)^n \chi_{\pm}\hat{s}, \varphi^{(n)} \rangle, \quad \varphi \in \mathcal{S},$$

that is equivalent to the exchange rule

$$\chi_{\pm} \left(\frac{d^*}{d\omega}\right)^n \hat{s} = \left(\frac{d^*}{d\omega}\right)^n (\widehat{s^{\pm}}), \tag{2.11}$$

it suffices to show

$$\langle h, \chi_{\pm}\varphi \rangle = \langle (-1)^n \hat{s}, (\chi_{\pm}\varphi)^{(n)} \rangle, \quad \varphi \in \mathcal{S}. \tag{2.12}$$

Note that $\chi_{\pm}\varphi \in L_n^2(\mathbb{R})$. Since the Schwarz class is dense in the Sobolev space $L_n^2(\mathbb{R})$ [24, p. 122], there exists a sequence of functions ψ_k in the Schwarz class $\mathcal{S}$ such that in the L^2 convergence sense

$$\psi_k \to \chi_{\pm}\varphi \quad \text{and} \quad \psi_k^{(n)} \to (\chi_{\pm}\varphi)^{(n)}$$

simultaneously. In (2.10) let $\varphi = \psi_k$ and take limit $k \to \infty$, we obtain (2.12). By taking Fourier transform and using the Plancherel Theorem on the right-hand-side of (2.11), we obtain $t^n s^+(t),\ t^n s^-(t) \in L^2(\mathbb{R})$. Conversely, if $t^n s^+(t),\ t^n s^-(t) \in L^2(\mathbb{R})$, by adding them up we obtain $t^n s(t) \in L^2(\mathbb{R})$. □

3 Definition of Phase and Amplitude Derivatives

Now we proceed to define the phase and amplitude derivatives. We first have the following observations. Let $s(z)$ be an analytic function in $\mathbb{C}^{\pm}$ and $s(z) = A_y(t)e^{i\varphi_y(t)},\ z = t + iy,\ \pm y > 0$. Taking the partial derivative with respect to t and dividing $s(t + iy)$ on both sides, we obtain

$$\mathrm{Im}\left(\frac{s'(z)}{s(z)}\right) = \frac{\partial \varphi_y(t)}{\partial t},$$

$$A_y(t)\mathrm{Re}\left(\frac{s'(z)}{s(z)}\right) = \frac{\partial A_y(t)}{\partial t}.$$

For a signal in the Sobolev space $L_1^2(\mathbb{R})$ these relations do not hold on $\mathbb{R}$ in general. However, it suggests the following formulation.

Let $s(z)$ be a function analytic in $\mathbb{C}^{\pm}$. If $t_0 \in \mathbb{R}$, we denote

$$D_p^{\pm}s(t_0) \triangleq \lim_{\Gamma: z \to t_0} \mathrm{Im}\left(\frac{s'(z)}{s(z)}\right),$$

and

$$D_a^{\pm}s(t_0) \triangleq \lim_{\Gamma: z \to t_0} A_y^{\pm}(t)\mathrm{Re}\left(\frac{s'(z)}{s(z)}\right),$$

provided that the limits exist.

Theorem 3.1 *If $s,\ s' \in H^2(\mathbb{C}^{\pm})$, then both $D_p^{\pm}s$ and $D_a^{\pm}s$ are well defined measurable functions*, *and*

$$D_p^{\pm}s(t) = \mathrm{Im}\left(\frac{s'(t)}{s(t)}\right), \qquad D_a^{\pm}s(t) = A^{\pm}(t)\mathrm{Re}\left(\frac{s'(t)}{s(t)}\right), \quad a.e. \tag{3.1}$$

Proof of Theorem We will only prove the result for $s,\ s' \in H^2(\mathbb{C}^+)$. If the non-tangential limit of $s(z)$ as z tends to t exists, then the non-tangential limit

$$A(t) = \lim_{\Gamma: z \to t} A_y(t) = \lim_{\Gamma: z \to t} |s_y(t)|$$

also exists. Noticing that s'/s is meromorphic, it suffices to show (i) the non-tangential boundary limits $s(t)$ and $s'(t)$ both exist a.e.; and (ii) $s(t)$ is a.e. non-zero. The assertion (i) follows upon the fact that $s(z)$ and $s'(z)$ are in the Hardy space $H^2(\mathbb{C}^+)$. The assertion (ii) is guaranteed by the result that the non-tangential boundary value of a function in the Hardy $H^2(\mathbb{C}^\pm)$ is a.e. non-zero [9, p. 65]. □

Definition 3.2 If $s,\ s' \in H^2(\mathbb{C}^\pm)$, then s has non-tangential boundary limit $s(t) = A^\pm(t)e^{i\varphi^\pm(t)}$. According to Theorem 3.1, $D_p^\pm s$ and $D_a^\pm s$ both exist as measurable functions. The analytic amplitude derivative (amplitude derivative) and the analytic phase derivative (phase derivative) are, respectively, defined by

$$A^{\pm'}(t) \triangleq D_a^\pm s(t), \qquad \varphi^{\pm'}(t) \triangleq D_p^\pm s(t). \tag{3.2}$$

Remark 3.3 Note that although the phase derivatives $\varphi^{\pm'}(t)$ have been defined, the phase functions $\varphi^\pm(t)$ are not yet defined. The unimodular functions $e^{i\varphi^\pm(t)}$ are defined by $s^\pm(t)/|s^\pm(t)|$. This does not mean there exist appropriate parametrizations $\varphi^\pm(t)$ such that $\frac{d\varphi^\pm(t)}{dt}$ are defined in the classical sense. Nevertheless, in the sense specified in Definition 3.2, Theorem 3.1 asserts the existence of the phase and amplitude derivatives of s^+ and s^-, if $s \in L_1^2(\mathbb{R})$. If it happens that the function s has an analytic continuation across an open interval containing t_0, then the above defined $A^{\pm'}(t_0)$ and $\varphi^{\pm'}(t_0)$ coincide with the respective classical derivatives at t_0 (also see [19]).

The results proved in this and the last section are summarized as follows.

