

Orthonormal Sequences in $L^2(\mathbf{R}^d)$ and Time Frequency Localization

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Abstract We prove that there does not exist an orthonormal basis $\{b_n\}$ for $L^2(\mathbf{R})$ such that the sequences $\{\mu(b_n)\}$, $\{\mu(\widehat{b}_n)\}$, and $\{\Delta(b_n)\Delta(\widehat{b}_n)\}$ are bounded. A higher dimensional version of this result that involves generalized dispersions is also obtained. The main tool is a time-frequency localization inequality for orthonormal sequences in $L^2(\mathbf{R}^d)$. On the other hand, for $d > 1$ we construct a basis $\{b_n\}$ for $L^2(\mathbf{R}^d)$ such that the sequences $\{\mu(b_n)\}$, $\{\mu(\widehat{b}_n)\}$, and $\{\Delta(b_n)\Delta(\widehat{b}_n)\}$ are bounded.

Keywords Uncertainty principle · Orthonormal basis · Time frequency localization

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1 Introduction

1.1 Preliminaries and Known Results

Let $f \in L^2(\mathbf{R})$, $\|f\|_2 = 1$, then

$$\mu(f) = \int_{\mathbf{R}} t|f(t)|^2 dt \quad \text{and} \quad \Delta(f) = \left(\int_{\mathbf{R}} (t - \mu(f))^2 |f|^2 dt \right)^{1/2}$$

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are called the time mean of f and the time dispersion of f respectively. The Fourier transform of $f \in L^2(\mathbf{R})$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbf{R}} f(x)e^{-2\pi i \xi x} dx.$$

Then $\mu(\widehat{f})$ and $\Delta(\widehat{f})$ are called the frequency mean and frequency dispersion of f . The classical Heisenberg uncertainty principle reads

$$\Delta(f)\Delta(\widehat{f}) \geq \frac{1}{4\pi} \tag{1}$$

for any $f \in L^2$ with $\|f\|_2 = 1$.

Clearly,

$$\|xf(x)\|_2^2 = \mu(f)^2 + \Delta^2(f).$$

The Heisenberg inequality may be also written in the form

$$\|xf(x)\|_2^2 + \|\xi \widehat{f}(\xi)\|_2^2 \geq \frac{1}{2\pi} \|f\|_2^2, \tag{2}$$

where $f \in L^2(\mathbf{R})$ is arbitrary. Various versions of the uncertainty principle can be found in the survey articles [1, 10] and in the monograph [12].

In this article we consider uncertainty inequalities for orthonormal sequences and bases. For some of the first results related to uncertainty inequalities for orthonormal bases we refer the reader to [17] and the references therein. The construction of Y. Meyer yields a wavelet basis $\{\phi_n\}_{n=1}^\infty$ for $L^2(\mathbf{R})$ such that

$$\sup_n \Delta(\phi_n)\Delta(\widehat{\phi}_n) < +\infty.$$

A similar basis is obtained for $L^2(\mathbf{R}^d)$ as well, see [17] for details. J. Bourgain proved that there is an orthonormal basis $\{b_n\}_{n=1}^\infty$ for $L^2(\mathbf{R})$ such that

$$\Delta(b_n), \Delta(\widehat{b}_n) < \frac{1}{2\sqrt{\pi}} + \epsilon,$$

see [4]. This result was generalized recently by J. Benedetto and A. Powell [2]. The technique was also used by A. Powell to construct orthonormal bases with other properties, see [18]. The result of J. Bourgain implies that for each $\epsilon > 0$ there is an orthonormal basis such that

$$\sup_n \Delta(b_n)\Delta(\widehat{b}_n) < \frac{1}{4\pi} + \epsilon,$$

so inequality (1) can not be improved for an orthonormal basis.

On the other hand H. Shapiro proved a number of uncertainty inequalities for orthonormal sequences that are stronger than corresponding inequalities for a single

function. For example, using compactness argument, see [19], one can conclude that for any orthonormal sequence $\{f_n\}_{n=1}^\infty$ in $L^2(\mathbf{R})$

$$\sup_n \|x f_n\|_2^2 + \|\widehat{\xi f_n}\|_2^2 = +\infty, \tag{3}$$

so inequality (2) can be refined for an orthonormal sequence. It is also proved in [19] that if $\phi, \psi \in L^2(\mathbf{R})$, $\phi, \psi \geq 0$, then any orthonormal sequence $\{f_n\}$ that satisfies

$$|f_n| \leq \phi, \quad |\widehat{f_n}| \leq \psi, \tag{4}$$

is finite. This statement is referred to as the Umbrella Theorem.

Quantitative versions of H. Shapiro’s results appeared in a recent article by Ph. Jaming and A. Powell [14], which contains in particular the following sharp Mean Dispersion inequality.

Let $\{e_k\}_{k=0}^n$ be an orthonormal sequence in $L^2(\mathbf{R})$ then

$$\sum_{k=0}^n \left(\mu(e_k)^2 + \Delta^2(e_k) + \mu(\widehat{e_k})^2 + \Delta^2(\widehat{e_k}) \right) \geq \frac{(n+1)^2}{2\pi}. \tag{5}$$

The equality is attained for the sequence of Hermite functions, see [14]. This inequality implies (3). Further, using results of D. Slepian, H.O. Pollak, and H.J. Landau on time-frequency localization, Ph. Jaming and A. Powell give a quantitative version of the Umbrella Theorem and obtain a number of inequalities for orthonormal basis and also for Riesz basis for $L^2(\mathbf{R})$.

1.2 Motivation

Our interest in the uncertainty principles for orthonormal bases was initiated by discussions with Yu. Lyubarskii and H. Führ that led to the following question:

Does there exist an orthonormal basis for $L^2(\mathbf{R})$ for which both time and frequency means are bounded and the products of dispersions are bounded?

In many instances, it is the product of dispersions that has some “physical meaning”, but we will not speculate on this here.

It is not difficult to construct an infinite orthonormal sequence with zero time and frequency means and bounded product of dispersions (see Example 1 in Sect. 4.1). However the following is true.

Theorem 1 *There does not exist an orthonormal basis $\{b_n\}_{n=1}^\infty$ for $L^2(\mathbf{R})$ such that the sequences $\{\mu(b_n)\}_{n=1}^\infty$, $\{\mu(\widehat{b_n})\}_{n=1}^\infty$, and $\{\Delta(b_n)\Delta(\widehat{b_n})\}_{n=1}^\infty$ are bounded.*

A number of results on time-frequency localization of orthonormal sequences and bases have been obtained by J. Benedetto in [1] and A. Powell in [18]. In particular, another example of a condition on means and dispersions which can be satisfied by an infinite orthonormal sequence but never by an orthonormal basis is due to A. Powell. It is proved in [18] that there is no orthonormal basis with bounded (both) dispersions and bounded time means. Theorem 1 can be derived from the Mean Dispersion

inequality. We will not do it, instead we consider a more general problem in higher-dimensional spaces.

