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Transplantation Theorems—A Survey

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Abstract In this survey article we overview transplantation theorems for several types of continuous and discrete orthogonal expansions. These include: Hankel and Dunkl transforms, and Fourier-Bessel, Jacobi and Laguerre expansions. We also discuss the idea of transference of transplantation and point out how a notion of conjugacy for orthogonal expansions may be interpreted as a generalized transplantation.

Keywords Transplantation · Hankel transform · Dunkl transform · Fourier-Bessel expansions · Jacobi expansions · Laguerre expansions · Transference · Conjugacy

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1 Introduction

What is *transplantation*? Let $\{\{\phi_n^\alpha\}_{n\in\mathbb{N}} : \alpha \in A\}$, card $A \geq 2$, be a family of orthonormal bases in $L^2(X, dm)$, where (X, dm) is a (nice) measure space. The transplantation operator $T_{\alpha\beta}$, $\alpha, \beta \in A$, is defined by the mapping

$$T_{\alpha\beta} : \phi_n^\beta \mapsto \phi_n^\alpha, \quad n \in \mathbb{N},$$

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and then extended to a linear bounded operator on $L^2(X,dm)$, which explicitly means that

$$T_{\alpha\beta} f = \sum_n \langle f, \phi_n^\beta\rangle \phi_n^\alpha, \qquad f = \sum_n \langle f, \phi_n^\beta\rangle \phi_n^\beta;$$

here $\langle\cdot,\cdot\rangle$ denotes the canonical inner product in $L^2(X,dm)$. Obviously, $T_{\alpha\beta}$ is an isometric isomorphism on $L^2(X,dm)$, and $T_{\alpha\beta}$ becomes the identity operator when $\alpha=\beta$ (thus to avoid trivialities we frequently assume that $\alpha\neq\beta$).

We say that for a given $p\in(1,\infty)$, $p\neq 2$ (or better, for every $p\in(1,\infty)$), and $\alpha\neq\beta$, a transplantation theorem holds provided we have

$$\|T_{\alpha\beta} f\|_p \le D_{\alpha\beta}\|f\|_p, \qquad f\in L^2\cap L^p(X,dm).$$

Then, $T_{\alpha\beta}$ uniquely extends to a bounded operator on $L^p(X,dm)$; to avoid a cumbersome notation we will sometimes denote this extension also by $T_{\alpha\beta}$.

How do we use transplantation theorems? Let $k=\{k_n\}_{n\in\mathbb{N}}$ be a bounded sequence of numbers. By a multiplier operator associated to k we mean the operator

$$f\mapsto M_k^\alpha f = \sum_n k_n \langle f, \phi_n^\alpha\rangle \phi_n^\alpha, \qquad f\in L^2(X,dm),$$

which is obviously bounded on $L^2(X,dm)$. We write $k\in\mathcal{M}_p^\alpha$, $1\le p<\infty$, provided

$$\|M_k^\alpha f\|_p \le C\|f\|_p, \qquad f\in L^2\cap L^p(X,dm),$$

i.e., M_k^α extends (uniquely) to a bounded operator on $L^p(X,dm)$.

Now, given $1<p<\infty$ and $\alpha,\beta\in A$, assume that $k\in\mathcal{M}_p^\alpha$ and transplantations for both pairs, (α,β) and (β,α), hold, i.e.,

$$\|T_{\alpha\beta} f\|_p \le D_{\alpha\beta}\|f\|_p, \qquad f\in L^2\cap L^p(X,dm),$$

$$\|T_{\beta\alpha} f\|_p \le D_{\beta\alpha}\|f\|_p, \qquad f\in L^2\cap L^p(X,dm).$$

Then we claim that $k\in\mathcal{M}_p^\beta$. Indeed, for every $f=\sum_n a_n\phi_n^\beta\in L^2\cap L^p(X,dm)$,

$$\Big\|\sum_n k_n a_n \phi_n^\beta\Big\|_p \le D_{\beta\alpha}\Big\|\sum_n k_n a_n\phi_n^\alpha\Big\|_p \le D_{\beta\alpha}C\Big\|\sum_n a_n\phi_n^\alpha\Big\|_p$$

$$\le D_{\beta\alpha}CD_{\alpha\beta}\Big\|\sum_n a_n\phi_n^\beta\Big\|_p.$$

Since the above reasoning may be reversed (with respect to α and β), it follows that

$$\mathcal{M}_p^\alpha = \mathcal{M}_p^\beta. \tag{1.1}$$

More generally, if a transplantation theorem holds for any pair (α,β), then the multiplier spaces $\mathcal{M}_p^\alpha$ are independent of $\alpha\in A$, i.e. (1.1) holds for any $\alpha,\beta\in A$.

Historically, the first transplantation theorem was proved by D.L. Guy [13] for the Hankel transform on the positive half-line. This theorem was a tool to prove a Marcinkiewicz' type multiplier theorem for the Hankel transform (it is perhaps interesting to note that this was not discrete but continuous case). Then the next step was done by Askey and Wainger [2] (ultraspherical expansions) and Gilbert [11] (a transplantation type theorem with assumptions that allow to include Fourier-Bessel expansions).

In this article we overview transplantation theorems with some emphasis on the role played in their proofs by the Calderón-Zygmund (frequently abbreviated to CZ) operator theory. (The author must admit that certainly this is somewhat subjective point of view.)

Throughout the paper we use a fairly standard notation. In the examples which we discuss, X will be one of the spaces: $\mathbb{R}$, $\mathbb{R}_+ = (0,\infty)$, $(0,\pi)$, $(0,1)$. The measure dm, denoted by dx, will be then Lebesgue measure on the corresponding space. By a weight on X we always mean a measurable and almost everywhere positive function on X. For a weight w on X we write $L^p(X,w)$, $1 \le p \le \infty$, and $L^{1,\infty}(X,w)$ to denote the weighted L^p and the weighted weak L^1 spaces (with respect to Lebesgue measure dx) that consist of all functions f on X for which

$$\|f\|_{p,w} = \left(\int_X |f(x)w(x)|^p \, dx \right)^{1/p} < \infty$$

(with obvious modification for $p = \infty$), or

$$\|f\|_{L^{1,\infty}(X,w)} = \sup_{t>0} \left(t \int_{\{x \in X : |f(x)| > t\}} w(x) \, dx \right) < \infty,$$

respectively. If $w \equiv 1$ we simplify the notation by writing $L^p(X)$ and $\|\cdot\|_p$, or $L^{1,\infty}(X)$ and $\|\cdot\|_{L^{1,\infty}}$. Sometimes, when it is completely clear from the context what X is, we write simply L^2 instead of $L^2(X)$. In the case when w is a power weight, $w(x) = x^a$, $a \in \mathbb{R}$, (this situation does not include $X = \mathbb{R}$), we write $\|\cdot\|_{p,a}$ instead of $\|\cdot\|_{p,x^a}$. Similar convention obeys multiplier spaces $\mathcal{M}^\alpha_{p,w}$ (to be defined later on): we will write $\mathcal{M}^\alpha_{p,a}$ when $w(x) = x^a$. Given $1 \le p \le \infty$, p' denotes its conjugate, $1/p + 1/p' = 1$. By $\langle f, g\rangle$ we always mean the canonical inner product in appropriate L^2 space under consideration. The symbol $\mathbb{N}$ is used to denote the set of nonnegative integers $\{0, 1, 2, \ldots\}$.

2 Transplantation Theorems

2.1 Hankel Transform

Given $\alpha > -1$ and a suitable function f on $(0,\infty)$, its Hankel transform of order α is defined by

$$\mathcal{H}_\alpha f(x) = \int_0^\infty (xy)^{1/2} J_\alpha(xy) f(y) dy, \quad x > 0.$$

Here $J_\alpha(x)$ denotes the Bessel function of the first kind of order α,

$$J_\alpha(x) = (x/2)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\Gamma(n+\alpha+1)},$$

see [19, (5.3.2)] or [36]. Then $(\mathcal{H}_\alpha \circ \mathcal{H}_\alpha) f = f$ and $\|\mathcal{H}_\alpha f\|_2 = \|f\|_2$, for any $f \in C_c^\infty(0,\infty)$, the space of C^∞ functions with compact support in $(0,\infty)$. These two facts were known in the literature for $\alpha \geq -1/2$; in [4, Lemma 2.6] a proof valid for any $\alpha > -1$ was furnished. We shall use the same symbol $\mathcal{H}_\alpha$ to denote the extension of $\mathcal{H}_\alpha$ to an isometric isomorphism of $L^2(\mathbb{R}_+)$. Since

$$J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z, \qquad J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos z, \tag{2.1}$$

thus $\mathcal{H}_{-1/2}$ and $\mathcal{H}_{1/2}$ become the cosine and the sine transform on $(0,\infty)$, respectively:

$$\mathcal{C}g(x) = (2/\pi)^{1/2} \int_0^\infty g(y)\cos(xy)\,dy, \quad x > 0,$$

and

$$\mathcal{S}g(x) = (2/\pi)^{1/2} \int_0^\infty g(y)\sin(xy)\,dy, \quad x > 0.$$

Guy [13] showed that the size of the Hankel transform of any suitable function, when measured in the power weighted L^p norm, remains the same whatever the order of the Hankel transform is. More precisely, given $\alpha, \gamma \geq -1/2$, $1 < p < \infty$ and $-1/p < a < 1 - 1/p$, there is a constant $C = C(\alpha, \gamma, p, a)$ such that for every appropriate function f

$$C^{-1}\|\mathcal{H}_\gamma f\|_{p,a} \leq \|\mathcal{H}_\alpha f\|_{p,a} \leq C\|\mathcal{H}_\gamma f\|_{p,a}. \tag{2.2}$$