Proposition 3.4 *Let n be a positive integer. We have*

(i) *If $s,\ s^{(n)} \in H^2(\mathbb{C}^\pm)$, then $s^{(n)}$, as boundary derivative defined by* (2.6), *exists, and coincides with the analytic derivative $s^{(n)}$ defined by* (2.7) *from inside $\mathbb{C}^\pm$. It also coincides with the distributional derivative $(\frac{d^*}{dt})^n s$. These derivatives are all in $L^2(\mathbb{R})$.*
(ii) *The conditions assumed in* (i) *are equivalent to the conditions* supp $\hat{s} \subset [0, \pm\infty)$, *and $\hat{s}(\omega)$, $\omega^n\hat{s}(\omega) \in L^2(\mathbb{R})$.*
(iii) *If the conditions of (i) hold, then the same conclusions hold for all positive integers less than n.*
(iv) *If $s,\ s' \in H^2(\mathbb{C}^\pm)$, then for the non-tangential boundary value $s(t) = A(t)e^{i\varphi(t)}$, the analytic amplitude and phase derivatives can be defined through non-tangential boundary values of the same quantities from inside of the domain $\mathbb{C}^\pm$, as*

$$A' = D_a s \quad \text{and} \quad \varphi' = D_p s. \tag{3.3}$$

(v) *$(\frac{d^*}{dt})^n s \in L^2(\mathbb{R})$ if and only if $(\frac{d^*}{dt})^n s^+ \in L^2(\mathbb{R})$ and $(\frac{d^*}{dt})^n s^- \in L^2(\mathbb{R})$. In the case, $(\frac{d^*}{dt})^n s = (\frac{d^*}{dt})^n s^+ + (\frac{d^*}{dt})^n s^-$.*

(vi) *$t^n s(t) \in L^2(\mathbb{R})$ if and only if $t^n s^+(t) \in L^2(\mathbb{R})$ and $t^n s^-(t) \in L^2(\mathbb{R})$.*

(vii) *If* $s \in L^2(\mathbb{R})$, *then* $(\frac{d^*}{dt})^n s$ *exists in* $\in L^2(\mathbb{R})$ *if and only if* $\omega^n \hat{s}(\omega) \in L^2(\mathbb{R})$, *and in the case* $((\frac{d^*}{dt})^n s)\hat{}(\omega) = (i\omega)^n \hat{s}(\omega)$.

4 Mean and Variance of Fourier Frequency in Terms of Analytic Phase and Amplitude Derivatives

Definition 4.1 Let s be a square-integrable signal and $|\hat{s}(\omega)|^2$ the density of the Fourier frequency, then we can define the mean of the Fourier frequency by

$$\langle \omega \rangle \triangleq \int_{-\infty}^{\infty} \omega |\hat{s}(\omega)|^2 d\omega, \tag{4.1}$$

the Fourier bandwidth by

$$B^2 = \sigma_\omega^2 \triangleq \int_{-\infty}^{\infty} (\omega - \langle \omega \rangle)^2 |\hat{s}(\omega)|^2 d\omega \tag{4.2}$$

$$= \langle \omega^2 \rangle - \langle \omega \rangle^2, \tag{4.3}$$

and the mean of any Fourier frequency function $g(\omega)$ by

$$\langle g(\omega) \rangle \triangleq \int_{-\infty}^{\infty} g(\omega) |\hat{s}(\omega)|^2 d\omega, \tag{4.4}$$

provided that the right-hand sides of (4.1), (4.2) and (4.4) are well defined integrals. Below we also use the notation

$$\langle g(\omega) \rangle_\pm \triangleq \int_{-\infty}^{\infty} g(\omega) |\hat{s}^\pm(\omega)|^2 d\omega, \tag{4.5}$$

where $\hat{s}^\pm(\omega)$ are defined through (5.7).

With the preparations made in the proceeding section the following theorem is straightforward.

Theorem 4.2 *Assume* s, $\frac{d^*}{dt} s \in L^2(R)$. *With the decomposition* $s = s^+ + s^-$, $(s^\pm)\hat{} = \chi_\pm \hat{s}$, $s^\pm(t) = A^\pm(t) e^{i\varphi^\pm(t)}$, *the mean Fourier frequency defined by* (4.1) *is identical with*

$$\langle \omega \rangle = \int_{-\infty}^{\infty} \varphi^{+\prime}(t) A^{+2}(t) dt + \int_{-\infty}^{\infty} \varphi^{-\prime}(t) A^{-2}(t) dt, \tag{4.6}$$

where $\varphi^{\pm\prime}(t)$ *are defined by* (3.2).

Proof of Theorem Since s, $\frac{d^*}{dt} s \in L^2(\mathbb{R})$, (vii) of Proposition 3.4 implies $\hat{s}(\omega)$, $\omega \hat{s}(\omega) \in L^2(\mathbb{R})$. Hölder's inequality implies $\omega |\hat{s}(\omega)|^2 \in L^1(\mathbb{R})$, and hence $\langle \omega \rangle$ is well defined. The assertions (v) and (ii) of Proposition 3.4 imply that $s^\pm$, $s^{\pm\prime}$ belong to

$H^2(\mathbb{C}^{\pm})$, respectively. The assertions (i) and (iv) of Proposition 3.4 further imply that the boundary values $s^{\pm'}$ and the phase derivatives $\varphi^{\pm'}$ all exist. There holds the decomposition

$$\langle\omega\rangle = \langle\omega\rangle^{+} + \langle\omega\rangle^{-},$$

where

$$\begin{aligned}\langle\omega\rangle^{\pm} \triangleq \pm\int_0^{\pm\infty}\omega|\hat{s}(\omega)|^2 d\omega &= -\int_{-\infty}^{\infty} i s^{\pm'}(t)\overline{s^{\pm}(t)}dt \\ &= -i\int_{-\infty}^{\infty}\frac{s^{\pm'}(t)}{s^{\pm}(t)}|s^{\pm}(t)|^2 dt \\ &= \int_{-\infty}^{\infty}\operatorname{Im}\left\{\frac{s^{\pm'}(t)}{s^{\pm}(t)}\right\}|s^{\pm}(t)|^2 dt \\ &= \int_{-\infty}^{\infty}\varphi^{\pm'}(t)|s^{\pm}(t)|^2 dt, \end{aligned} \tag{4.7}$$

where the fact that the boundary values $s^{\pm}$ are a.e. non zero [9, p. 65] justifies the division by $s^{\pm}(t)$. As a consequence,

$$\langle\omega\rangle = \int_{-\infty}^{\infty}\varphi^{+'}(t)|s^{+}(t)|^2 dt + \int_{-\infty}^{\infty}\varphi^{-'}(t)|s^{-}(t)|^2 dt. \qquad \square$$

Example 4.3 Let $s(t) = \frac{2}{1+t^2}$. It has the decomposition $s = s^+ + s^- = \frac{1}{1-it} + \frac{1}{1+it}$. Then

$$A^{\pm 2}(t) = |s^{\pm}(t)|^2 = \frac{1}{1+t^2} \in L^2(\mathbb{R}), \qquad s^{\pm'}(t) = \frac{\pm i}{(1\mp it)^2} \in L^2(\mathbb{R}),$$

$$\hat{s^{\pm}}(\omega) = \sqrt{2\pi}\,\chi_{\pm}(\omega)e^{-|\omega|}, \qquad \omega\hat{s^{\pm}}(\omega) = \sqrt{2\pi}\,\omega\chi_{\pm}(\omega)e^{-|\omega|} \in L^2(\mathbb{R}),$$

and $s^{\pm}$ as analytic functions in $\mathbb{C}^{\pm}$, respectively, have analytic continuations, and thus have smooth phase derivatives. Precisely,

$$s^{\pm}(t) = \frac{1}{1\mp it} = \frac{1}{\sqrt{1+t^2}}e^{\pm i\arctan t},$$

with

$$\varphi^{\pm}(t) = \pm\arctan t \quad \text{and} \quad \varphi^{\pm'}(t) = \frac{\pm 1}{1+t^2}.$$