1.3 Main Results

The main goal of this work is to give a version of time-frequency localization that yields a number of precise uncertainty inequalities for orthonormal sequences and bases. The results complement those in [13, 14, 18]; our approach is simple and works in \mathbf{R}^d for any d . We consider the operator that first time-limits the function and then frequency-limits it, following [20]. However we don't need the theory of Prolate Spheroidal Wave Functions and the celebrated $2WT$ approximation theorem that was used in [14]. Instead we use an elementary calculation of the trace of the corresponding self-adjoint operator, which can be found for example in [9, 10]. We obtain the following localization inequality.

Theorem 2 *Let $\{\phi_n\}_{n=1}^N$ be an orthonormal system in $L^2(\mathbf{R}^d)$ and let T and W be measurable subsets of \mathbf{R}^d . Assume that*

$$\int_T |\phi_n|^2 = 1 - a_n^2, \quad \int_W |\widehat{\phi}_n|^2 = 1 - b_n^2.$$

Then

$$\sum_{n=1}^N \left(1 - \frac{3}{2}a_n - \frac{3}{2}b_n\right) \leq |T||W|.$$

This result provides a quantitative estimate for the Umbrella Theorem in \mathbf{R}^d as well as a number of inequalities for orthonormal sequences. For any $p > 0$ and $\phi \in L^2(\mathbf{R}^d)$ we define

$$\tau_p^p(\phi) = \int_{\mathbf{R}^d} |x|^p |\phi(x)|^2 dx.$$

Clearly $\tau_p(\phi) \in [0, +\infty]$ and $\tau_p(\phi) > 0$ when $\phi \neq 0$. Hölder's inequality implies that $\tau_p(\phi) \leq \tau_q(\phi)$ when $p < q$ and $\|\phi\|_2 = 1$. The localization inequality implies the following generalization of the Mean Dispersion inequality.

Theorem 3 *Let p be positive and let $\{\phi_n\}_n$ be an orthonormal sequence in $L^2(\mathbf{R}^d)$. Then*

$$\sum_{n=1}^N (\tau_p^p(\phi_n) + \tau_p^p(\widehat{\phi}_n)) \geq CN^{1+p/2d}, \tag{6}$$

where C depends on d and p only. Further,

$$\sum_n (\tau_p(\phi_n) + \tau_p(\widehat{\phi}_n))^{-2d-\epsilon} < +\infty, \tag{7}$$

for any $\epsilon > 0$.

It is well known that the Hermite functions are extremal in many problems concerning the uncertainty principle, see for example [3, 10], and [14]. In Sect. 2.4 we will use products of the Hermite functions to show that (6) is sharp up to a (multiplicative) constant and that (7) does not hold in general when $\epsilon = 0$.

Another application of the localization principle gives a higher dimensional version of Theorem 1. We prove the following:

Theorem 1' *Let $p > d$ and let $\{b_n\}_{n=1}^\infty$ be an orthonormal basis for $L^2(\mathbf{R}^d)$. If sequences $\{q_n\}_{n=1}^\infty \subset \mathbf{R}^d$ and $\{r_n\}_{n=1}^\infty \subset \mathbf{R}^d$ are bounded then*

$$\sup_n \int_{\mathbf{R}^d} |x - q_n|^p |b_n|^2 dx \int_{\mathbf{R}^d} |\xi - r_n|^p |\widehat{b_n}|^2 d\xi = +\infty.$$

Clearly Theorem 1 follows from Theorem 1'. The next theorem shows that the restriction $p > d$ in Theorem 1' is necessary.

Theorem 4 *For $p \leq d$ there exists an orthonormal basis $\{b_n\}_{n=1}^\infty$ for $L^2(\mathbf{R}^d)$ and bounded sequences $\{q_n\}_{n=1}^\infty \subset \mathbf{R}^d$ and $\{r_n\}_{n=1}^\infty \subset \mathbf{R}^d$ such that*

$$\sup_n \int_{\mathbf{R}^d} |x - q_n|^p |b_n|^2 dx \int_{\mathbf{R}^d} |\xi - r_n|^p |\widehat{b_n}|^2 d\xi < +\infty. \tag{8}$$

It follows from the prove that for any $a > 0$ we may choose such a basis with $q_n = 0$ and $|r_n| \leq a$; however we will see below that no orthonormal basis satisfies (8) with $q_n = r_n = 0$. We use an argument similar to one in [4] to prove the theorem for $p < d$, some additional technical details are needed to make the argument work for $p = d$. The proof of the last theorem implies that

For $d \geq 2$ there exists a basis for $L^2(\mathbf{R}^d)$ with bounded time and frequency means and bounded products of dispersions.

1.4 Other Uncertainty Inequalities

Various versions of the uncertainty principle are known for functions in $L^2(\mathbf{R}^d)$. We consider two particular inequalities. The first one is a multidimensional version of the inequality of M.G. Cowling and J.F. Price and is due to J. Bnedetto [1]. For any $a > 0$ there exists $K(a) > 0$ such that

$$\| |x|^a f(x) \|_2 \| |\xi|^a \widehat{f}(\xi) \|_2 \geq K(a) \| f \|_2^2, \tag{9}$$

whenever $f \in L^2(\mathbf{R}^d)$. The second is a recent inequality of B. Demange [8]. Let

$$v(x) = |x_1|^{\alpha_1} \dots |x_d|^{\alpha_d}, \tag{10}$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$, and $\alpha_j > 0$ for $j = 1, \dots, d$. There exists $K(\alpha) > 0$ such that

$$\| v(x) f(x) \|_2 \| v(\xi) \widehat{f}(\xi) \|_2 \geq K(\alpha) \| f \|_2^2, \tag{11}$$

for any $f \in L^2(\mathbf{R}^d)$.

As we will show in Theorem 5, the localization inequality implies that for any orthonormal basis $\{b_n\}_{n=1}^\infty$ for $L^2(\mathbf{R}^d)$ and any $a > 0$

$$\sup_n \| |x|^a b_n \|_2 \| |\xi|^a \widehat{b_n} \|_2 = +\infty. \tag{12}$$

We remark that the above statement holds for any $a > 0$ in contrast to Theorem 1'. The reason is that we don't allow any time-frequency shifts now, while in Theorem 1' bounded shifts (q_n, r_n) are allowed.

Inequality (12) can be regarded as a version of the uncertainty inequality (9) for orthonormal bases. If we consider (11) instead of (9) the situation becomes different. We show that for any v of the form (10) there is an orthonormal basis $\{b_n\}_{n=1}^\infty$ for $L^2(\mathbf{R}^d)$, $d > 1$, such that

$$\sup_n \| v(x) b_n \|_2 \| v(\xi) \widehat{b_n} \|_2 < +\infty. \tag{13}$$

Here our argument is a simple version of that of J. Bourgain, see [4].