In a different way, (2.2) may be expressed as

$$\|(\mathcal{H}_\alpha \circ \mathcal{H}_\gamma) f\|_{p,a} \leq C\|f\|_{p,a}.$$

Another proof of Guy's transplantation theorem was delivered by Schindler [27]. She found an explicit expression for the integral kernel of the transplantation operator

$$T_{\alpha\gamma} = \mathcal{H}_\alpha \circ \mathcal{H}_\gamma.$$

Due to a singularity along the diagonal, the corresponding integral was understood in the principal value sense. More precisely, in the case $\alpha, \gamma \geq -1/2$, $\alpha \neq \gamma$,

$$T_{\alpha\gamma} f(x) = \text{P. V.} \int_0^\infty K_{\alpha\gamma}(x,y) f(y)dy + C_{\alpha\gamma} f(x), \quad f \in C_c^\infty(0,\infty), \tag{2.3}$$

where $C_{\alpha\gamma} = \cos((\alpha - \gamma)\pi/2)$ and, for $0 < y < x$, $K_{\alpha\gamma}(x, y)$ is given by

$$\frac{2\Gamma((\alpha+\gamma+2)/2)}{\Gamma(\gamma+1)\Gamma((\alpha-\gamma)/2)} x^{-(\gamma+3/2)} y^{\gamma+1/2} \cdot {}_2F_1\left(\frac{\alpha+\gamma+2}{2}, \frac{\gamma-\alpha+2}{2}; \gamma+1; \left(\frac{y}{x}\right)^2\right),$$

while, for $0 < x < y$, $K_{\alpha\gamma}(x, y)$ equals

$$\frac{2\Gamma((\alpha+\gamma+2)/2)}{\Gamma(\alpha+1)\Gamma((\gamma-\alpha)/2)} x^{\alpha+1/2} y^{-(\alpha+3/2)} \cdot {}_2F_1\left(\frac{\alpha+\gamma+2}{2}, \frac{\alpha-\gamma+2}{2}; \alpha+1; \left(\frac{x}{y}\right)^2\right)$$

here ${}_2F_1$ denotes the Gauss hypergeometric function.

In [29] Guy's result was enhanced by enlarging the range of admissible parameters α and γ to $\alpha > -1$ and $\gamma > -1$, and extending the range of power weight exponent a to $-(\alpha+1/2) - 1/p < a < (\gamma+3/2) - 1/p$. The result was obtained by transferring Muckenhoupt's transplantation theorem for Jacobi expansions to the Hankel transform setting. In the restricted range $\alpha \geq -1/2$, $\gamma \geq -1/2$, Schindler's explicit kernel representation was used to obtain the same conclusion. This was done by splitting the integration into the three regions: $0 < y < x/2$, $x/2 < y < 3x/2$ and $3x/2 < y < \infty$. The splitting brought an advantage: while on both outer regions Hardy's integral inequalities were applied, the integration on the inner region was treated by using local versions of the Hardy-Littlewood maximal function and the Hilbert transform.

Nowak and Stempak [24] investigated the Hankel transform transplantation operator $T_{\alpha\gamma}$ by means of a suitably established local version of the CZ operator theory. This approach delivered weighted norm inequalities with weights more general than previously considered power weights. Moreover, it also allowed to obtain weighted weak type $(1, 1)$ inequalities, which were new even in the unweighted setting.

It is easy to check by a direct computation that the (Schindler's) kernel $K_{\alpha\gamma}(x, y)$ (associated to $T_{\alpha\gamma}$ in an appropriate sense for any $\alpha, \gamma > -1$) is a standard Calderón-Zygmund kernel if $\alpha, \gamma \geq 1/2$, but it fails to satisfy the appropriate smoothness condition (leading to Hörmander's condition) when either $\alpha < 1/2$ or $\gamma < 1/2$. In these cases problems occur on the regions $0 < y < x/2$ and $3x/2 < y < \infty$. Therefore in [24] the operator $T_{\alpha\gamma}$ was split according to these regions:

$$T_{\alpha\gamma} = T^1_{\alpha\gamma} + T^2_{\alpha\gamma} + T^3_{\alpha\gamma},$$

where the kernels $K^i_{\alpha\gamma}$ defining the integral operators $T^i_{\alpha\gamma}$, $i = 1, 2$, are given by

$$K^1_{\alpha\gamma}(x, y) = \chi_{\{(x,y):0<y<x/2\}} K_{\alpha\gamma}(x, y),$$
$$K^2_{\alpha\gamma}(x, y) = \chi_{\{(x,y):0<3x/2<y\}} K_{\alpha\gamma}(x, y).$$

Then it occurred that $T^1_{\alpha\gamma}$ and $T^2_{\alpha\gamma}$ were easy to handle by means of weighted Hardy's inequalities. To treat $T^3_{\alpha\gamma}$ a notion of a local Calderón-Zygmund operator was introduced.

To state the main results of [24] we need some additional preparations. Given a weight function $w(x)$ on $(0,\infty)$, consider the following set of conditions:

$$\sup_{r>0}\left(\int_r^\infty w(x)^p x^{-p(\gamma+3/2)}\,dx\right)^{1/p}\left(\int_0^r w(x)^{-p'} x^{p'(\gamma+1/2)}\,dx\right)^{1/p'}<\infty, \tag{2.4}$$

$$\sup_{r>0}\left(\int_0^r w(x)^p x^{p(\alpha+1/2)}\,dx\right)^{1/p}\left(\int_r^\infty w(x)^{-p'} x^{-p'(\alpha+3/2)}\,dx\right)^{1/p'}<\infty, \tag{2.5}$$

$$\sup_{0<u<v<2u}\frac{1}{v-u}\left(\int_u^v w(x)^p\,dx\right)^{1/p}\left(\int_u^v w(x)^{-p'}\,dx\right)^{1/p'}<\infty. \tag{2.6}$$

We admit $1\le p\le\infty$ when considering (2.4) and (2.5), and $1\le p<\infty$ when considering (2.6). Here and later on, for $p=1$ or $p=\infty$, integrals of the form appearing in (2.4)–(2.6) have the usual interpretation. For example, when $p=1$, the second factor in (2.4) is taken as $\operatorname{ess\,sup}_{x\in(0,r)}[w(x)^{-1}x^{\gamma+1/2}]$.

Theorem 2.1 ([24]) *Let $\alpha,\gamma>-1$, $\alpha\neq\gamma$, and $1<p<\infty$ if $|\alpha-\gamma|\neq 2k$ for every $k\in\mathbb{N}$, or $1\le p\le\infty$ if $|\alpha-\gamma|=2k$ for some $k\in\mathbb{N}$. Let $w(x)$ be a weight on $\mathbb{R}_+$ that satisfies: condition (2.4) if $\alpha=\gamma+2k$ for some $k\in\mathbb{N}$; condition (2.5) if $\gamma=\alpha+2k$ for some $k\in\mathbb{N}$; conditions (2.4), (2.5) and (2.6) if $|\alpha-\gamma|\neq 2k$ for every $k\in\mathbb{N}$. Then*

$$\|T_{\alpha\gamma}f\|_{p,w}\le C\|f\|_{p,w},\qquad f\in L^2\cap L^p(\mathbb{R}_+,w). \tag{2.7}$$

Consequently, for $1\le p<\infty$, $T_{\alpha\gamma}$ extends to a bounded linear operator on $L^p(\mathbb{R}_+,w)$.

It may be seen that for a power weight function $w(x)=x^a$, $a\in\mathbb{R}$, (2.4) is satisfied if and only if $a<-\frac{1}{p}+(\gamma+\frac{3}{2})$, (2.5) is satisfied if and only if $a>-(\alpha+\frac{1}{2})-\frac{1}{p}$ and (2.6) is satisfied for each $a\in\mathbb{R}$.

Corollary 2.2 *Let $\alpha,\gamma>-1$, $\alpha\neq\gamma$, and $1<p<\infty$. If $-\min\{\alpha,\gamma\}-1/p-1/2<a<3/2-1/p+\min\{\alpha,\gamma\}$, then the inequalities*

$$\|T_{\alpha\gamma}f\|_{p,a}\le C\|f\|_{p,a}\quad\text{and}\quad \|T_{\gamma\alpha}f\|_{p,a}\le C\|f\|_{p,a}$$

hold with C independent of $f\in L^2\cap L^p(\mathbb{R}_+,x^a)$.

In order to obtain weak type $(1,1)$ inequalities for the transplantation operator $T_{\alpha,\gamma}$ assumptions similar to those in (2.4) and (2.5) must be imposed. We refer the reader to [24] where a relevant theorem is stated and proved.