At the same time, we also have

$$\lim_{y\to 0+}\varphi_y^{\pm'}(t) = \lim_{\Gamma:z\to t}\operatorname{Im}\left(\frac{s^{\pm'}(z)}{s^{\pm}(z)}\right) = \lim_{\Gamma:z\to t}\operatorname{Im}\left(\frac{\pm i}{1\mp iz}\right) = \frac{\pm 1}{1+t^2},$$

justifying the existence of the phase derivative defined in Definition 3.2. Since s itself is real-valued, $s(t) = \frac{2}{1+t^2}e^{i\varphi(t)}$ with $\varphi(t) = 2\pi k$ for any integer k. Therefore,

s satisfies all the conditions assumed in Theorem 1.1 and Theorem 4.2. By using Theorem 1.1,

$$\langle\omega\rangle = \int_{-\infty}^{\infty} \varphi'(t)A^2(t)dt = \int_{-\infty}^{\infty} 0\left(\frac{2}{1+t^2}\right)^2 dt = 0.$$

This can also be obtained directly from the definition of $\langle\omega\rangle$ and the property $\hat{s}(-\omega) = \overline{\hat{s}(\omega)}$ for real valued signals. By using Theorem 4.2, we have

$$\begin{aligned}\langle\omega\rangle &= \langle\omega\rangle^+ + \langle\omega\rangle^- \\ &= \int_{-\infty}^{\infty} \varphi^{+\prime}(t)A^{+2}(t)dt + \int_{-\infty}^{\infty} \varphi^{-\prime}(t)A^{-2}(t)dt \\ &= 0.\end{aligned}$$

The results obtained from Theorem 1.1 and Theorem 4.2 coincide.

The following theorem gives a similar formula for $\langle\omega^2\rangle$.

Theorem 4.4 *Assume* $s, \frac{d^*}{dt}s \in L^2(\mathbb{R})$. *With the decomposition* $s = s^+ + s^-$, $(s^\pm)\hat{} = \chi_\pm \hat{s}$, $s^\pm(t) = A^\pm(t)e^{i\varphi^\pm(t)}$, *there follows*

$$\langle\omega^2\rangle = \int_{-\infty}^{\infty} [(A^{+\prime}(t))^2 + (A^{-\prime}(t))^2]dt + \int_{-\infty}^{\infty} [(A^+(t)\varphi^{+\prime}(t))^2 + (A^-(t)\varphi^{-\prime}(t))^2]dt. \tag{4.8}$$

Proof of Theorem Since $s, \frac{d^*}{dt}s \in L^2(\mathbb{R})$, (ii) of Proposition 3.4 implies $\omega\hat{s}(\omega) \in L^2(\mathbb{R})$, and thus $\omega^2|\hat{s}(\omega)|^2 \in L^1(\mathbb{R})$. $\langle\omega^2\rangle$ is therefore well defined. The assertion (v) and (ii) of Proposition 3.4 imply that $s^\pm$, $s^{\pm\prime}$ belong to $H^2(\mathbb{C}^\pm)$, respectively. The assertion (i) and (iv) of Proposition 3.4 further imply that the boundary values $s^{\pm\prime}$ and the phase derivatives $\varphi^{\pm\prime}$ all exist. There then holds the decomposition

$$\langle\omega^2\rangle = \langle\omega^2\rangle^+ + \langle\omega^2\rangle^-$$

with

$$\begin{aligned}\langle\omega^2\rangle^\pm &\triangleq \pm\int_0^{\pm\infty} \omega^2|\hat{s}(\omega)|^2 d\omega \\ &= -\int_{-\infty}^{\infty} is^{\pm\prime}(t)\overline{-is^{\pm\prime}(t)}dt \\ &= \int_{-\infty}^{\infty} \left|\frac{s^{\pm\prime}(t)}{s^\pm(t)}\right|^2 |s^\pm(t)|^2 dt\end{aligned}$$

$$= \int_{-\infty}^{\infty} \mathrm{Re}^2\left\{\frac{s^{\pm\prime}(t)}{s^{\pm}(t)}\right\}|s^{\pm}(t)|^2 dt + \int_{-\infty}^{\infty} \mathrm{Im}^2\left\{\frac{s^{\pm\prime}(t)}{s^{\pm}(t)}\right\}|s^{+}(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} \left[A^{\pm\prime}(t)\right]^2 dt + \int_{-\infty}^{\infty} \left[A^{\pm}(t)\varphi^{\pm\prime}(t)\right]^2 dt,$$

where, again, the fact that the boundary values $s^{\pm}$ are a.e. non zero [9, p. 65] justifies the division by $s^{\pm}(t)$. Therefore,

$$\langle \omega^2 \rangle = \int_{-\infty}^{\infty} [(A^{+\prime}(t))^2 + (A^{-\prime}(t))^2] dt$$
$$+ \int_{-\infty}^{\infty} [(A^{+}(t)\varphi^{+\prime}(t))^2 + (A^{-}(t)\varphi^{-\prime}(t))^2] dt. \qquad \square$$

Corollary 4.5 *Assume* $s, \frac{d^*}{dt}s \in L^2(R)$. *With the decomposition* $s = s^+ + s^-$, $(s^{\pm})\hat{} = \chi_{\pm}\hat{s}$, $s^{\pm}(t) = A^{\pm}(t)e^{i\varphi^{\pm}(t)}$, *the bandwidth*

$$\begin{aligned} B^2 &= \langle \omega^2 \rangle - \langle \omega \rangle^2 \\ &= \int_{-\infty}^{\infty} [(A^{+\prime}(t))^2 + (A^{-\prime}(t))^2] dt \\ &\quad + \int_{-\infty}^{\infty} [(\varphi^{+\prime}(t) - \langle \omega \rangle)^2 A^{+2}(t) + (\varphi^{-\prime}(t) - \langle \omega \rangle)^2 A^{-2}(t)] dt. \end{aligned} \tag{4.9}$$