The article is organized as follows. Time-frequency localization is discussed in the next section, we prove Theorem 2 and obtain its various applications including Theorem 3; at the end of the section we use the Hermite functions to show that Theorem 3 is sharp. Section 3 is devoted to Theorem 1'; we use localization result to show that there is no orthonormal basis with given properties, we also prove (12). In the last section various orthonormal bases are constructed, we prove Theorem 4 and show that there is a basis with bounded means and bounded products of dispersions for $L^2(\mathbf{R}^d)$ when $d > 1$; finally we construct a basis that satisfies (13).

2 Time-Frequency Localization

2.1 Proof of Theorem 2

Let T and W be two measurable subsets of \mathbf{R}^d and $\{\phi_n\}_{n=1}^N$ be an orthonormal sequence in $L^2(\mathbf{R}^d)$. Denote by χ_T and χ_W the characteristic functions of T and W ; we consider the operators P_T and P_W on $L^2(\mathbf{R}^d)$ defined by

$$P_T(f) = f \chi_T, \quad \text{and} \quad P_W(f) = \mathcal{F}^{-1}(\chi_W \mathcal{F} f),$$

where \mathcal{F} stands for the Fourier transform, $\mathcal{F} : L^2(\mathbf{R}^d) \rightarrow L^2(\mathbf{R}^d)$. Then $P_W P_T$ is an integral operator with the kernel (see [9, 10])

$$q(s, t) = \chi_T(s) \int_W e^{2\pi i(s-t) \cdot w} dw.$$

A standard calculation in [10] shows that $P_W P_T$ is a Hilbert-Schmidt operator and $\|P_W P_T\|_{HS}^2 = |W||T|$. The corresponding self-adjoint operator

$$Q = (P_W P_T)^* P_W P_T = P_T P_W P_T$$

is of trace class (see also [9]) and

$$\text{tr}(Q) = \|P_W P_T\|_{HS}^2 = |W||T|.$$

Applying Theorem 5.6 from Chap. IV, [11], we obtain

$$\sum_{n=1}^N \langle Q\phi_n, \phi_n \rangle \leq \text{tr}(Q) = |W||T|.$$

On the other hand,

$$\begin{aligned} \langle Q\phi_n, \phi_n \rangle &= \langle P_W P_T \phi_n, P_T \phi_n \rangle \\ &= \langle \phi_n, \phi_n \rangle - \langle \phi_n - P_T \phi_n, \phi_n \rangle - \langle P_T \phi_n, \phi_n - P_W \phi_n \rangle \\ &\quad - \langle P_W P_T \phi_n, \phi_n - P_T \phi_n \rangle. \end{aligned}$$

Hence $\langle Q\phi_n, \phi_n \rangle \geq 1 - 2a_n - b_n$ and

$$\sum_{n=1}^N (1 - 2a_n - b_n) \leq |W||T|.$$

If we consider the operator $\tilde{Q} = (P_T P_W)^* P_T P_W$, we get similarly

$$\sum_{n=1}^N (1 - a_n - 2b_n) \leq |T||W|.$$

And the desired time-frequency localization inequality follows.

2.2 Inequalities for Orthonormal Sequences

In this subsection we follow the ideas of [14], where various inequalities for orthonormal sequences were derived from a one-dimensional localization principles. We apply the time-frequency localization proved in the previous section to obtain rather accurate inequalities.

The following corollary is an immediate consequence of Theorem 2.

Corollary 1 *Let $\{\phi_n\}_{n=1}^N$ be an orthonormal system in $L^2(\mathbf{R}^d)$ such that ϕ_n is ϵ -concentrated on a ball $\{|x| < r_0\}$ and $\widehat{\phi}_n$ is ϵ -concentrated on a ball $\{|\xi| < \rho_0\}$, for each $n = 1, \dots, N$, i.e.*

$$\int_{|x| < r_0} |\phi_n|^2 \geq 1 - \epsilon^2, \quad \int_{|\xi| < \rho_0} |\widehat{\phi}_n|^2 \geq 1 - \epsilon^2.$$

Then

$$N \leq \frac{\pi^d r_0^d \rho_0^d}{(1 - 3\epsilon)\Gamma(\frac{d}{2} + 1)^2}.$$

Another immediate application of the localization inequality is a quantitative version of Shapiro’s Umbrella Theorem. We employ localization on arbitrary measurable subsets. Let ϵ be positive and $\omega \in L^2(\mathbf{R}^d)$, define

$$K_\omega(\epsilon) = \inf \left\{ |T| : \int_{\mathbf{R}^d \setminus T} |\omega|^2 \leq \epsilon^2 \right\}.$$

Corollary 2 *Let ϕ, ψ be non-negative functions in $L^2(\mathbf{R}^d)$ and let $\{f_n\}_{n=1}^N$ be an orthonormal sequence that satisfies*

$$|f_n| \leq \phi, \quad |\widehat{f}_n| \leq \psi.$$

Then $N \leq (1 - 3\epsilon)^{-1} K_\phi(\epsilon) K_\psi(\epsilon)$ for each $\epsilon \in (0, \frac{1}{3})$.

Proof Let ω be a non-negative function in $L^2(\mathbf{R}^d)$, we denote by ω^* its non-increasing rearrangement defined on $[0, +\infty)$. For each $\epsilon > 0$ there exist $T_\omega(\epsilon)$,

$$\{\omega > \omega^*(K_\omega(\epsilon))\} \subset T_\omega(\epsilon) \subset \{\omega \geq \omega^*(K_\omega(\epsilon))\},$$

such that $|T_\omega(\epsilon)| = K_\omega(\epsilon)$ and

$$\int_{\mathbf{R}^d \setminus T_\omega(\epsilon)} |\omega|^2 = \epsilon^2.$$

Then for each n we obtain

$$\int_{T_\phi(\epsilon)} |f_n|^2 \geq 1 - \epsilon^2, \quad \int_{T_\psi(\epsilon)} |\widehat{f}_n|^2 \geq 1 - \epsilon^2.$$

Thus by Theorem 2, $N(1 - 3\epsilon) \leq K_\phi(\epsilon) K_\psi(\epsilon)$. □

2.3 Proof of Theorem 3

Let $\{\phi_n\}_n$ be an orthonormal sequence in $L^2(\mathbf{R}^d)$. For each $k \in \mathbf{Z}$ we define

$$P_k = \{n : \max\{\tau_p(\phi_n), \tau_p(\widehat{\phi}_n)\} \in [2^{k-1}, 2^k)\}.$$

Then

$$\int_{\mathbf{R}^d} |x|^p |\phi_n(x)|^2 dt \leq 2^{kp} \quad \text{and} \quad \int_{\mathbf{R}^d} |\xi|^p |\widehat{\phi}_n(\xi)|^2 d\xi \leq 2^{kp},$$

whenever $n \in P_k$. This implies that ϕ_n is $\frac{1}{4}$ -concentrated on the ball $B(0, 2^{k+\frac{2}{p}})$ both in time and frequency. From Corollary 1 we get that the number of elements in $\bigcup_{j=1}^k P_j$ is less than $c_1(p, d)4^{dk}$, where $c_1(p, d)$ is a constant that does not depend on k . In particular, we see that there exists k_0 such that P_k is empty for all $k < k_0$. (The last statement follows also from a theorem of M. Cowling and J. Price, see [7].)