As was already pointed out, $K_{\alpha\gamma}$ is a standard CZ kernel whenever $\alpha,\gamma \ge 1/2$, $|\alpha-\gamma| \neq 2k$, $k = 0,1,2,\dots$. Consequently, in such a case further mapping properties of $T_{\alpha\gamma}$ follow by a general theory, cf. [8, Chap. 6]. For instance, $T_{\alpha\gamma}$ extends to a bounded operator from $H^1(\mathbb{R}_+)$ to $L^1(\mathbb{R}_+)$, where $H^1(\mathbb{R}_+)$ denotes the real Hardy space on $\mathbb{R}_+$. It is worth noting that Kanjin [16] has recently proved a stronger result: $T_{\alpha\gamma}$ extends to a bounded operator on $H^1(\mathbb{R}_+)$ whenever $\alpha \ge -1/2$ and $\gamma > -1/2$. His proof was based on Schindler's explicit integral representation of $T_{\alpha\gamma}$, atomic decomposition of H^1-functions and a molecular characterization of $H^1(\mathbb{R}_+)$.

Moreover, Kanjin [17] applied the Hankel transform transplantation operator $T_{\alpha\gamma}$ to define and investigate Cesàro operators C^{α}_{γ} for the Hankel transform proving their boundedness on L^p and H^1 spaces under appropriate assumptions on α and γ.

As was already mentioned in the introductory section, a typical application of transplantation theorems is for multipliers. In the context of Hankel transform we say that a bounded measurable function m on $\mathbb{R}_+$ is an $L^p(\mathbb{R}_+, w)$ multiplier for $\mathcal{H}_\alpha$, $\alpha > -1$, and we write $m \in \mathcal{M}^{\alpha}_{p,w}$, provided

$$\|\mathcal{H}_\alpha(m\mathcal{H}_\alpha f)\|_{p,w} \le C\|f\|_{p,w}, \qquad f \in L^2 \cap L^p(\mathbb{R}_+, w). \tag{2.8}$$

Repeating an argument from Sect. 1 that led to justification of (1.1) and using Corollary 2.7 shows the following.

Proposition 2.3 *Let $\alpha > -1$ and $1 < p < \infty$. If*

$$-\alpha - 1/p - 1/2 < a < 3/2 - 1/p + \alpha,$$

then

$$\mathcal{M}^{\alpha}_{p,a} = \mathcal{M}^{\gamma}_{p,a}, \qquad \gamma > \alpha.$$

In particular, for $\alpha = -1/2$ one has $\mathcal{M}^{-1/2}_{p,a} = \mathcal{M}^{\gamma}_{p,a}$ for any $\gamma > -\frac{1}{2}$ and $-\frac{1}{p} < a < 1 - \frac{1}{p}$. Consequently, we are enabled to derive weighted L^p multiplier results with power weight x^a, $-\frac{1}{p} < a < 1 - \frac{1}{p}$, for the Hankel transform of an arbitrary order $\gamma > -\frac{1}{2}$, by applying known results (for instance those in [23]) for the Fourier transform, modified in an obvious way to the cosine transform that corresponds to $\alpha = -\frac{1}{2}$, see [23, Corollary 2.3].

The modified Hankel transform H_α, $\alpha > -1$, is defined for any suitable function f on $(0,\infty)$ by

$$H_\alpha f(x) = \int_0^\infty \frac{J_\alpha(xy)}{(xy)^\alpha} f(y) y^{2\alpha+1} dy, \qquad x > 0.$$

When $\alpha = \frac{n-2}{2}$, $n \ge 1$, H_α becomes the radial part of the Fourier transform on $\mathbb{R}^n$,

$$\mathcal{F}_n g(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} g(y) e^{-ixy} dy, \qquad x \in \mathbb{R}^n,$$

in the sense that $H_{(n-2)/2} f(|x|) = \mathcal{F}_n g(x)$, $0 \neq x \in \mathbb{R}^n$, where $g(y) = f(|y|)$, $y \in \mathbb{R}^n$. In particular,

$$H_{-1/2} f(|x|) = \mathcal{H}_{-1/2} f(|x|) = \mathcal{F}_1 g(x) = \mathcal{C} g(x), \quad x \in \mathbb{R},\ x \neq 0.$$

For any $\alpha > -1$ one has

$$\mathcal{H}_\alpha f(x) = x^{\alpha+1/2} H_\alpha((\cdot)^{-(\alpha+1/2)} f(\cdot))(x),$$

for any appropriate f, and H_α extends to an isometric isomorphism of $L^2(\mathbb{R}_+, d\mu_\alpha)$, $d\mu_\alpha(x) = x^{2\alpha+1} dx$.

Multiplier spaces $M^\alpha_{p,w}$ associated to the transform H_α may be defined in a similar way as $\mathcal{M}^\alpha_{p,w}$, i.e. by (2.8) with replacement of $\mathcal{H}_\alpha$ by H_α and $L^p(\mathbb{R}_+, w)$ by $L^p(\mathbb{R}_+, w d\mu_\alpha)$. It occurs, however, that in order to relate both classes of multipliers a modification has to be done in the definition (2.8) and its H_α counterpart, by replacing the testing function space $L^2 \cap L^p(\mathbb{R}_+, w)$ by $\mathcal{H}_\alpha(C_0^\infty)$, the image of C_0^∞ under $\mathcal{H}_\alpha$, and similarly, $L^2 \cap L^p(\mathbb{R}_+, w d\mu_\alpha)$ by $H_\alpha(C_0^\infty)$. This approach was undertaken by Stempak and Trebels [32]. With modifications described above, one has

$$M^\alpha_{p,b} = \mathcal{M}^\alpha_{p,b+(2\alpha+1)(\frac{1}{p}-\frac{1}{2})},$$

$\alpha \geq -1/2$, $b \in \mathbb{R}$, $1 < p < \infty$, and, what is also important, $\mathcal{H}_\alpha(C_0^\infty)$ is dense in $L^p(\mathbb{R}_+, x^b)$ provided $b > -\frac{1}{p} - (\alpha + \frac{1}{2})$, while $H_\alpha(C_0^\infty)$ is dense in $L^p(\mathbb{R}_+, x^b d\mu_\alpha)$ provided $b > -\frac{2}{p}(\alpha + 1)$. (See Theorem 4.7 and Corollary 4.8 in [32] with regard to different notation used there.)

2.2 Dunkl Transform

The Dunkl transform is an integral transform that generalizes the Euclidean Fourier transform in a framework of some symmetries on $\mathbb{R}^d$ related to a finite reflection group. For details see references in [26]. The simplest case of such a situation occurs when $d = 1$ and the reflection group is isomorphic with $\mathbb{Z}_2$. The only (nontrivial) reflection is then given by $x \mapsto -x$ and the so called multiplicity function is represented by a number parameter. In what follows we restrict our attention to this simplest case of the Dunkl transform.

Given $\alpha > -1$ and a suitable function f on $\mathbb{R}$, its Dunkl transform $D_\alpha f$ is defined by

$$D_\alpha f(\lambda) = \int_{\mathbb{R}} f(x) \frac{1}{2(x\lambda)^\alpha} \big(J_\alpha(x\lambda) - i J_{\alpha+1}(x\lambda)\big) |x|^{2\alpha+1} dx, \quad \lambda \in \mathbb{R}. \tag{2.9}$$

See [26] for details concerning this convenient representation of the discussed transform and for an explanation why $\alpha > -1$ (rather than $\alpha \geq -1/2$) is admitted. A simple calculation based on the identities (2.1) shows that $D_{-1/2} = \mathcal{F}_1$.

We will express D_α in terms of Hankel transforms of orders α and $\alpha + 1$. For a function f on $\mathbb{R}$ we denote by f_e and f_o the restrictions to $\mathbb{R}_+$ of its even and odd parts, respectively, i.e. the functions on $\mathbb{R}_+$ defined by

$$f_e(x) = \frac{1}{2}\big(f(x) + f(-x)\big), \qquad f_o(x) = \frac{1}{2}\big(f(x) - f(-x)\big), \quad x > 0.$$

A short computation then shows that given $\alpha > -1$, we have

$$D_\alpha f(\lambda) = |\lambda|^{-(\alpha+1/2)}\big(\mathcal{H}_\alpha\big((\cdot)^{\alpha+1/2} f_e\big)(|\lambda|) \\ - i(\operatorname{sgn}\lambda)\mathcal{H}_{\alpha+1}\big((\cdot)^{\alpha+1/2} f_o\big)(|\lambda|)\big). \tag{2.10}$$

Nowak and Stempak [26] introduced the following definition suggested by (2.10).

Definition 2.1 For $\alpha > -1$ and a suitable function f on $\mathbb{R}$ we define $\mathcal{D}_\alpha f$ by

$$\mathcal{D}_\alpha f(\lambda) = \mathcal{H}_\alpha(f_e)(|\lambda|) - i(\operatorname{sgn}\lambda)\mathcal{H}_{\alpha+1}(f_o)(|\lambda|), \quad \lambda \in \mathbb{R}, \tag{2.11}$$

and call it *the modified Dunkl transform* of f of order α.