Proof of Corollary Based on the expressions of $\langle \omega \rangle$, $\langle \omega^2 \rangle$ obtained in Theorem 4.2 and Theorem 4.4, we have

$$\begin{aligned} B^2 &= \langle \omega^2 \rangle - \langle \omega \rangle^2 \\ &= \langle \omega^2 \rangle - 2\langle \omega \rangle(\langle \omega \rangle^+ + \langle \omega \rangle^-) + \langle \omega \rangle^2 \int_{-\infty}^{\infty} |s(t)|^2 dt \\ &= \int_{-\infty}^{\infty} [(A^{+\prime}(t))^2 + (A^{-\prime}(t))^2] dt + \int_{-\infty}^{\infty} [(A^{+}(t)\varphi^{+\prime}(t))^2 + (A^{-}(t)\varphi^{-\prime}(t))^2] dt \\ &\quad - \left[2\int_{-\infty}^{\infty} \langle \omega \rangle \varphi^{+\prime}(t)|s^+(t)|^2 dt + 2\int_{-\infty}^{\infty} \langle \omega \rangle \varphi^{-\prime}(t)|s^-(t)|^2 dt\right] \\ &\quad + \left[\int_{-\infty}^{\infty} \langle \omega \rangle^2 |s^+(t)|^2 dt + \int_{-\infty}^{\infty} \langle \omega \rangle^2 |s^-(t)|^2 dt\right] \\ &= \int_{-\infty}^{\infty} [(A^{+\prime}(t))^2 + (A^{-\prime}(t))^2] dt \\ &\quad + \int_{-\infty}^{\infty} [(\varphi^{+\prime}(t) - \langle \omega \rangle)^2 A^{+2}(t) + (\varphi^{-\prime}(t) - \langle \omega \rangle)^2 A^{-2}(t)] dt. \qquad \square \end{aligned}$$

Alternatively, the bandwidth can be obtained directly by the same steps as in the proof of Theorem 4.4.

Remark 4.6 If the signal $s(t)$ satisfies the assumptions of Theorem 1.1, then the decomposition $s = s^+ + s^-$ can be performed, and the steps in the proofs of Theorem 4.2, Theorem 4.4 and Corollary 4.5 can be followed. The steps to prove the classical results (1.2), (1.9) and

$$\langle \omega^2 \rangle = \int_{-\infty}^{\infty} \left[\frac{dA(t)}{dt}\right]^2 dt + \int_{-\infty}^{\infty} \left[A(t)\frac{d\varphi(t)}{dt}\right]^2 dt \tag{4.10}$$

can also be followed (see [4]). Therefore, in the case, the quantities $\langle \omega \rangle$, $\langle \omega^2 \rangle$ and B^2 have the alternative representations, and Theorem 4.2, Theorem 4.4 and Corollary 4.5 are indeed generalizations of (1.2), (4.10) and (1.9).

5 Mean and Variance of Time in Analytic Phase and Amplitude Derivatives of Fourier Transform of the Signal

We have written the bandwidth and the mean of the Fourier frequency in terms of the phase and amplitude derivatives of the signal. The same idea can be used to derive the mean of time and the duration in terms of the phase and amplitude derivatives of the Fourier transform of the signal.

Definition 5.1 Assume $s \in L^2(\mathbb{R})$. Define the mean of time by

$$\langle t \rangle \triangleq \int_{-\infty}^{\infty} t|s(t)|^2 dt, \tag{5.1}$$

the duration by

$$T^2 = \sigma_t^2 \triangleq \int_{-\infty}^{\infty} (t - \langle t \rangle)^2 |s(t)|^2 dt, \tag{5.2}$$

and the mean of any time function $g(t)$ by

$$\langle g(t) \rangle \triangleq \int_{-\infty}^{\infty} g(t)|s(t)|^2 dt, \tag{5.3}$$

provided that the right-hand-sides of (5.1), (5.2) and (5.3) are, respectively, well defined integrals. Below we will also use the notation

$$\langle g(t) \rangle_{\pm} \triangleq \int_{-\infty}^{\infty} g(t)|s^{\pm}(t)|^2 dt. \tag{5.4}$$

Classical results associated with the amplitude-phase representation of the Fourier spectrum $\hat{s}$ include (see [4])

$$\langle t \rangle = -\int_{-\infty}^{\infty} \frac{d\psi(\omega)}{d\omega} |\hat{s}(\omega)|^2 d\omega \tag{5.5}$$

and

$$T^2=\sigma_t^2=\int_{-\infty}^{\infty}\left(\frac{dB(\omega)}{d\omega}\right)^2 d\omega+\int_{-\infty}^{\infty}\left(\frac{d\psi(\omega)}{d\omega}+\langle t\rangle\right)^2 B^2(\omega)d\omega, \qquad (5.6)$$

where $\hat{s}(\omega)=B(\omega)e^{i\psi(\omega)}$. Under the classical setting, these results are proved based on the pointwise (strong) differentiability.

Assume $s(t)\in L^2(R)$ and $ts(t)\in L^2(R)$. Then $\hat{s}(\omega)\in L^2(R)$ and $\hat{s}(\omega)=\hat{s}^+(\omega)+\hat{s}^-(\omega)$, where

$$\hat{s}^+(\omega)=\left[\chi_- s\right]^\wedge(\omega), \qquad \hat{s}^-(\omega)=\left[\chi_+ s\right]^\wedge(\omega). \qquad (5.7)$$

Being similar to the case of the amplitude-phase representation of the signal s, we now have $\hat{s}^\pm(\omega)=B^\pm(\omega)e^{i\psi^\pm(\omega)}\in L^2(\mathbb{R})$, where $B^{\pm'}(\omega)$ and $\psi^{\pm'}(\omega)$ are defined in the same way as in Theorem 3.1 and Definition 3.2. We have the following two results.

Theorem 5.2 *Assume $s(t)\in L^2(R)$ and $ts(t)\in L^2(R)$. With the decomposition $\hat{s}(\omega)=\hat{s}^+(\omega)+\hat{s}^-(\omega)$, $\hat{s}^\pm(\omega)=[\chi_\mp s]^\wedge(\omega)$, $\hat{s}^\pm(\omega)=B^\pm(\omega)e^{i\psi^\pm(\omega)}$, the mean time defined by* (5.1) *is identical with*

$$\langle t\rangle=-\int_{-\infty}^{\infty}\psi^{-'}(\omega)B^{-2}(\omega)d\omega-\int_{-\infty}^{\infty}\psi^{+'}(\omega)B^{+2}(\omega)d\omega.$$

Theorem 5.3 *Assume $s(t)\in L^2(R)$ and $ts(t)\in L^2(R)$. With the decomposition $\hat{s}(\omega)=\hat{s}^+(\omega)+\hat{s}^-(\omega)$, $\hat{s}^\pm(\omega)=[\chi_\mp s(\cdot)]^\wedge(\omega)$, $\hat{s}^\pm(\omega)=B^\pm(\omega)e^{i\psi^\pm(\omega)}$, the duration defined by* (5.2) *is identical with*

$$\begin{aligned}\sigma_t^2=&\int_{-\infty}^{\infty}[(B^{-'}(\omega))^2+(B^{+'}(\omega))^2]d\omega+\int_{-\infty}^{\infty}[(\psi^{-'}(\omega)\\&+\langle t\rangle)^2B^{-2}(\omega)+(\psi^{+'}(\omega)+\langle t\rangle)^2B^{+2}(\omega)]d\omega.\end{aligned}$$

The proofs of Theorem 5.2 and Theorem 5.3 are omitted as they are similar to those of Theorem 4.2 and Theorem 4.4.