For given $N > 2c_1(p, d)$ choose k such that $2c_1(p, d)4^{dk} \geq N > 2c_1(p, d)4^{d(k-1)}$. Then at least half of $\{1, \dots, N\}$ does not belong to $\bigcup_{j=1}^{k-1} P_j$ and we obtain

$$\sum_{n=1}^N (\tau_p^p(\phi_n) + \tau_p^p(\widehat{\phi}_n)) \geq \frac{N}{2} 2^{(k-1)p} \geq a(p, d)N^{1+p/2d}.$$

For $N < 2c_1(p, d)$ we have $\sum_{n=1}^N (\tau_p^p(\phi_n) + \tau_p^p(\widehat{\phi}_n)) \geq N2^{(k_0-1)p}$ and (6) follows. In order to prove (7) we note that

$$\begin{aligned} \sum_n (\tau_p(\phi_n) + \tau_p(\widehat{\phi}_n))^{-2d-\epsilon} &\leq \sum_{k=k_0}^{\infty} \sum_{n \in P_k} 2^{(1-k)(2d+\epsilon)} \\ &\leq \sum_{k=k_0}^{\infty} c(p, d)4^{dk} 2^{(-k)(2d+\epsilon)} < +\infty. \end{aligned}$$

2.4 Hermite Functions and Sharpness of Theorem 3

The Hermite functions are defined by

$$h_k(t) = \frac{2^{1/4}}{\sqrt{k!}} \left(-\frac{1}{\sqrt{2\pi}} \right)^k e^{\pi t^2} \frac{d^k}{dt^k} e^{-2\pi t^2}, \quad k = 0, 1, 2, \dots$$

These functions form an orthonormal basis for $L^2(\mathbf{R})$ and satisfy $\widehat{h}_k = i^{-k}h_k$,

$$\mu(h_k) = \mu(\widehat{h}_k) = 0, \quad \Delta(h_k) = \Delta(\widehat{h}_k) = \sqrt{\frac{2k+1}{4\pi}}.$$

Remind that (see, for example, [10])

$$xh_k(x) = \frac{\sqrt{k+1}}{2\sqrt{\pi}}h_{k+1}(x) + \frac{\sqrt{k}}{2\sqrt{\pi}}h_{k-1}(x).$$

Note that $\tau_{2n}^{2n}(h_k) = \|x^n h_k(x)\|_2^2$ for each positive integer n . Then using the orthogonality of the Hermite functions, we obtain that there exists c_n such that

$$c_n^{-1}(k+1)^n \leq \tau_{2n}^{2n}(h_k) \leq c_n(k+1)^n, \tag{14}$$

for each $k \geq 0$. By Hölder’s inequality, we have also $\tau_p(h_k) \leq \tau_{2n}(h_k)$ for $0 < p < 2n$. Thus for each $p > 0$ there exist κ_p such that

$$\tau_p(h_k) \leq \kappa_p \sqrt{k+1}, \quad k \geq 0.$$

We consider the following orthonormal sequence in $L^2(\mathbf{R}^d)$

$$\phi_I(x) = h_{i_1}(x_1)h_{i_2}(x_2) \dots h_{i_d}(x_d), \quad \text{where } I = (i_1, \dots, i_d), \quad i_m \geq 0.$$

For $p > 0$ we have

$$\tau_p^p(\phi_I) = \tau_p^p(\widehat{\phi}_I) \leq c \sum_{m=1}^d \tau_p^p(h_{i_m}) \leq C \left(\sum_{m=1}^d (i_m + 1) \right)^{p/2},$$

where C depends on p and d only. We consider

$$\mathcal{M}_j = \{I = (i_1, \dots, i_d) \in \mathbf{N}^d : |I| = i_1 + \dots + i_d = j\}$$

and note that

$$N_j = |\mathcal{M}_j| = \binom{j + d - 1}{d - 1} \leq c_1 j^{d-1}.$$

We see also that

$$N = \sum_{j=1}^K N_j = \sum_{j=1}^K \binom{j + d - 1}{d - 1} = \binom{K + d}{d} \geq c_2 K^d.$$

Then we obtain

$$\begin{aligned} \sum_{|I| \leq K} (\tau_p^p(\phi_I) + \tau_p^p(\widehat{\phi}_I)) &\leq 2C \sum_{j=1}^K \binom{j + d - 1}{d - 1} (j + d)^{p/2} \leq C_1 K^{d+p/2} \\ &\leq C_2 N^{1+p/2d}. \end{aligned}$$

Thus inequality (6) is sharp up to a multiplicative constant.

Now we look at (7). Let $p < 2n$ then

$$\sum_I (\tau_p(\phi_I) + \tau_p(\widehat{\phi}_I))^{-a} \geq \sum_I (\tau_{2n}(\phi_I) + \tau_{2n}(\widehat{\phi}_I))^{-a}.$$

By (14) the last sum is finite if and only if

$$\sum_I (|I| + d)^{-a/2} = \sum_j \binom{j + d - 1}{d - 1} (j + d)^{-a/2}$$

is finite, which holds if and only if $a > 2d$. Thus

$$\sum_I (\tau_p(\phi_I) + \tau_p(\widehat{\phi}_I))^{-2d} = +\infty.$$

3 Unbounded Product of Dispersions

3.1 Preliminary Lemmas

Our proof of Theorem 1' formulated in the Introduction is based on the following lemmas.

Lemma 1 *Let p be a positive number and $\{\phi_n\}_{n=1}^N$ be an orthonormal system in $L^2(\mathbf{R}^d)$ that satisfies $\tau_p(\phi_n) \leq J$ and $\tau_p(\widehat{\phi}_n) \leq K$. Then*

$$N \leq c_0(p, d)(JK)^d. \tag{15}$$

Proof Clearly, each ϕ_n is ϵ -concentrated on the ball $\{|x|^p \leq \epsilon^{-2}J^p\}$ and each $\widehat{\phi}_n$ is ϵ -concentrated on the ball $\{|\xi|^p \leq \epsilon^{-2}K^p\}$. Applying Corollary 1 with $\epsilon = \frac{1}{4}$, we obtain $N \leq c_0(p, d)(JK)^d$. \square

We note that for $d = 1$ and $p = 2$ the Mean Dispersion inequality (5), see [14], implies (15) with $c_0(2, 1) = 2\pi$. The results of Ph. Jaming and A. Powell on one-dimensional time-frequency localization give also an estimate on N when $d = 1$ and $p > 0$.

Another lemma we need is known, it follows for example from Chap. 3.2.5B in [12] or [5, pp. 822–823], we give a proof here for the convenience of the reader.

Lemma 2 *Let b and c be positive numbers, there exists a nonzero function f in $L^2(\mathbf{R}^d)$ such that $f(x) = 0$ when $|x| \leq b$, and $\widehat{f}(\xi) = 0$ when $|\xi| \leq c$.*

Proof It is enough to consider $d = 1$, if g is a required function for $d = 1$ (and appropriate b_1 and c_1) we take $f(x) = g(x_1) \dots g(x_n)$.