Observe that for $\alpha = -1/2$ we have $\mathcal{D}_{-1/2} = D_{-1/2} = \mathcal{F}_1$ and one recovers in (2.11) the decomposition

$$\mathcal{F}_1 f(\lambda) = \mathcal{C} f_e(|\lambda|) - i(\operatorname{sgn}\lambda)\mathcal{S} f_o(|\lambda|),$$

where $\mathcal{C}$ and $\mathcal{S}$ are the cosine and sine transforms on $\mathbb{R}_+$, respectively. The inverse Dunkl and inverse modified Dunkl transforms of order α is defined by

$$\check{D}_\alpha f(\lambda) = D_\alpha f(-\lambda), \qquad \check{\mathcal{D}}_\alpha f(\lambda) = \mathcal{D}_\alpha f(-\lambda), \quad \lambda \in \mathbb{R}.$$

It is known that for $\alpha > -1$ and any $f \in C_c^\infty(\mathbb{R} \setminus \{0\})$ we have

$$\|\mathcal{D}_\alpha f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}, \qquad \|D_\alpha f\|_{L^2(\mathbb{R}, d\mu_\alpha)} = \|f\|_{L^2(\mathbb{R}, d\mu_\alpha)}, \tag{2.12}$$

and

$$\check{\mathcal{D}}_\alpha(\mathcal{D}_\alpha f) = \mathcal{D}_\alpha(\check{\mathcal{D}}_\alpha f) = f, \qquad \check{D}_\alpha(D_\alpha f) = D_\alpha(\check{D}_\alpha f) = f. \tag{2.13}$$

In the sequel we will use the same symbols $D_\alpha, \check{D}_\alpha, \mathcal{D}_\alpha, \check{\mathcal{D}}_\alpha$, to denote the extensions of the relevant transforms and inverse transforms to isometric isomorphisms on $L^2(\mathbb{R})$ or $L^2(\mathbb{R}, d\mu_\alpha)$, respectively.

In this subsection we work simultaneously with functions on $\mathbb{R}$ and $\mathbb{R}_+$ therefore to avoid possible collisions we shall write $L^p(\mathbb{R}, w)$, $L^p(\mathbb{R}, w d\mu_\alpha)$ and $L^p(\mathbb{R}_+, w)$, $L^p(\mathbb{R}_+, w d\mu_\alpha)$ to distinguish L^p weighted spaces on both domains.

Similarly to the Hankel transform setting, given $\alpha, \gamma > -1$, $\alpha \neq \gamma$, we shall call the operator

$$\mathcal{T}_{\alpha\gamma} = \check{\mathcal{D}}_\alpha \circ \mathcal{D}_\gamma$$

the Dunkl transform transplantation operator. Note that $\mathcal{T}_{\alpha\gamma}$ is a well defined isometric isomorphism on $L^2(\mathbb{R})$. Nowak and Stempak [26] related a weighted inequality for the Hankel transform transplantation operator

$$\|T_{\alpha\gamma} g\|_{L^p(\mathbb{R}_+, w)} \le C_{\alpha\gamma}\|g\|_{L^p(\mathbb{R}_+, w)}, \quad g \in L^2 \cap L^p_{L^p(\mathbb{R}_+, w)}, \tag{2.14}$$

with that for $\mathcal{T}_{\alpha\gamma}$.

Proposition 2.4 ([26]) *Let $\alpha, \gamma > -1$, $\alpha \neq \gamma$, $1 \leq p < \infty$ and a weight w on $\mathbb{R}_+$ be given. Then*

$$\|\mathcal{T}_{\alpha\gamma} f\|_{L^p(\mathbb{R}, w(|\cdot|))} \leq C\|f\|_{L^p(\mathbb{R}, w(|\cdot|))}, \qquad f \in L^2 \cap L^p(\mathbb{R}, w(|\cdot|)), \tag{2.15}$$

holds if and only if (2.14) *and the analogous inequality with α and γ replaced by $\alpha + 1$ and $\gamma + 1$ are satisfied* ($w(|\cdot|)$ *denotes the weight* $w(|x|)$ *on* $\mathbb{R}$).

The result of [24, Theorem 2.1] specified to power weights $w(x) = x^a$, $x > 0$, $a \in \mathbb{R}$, one of the indices $\alpha \neq \gamma$ equal to $\pm 1/2$ and $p > 1$ gives the following: if $\alpha = -1/2$, then $-\frac{1}{p} < a < \gamma + 3/2 - \frac{1}{p}$ is sufficient for (2.14) to hold with $w(x) = x^a$ and, similarly, if $\alpha = 1/2$, then $-1 - \frac{1}{p} < a < \gamma + 5/2 - \frac{1}{p}$ is sufficient for (2.14) to hold with $w(x) = x^a$ and γ replaced by $\gamma + 1$; if $\gamma = -1/2$, then $-(\alpha + 1/2) - \frac{1}{p} < a < 1 - \frac{1}{p}$ is sufficient for (2.14) to hold with $w(x) = x^a$ and, similarly, if $\gamma = 1/2$, then $-(\alpha + 3/2) - \frac{1}{p} < a < 2 - \frac{1}{p}$ is sufficient for (2.14) to hold with $w(x) = x^a$ and $\alpha + 1$ replacing α. Therefore, as a consequence of Proposition 2.4 we obtain the following (note that the case of $\alpha = -1/2$ is trivially included).

Corollary 2.5 *Let $1 < p < \infty$, $\alpha > -1$. Then*

$$\|\mathcal{T}_{\alpha,-1/2} f\|_{L^p(\mathbb{R},|x|^a)} \leq C\|f\|_{L^p(\mathbb{R},|x|^a)}, \qquad f \in L^2 \cap L^p(\mathbb{R}, |x|^a),$$

provided that $-(\alpha + 1/2) - \frac{1}{p} < a < 1 - \frac{1}{p}$, and

$$\|\mathcal{T}_{-1/2,\alpha} f\|_{L^p(\mathbb{R},|x|^a)} \leq C\|f\|_{L^p(\mathbb{R},|x|^a)}, \qquad f \in L^2 \cap L^p(\mathbb{R}, |x|^a),$$

provided that $-\frac{1}{p} < a < \alpha + 3/2 - \frac{1}{p}$.

Analogously to the situation of Hankel multipliers we incorporate the following definition of Dunkl multipliers. Let $\alpha > -1$, $1 \leq p < \infty$ and a weight w on $\mathbb{R}$ be given. We say that a bounded measurable function m on $\mathbb{R}$ is an:

(i) $L^p(\mathbb{R}, w\,d\mu_\alpha)$ multiplier for D_α provided that

$$\left\|\check{D}_\alpha(m D_\alpha f)\right\|_{L^p(R,\, wd\mu_\alpha)} \leq A\|f\|_{L^p(\mathbb{R}, wd\mu_\alpha)}, \qquad f \in L^2 \cap L^p(\mathbb{R}, wd\mu_\alpha); \tag{2.16}$$

(ii) $L^p(\mathbb{R}, w)$ multiplier for $\mathcal{D}_\alpha$ provided that

$$\left\|\check{\mathcal{D}}_\alpha(m\mathcal{D}_\alpha f)\right\|_{L^p(\mathbb{R},w)} \leq B\|f\|_{L^p(\mathbb{R},w)}, \qquad f \in L^2 \cap L^p(\mathbb{R}, w). \tag{2.17}$$

It is now clear that the spaces of weighted (modified) Dunkl multipliers are independent of $\alpha > -1$ for any given $1 < p < \infty$ and an appropriate weight w. Consequently, Corollary 2.5 leads to the following.

Corollary 2.6 *Let $\alpha > -1$, $1 < p < \infty$ and*

$$-\min\{0, \alpha + 1/2\} - 1/p < a < 1 - 1/p + \min\{0, \alpha + 1/2\}.$$

Then m is an $L^p(\mathbb{R}, |x|^a)$ multiplier for $\mathcal{D}_\alpha$ if and only if m is an $L^p(\mathbb{R}, |x|^a)$ multiplier for the Fourier transform $\mathcal{F}_1$.

Finally, it may be easily noted that m is an $L^p(\mathbb{R}, w)$ multiplier for $\mathcal{D}_\alpha$ if and only if m is an $L^p(\mathbb{R}, w^* d\mu_\alpha)$ multiplier for D_α, where $w^*(x) = w(x)|x|^{-(2\alpha+1)(1/p-1/2)}$. Thus, known multiplier results for the Fourier transform deliver also multiplier results the Dunkl transform D_α, $\alpha > -1$.

2.3 Fourier-Bessel Expansions

Given $\nu > -1$ let $\lambda_{n,\nu}$, $n = 1, 2, \ldots$, denote the sequence of successive positive zeros of the Bessel function $J_\nu(z)$. Then the functions

$$\psi_n^\nu(x) = d_{n,\nu}(\lambda_{n,\nu}x)^{1/2} J_\nu(\lambda_{n,\nu}x), \quad d_{n,\nu} = \sqrt{2}|\lambda_{n,\nu}^{1/2} J_{\nu+1}(\lambda_{n,\nu})|^{-1},$$

$n = 1, 2, \ldots$, form a complete orthonormal system in $L^2((0, 1), dx)$. In particular,

$$\psi_n^{-1/2}(x) = \sqrt{2}\cos(\pi(n - 1/2)x), \quad \psi_n^{1/2}(x) = \sqrt{2}\sin(\pi n x),$$

for $n = 1, 2, \ldots$. Given a function f on $(0, 1)$, we associate its Fourier-Bessel series

$$f(x) \sim \sum_1^\infty c_n^\nu(f)\psi_n^\nu(x), \quad c_n^\nu(f) = \int_0^1 f(x)\psi_n^\nu(x)\,dx,$$

provided that the coefficients $c_n^\nu(f)$ exist (i.e., the defining integrals are absolutely convergent). A comprehensive study of Fourier-Bessel expansions is contained in Chap. XVII of Watson's monograph [36].