6 Covariance Under the Hardy-Sobolev Spaces Decomposition

It is well known that the correlation between frequency and time is measured by the covariance given in the following definition.

Definition 6.1 [4] Let $s(t)=A(t)e^{i\varphi(t)}$ and $ts(t)\in L^2(\mathbb{R})$, where $A(t)=|s(t)|$. Assume that the classical derivatives $\frac{dA(t)}{dt}$, $\frac{d\varphi(t)}{dt}$ and $\frac{ds(t)}{dt}$ all exist as Lebesgue measurable functions, and $\frac{ds}{dt}$ is in $L^2(\mathbb{R})$. The covariance is defined by

$$\mathrm{Cov}_{t\omega}=\left\langle t\frac{d\varphi(t)}{dt}\right\rangle-\langle t\rangle\langle\omega\rangle. \qquad (6.1)$$

This definition is based on the existence of the classical derivatives $\frac{dA}{dt}$, $\frac{ds}{dt}$ and $\frac{d\varphi}{dt}$. The following is an extension of the above definition to general cases.

Definition 6.2 Assume s, $\frac{d^*}{dt}s$ and $ts(t) \in L^2(R)$. With the decomposition $s = s^+ + s^-$, $(s^\pm)\hat{}(\omega) = \chi_\pm \hat{s}(\omega)$, $s^\pm(t) = A^\pm(t)e^{i\varphi^\pm(t)}$, the covariance is defined by

$$\mathrm{Cov}_{t\omega} = \langle t{\varphi^+}'(t)\rangle_+ + \langle t{\varphi^-}'(t)\rangle_- - \langle t\rangle\langle\omega\rangle. \tag{6.2}$$

Proposition 6.3 *Assume that signal s satisfies the conditions in Definition 6.1. Then it also satisfies the conditions in Definition 6.2, and the right-hand-side of (6.2) is reduced to the right-hand-side of (6.1).*

Proof of Proposition Since strong differentiability implies weak differentiability (see Remark 8.3), the signal $s(t)$ also satisfies the conditions in Definition 6.2. The phase derivatives ${\varphi^\pm}'(t) = \mathrm{Im}\{\frac{{s^\pm}'(t)}{s^\pm(t)}\}$ are well defined, and

$$\begin{aligned}\langle t{\varphi^\pm}'(t)\rangle_\pm &= \int_{-\infty}^{\infty} t{\varphi^\pm}'(t)|s^\pm(t)|^2 dt \\ &= \mathrm{Im}\int_{-\infty}^{\infty} {s^\pm}'(t)\overline{ts^\pm(t)}dt \\ &= \mathrm{Im}\int_{-\infty}^{\infty} i\omega(s^\pm)\hat{}(\omega)\overline{i(s^\pm)\hat{}'(\omega)}d\omega \\ &= \mathrm{Im}\int_{-\infty}^{\infty} \omega(s^\pm)\hat{}(\omega)\overline{(s^\pm)\hat{}'(\omega)}d\omega.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle t{\varphi^+}'(t)\rangle_+ + \langle t{\varphi^-}'(t)\rangle_- &= \mathrm{Im}\int_{-\infty}^{\infty} \omega(s^+)\hat{}(\omega)\overline{(s^+)\hat{}'(\omega)}d\omega \\ &\quad + \mathrm{Im}\int_{-\infty}^{\infty} \omega(s^-)\hat{}(\omega)\overline{(s^-)\hat{}'(\omega)}d\omega \\ &= \mathrm{Im}\int_0^{\infty} \omega\hat{s}(\omega)\overline{\hat{s}'(\omega)}d\omega + \mathrm{Im}\int_{-\infty}^{0} \omega\hat{s}(\omega)\overline{\hat{s}'(\omega)}d\omega \\ &= \mathrm{Im}\int_{-\infty}^{\infty} \omega\hat{s}(\omega)\overline{\hat{s}'(\omega)}d\omega.\end{aligned}$$

Since s satisfies the assumptions in Definition 6.1, $\frac{d\varphi(t)}{dt}$ is well defined and identical with $\mathrm{Im}\{\frac{s'(t)}{s(t)}\}$. Therefore,

$$\left\langle t\frac{d\varphi(t)}{dt}\right\rangle = \int_{-\infty}^{\infty} t\frac{d\varphi(t)}{dt}|s(t)|^2 dt$$

$$= \mathrm{Im} \int_{-\infty}^{\infty} s'(t)\overline{ts(t)}dt$$

$$= \mathrm{Im} \int_{-\infty}^{\infty} i\omega\hat{s}(\omega)\overline{i\hat{s}(\omega)'(\omega)}d\omega$$

$$= \mathrm{Im} \int_{-\infty}^{\infty} \omega\hat{s}(\omega)\overline{\hat{s}'(\omega)}d\omega$$

and

$$\langle t\varphi^{+'}(t)\rangle_+ + \langle t\varphi^{-'}(t)\rangle_- = \left\langle t\frac{d\varphi(t)}{dt}\right\rangle.$$

Thus the right-hand-side of (6.2) is identical with the right-hand-side of (6.1). □

The covariance defined in Definition 6.2 is in the time domain, where the Fourier frequency ω is replaced with the analytic phase derivative $\varphi^{\pm'}(t)$. One can alternatively define the covariance in the frequency domain, where the time t should be replaced with the group delay $-\psi^{\pm'}(\omega)$. The following question arises: If we define the covariance by

$$\mathrm{Cov}_{t\omega} = -\langle\omega\psi^{+'}(\omega)\rangle_+ - \langle\omega\psi^{-'}(\omega)\rangle_- - \langle t\rangle\langle\omega\rangle, \tag{6.3}$$

are the two definitions equivalent? The following theorem gives the positive answer.

Theorem 6.4 *Assume $s(t)$, $\frac{d^*}{dt}s$ and $ts(t) \in L^2(R)$. With the decomposition $s = s^+ + s^-$, $(s^\pm)\hat{} = \chi_\pm\hat{s}$, $s^\pm(t) = A^\pm(t)e^{i\varphi^\pm(t)}$, $\hat{s}(\omega) = \hat{s}^+(\omega) + \hat{s}^-(\omega)$, $\hat{s}^\pm(\omega) = [\chi_\mp s]^\wedge(\omega)$ and $\hat{s}^\pm(\omega) = B^\pm(\omega)e^{i\psi^\pm(\omega)}$, there holds*

$$\langle t\varphi^{+'}(t)\rangle_+ + \langle t\varphi^{-'}(t)\rangle_- = -\langle\omega\psi^{+'}(\omega)\rangle_+ - \langle\omega\psi^{-'}(\omega)\rangle_-.$$

Proof of Theorem 6.4 Since $s(t)$, $\frac{d^*}{dt}s$, and $ts(t) \in L^2(R)$, the phase derivatives of $s^\pm(t)$ and $\hat{s}^\pm(\omega)$, viz. $\varphi^{\pm'}(t)$ and $\psi^{\pm'}(\omega)$, all exist, and $\varphi^{\pm'}(t) = \mathrm{Im}\{\frac{s^{\pm'}(t)}{s^\pm(t)}\}$, $\psi^{\pm'}(\omega) = \mathrm{Im}\{\frac{\hat{s}^{\pm'}(\omega)}{\hat{s}^\pm(\omega)}\}$.