Let PW_c be the space of $f \in L^2(\mathbf{R})$ such that $\widehat{f}(\xi) = 0$ when $|\xi| \geq c$. There exists a such that

$$\|f\|_2 \leq a\|f\chi_{\{|x|>b\}}\|_2, \tag{16}$$

for any $f \in PW_c$, see e.g. [15]. The last inequality implies that the traces of functions from PW_c on $\{|x| > b\}$ form a closed subspace in $L^2(\{|x| > b\})$ which is obviously not the whole space. Thus there exists $f \in L^2(\{|x| > b\})$ such that

$$\int_{|x|>b} f(x)\overline{g(x)}dx = 0,$$

for any $g \in PW_c$. We extend f by zero on $\{|x| \leq b\}$ in order to get the required function. \square

Functions discussed in Lemma 2 are called the Sonine functions (at least when $d = 1$) and explicit constructions of such functions are due to J.-P. Kahane and J.-F. Burnol, we refer the reader to [5] and [6] for details. Inequality (16) can be also considered as a simple case of the Logvinenko–Sereda theorem, see [12] and references therein. An improvement of constants in Logvinenko–Sereda theorem is due to O. Kovrijkine, [16].

3.2 Proof of Theorem 1'

Let $f \in L^2(\mathbf{R}^d)$, $p > 0$, and $a \in \mathbf{R}^d$. We define

$$\tau_p^p(f, a) = \int_{\mathbf{R}^d} |x - a|^p |f(x)|^2 dx.$$

Assume that $\{b_n\}_{n=1}^\infty$ is an orthonormal basis, and the sequences $\{q_n\}_{n=1}^\infty$, $\{r_n\}_{n=1}^\infty$, and $\{\tau_p(b_n, q_n)\tau_p(\widehat{b}_n, r_n)\}_{n=1}^\infty$ are bounded for some $p > d$. Let

$$D^2 = \sup_n \tau_p(b_n, q_n)\tau_p(\widehat{b}_n, r_n), \quad \text{and} \quad M = \max\left\{\sup_n |q_n|, \sup_n |r_n|\right\}.$$

We consider

$$S_k = \{b_n : \tau_p(b_n, q_n) \in (D2^{-k}, D2^{-k+1}]\},$$

where $k \in \mathbf{Z}$. Clearly, $\{b_n\}_{n=1}^\infty = \bigcup_k S_k$. Note that $\tau_p(\widehat{b}_n, r_n) \leq 2^k D$ for $b_n \in S_k$. For $b_n \in S_k$ we have

$$\tau_p(b_n) \leq \tau_p(b_n, q_n) + |q_n| \leq 2^{-k+1} D + M \quad \text{and} \quad \tau_p(\widehat{b}_n) \leq 2^k D + M.$$

It follows from Lemma 1 that S_k is finite, and if N_k is the number of elements in S_k then

$$N_k \leq c_0(p, d)(2^{|k|+1} D + M)^d (D + M)^d \leq a(p, d)2^{d|k|} (D + M)^{2d}. \tag{17}$$

Let R be a positive number, we take a function $f \in L^2(\mathbf{R}^d)$, $\|f\|_2 = 1$, that vanishes on $\{|x| < M + R\}$, and whose Fourier transform vanishes on $\{|\xi| < M + R\}$, see Lemma 2. Then we have

$$1 = \|f\|^2 = \sum_k \sum_{b_n \in S_k} |\langle f, b_n \rangle|^2. \tag{18}$$

Now if $b_n \in S_k, k > 0$ then

$$\begin{aligned} |\langle f, b_n \rangle| &\leq \int_{|x| > M+R} |f(x)| |b_n(x)| dx \leq R^{-p/2} \int_{\mathbf{R}^d} |x - q_n|^{p/2} |f(x)| |b_n(x)| dx \\ &\leq R^{-p/2} \|f\|_2 \tau_p^{p/2}(b_n, q_n) \leq \left(\frac{D}{2^{k-1} R}\right)^{p/2}. \end{aligned} \tag{19}$$

Similarly, for $b_{n'} \in S_{k'}, k' \leq 0$, we have

$$|\langle f, b_{n'} \rangle| = |\langle \widehat{f}, \widehat{b}_{n'} \rangle| \leq \left(\frac{2^{k'} D}{R}\right)^{p/2}. \tag{20}$$

Combining the inequalities (17), (18), (19), and (20), we obtain

$$1 \leq \sum_{k=1}^\infty \left(\frac{D}{2^{k-1} R}\right)^p N_k + \sum_{k=0}^\infty \left(\frac{D}{2^k R}\right)^p N_{-k} \leq \frac{A(p, d, D, M)}{R^p} \sum_{k=0}^\infty 2^{k(d-p)}.$$

Choosing R large enough, we get a contradiction. The theorem is proved.

3.3 Another Unbounded Product

We complete this section by proving (12).

Theorem 5 *If $\{b_n\}_{n=1}^\infty$ is an orthonormal basis for $L^2(\mathbf{R}^d)$ and p is positive then*

$$\sup_n \tau_p(b_n)\tau_p(\widehat{b}_n) = \infty.$$

Proof Assume that there exists an orthonormal basis such that $\tau_p(b_n)\tau_p(\widehat{b}_n) \leq C^2$. Let

$$A_k = \{b_n : \tau_p(b_n) \in (2^{-k}C, 2^{-k+1}C]\},$$

where k is integer. Clearly for $b_n \in A_k$ we have $\tau_p(\widehat{b}_n) \leq C2^k$. Then each $b_n \in A_k$ is $\frac{1}{4}$ -concentrated on the ball $\{|x|^p < C2^{-k}\}$ and \widehat{b}_n is $\frac{1}{4}$ -concentrated on the ball $\{|\xi|^p < C2^k\}$. Thus, by Corollary 1, the number of elements in A_k is bounded by a constant that does not depend on k . Let once again use Lemma 2, we take a function f in $L^2(\mathbf{R}^d)$, $\|f\|_2 = 1$, that vanishes on $B(0, R)$ with its Fourier transform. When $k \geq 0$ and $b_n \in A_k$ we get

$$|\langle f, b_n \rangle|^2 \leq R^{-p} \tau_p^p(b_n) \leq 2CR^{-p}2^{-kp}.$$

When $k < 0$ and $b_n \in A_k$ similarly

$$|\langle f, b_n \rangle|^2 = |\langle \widehat{f}, \widehat{b}_n \rangle|^2 \leq CR^{-p}2^{kp}.$$

As above $\sum_n |\langle f, b_n \rangle|^2 = 1$ and choosing R large enough we get a contradiction. \square

4 Existence of Some Orthonormal Bases for $L^2(\mathbf{R}^d)$

4.1 Orthonormal Sequences in One Dimension

We start with two examples of orthonormal sequences in $L^2(\mathbf{R})$.