Given $\mu > -1$, $\nu > -1$, we define the transplantation operator $T_{\mu\nu}$ on $L^2((0, 1), dx)$ by

$$T_{\mu\nu} f = \sum_{n=1}^\infty \langle f, \psi_n^\mu\rangle \psi_n^\nu, \quad f \in L^2((0, 1), dx).$$

A transplantation theorem for Fourier-Bessel expansions is contained in Theorem A of Gilbert's paper [11]. This theorem states a general result of transplantation type for operators with kernels satisfying a number of "natural" conditions. The Fourier-Bessel expansions fit into that frame and, moreover, a proper modification of Gilbert's argument leads to a more general weighted result with A_p weights involved, cf. [12].

In [5] Ciaurri and Stempak proved a transplantation result for Fourier-Bessel series by following Muckenhoupt's approach from [22]. This approach allowed to consider power weights. In [6] the authors used the theory of CZ operators to include general weights. To be precise, the transplantation operator $T_{\mu\nu}$ occurred to be a Calderón–Zygmund operator only for $\mu, \nu \ge 1/2$. Since the Fourier-Bessel expansions should be treated as discrete analogues of the Hankel transforms it is also interesting to note that the approach used in [6] was a natural counterpart to the approach developed in [24] for the Hankel transform.

The main results of [6] reads as follows.

Theorem 2.7 ([6]) *Let $\mu, \nu > -1$, $\mu \neq \nu$, and $1 < p < \infty$. Let $w(x)$ be a weight on $(0, 1)$ that satisfies the conditions (2.4) and (2.5) modified by replacing ∞ by 1, and (2.6) modified by replacing $2u$ by $\min\{1, 2u\}$. Then*

$$\|T_{\mu\nu} f\|_{p,w} \le C\|f\|_{p,w}, \quad f \in L^2 \cap L^p((0,1), w). \tag{2.18}$$

Consequently, $T_{\mu\nu}$ extends uniquely to a bounded linear operator on $L^p((0,1), w)$ and, using the same symbol $T_{\mu\nu}$ to denote this extension, for $f \in L^p((0,1), w)$,

$$\langle T_{\mu\nu} f, \psi_n^\nu\rangle = \langle f, \psi_n^\mu\rangle, \quad n = 1, 2, \ldots. \tag{2.19}$$

2.4 Jacobi expansions

Given $\alpha > -1$, $\beta > -1$, consider the orthonormalized Jacobi polynomials

$$\phi_n^{(\alpha,\beta)}(\theta) = t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos\theta) \sin^{\alpha+1/2}(\theta/2) \cos^{\beta+1/2}(\theta/2), \quad n \in \mathbb{N},$$

(seemingly more appropriate term "Jacobi functions" would be, alas, confusing), where

$$t_n^{(\alpha,\beta)} = \left(\frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right)^{1/2}$$

(for $n = 0$ and $\alpha + \beta = -1$ the product $(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)$ must be replaced by $\Gamma(\alpha+\beta+2)$). The functions $\phi_n^{(\alpha,\beta)}(\theta)$, $n = 0, 1, \ldots$, form a complete orthonormal system in $L^2((0,\pi), d\theta)$. They are also the eigenfunctions of the symmetric in $L^2((0,\pi), d\theta)$ differential operator

$$L_{\alpha,\beta} = \frac{d^2}{d\theta^2} + \left\{\frac{1/4 - \alpha^2}{4\sin^2(\theta/2)} + \frac{1/4 - \beta^2}{4\cos^2(\theta/2)}\right\}, \tag{2.20}$$

cf. [33, (4.24.2)], with eigenvalues $-\mu_n^2$, $\mu_n = n + \frac{\alpha+\beta+1}{2}$, i.e.,

$$L_{\alpha,\beta}\phi_n^{(\alpha,\beta)}(\theta) = -\mu_n^2 \phi_n^{(\alpha,\beta)}(\theta). \tag{2.21}$$

Given (α, β) and (γ, δ) with $\alpha, \beta, \gamma, \delta \in (-1, \infty)$, we define the transplantation operator $T = T^{(\alpha,\beta),(\gamma,\delta)}$ on $L^2((0,\pi), dx)$ by the convergent in $L^2((0,\pi), dx)$ series

$$Tf = \sum_{n=0}^{\infty} \langle f, \phi_n^{(\gamma,\delta)}\rangle \phi_n^{(\alpha,\beta)}.$$

Clearly, T is an L^2 isometry which becomes the identity operator when $(\alpha, \beta) = (\gamma, \delta)$.

A transplantation theorem for Jacobi expansions was first obtained by Askey [1] following an earlier result for ultraspherical expansions [2]. Then Muckenhoupt [22] substantially enhanced Askey's result in several directions: by considering the largest possible range of Jacobi parameters, admitting fairly general class of weights for L^p estimates (a class which is different from the usual A_p class), introducing a shift in the order parameter of Jacobi orthonormalized polynomial, adding a multiplier sequence, and, eventually, by assuming moment conditions.

For the sake of convenience we now state a simplified version of Muckenhoupt's transplantation theorem; in [22, Theorem (1.14)] we choose $s = d = M = N = 0$ and $g(n) \equiv 1$.

Theorem 2.8 ([22]) *Let $\alpha, \beta, \gamma, \delta \in (-1, \infty)$, $1 < p < \infty$, and $w(x)$ be a weight on $(0, \pi)$ such that*

$$\left(\int_u^v \left[w(x) x^{\alpha+1/2} (\pi - x)^{\beta+1/2} \right]^p dx \right)^{1/p}$$
$$\times \left(\int_u^v \left[w(x)^{-1} x^{\gamma+1/2} (\pi - x)^{\delta+1/2} \right]^{p'} dx \right)^{1/p'}$$
$$\leq C(v-u) v^{\alpha+\gamma+1} (\pi - u)^{\beta+\delta+1}, \quad 0 \leq u < v \leq \pi. \tag{2.22}$$

Then, given $f \in L^p((0,\pi), w)$ and $0 < r < 1$, the series

$$T_r f(x) = \sum_{n=0}^{\infty} r^n \langle f, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)}(x)$$

converges for every $x \in (0, \pi)$, the inequality

$$\left(\int_0^\pi \left| T_r f(x) w(x) \right|^p dx \right)^{1/p} \leq C \left(\int_0^\pi |f(x) w(x)|^p dx \right)^{1/p}$$

holds with C independent of r and f, and there is a function $Tf \in L^p((0,\pi), w)$ such that $T_r f$ converges to Tf in $L^p((0,\pi), w)$ as $r \to 1^-$. If it is also assumed that

$$\int_0^\pi \left[w(x)^{-1} x^{\alpha+1/2} (\pi - x)^{\beta+1/2} \right]^{p'} dx < \infty, \tag{2.23}$$

then

$$\langle Tf, \phi_n^{(\alpha,\beta)} \rangle = \langle f, \phi_n^{(\gamma,\delta)} \rangle.$$

Note that if $w_{a,b}$ is a double power weight of the form

$$w_{a,b}(x) = x^a (\pi - x)^b \quad \text{or} \quad w_{a,b}(x) = \sin^a(x/2) \cos^b(x/2),$$

a, b real, then for such a weight condition (2.22) is equivalent to

$$\begin{aligned} &-\alpha - 1/2 - 1/p < a < \gamma + 3/2 - 1/p, \\ &-\beta - 1/2 - 1/p < b < \delta + 3/2 - 1/p, \end{aligned} \tag{2.24}$$

see [22, Corollary 17.11], whereas condition (2.23) holds if and only if $a < \alpha + 3/2 - 1/p$ and $b < \beta + 3/2 - 1/p$. Note also that for $\alpha = \beta = \gamma = \delta = -1/2$, condition (2.22) becomes simply the usual A_p condition for w^p.

Recently Miyachi [20] extended Muckenhoupt's result to the setting of weighted Hardy spaces $H^p_{a,b}(0,1)$ on the interval $(0,1)$, $0 < p \le 1$, $a, b \in \mathbb{R}$. He proved that for $0 < p \le 1$, $-1 < a, b < p - 1$ and $\alpha, \beta, \gamma, \delta > -1/2 + 2([1/p] - 1)$, the operator $T^{(\alpha,\beta),(\gamma,\delta)}$ extends to a bounded operator on $H^p_{a,b}(0,1)$. In [21] Miyachi enhanced his previous result by introducing a shift in the order parameter of Jacobi orthonormalized polynomial, adding a multiplier sequence, by assuming moment conditions (according to Muckenhoupt's general theorem) and, eventually, by simplifying arguments from the former proof.