We have

$$\int_{-\infty}^{\infty} t\varphi^{\pm'}(t)|s^\pm(t)|^2dt$$

$$= \mathrm{Im} \int_{-\infty}^{\infty} s^{\pm'}(t)\overline{ts^\pm(t)}dt$$

$$= \mathrm{Im} \int_{-\infty}^{\infty} [\chi_+(t)s^\pm(t) + \chi_-(t)s^\pm(t)]'\,\overline{t[\chi_+(t)s^\pm(t) + \chi_-(t)s^\pm(t)]}dt$$

$$= \mathrm{Im} \int_{-\infty}^{\infty} \chi_+(t)s^{\pm'}(t)\overline{t[\chi_+(t)s^\pm(t)]}dt + \mathrm{Im} \int_{-\infty}^{\infty} \chi_+(t)s^{\pm'}(t)\overline{t[\chi_-(t)s^\pm(t)]}dt$$

$$
\begin{aligned}
&+\operatorname{Im}\int_{-\infty}^{\infty}\chi_-(t)s^{\pm\prime}(t)\overline{t[\chi_+(t)s^{\pm}(t)]}dt+\operatorname{Im}\int_{-\infty}^{\infty}\chi_-(t)s^{\pm\prime}(t)\overline{t[\chi_-(t)s^{\pm}(t)]}dt\\
&=\operatorname{Im}\int_{-\infty}^{\infty}\chi_+(t)s^{\pm\prime}(t)\overline{t[\chi_+(t)s^{\pm}(t)]}dt+\operatorname{Im}\int_{-\infty}^{\infty}\chi_-(t)s^{\pm\prime}(t)\overline{t[\chi_-(t)s^{\pm}(t)]}dt\\
&=\operatorname{Im}\int_{-\infty}^{\infty}i\omega(\chi_+s^{\pm})^{\wedge}(\omega)\overline{i(\chi_+s^{\pm})^{\wedge\prime}(\omega)}d\omega\\
&\quad+\operatorname{Im}\int_{-\infty}^{\infty}i\omega(\chi_-s^{\pm})^{\wedge}(\omega)\overline{i(\chi_-s^{\pm})^{\wedge\prime}(\omega)}d\omega\\
&=\operatorname{Im}\int_{-\infty}^{\infty}\omega(\chi_+s^{\pm})^{\wedge}(\omega)\overline{(\chi_+s^{\pm})^{\wedge\prime}(\omega)}d\omega\\
&\quad+\operatorname{Im}\int_{-\infty}^{\infty}\omega(\chi_-s^{\pm})^{\wedge}(\omega)\overline{(\chi_-s^{\pm})^{\wedge\prime}(\omega)}d\omega,
\end{aligned}
$$

where the cross terms vanish because of the relation $\chi_+\chi_-=0$. Denote

$$
\int_{-\infty}^{\infty}\omega(\chi_+s^{\pm})^{\wedge}(\omega)\overline{(\chi_+s^{\pm})^{\wedge\prime}(\omega)}d\omega+\int_{-\infty}^{\infty}\omega(\chi_-s^{\pm})^{\wedge}(\omega)\overline{(\chi_-s^{\pm})^{\wedge\prime}(\omega)}d\omega\triangleq f^{\pm}.
$$

Note that,

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\omega\psi^{\pm\prime}(\omega)|\hat{s}^{\pm}(\omega)|^2d\omega\\
&\quad=\operatorname{Im}\int_{-\infty}^{\infty}\omega(\hat{s}^{\pm})'(\omega)\overline{\hat{s}^{\pm}(\omega)}d\omega\\
&\quad=\operatorname{Im}\int_{-\infty}^{\infty}\omega[(\chi_{\mp}s^{+})^{\wedge}(\omega)+(\chi_{\mp}s^{-})^{\wedge}(\omega)]'\overline{(\chi_{\mp}s^{+})^{\wedge}(\omega)+(\chi_{\mp}s^{-})^{\wedge}(\omega)}d\omega\\
&\quad=\operatorname{Im}\int_{-\infty}^{\infty}\omega(\chi_{\mp}s^{+})^{\wedge\prime}(\omega)\overline{(\chi_{\mp}s^{+})^{\wedge}(\omega)}d\omega\\
&\qquad+\operatorname{Im}\int_{-\infty}^{\infty}\omega(\chi_{\mp}s^{+})^{\wedge\prime}(\omega)\overline{(\chi_{\mp}s^{-})^{\wedge}(\omega)}d\omega\\
&\qquad+\operatorname{Im}\int_{-\infty}^{\infty}\omega(\chi_{\mp}s^{-})^{\wedge\prime}(\omega)\overline{(\chi_{\mp}s^{+})^{\wedge}(\omega)}d\omega\\
&\qquad+\operatorname{Im}\int_{-\infty}^{\infty}\omega(\chi_{\mp}s^{-})^{\wedge\prime}(\omega)\overline{(\chi_{\mp}s^{-})^{\wedge}(\omega)}d\omega.
\end{aligned}
$$

By adding

$$
\int_{-\infty}^{\infty}\omega\psi^{+\prime}(\omega)|\hat{s}^{+}(\omega)|^2d\omega\quad\text{and}\quad\int_{-\infty}^{\infty}\omega\psi^{-\prime}(\omega)|\hat{s}^{-}(\omega)|^2d\omega,
$$

the cross terms are canceled out. In fact,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \omega(\chi_- s^+)^{\wedge\prime}(\omega)\overline{(\chi_- s^-)^\wedge(\omega)}d\omega + \int_{-\infty}^{\infty} \omega(\chi_+ s^+)^{\wedge\prime}(\omega)\overline{(\chi_+ s^-)^\wedge(\omega)}d\omega \\
&\quad = \int_{-\infty}^{\infty} [\omega(\chi_- s^+)^{\wedge\prime}(\omega) + \omega(\chi_+ s^+)^{\wedge\prime}(\omega)][\overline{(\chi_- s^-)^\wedge(\omega)} + \overline{(\chi_+ s^-)^\wedge(\omega)}]d\omega \\
&\qquad - \int_{-\infty}^{\infty} \omega(\chi_- s^+)^{\wedge\prime}(\omega)\overline{(\chi_+ s^-)^\wedge(\omega)}d\omega - \int_{-\infty}^{\infty} \omega(\chi_+ s^+)^{\wedge\prime}(\omega)\overline{(\chi_- s^-)^\wedge(\omega)}d\omega \\
&\quad = \int_{-\infty}^{\infty} \omega(s^+)^{\wedge\prime}(\omega)\overline{(s^-)^\wedge(\omega)}d\omega \\
&\qquad - \int_{-\infty}^{\infty} -it\chi_-(t)s^+(t)\overline{-i\chi_+(t)s^{-\prime}(t)}dt \\
&\qquad - \int_{-\infty}^{\infty} -it\chi_+(t)s^+(t)\overline{-i\chi_-(t)s^{-\prime}(t)}dt \\
&\quad = 0.
\end{aligned}$$