Example 1 Let ϕ be a real-valued even C^∞ -function, $\text{supp}(\phi) \subset [-2, -1] \cup [1, 2]$ and $\|\phi\|_2 = 1$. Then $\Delta(\phi), \Delta(\widehat{\phi}) < +\infty$. Consider $\phi_n(x) = 2^{n/2}\phi(2^n x)$, where n is integer, then $\{\phi_n\}_n$ form an orthonormal sequence such that

$$\mu(\phi_n) = \mu(\widehat{\phi}_n) = 0, \quad \Delta(\phi_n)\Delta(\widehat{\phi}_n) = c.$$

This is an example of an infinite orthonormal sequence with zero means and bounded products of dispersions.

Example 2 There exists a real function ψ and a corresponding wavelet basis $\psi_{m,n}(t) = 2^{m/2}\psi(2^m t - n)$; such that $\Delta(\psi) < +\infty$ and $\Delta(\widehat{\psi}) < +\infty$, see [1, 17].

One has

$$\begin{aligned} \mu(\psi_{m,n}) &= 2^{-m}(\mu(\psi) + n); & \mu(\widehat{\psi_{m,n}}) &= 2^m \mu(\widehat{\psi}); \\ \Delta(\psi_{m,n}) &= 2^{-m} \Delta(\psi); & \Delta(\widehat{\psi_{m,n}}) &= 2^m \Delta(\widehat{\psi}). \end{aligned}$$

Thus $\Delta(\psi_{m,n})\Delta(\widehat{\psi_{m,n}}) = c$ and $\mu(\widehat{\psi_{m,n}}) = 0$, since $\widehat{\psi}(-\xi) = \overline{\widehat{\psi}(\xi)}$. So we have an example of an orthonormal basis for $L^2(\mathbf{R})$ with zero frequency means and bounded products of dispersions. However the time means $\mu(\psi_{m,n})$ are unbounded.

4.2 Some Orthonormal Sequences in Higher Dimensions

In this section we obtain preliminary results that we use later to prove Theorem 4. First we construct an orthonormal sequence with required properties that is large in some sense.

Let χ be the characteristic function of the cube

$$\{x = (x_1, \dots, x_d) : 5/2 < x_m < 7/2, m = 1, \dots, d\}$$

and ω be a smooth radial function supported in $B(0, 1/2)$. Then $\phi = \chi * \omega$ is a smooth non-negative function, define $\psi(x) = \phi(x)^{1/2}$. Then

$$\langle \psi(x), e^{2\pi i x \cdot b} \psi(x) \rangle = \widehat{\phi}(b) = \widehat{\chi}(b)\widehat{\omega}(b) = 0$$

whenever $b \in \mathbf{Z}^d$, $b \neq 0$. Further define $\Psi(x) = a\psi(4x)$, where a is chosen such that $\|\Psi\|_2 = 1$. Clearly,

$$\text{supp}(\Psi) \subset \{x = (x_1, \dots, x_d) : 1/2 < x_m < 1\}.$$

We have

$$\langle \Psi(x), e^{2\pi i x \cdot b} \Psi(x) \rangle = a^2 4^{-d} \langle \psi(t), e^{2\pi i t \cdot b/4} \psi(t) \rangle = a^2 4^{-d} \widehat{\phi}(b/4) = 0 \tag{21}$$

when $b \in 4\mathbf{Z}^d$, $b \neq 0$. For every positive integer s we define

$$\Psi_{j,s}(x) = 2^{-ds/2} e^{2\pi i j \cdot 2^{-s} x} \Psi(2^{-s} x), \quad \text{where } j = (j_1, \dots, j_d) \in 4\mathbf{Z}^d, |j_l| \leq 2^s. \tag{22}$$

Lemma 3 *Let $\Psi_{j,s}$ be defined as above. Then*

$$\text{supp}(\Psi_{j,s}) \subset \{x = (x_1, \dots, x_d) : 2^{s-1} < x_m < 2^s, m = 1, \dots, d\}, \tag{23}$$

the sequence $\{\Psi_{j,s}\}_{j,s}$ is orthonormal, and for each $p > 0$ there exist $C_1, C_2 > 0$ such that

$$\tau_p(\Psi_{j,s}) = 2^s C_1, \quad \tau_p(\widehat{\Psi_{j,s}}, 2^{-s} j) = 2^{-s} C_2. \tag{24}$$

Proof The supports of $\Psi_{j,s}$ and $\Psi_{j',s'}$ are disjoint when $s \neq s'$. When $s = s'$ and $j \neq j'$ we have by (21)

$$\langle \Psi_{j,s}, \Psi_{j',s} \rangle = 2^{-ds} \langle \Psi(2^{-s}x), e^{2\pi i(j'-j) \cdot 2^{-s}x} \Psi(2^{-s}x) \rangle = 0.$$

Further, for any $p > 0$,

$$\tau_p^p(\Psi_{j,s}) = \int_{\mathbf{R}^d} |x|^p 2^{-ds} |\Psi(2^{-s}x)|^2 dx = 2^{sp} \tau_p^p(\Psi) = 2^{sp} C_1^p.$$

Clearly $\widehat{\Psi}_{j,s} = 2^{ds/2} \widehat{\Psi}(2^s \xi - j)$ and

$$\begin{aligned} \tau_p^p(\widehat{\Psi}_{j,s}, 2^{-s}j) &= \int_{\mathbf{R}^d} |\xi - 2^{-s}j|^p |\widehat{\Psi}_{j,s}(\xi)|^2 d\xi = 2^{-sp} \int_{\mathbf{R}^d} |\eta|^p |\widehat{\Psi}(\eta)|^2 d\eta \\ &= 2^{-sp} \tau_p^p(\widehat{\Psi}). \end{aligned} \quad \square$$

Remark We enumerate j for fixed $s \geq 2$ as $\{j(n, s)\}_{n=1}^{J_s}$, where

$$J_s = (2^{s-1} + 1)^d > 2^{-d} 2^{sd}$$

and write $\Psi_{n,s} = \Psi_{j(n,s),s}$ for $n = 1, \dots, J_s$.

Lemma 4 *Let $\Psi_{j,s}$ satisfy (22), where $\text{supp} \Psi \subset [-1, 1]^d$ and let q be a positive integer. Then there exists $A(q, \Psi)$ such that for any s and any $R(x) = \sum_j \alpha_j \Psi_{j,s}$ the following inequality holds*

$$\|\partial_m^q R\|_2^2 \leq A(\Psi, q) \sum_j |\alpha_j|^2,$$

here $m \in \{1, \dots, d\}$ and ∂_m denotes the partial derivative with respect to x_m .