In [7] Ciaurri, Nowak and Stempak reinvestigated Muckenhoupt's transplantation theorem by means of a suitable variant of Calderón-Zygmund operator theory. An essential novelty of that paper was weak type (1,1) estimate for the Jacobi transplantation operator, located in a fairly general weighted setting. Moreover, L^p estimates were proved for a class of weights that contains the class admitted in Muckenhoupt's theorem.

The procedure applied in [7] consisted of the following. It is easily seen that the operator

$$T_r f(x) = \sum_{n=0}^{\infty} r^n \langle f, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)}(x), \qquad f \in L^2((0,\pi), dx), x \in (0,\pi),$$

$0 < r < 1$, is an integral operator with the kernel

$$L_r(x,y) = \sum_{n=0}^{\infty} r^n \phi_n^{(\alpha,\beta)}(x) \phi_n^{(\gamma,\delta)}(y), \qquad x, y \in (0,\pi),$$

that is

$$T_r f(x) = \int_0^{\pi} L_r(x,y) f(y)\, dy, \qquad x \in (0,\pi).$$

It was then proved in [7, Proposition 3.3] that the limit

$$L(x,y) = \lim_{r \to 1^-} L_r(x,y)$$

exists for every $x \ne y$, $0 < x, y < \pi$ (a Darboux type asymptotic formula of higher order for Jacobi polynomials was used as a crucial tool) and satisfies relevant estimates (inherited from those proved earlier for $L_r(x,y)$). These estimates consist of the usual growth and smoothness conditions for comparable x and y, and another estimates (which fit into Hardy's integral operators) in the regions of $0 < x, y < \pi$, where x and y are not comparable. Finally, it was shown, [7, Proposition 3.4] that $L(x,y)$ is associated with T in the sense of CZ theory. It has to be stressed that for $\alpha, \beta, \gamma, \delta \ge 1/2$, $L(x,y)$ is a usual CZ kernel and then T is a (usual) CZ operator. But if one of the parameters $\alpha, \beta, \gamma, \delta$ is in $(-1, 1/2)$, then T is only a *double local* CZ operator (a notion introduced in [7]). Nevertheless treating T as a double local CZ operator brings an advantage by allowing more weights.

It is instructive to see an explicit formula of $L(x,y)$ in some particular cases of the parameters $\alpha,\beta,\gamma,\delta$. For instance, for $(\alpha,\beta)=(-1/2,-1/2)$ and $(\gamma,\delta)=(1/2,1/2)$ we have

$$L(x,y)=\frac{\sqrt{2}-1}{\pi}\sin y+\frac{1}{\pi}\cos y\frac{\sin y}{\cos y-\cos x}.$$

Also, for $(\gamma,\delta)=(\beta,\alpha)$, $\alpha,\beta>-1$, one has

$$L^{(\alpha,\beta),(\beta,\alpha)}(x,y)=\frac{2\Gamma((\alpha+\beta+2)/2)}{\Gamma(\alpha+1)\Gamma((\beta-\alpha)/2)}\big(\Phi(x,y)\big)^{\alpha+1/2}\big(\Phi(y,x)\big)^{-(\alpha+3/2)}$$
$$\times\,{}_2F_1\left(\frac{\alpha+\beta+2}{2},\frac{\alpha-\beta+2}{2};\alpha+1;\left(\frac{\Phi(x,y)}{\Phi(y,x)}\right)^2\right),$$

where $\Phi(x,y)=2\sin(x/2)\cos(y/2)$. It is also worth to point out that there is a striking coincidence between $L^{(\alpha,\beta),(\beta,\alpha)}$ and the Hankel transform transplantation kernel $K_{\alpha\beta}$. Namely, we have

$$L^{(\alpha,\beta),(\beta,\alpha)}(x,y)=K_{\alpha\beta}\big(\Phi(x,y),\Phi(y,x)\big).$$

2.5 Laguerre Expansions

Let L_n^α denote the nth Laguerre polynomial of order $\alpha>-1$, see [19, p. 76]. The Laguerre functions $\mathcal{L}_n^\alpha(x)$ are then defined by

$$\mathcal{L}_n^\alpha(x)=\left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{1/2}e^{-x/2}x^{\alpha/2}L_n^\alpha(x),\qquad n\in\mathbb{N}.$$

$\{\mathcal{L}_n^\alpha\}_{n\in\mathbb{N}}$ is a complete orthonormal system in $L^2(\mathbb{R}_+,dx)$. According to general procedure, for $\alpha,\gamma\in(-1,\infty)$ consider the Laguerre transplantation operator

$$T_{\alpha\gamma}f=\sum_{n=0}^{\infty}\langle f,\mathcal{L}_n^\gamma\rangle\mathcal{L}_n^\alpha.$$

In the sequel, given $\alpha,\gamma>-1$ we denote $\tau=\min(\alpha,\gamma)$.

The following theorem was proved by Kanjin.

Theorem 2.9 ([14]) *Let $\alpha,\gamma>-1$ and $1<p<\infty$. If $\tau\ge 0$, then*

$$\|T_{\alpha\gamma}f\|_p\le C\|f\|_p,\qquad f\in L^2\cap L^p(\mathbb{R}_+).$$

In the case $\tau<0$ the above inequality holds in the restricted range $(1+\frac{\tau}{2})^{-1}<p<-\frac{2}{\tau}$.

A comment is probably in order to explain why each of the two inequalities in the assumption $(1+\tau/2)^{-1}<p<-2/\tau$ is natural in the case when $\tau<0$. Firstly, if $\alpha\ge$

0, then $\{\mathcal{L}_n^\alpha : n \in \mathbb{N}\} \subset L^p(\mathbb{R}_+)$ for all $1 \le p \le \infty$, but for $-1 < \alpha < 0$ the inclusion holds if and only if $1 \le p < -2/\alpha$. Secondly, if $\gamma \ge 0$ and $1 \le p \le \infty$, then the coefficients $\langle f, \mathcal{L}_n^\gamma \rangle$, $n \in \mathbb{N}$, do exist for every $f \in L^p(\mathbb{R}_+)$, but if $-1 < \gamma < 0$, then the last statement remains true if and only if $p > (1+\gamma/2)^{-1}$. Now, fix $1 < p < \infty$. Assuming $T_{\alpha\gamma}$ has a bounded extension on $L^p(\mathbb{R}_+)$ it would be natural to expect that the extension still sends $\mathcal{L}_n^\gamma$ to $\mathcal{L}_n^\alpha$. But this simply requires the inclusion of both systems, $\mathcal{L}_n^\alpha$ and $\mathcal{L}_n^\gamma$, $n \in \mathbb{N}$, in $L^p(\mathbb{R}_+)$. Hence $p < -2/\tau$ is demanded when $\tau < 0$. Moreover, it would be desirable to know that the extended operator still possesses the property

$$\langle T_{\alpha\gamma} f, \mathcal{L}_n^\alpha \rangle = \langle f, \mathcal{L}_n^\gamma \rangle, \quad n \in \mathbb{N}, \tag{2.25}$$

for every $f \in L^p(\mathbb{R}_+)$, as it was (trivially) in the initial case for every $f \in L^2(\mathbb{R}_+)$. Proving (2.25) requires the assumption $(1+\tau/2)^{-1} < p$ when $\tau < 0$. Indeed, fix $f \in L^p(\mathbb{R}_+)$ with p satisfying $(1+\tau/2)^{-1} < p$ when $\tau < 0$, and choose $f_k \in L^2 \cap L^p(\mathbb{R}_+)$ such that $f_k \to f$ in $L^p(\mathbb{R}_+)$, $k \to \infty$. Then, for any $n \in \mathbb{N}$, $\langle f_k, \mathcal{L}_n^\gamma \rangle \to \langle f, \mathcal{L}_n^\gamma \rangle$ ($\mathcal{L}_n^\gamma \in L^{p'}(\mathbb{R}_+)$!) and $\langle T_{\alpha\gamma} f_k, \mathcal{L}_n^\alpha \rangle \to \langle T_{\alpha\gamma} f, \mathcal{L}_n^\alpha \rangle$ ($\mathcal{L}_n^\alpha \in L^{p'}(\mathbb{R}_+)$!), where $T_{\alpha\gamma} f$ is, by the very definition, the limit of $T_{\alpha\gamma} f_k$ in $L^p(\mathbb{R}_+)$. The claim follows.

Kanjin's theorem was enhanced by Stempak and Trebels to a weighted setting.

Theorem 2.10 ([31]) *Let $\alpha, \gamma > -1$ and $1 < p < \infty$. Then*

$$\|T_{\alpha\gamma} f\|_{p,a} \le C \|f\|_{p,a}, \quad f \in L^2 \cap L^p(\mathbb{R}_+, x^a),$$

where $-\frac{1}{p} < a < 1 - \frac{1}{p}$ if $\tau \ge 0$, and $-\frac{\tau}{2} - \frac{1}{p} < a < 1 - \frac{1}{p} + \frac{\tau}{2}$ if $\tau < 0$.

Finally, the above result was refined by Garrigos, Harboure, Signes, Torrea and Viviani and took its final form in the power weight setting.