Similarly,

$$\int_{-\infty}^{\infty} \omega(\chi_- s^-)^{\wedge\prime}(\omega)\overline{(\chi_- s^+)^\wedge(\omega)}d\omega + \int_{-\infty}^{\infty} \omega(\chi_+ s^-)^{\wedge\prime}(\omega)\overline{(\chi_+ s^+)^\wedge(\omega)}d\omega = 0.$$

Therefore, with the notation

$$\int_{-\infty}^{\infty} \omega(\chi_\mp s^+)^{\wedge\prime}(\omega)\overline{(\chi_\mp s^+)^\wedge(\omega)}d\omega + \int_{-\infty}^{\infty} \omega(\chi_\mp s^-)^{\wedge\prime}(\omega)\overline{(\chi_\mp s^-)^\wedge(\omega)}d\omega \triangleq g^\pm,$$

we have

$$\int_{-\infty}^{\infty} \omega\psi^{+\prime}(\omega)|\hat{s}^+(\omega)|^2 d\omega + \int_{-\infty}^{\infty} \omega\psi^{-\prime}(\omega)|\hat{s}^-(\omega)|^2 d\omega = \mathrm{Im}(g^+ + g^-).$$

It then follows

$$f^+ + f^- = \overline{g^+ + g^-}.$$

Therefore,

$$\mathrm{Im}(f^+ + f^-) = -\mathrm{Im}(g^+ + g^-).$$

As a consequence,

$$\begin{aligned}
&\int_{-\infty}^{\infty} t\varphi^{+\prime}(t)|s^+(t)|^2 dt + \int_{-\infty}^{\infty} t\varphi^{-\prime}(t)|s^-(t)|^2 dt \\
&\quad = -\int_{-\infty}^{\infty} \omega\psi^{+\prime}(\omega)|\hat{s}^+(\omega)|^2 d\omega - \int_{-\infty}^{\infty} \omega\psi^{-\prime}(\omega)|\hat{s}^-(\omega)|^2 d\omega,
\end{aligned}$$

that is,

$$\langle t\varphi^{+\prime}(t)\rangle_+ + \langle t\varphi^{-\prime}(t)\rangle_- = -\langle \omega\psi^{+\prime}(\omega)\rangle_+ - \langle \omega\psi^{-\prime}(\omega)\rangle_-. \qquad \square$$

7 Uncertainty Principle Under the Hardy-Sobolev Spaces Decomposition

Next we present a proof of the uncertainty principle by using the Hardy-Sobolev spaces decomposition.

Theorem 7.1 *Assume* $s(t)$, $\frac{d^*}{dt}s$ *and* $ts(t) \in L^2(R)$. *Then there holds*

$$\sigma_t\sigma_\omega \geq \frac{1}{2}\sqrt{1+4\mathrm{Cov}_{t\omega}^2}.$$

Proof of Theorem 7.1 For the signal $s(t)$, the new signal defined by

$$s_{new}(t) = e^{-i\langle\omega\rangle(t+\langle t\rangle)}s(t+\langle t\rangle)$$

has the same shape in both time and frequency as $s(t)$ does, except that it has been translated in time and frequency such that the means become zero. Conversely, if we have a signal s_{new} that has zero mean time and zero mean frequency and we want a signal $s(t)$ of the same shape but with the particular mean time $\langle t\rangle$ and the mean frequency $\langle\omega\rangle$, then

$$s(t) = e^{i\langle\omega\rangle t}s_{new}(t-\langle t\rangle).$$

So we may assume $\langle t\rangle = 0$, $\langle\omega\rangle = 0$, and the bandwidth is

$$\sigma_\omega^2 = \int_{-\infty}^{\infty}\omega^2|\hat{s}(\omega)|^2 d\omega = \int_{-\infty}^{\infty}\left|\frac{d^*}{dt}s(t)\right|^2 dt$$

and the duration is

$$\sigma_t^2 = \int_{-\infty}^{\infty} t^2|s(t)|^2 dt.$$

By using Hölder's inequality and (v) and (vi) of Proposition 3.4, we have

$$\begin{aligned}
\sigma_t^2\sigma_\omega^2 &= \int_{-\infty}^{\infty} t^2|s(t)|^2 dt \times \int_{-\infty}^{\infty}\left|\frac{d^*}{dt}s(t)\right|^2 dt\\
&\geq \left|\int_{-\infty}^{\infty}\overline{ts(t)}\frac{d^*}{dt}s(t)dt\right|^2\\
&= \left|\int_{-\infty}^{\infty}\overline{t[s^+(t)+s^-(t)]}[s^{+\prime}(t)+s^{-\prime}(t)]dt\right|^2\\
&= \left|\int_{-\infty}^{\infty}\overline{ts^+(t)}s^{+\prime}(t)dt + \int_{-\infty}^{\infty}\overline{ts^-(t)}s^{-\prime}(t)dt\right|^2.
\end{aligned}$$

With $s^+(t+iy) = s_y^+(t) = A_y^+(t)e^{i\varphi_y^+(t)}$, $y>0$, the product rule of classical derivative gives

$$\overline{ts_y^+(t)}s_y^{+\prime}(t) = tA_y^+(t)A_y^{+\prime}(t) + itA_y^{+2}(t)\varphi_y^{+\prime}(t)$$

$$= \frac{1}{2}(tA_y^{+2}(t))' - \frac{1}{2}A_y^{+2}(t) + itA_y^{+2}(t){\varphi^+}'_y(t). \tag{7.1}$$

Below the main effort is devoted to justifying the integrability of (7.1) and passing to the limit $y \to 0$. To show that ts_y^+ is dominated by a $L^2(\mathbb{R})$ function independent of y, it suffices to show that $zs^+(z) = (t+iy)s_y^+(t)$ is a function in $H^2(\mathbb{C}^+)$. To prove this we start by showing that the Fourier transform of $ts^+(t)$ is supported in $[0,\infty)$. A basic result of Fourier transform gives

$$[(\cdot)s^+(\cdot)]\hat{}\,(\omega) = i\frac{d^*}{d\omega}(\chi_+ s)\hat{}\,(\omega) = i\chi_+(\omega)\frac{d^*}{d\omega}\hat{s}(\omega),$$

where the validity of interchanging the order of the multiplication by χ_+ and the operation of taking distributional derivative is proved in the proof of Lemma 2.6(iii). Thus the Fourier transform of $ts^+(t)$ is supported in $[0,\infty)$, and hence $ts^+(t)$ is the boundary value of a function $g(z)$ in $H^2(\mathbb{C}^+)$. We are to show $g(z) = zs^+(z)$. The Hardy function $g(z)$ is given by

$$\begin{aligned} g(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{xs^+(x)}{x-(t+iy)}dx \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\sqrt{2\pi}ie^{i\omega t}e^{-\omega y}\chi_+(\omega)[(\cdot)s^+(\cdot)]\hat{}\,(\omega)d\omega \\ &= \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\omega t}e^{-\omega y}\chi_+(\omega)\frac{d^*}{d\omega}[s^+(\cdot)]\hat{}\,(\omega)d\omega. \end{aligned}$$