Proof By (22) we have $R(x) = 2^{-sd/2} P(2^{-s}x) \Psi(2^{-s}x)$, where $P(y) = \sum_j \alpha_j e^{2\pi i j \cdot y}$ is a trigonometric polynomial. We have

$$\|\partial_m^q R\|_2^2 = 2^{-2sq} \|\partial_m^q (P\Psi)\|_2^2 \leq 2^{-2sq} A(q) \sum_{r=0}^q \|\partial_m^r P \partial_m^{q-r} \Psi\|_2^2.$$

Now $\text{supp}(\Psi) \subset Q = [-1, 1]^d$, the functions $e^{2\pi i j \cdot y}$ are orthogonal on this cube and have the same norms. We obtain

$$\|\partial_m^r P \partial_m^{q-r} \Psi\|_2^2 \leq A_1 \|\chi_Q \partial_m^r P\|_2^2 \leq A_2 \sum_j |\alpha_j (2\pi j_m)^r|^2 \leq A_3 2^{2sr} \sum_j |\alpha_j|^2,$$

where A_1, A_2, A_3 depend on q and Ψ . The required inequality follows. □

4.3 Proof of Theorem 4. Case I: $p < d$

We use the construction described in [4] to replace an orthonormal sequence by a basis, we repeat the details of the construction for the convenience of the reader.

Let a sequence $\{f_k\}_{k=1}^\infty$ of smooth functions with compact supports be dense on the unit sphere in $L^2(\mathbf{R}^d)$. The basis $\{b_n\}_{n=1}^\infty$ is obtained as $\bigcup_k B_k$, where each B_k is a finite orthonormal system. Suppose that a finite orthonormal system of smooth functions B_1, \dots, B_{k-1} with compact supports is already obtained, let B_{k-1} be the linear span of these functions, we also put $B_0 = \{0\}$. We define

$$f = f_k - P_{B_{k-1}} f_k.$$

Then f is a smooth function with compact support and $I_p(f), I_p(\widehat{f}) < +\infty$, where

$$I_p(f) = \int_{\mathbf{R}^d} (|x| + 1)^p |f(x)|^2 dx. \tag{25}$$

We take s large enough such that (i) the supports of $\Psi_{j,s}$ defined by (22) do not intersect the supports of f and of functions in B_{k-1} , (ii) the following inequality holds

$$J_s \geq 2^{-d} 2^{ds} > \max\{I_p(f), 2^{sp} I_p(\widehat{f})\}. \tag{26}$$

Remark The condition $p < d$ is used here.

We enumerate $\Psi_{j,s}$ as in the Remark before Lemma 4. Following [4] further, define

$$\begin{aligned} \beta_1 &= \frac{\theta}{\sqrt{J_s}} f + \gamma_1 \Psi_{1,s}, \\ \beta_2 &= \frac{\theta}{\sqrt{J_s}} f + \sigma_1 \Psi_{1,s} + \gamma_2 \Psi_{2,s}, \\ \beta_3 &= \frac{\theta}{\sqrt{J_s}} f + \sigma_1 \Psi_{1,s} + \sigma_2 \Psi_{2,s} + \gamma_3 \Psi_{3,s}, \\ &\dots \\ \beta_{J_s} &= \frac{\theta}{\sqrt{J_s}} f + \sigma_1 \Psi_{1,s} + \dots + \sigma_{J_s-1} \Psi_{J_s-1,s} + \gamma_{J_s} \Psi_{J_s,s}, \end{aligned} \tag{27}$$

here $\theta \in (0, 1/3)$ will be specified later. Clearly, β_l are orthogonal to B_{k-1} . The constants $\sigma_1, \dots, \sigma_{J_s-1}$ and $\gamma_1, \dots, \gamma_{J_s}$ are chosen to make $\{\beta_l\}_{l=1}^{J_s}$ an orthonormal sequence. Thus

$$\gamma_l^2 = 1 - \frac{\theta^2}{J_s} \|f\|_2^2 - \sum_{n=1}^{l-1} \sigma_n^2, \quad \sigma_l \gamma_l = -\frac{\theta^2}{J_s} \|f\|_2^2 - \sum_{n=1}^{l-1} \sigma_n^2. \tag{28}$$

Clearly, $|\gamma_l| < 1$ and, by induction, one has

$$|\gamma_l| \geq |\gamma_l|^2 \geq 1 - \frac{2\theta^2}{J_s} \quad \text{and} \quad |\sigma_l| \leq \frac{\theta}{J_s}. \tag{29}$$

We take $\theta = 1/4$ and estimate $\tau_p(\beta_l)$ first

$$\tau_p^p(\beta_l) \leq \frac{3}{16J_s} \int_{\mathbf{R}^d} |x|^p |f|^2 dx + 3 \int_{\mathbf{R}^d} |x|^p |\Psi_{l,s}|^2 dx + 3 \int_{\mathbf{R}^d} |x|^p \left| \sum_{n=1}^{l-1} \sigma_n \Psi_{n,s} \right|^2 dx.$$

The first term is bounded by $I_p(f)J_s^{-1} < 1 < 2^{sp}$, the second term is less than 2^{sp}const , see (24). To estimate the last term, note that $\text{supp}(\Psi) \in [-1, 1]^d$ and (22) implies

$$\begin{aligned} \int_{\mathbf{R}^d} |x|^p \left| \sum_{n=1}^{l-1} \sigma_n \Psi_{n,s} \right|^2 dx &= 2^{sp} \int_{\mathbf{R}^d} |y|^p \left| \sum_{n=1}^{l-1} \sigma_n e^{2\pi i j(n,s) \cdot y} \right|^2 |\Psi(y)|^2 dy \\ &\leq 2^{sp} d^{p/2} \int_{\mathbf{R}^d} \left| \sum_{n=1}^{l-1} \sigma_n e^{2\pi i j(n,s) \cdot y} \right|^2 |\Psi(y)|^2 dy \\ &= 2^{sp} d^{p/2} \sum_{n=1}^{l-1} |\sigma_n|^2 < 2^{sp} d^{p/2}. \end{aligned}$$

Thus $\tau_p(\beta_l) \leq C2^s$, where C depends on p, d , and on Ψ .

For the Fourier transform we estimate $\tau_p(\widehat{\beta}_l, 2^{-s} j_l)$, where $j_l = j(l, s)$ was defined in Remark in Sect. 4.2. We have

$$\begin{aligned} \tau_p^p(\widehat{\beta}_l, 2^{-s} j_l) &= \int_{\mathbf{R}^d} |\xi - 2^{-s} j_l|^p |\widehat{\beta}_l|^2 d\xi \leq \frac{3}{16J_s} \int_{\mathbf{R}^d} (|\xi| + 1)^p |\widehat{f}|^2 d\xi \\ &\quad + 3 \int_{\mathbf{R}^d} |\xi - 2^{-s} j_l|^p |\widehat{\Psi}_{l,s}|^2 d\xi \\ &\quad + 3 \int_{\mathbf{R}^d} |\xi - 2^{-s} j_l|^p \left| \sum_{n=1}^{l-1} \sigma_n \widehat{\Psi}_{n,s} \right|^2 d\xi. \end{aligned} \tag{30}$$