Theorem 2.11 ([9]) *Let $\alpha, \gamma > -1$, $1 < p < \infty$ and $-\frac{\tau}{2} - \frac{1}{p} < a < 1 - \frac{1}{p} + \frac{\tau}{2}$. Then*

$$\|T_{\alpha\gamma} f\|_{p,a} \le C \|f\|_{p,a}, \quad f \in L^2 \cap L^p(\mathbb{R}_+, x^a).$$

We now briefly comment why the restriction on a in the above theorem is natural and optimal at the same time. Firstly, given $1 < p < \infty$, the inclusion $\{\mathcal{L}_n^\alpha : n \in \mathbb{N}\} \subset L^p(\mathbb{R}_+, x^a)$ holds true provided $-\frac{\alpha}{2} - \frac{1}{p} < a$. Together with the analogous inclusion for $\mathcal{L}_n^\gamma$, this requires $-\frac{\tau}{2} - \frac{1}{p} < a$. Secondly, to guarantee (2.25) for the bounded extension of $T_{\alpha\gamma}$ onto $L^p(\mathbb{R}_+, x^a)$ to hold, requires $\mathcal{L}_n^\alpha, \mathcal{L}_n^\gamma \in L^{p'}(\mathbb{R}_+, x^{-a})$, $n \in \mathbb{N}$, to be satisfied. This forces the inequality $a < 1 - \frac{1}{p} + \frac{\tau}{2}$.

Consider the system of Laguerre functions of Hermite type,

$$\varphi_n^\alpha(x) = \mathcal{L}_n^\alpha(x^2)\sqrt{2x} = \left(\frac{2n!}{\Gamma(n+\alpha+1)}\right)^{1/2} e^{-x^2/2} x^{\alpha+1/2} L_n^\alpha(x^2),$$

which is a complete orthonormal system in $L^2(\mathbb{R}_+, dx)$. There are several reasons to claim that this system of Laguerre functions is more friendly in dealing with than the previous one.

Let $Sf(x)=\sqrt{2x}f(x^2)$ be the operator defined on functions living on $(0,\infty)$. Then S intertwines the settings of $\{\mathcal{L}_n^\alpha\}$ and $\{\varphi_n^\alpha\}$ expansions in the sense that S is an isometric isomorphism of $L^2(\mathbb{R}_+,dx)$ such that $S\mathcal{L}_n^\alpha=\varphi_n^\alpha$, $n\in\mathbb{N}$. Moreover, if $T_{\alpha\gamma}^{\mathcal{L}}$ and $T_{\alpha\gamma}^{\varphi}$ denote the transplantation operators for $\{\mathcal{L}_n^\alpha\}$ and $\{\varphi_n^\alpha\}$ systems, respectively, then

$$T_{\alpha\gamma}^{\varphi}\circ S=S\circ T_{\alpha\gamma}^{\mathcal{L}}.$$

Consequently, the estimate

$$\|T_{\alpha\gamma}^{\varphi}f\|_{L^p(\mathbb{R}_+,U)}\le C\|f\|_{L^p(\mathbb{R}_+,U)},\qquad f\in L^p(\mathbb{R}_+,U),$$

is equivalent to the bound

$$\|T_{\alpha\gamma}^{\mathcal{L}}f\|_{L^p(\mathbb{R}_+,\widetilde{U})}\le C\|f\|_{L^p(\mathbb{R}_+,\widetilde{U})},\qquad f\in L^p(\mathbb{R}_+,\widetilde{U}),$$

where $\widetilde{U}(x^2)=U(x)x^{\frac{1}{2}-\frac{1}{p}}$, $x>0$.

Thus, the result of Theorem 2.11 is equivalent with the following statement concerning the $\{\varphi_n^\alpha\}$-expansions.

Proposition 2.12 *Assume $\alpha,\gamma>-1$, $1<p<\infty$, and $-\tau-\frac{1}{p}-\frac{1}{2}<a<\frac{3}{2}-\frac{1}{p}+\tau$. Then*

$$\|T_{\alpha\gamma}^{\varphi}f\|_{p,a}\le C\|f\|_{p,a},\qquad f\in L^2\cap L^p(\mathbb{R}_+,x^a).$$

Note that there is a striking coincidence of the restriction imposed above on a with that required for a for the weighted Hankel transplantation theorem to hold with power weight x^a, see Corollary 2.2. This coincidence together with the transference result contained in Theorem 3.2, strongly suggested a necessity of enlarging the usual A_p interval $(-\frac{1}{p},1-\frac{1}{p})$ of admissible exponents to the larger one, $(-\frac{\tau}{2}-\frac{1}{p},1-\frac{1}{p}+\frac{\tau}{2})$ (when $\tau>0$) in Proposition 2.12 and thus in Theorem 2.11.

It would be challenging to furnish a proof of the Laguerre transplantation theorem in the context of $\{\varphi_n^\alpha\}$-expansions, $\alpha>-1$, based on kernel estimates.

Finally we add that a multi-dimensional (unweighted) transplantation theorem for expansions with respect to tensor products of Laguerre functions of Hermite type was considered by Thangavelu [34, 35]. The multi-dimensional case of L^p estimate was first reduced to the one-dimensional situation and then further reduced to a weighted setting of $\{\mathcal{L}_n^\alpha\}$-expansions with power weight $x^{\frac{1}{4}-\frac{1}{2p}}$ involved; all this was done with the restriction $\alpha,\gamma\ge-1/2$. Note that the case of the weight $x^{\frac{1}{4}-\frac{1}{2p}}$ was later included as a special case in weighted transplantation theorems mentioned above.

For applications of weighted Laguerre transplantation see [10].

3 Transference of Transplantation

It was interesting to observe that there are relations between different transplantations. This is not surprising once we realize that a transplantation theorem is something "more" than a multiplier theorem. Transference of multiplier results was known for a long time; for example, a transference of multipliers from either Jacobi or Laguerre expansion setting to Hankel transform context was proved in [15, 28].

With the notation $\|\cdot\|_{p,a,b} = \|f\|_{L^p(w_{a,b})}$, where $w_{a,b}(x) = \sin^a(x/2)\cos^b(x/2)$, the following transference result between Jacobi and Hankel transplantations was proved.

Theorem 3.1 ([29]) *Let* $1 < p < \infty$, $a, b \in \mathbb{R}$, *and* $\alpha, \beta, \gamma, \delta > -1$. *If the Jacobi transplantation inequality*

$$\left\| \sum_0^\infty \langle f, \phi_n^{(\gamma,\delta)} \rangle \phi_n^{(\alpha,\beta)} \right\|_{p,a,b} \le C\|f\|_{p,a,b}, \quad f \in C_c^\infty(0,\pi),$$

holds, *then the Hankel transplantation inequality*

$$\|(\mathcal{H}_\alpha \circ \mathcal{H}_\gamma) f\|_{p,a} \le C\|f\|_{p,a}, \quad f \in C_c^\infty(0,\infty),$$

is also satisfied (*with the same constant* C).

Similarly, transference between Laguerre and Hankel transplantations was established.

Theorem 3.2 ([29]) *Let* $1 < p < \infty$, $a \in \mathbb{R}$, *and* $\alpha, \gamma > -1$. *If the Laguerre transplantation inequality*

$$\left\| \sum_0^\infty \langle f, \varphi_n^\gamma \rangle \varphi_n^\alpha \right\|_{p,a} \le C\|f\|_{p,a}, \quad f \in C_c^\infty(0,\infty),$$

holds, *then the Hankel transplantation inequality*

$$\|(\mathcal{H}_\alpha \circ \mathcal{H}_\gamma) f\|_{p,a} \le C\|f\|_{p,a}, \quad f \in C_c^\infty(0,\infty),$$

is also satisfied (*with the same constant* C).

Finally, transference between Fourier-Bessel and Hankel transplantations was derived by Betancor and Stempak.

Theorem 3.3 ([4]) *Let* $1 < p < \infty$, $a \in \mathbb{R}$, *and* $\nu, \mu > -1$. *If the Fourier-Bessel transplantation inequality*

$$\left\| \sum_{n=1}^\infty \langle f, \psi_n^\mu \rangle \psi_n^\nu \right\|_{p,a} \le C\|f\|_{p,a}, \quad f \in C_c^\infty(0,1),$$

holds, *then the Hankel transplantation inequality*

$$\|(\mathcal{H}_\nu \circ \mathcal{H}_\mu) f\|_{p,a} \le C\|f\|_{p,a}, \quad f \in C_c^\infty(0,\infty),$$

is also satisfied (*with the same constant* C).

It should be stressed that functions defined by the series appearing in the right-hand sides of the transplantation inequalities in Theorems 3.1–3.3 are understood as pointwise sums of the relevant series: smoothness and support compactness of f ensure everywhere convergence. Also $\mathcal{H}_\nu(\mathcal{H}_\mu f)$ is well defined for $f \in C_c^\infty(0,\infty)$; for details concerning an explanation of the last claim, see [4, 28].