Since the multiplication by χ_+ can commute with $\frac{d^*}{d\omega}$, and since $e^{i\omega t}e^{-\omega y}$, as a function of ω, is in the Schwarz class, we have

$$\begin{aligned} g(z) &= \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\omega t}e^{-\omega y}\frac{d^*}{d\omega}[s^+(\cdot)]\hat{}\,(\omega)d\omega \\ &= -\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[e^{i\omega t}e^{-\omega y}]'_\omega[s^+(\cdot)]\hat{}\,(\omega)d\omega \\ &= \frac{z}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\omega t}e^{-\omega y}[s^+(\cdot)]\hat{}\,(\omega)d\omega \\ &= zs^+(z). \end{aligned}$$

Therefore, $zs^+(z)$, and so $ts_y^+(t)$ as well, are dominated by a non-tangential maximal function in $L^2(\mathbb{R})$ independent of y. The function ${s^+}'(z) = {s_y^+}'(t)$ belongs to $H^2(\mathbb{C}^+)$ and is also dominated by the corresponding non-tangential maximal functions in $L^2(\mathbb{R})$. Therefore, the product $\overline{ts_y^+(t)}{s_y^+}'(t)$ on the left-end of (7.1) is dominated by a function in $L^1(\mathbb{R})$ that is independent of y. It is easy to show that the terms

$$A_y^{+2}(t) \quad \text{and} \quad tA_y^{+2}(t){\varphi^+}'_y(t)$$

are also dominated by $L^1(\mathbb{R})$ functions independent of y. In fact,

$$A_y^{+2}(t) = |s_y^+|^2 \quad \text{and} \quad tA_y^{+2}(t){\varphi^+}'_y(t) = t|s_y^+(t)|^2 \mathrm{Im}\left(\frac{(s_y^+)'(t)}{s_y^+(t)}\right).$$

Therefore, all the terms on the left-end and the right-end of the equation chain (7.1) are dominated by $L^1(\mathbb{R})$ functions independent of y. Now the integrability of $tA_y^{+2}(t)$ implies

$$\int_{-N}^{M} (tA_y^{+2}(t))' dt = (tA_y^{+2}(t))_{-N}^{M} \to 0$$

for suitably choosing $M, N \to \infty$. From (7.1) we have

$$\int_{-\infty}^{\infty} \overline{ts_y^+(t)} {s_y^+}'(t) dt = -\frac{1}{2}\int_{-\infty}^{\infty} A_y^{+2}(t) dt + i\int_{-\infty}^{\infty} tA_y^{+2}(t){\varphi^+}'_y(t) dt.$$

By using Lebesgue's dominated convergence theorem, we have

$$\int_{-\infty}^{\infty} \overline{ts^+(t)} {s^+}'(t) dt = -\frac{1}{2}\int_{-\infty}^{\infty} A^{+2}(t) dt + i\langle t{\varphi^+}'(t)\rangle_+.$$

Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} \overline{ts^+(t)} {s^+}'(t) dt + \int_{-\infty}^{\infty} \overline{ts^-(t)} {s^-}'(t) dt &= -\frac{1}{2}\int_{-\infty}^{\infty} A^{+2}(t) dt + i\langle t{\varphi^+}'(t)\rangle_+ \\ &\quad - \frac{1}{2}\int_{-\infty}^{\infty} A^{-2}(t) dt + i\langle t{\varphi^-}'(t)\rangle_- \\ &= -\frac{1}{2} + i[\langle t{\varphi^+}'(t)\rangle_+ + \langle t{\varphi^-}'(t)\rangle_-] \\ &= -\frac{1}{2} + i\mathrm{Cov}_{t\omega}.\end{aligned}$$

Thus,

$$\sigma_t^2 \sigma_\omega^2 \geq \frac{1}{4} + \mathrm{Cov}_{t\omega}^2. \qquad \square$$

8 Conclusions and Remarks

In this paper we show that the analytic phase and amplitude derivatives may be defined for signals in the Hardy-Sobolev spaces $H^\pm L_1^2(\mathbb{R})$. As application, formulas (1.2), (1.9) and (1.10) can be extended in the appropriate forms to signals in the Sobolev space $L_1^2(\mathbb{R})$. The classical uncertainty principle has a natural proof by means of the Hardy-Sobolev spaces decomposition.

Remark 8.1 There has been unfaded interest in the concept analytic signal since Gabor introduced it [8]. There have been temptations to define analytic instantaneous frequency based on this concept. Any new clarification and development in relation to the phase derivative and analytic instantaneous frequency would lead to a deeper understanding of the subject.

Remark 8.2 The developed theory and results are valid for both complex- and real-valued signals. Since for real-valued signals there holds $\hat{s}(-\omega) = \overline{\hat{s}}(\omega)$, we always have $\langle\omega\rangle = 0$ (see Example 4.3). For a real-valued signal the mean $\langle\omega\rangle$ is often replaced by $\langle\omega\rangle^+$ that presents a meaningful mean of the Fourier frequency. The latter corresponds to s^+ in the Hardy-Sobolev spaces decomposition $s = s^+ + s^-$.

Remark 8.3 In the generalized function theory it is well known that the existence of the strong derivative $\frac{ds}{dt}$ as a function in $L^1_{loc}(\mathbb{R})$ implies the existence of the weak derivative $\frac{d^*s}{dt}$. The converse also holds: if $s, \frac{d^*s}{dt} \in L^2(\mathbb{R})$, then there exists an absolutely continuous function $\tilde{s}$ such that $s(t) = \tilde{s}(t)$ a.e., and $\frac{d\tilde{s}}{dt} = \frac{d^*s}{dt}$ [25]. Therefore, the absolutely continuous function $\tilde{s}(t)$ is a representative of the Lebesgue equivalent class of s in the Sobolev space. There is then the question whether we can use such an absolutely continuous function as an representative for the equivalent class s for which $\frac{d^*s}{dt} \in L^2(\mathbb{R})$? The answer is "no", for a signal analysis result should be based on the equivalent classes but not on particular representatives. Indeed, the amplitude and phase derivatives are based on $s^\pm$ being defined through formulas (2.3) that are independent of particular representatives s in the equivalent class.

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