Main tools used in the proofs of Theorems 3.1 and 3.2 were Hilb's asymptotic formulae, [33, Theorems 8.22.4, 8.21.12], written in the form:

- $\varphi_n^\alpha(t) = \sqrt{2t}\, J_\alpha(2n^{1/2}t) + \begin{cases} O(tn^{-3/4}), & cn^{-1/2} \le t \le \omega, \\ O(t^{\alpha+1/2}n^{\alpha/2-1}), & 0 < t < cn^{-1/2}; \end{cases}$
- $\psi_n^{(\alpha,\beta)}(t) = (nt)^{1/2} J_\alpha(nt) + \begin{cases} O(t), & cn^{-1} \le t \le \pi - \varepsilon, \\ O(t^{\alpha+1/2}n^{\alpha-1/2}), & 0 < t < cn^{-1}. \end{cases}$

In the first formula above $\alpha > -1$, and c and ω are arbitrarily fixed positive constants, while in the second one, $\alpha, \beta > -1$, and c and $\varepsilon < \pi$ are fixed positive constants. In the proof of Theorem 3.3 a natural connection between the functions ψ_n^ν and the Bessel function J_ν, together with an asymptotic of the sequence of zeros $\lambda_{n,\nu}$, was used.

A detailed analysis of different aspects of transference between Fourier-Bessel expansions and Hankel transform was presented by Betancor in the survey article [3].

4 Generalized Transplantation and Conjugacy

In this section we shall consider an extension of the notion of transplantation discussed in Sect. 1. As in the beginning of the article let $\{\{\phi_n^\alpha\}_{n\in\mathbb{N}} : \alpha \in A\}$ be a family of orthonormal bases in $L^2(X, dm)$. Assume also that $d \in \mathbb{Z}$ and a bounded sequence $g(n)$ (usually satisfying in addition a smoothness condition) are fixed. Consider the generalized transplantation operator $T_{\alpha\beta}^{d,g}$ given by the mapping

$$T_{\alpha\beta}^{d,g} : \phi_n^\beta \mapsto g(n)\phi_{n+d}^\alpha, \quad n \in \mathbb{N},$$

and then extended by linearity to a bounded operator on $L^2(X, dm)$. Here we use the convention: $\phi_{n+d}^\alpha \equiv 0$ for $n + d \notin \mathbb{N}$. This is indeed a rich generalization that includes:

- usual transplantation operator if $g(n) \equiv 1$ and $d = 0$;
- shift operators if $g(n) \equiv 1$, $d = \pm 1$ and $\alpha = \beta$;
- multiplier operator if $d = 0$ and $\alpha = \beta$;
- some conjugacy operators if $d = \pm 1$ and a relation between α and β holds.

The first generalized transplantation theorem was proved in the context of Jacobi expansions by Muckenhoupt, cf. [22, Theorem (1.14)]. The theorem says that given $\alpha, \beta, \gamma, \delta > -1$, $d \in \mathbb{Z}$, and a sequence $g(n)$ that satisfies

$$g(n) = \sum_{j=0}^{J-1} c_j(n+1)^{-j} + O((n+1)^{-J}),$$

with some constants c_j and a sufficiently large J, the operator given by the mapping

$$T^{d,g}_{(\alpha,\beta),(\gamma,\delta)} : \phi_n^{(\gamma,\delta)} \mapsto g(n)\phi_{n+d}^{(\alpha,\beta)}, \quad n \in \mathbb{N},$$

extends to a bounded operator on $L^p((0,\pi), w)$, $1 < p < \infty$, for a fairly large class of admissible weights w.

As a (basic) example of a situation where a conjugacy operator may be seen as a generalized transplantation operator, consider a family of the two complete orthonormal systems in $L^2(0,\pi)$:

- $\phi_n^{(1)}(x) = (2/\pi)^{1/2}\sin((n+1)x)$, $n \in \mathbb{N}$;
- $\phi_0^{(2)}(x) = (1/\pi)^{1/2}$, $\phi_n^{(2)}(x) = (2/\pi)^{1/2}\cos nx$, $n \ge 1$.

The M. Riesz inequality for the classic conjugate operator given by the mapping $\sin nx \mapsto \cos nx$, $n \ge 1$,

$$\int_0^\pi \left|\sum_{n\ge1} b_n \cos nx\right|^p dx \le C_p^p \int_0^\pi \left|\sum_{n\ge1} b_n \sin nx\right|^p dx,$$

$1 < p < \infty$, may be viewed as the transplantation inequality

$$\left\|\sum_{n=0}^\infty a_n \phi_{n+1}^{(2)}\right\|_p \le C_p \left\|\sum_{n=0}^\infty a_n \phi_n^{(1)}\right\|_p$$

for the generalized transplantation operator determined by the mapping $\phi_n^{(1)} \mapsto \phi_{n+1}^{(2)}$.

As an other example consider the conjugacy operator for Jacobi expansions determined by

$$\phi_n^{(\alpha,\beta)} \mapsto \frac{(n(n+\alpha+\beta+1))^{1/2}}{n+(\alpha+\beta+1)/2}\phi_{n-1}^{(\alpha+1,\beta+1)}, \quad n \in \mathbb{N}, \tag{4.1}$$

which was defined and investigated in [30]. Definition (4.1) was motivated by an attempt to relate both, the appropriately defined Poisson and the conjugate Poisson integrals of a given function f, into generalized Cauchy-Riemann type equations. It was proved in [30], by using Muckenhoupt's general transplantation theorem, that for $\alpha, \beta > -1$ and $1 < p < \infty$, the mapping given by (4.1) extends to a bounded operator on $L^p((0,\pi), w_{a,b})$,

$$\left\|\sum_{n=0}^\infty \frac{(n(n+\alpha+\beta+1))^{1/2}}{n+(\alpha+\beta+1)/2} a_n \phi_{n-1}^{(\alpha+1,\beta+1)}\right\|_{L^p(w_{a,b})} \le C_p^{(\alpha,\beta)} \left\|\sum_{n=0}^\infty a_n \phi_n^{(\alpha,\beta)}\right\|_{L^p(w_{a,b})}, \tag{4.2}$$

provided a and b satisfy

$$-(\alpha+3/2) - 1/p < a < (\alpha+3/2) - 1/p,$$
$$-(\beta+3/2) - 1/p < b < (\beta+3/2) - 1/p.$$

Note that

$$\phi_n^{(-1/2,-1/2)} = \phi_n^{(2)}, \qquad \phi_n^{(1/2,1/2)} = \phi_n^{(1)}, \quad n \in \mathbb{N},$$

hence for $(\alpha, \beta) = (-1/2, -1/2)$ one recovers the complementary classic conjugacy operator given by the mapping

$$\cos nx \mapsto \sin nx, \quad n \geq 0,$$

that acts on functions living on $(0, \pi)$. More precisely, this operator should be considered (up to the multiplicative constant $-i$) as a restriction to the space of odd functions on $(-\pi, \pi)$ of the indeed *classic* conjugate operator given by $e^{in\theta} \mapsto (-i)\,\mathrm{sgn}\, n\, e^{in\theta}$. (In the same way, the abovementioned mapping $\sin nx \mapsto \cos nx$, $n \geq 1$, should be considered as a restriction to the space of even functions on $(-\pi, \pi)$ of the *classic* conjugate operator.) Moreover, (4.2) specified to $\alpha = \beta = -1/2$, $a = b = 0$, reads

$$\left\| \sum_{n=0}^{\infty} a_n \phi_{n-1}^{(1)} \right\|_p \leq C_p \left\| \sum_{n=0}^{\infty} a_n \phi_n^{(2)} \right\|_p,$$

which includes the complementary M. Riesz inequality for the classic conjugate operator given by the mapping $\cos nx \mapsto \sin nx$, $n \geq 1$,

$$\int_0^{\pi} \left| \sum_{n\geq 1} b_n \sin nx \right|^p dx \leq C_p^p \int_0^{\pi} \left| \sum_{n\geq 1} b_n \cos nx \right|^p dx.$$

In [25] a conjugacy operator for expansions with respect to the Laguerre function system $\{\varphi_n^\alpha\}$, and determined by

$$\varphi_n^\alpha \mapsto -2\left(\frac{n}{4n + 2\alpha + 2}\right)^{1/2} \varphi_{n-1}^{\alpha+1}, \tag{4.3}$$

was defined and investigated (actually, in [25] the multi-dimensional situation was considered). Again this may be viewed as generalized transplantation operator with $d = -1$ and $g(n) = -2(n/(4n + 2\alpha + 2))^{1/2}$. It was proved in [25, Theorem 3.4] that for $\alpha \geq -1/2$ the operator initially defined on $L^2(\mathbb{R}_+)$ by (4.3), extends to a bounded operator on $L^p(\mathbb{R}_+, w)$ if $1 < p < \infty$, and to a bounded operator from $L^1(\mathbb{R}_+, w)$ to $L^{1,\infty}(\mathbb{R}_+, w)$ if $p = 1$, for a large class of weights w.

It would be desirable to furnish a proof of a general transplantation theorem for the systems $\{\varphi_n^\alpha\}$, $\alpha > -1$, with the above result as a consequence.

We hope this article showed transplantation theorems as a beautiful and active area of pure mathematics. Nevertheless, in the authors opinion it is unlikely to find applications in "real life", in particular in transplantology [18].

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