The first term is bounded by $I_p(\widehat{f})J_s^{-1}$ and is less than 2^{-sp} due to our choice of s . Identity (24) implies that the second term is less than $2^{-sp}C_2$. We want to show that the third term is small enough. We have

$$\int_{\mathbf{R}^d} |\xi - 2^{-s} j_l|^p \left| \sum_{n=1}^{l-1} \sigma_n \widehat{\Psi}_{n,s} \right|^2 d\xi \leq A(d) \int_{\mathbf{R}^d} \left(1 + \sum_{m=1}^d \xi_m^{2d} \right) |\widehat{R}_l|^2 d\xi, \tag{31}$$

where $R_l(x) = \sum_{n=1}^{l-1} \sigma_n \Psi_{n,s}(x)$. Clearly,

$$\|\widehat{R}_l\|_2^2 = \|R_l\|_2^2 = \sum_{n=1}^{l-1} |\sigma_n|^2 \leq J_s^{-1} < 2^{d-sd}. \tag{32}$$

Further,

$$\sum_{m=1}^d \int_{\mathbf{R}^d} \xi_m^{2d} |\widehat{R}_l|^2 d\xi = (2\pi)^{-2d} \sum_{m=1}^d \|\partial_m^d R_l\|_2^2. \tag{33}$$

Using Lemma 4, we obtain

$$\|\partial_m^d R_l\|_2^2 \leq A(\Psi, d) \sum_{n=1}^{l-1} |\sigma_n|^2 \leq A_1(\Psi, d) 2^{-sd}. \tag{34}$$

Finally we get

$$\tau_p^p(\widehat{\beta}_l, 2^{-s} j) \leq C(\Psi, d) 2^{-sp}.$$

Thus $\tau_p(\beta_l) \tau_p(\widehat{\beta}_l, 2^{-s} j) < C$. We set $\mathcal{B}_k = \{\beta_1, \dots, \beta_{J_s}\}$ and continue the procedure.

We want to check that the resulting orthonormal sequence is complete, once again we follow [4]. First,

$$P_{B_k} f_k = P_{B_{k-1}} f_k + \sum_{l=1}^{J_s} \langle f, \beta_l \rangle \beta_l$$

and

$$\|P_{B_k} f_k\|^2 \geq \|f_k - f\|_2^2 + \frac{1}{16} \|f\|_2^4 \geq \frac{1}{16}.$$

Suppose that the orthonormal sequence $\bigcup_{k=1}^\infty \mathcal{B}_k$ is not complete; let B be its closed span. Then there exists $g \in L^2(\mathbf{R}^d)$ such that $\|g\|_2 = 1$ and g is orthogonal to B . For some k we have $\|g - f_k\|_2 < 1/4$ since $\{f_k\}$ is a dense sequence on the unit sphere of $L^2(\mathbf{R}^d)$. Then we obtain a contradiction

$$\frac{1}{16} \leq \|P_{B_k}(f_k)\|_2^2 \leq \|P_B f_k\|_2^2 = \|P_B(f_k - g)\|_2^2 \leq \|f_k - g\|_2^2 < \frac{1}{16}.$$

4.4 Proof of Theorem 4. Case II: $p = d$

Our argument in the preceding section does not work when $p = d$. We use strict inequality when on each step we take a function f_k , construct $f = f_k - P_{B_{k-1}} f_k$ and choose s that satisfies (26). We will adjust the argument to prove the result for $p = d$.

As before, we start with a sequence $\{f_k\}_{k=1}^\infty$ of smooth functions with compact supports that is dense on the unit sphere in $L^2(\mathbf{R}^d)$. The basis $\{b_n\}$ is obtained as $\bigcup_k \mathcal{B}_k$, where \mathcal{B}_k is a finite orthonormal system; but this time we will have

$$\mathcal{B}_k = \bigcup_{t=1}^{T_k} \mathcal{C}_{t,k}.$$

Suppose that an orthonormal system of functions $\mathcal{B}_1, \dots, \mathcal{B}_{k-1}, \mathcal{C}_{1,k}, \dots, \mathcal{C}_{t-1,k}$ with compact supports is obtained, let $B_{t,k-1}$ be the linear span of these functions. We define

$$g_t = f_k - P_{B_{t,k-1}} f_k.$$

As before, g_t is a smooth function with compact support and using notation (25) we have $I_d(g_t), I_d(\widehat{g}_t) < +\infty$. We define also

$$I(g_t) = \sum_{m=1}^d \int_{\mathbf{R}^d} |\xi_m|^{2d} |\widehat{g}_t(\xi)|^2 d\xi,$$

then we have $I_d(\widehat{g}_t) \leq c(d)(1 + I(g_t))$.

We take s big enough so that the supports of $\Psi_{j,s}$ do not intersect the supports of g_t and of functions in $B_{t,k-1}$ and also

$$J_s \geq 2^{sd-d} > I_d(g_t).$$

Further we repeat construction (27–29) with $f = g_t$ and

$$\theta = \theta_t = (9 + I(g_t))^{-1/2} \in (0, 1/3).$$

We set $\mathcal{C}_{t,k} = \{\beta_1, \dots, \beta_{J_s}\}$.

As earlier β_l are orthogonal to $B_{t,k-1}$. We estimate $\tau_d(\beta_l)$ as above, $\tau_d(\beta_l) \leq C2^s$. For the Fourier transform we once again estimate $\tau_d(\widehat{\beta}_l, 2^{-s} j_l)$. We have as before

$$\begin{aligned} \tau_d^d(\widehat{\beta}_l, 2^{-s} j_l) &= \int_{\mathbf{R}^d} |\xi - 2^{-s} j_l|^d |\widehat{\beta}_l|^2 d\xi \leq \frac{3\theta^2}{J_s} \int_{\mathbf{R}^d} (|\xi| + 1)^d |\widehat{g}_t|^2 d\xi \\ &+ 3 \int_{\mathbf{R}^d} |\xi - 2^{-s} j_l|^d |\widehat{\Psi}_{l,s}|^2 d\xi \\ &+ 3 \int_{\mathbf{R}^d} |\xi - 2^{-s} j_l|^d \left| \sum_{n=1}^{l-1} \sigma_n \widehat{\Psi}_{n,s} \right|^2 d\xi. \end{aligned}$$

The first term is bounded by $3\theta_t^2 I_d(\widehat{g}_t) J_s^{-1}$. Using our choice of θ_t and repeating estimates for the second and third terms from Case I, we get

$$\tau_d(\widehat{\beta}_l, 2^{-s} j) \leq C(\Psi)2^{-s}.$$

Thus $\tau_d(\beta_l)\tau_d(\widehat{\beta}_l, 2^{-s} j) < C$.

Now, in contrast to Case I, the projection of f_k onto the subspace spanned by $\mathcal{B}_1, \dots, \mathcal{B}_{k-1}, \mathcal{C}_{1,k}, \dots, \mathcal{C}_{t-1,k}, \mathcal{C}_{t,k}$ could be small if θ_t is small. So we use the same function f_k again to continue the procedure. Note that

$$g_{t+1} = f_k - P_{B_{t+1,k-1}} f_k = g_t - \sum_{n=1}^{J_s} \langle g_t, \beta_n \rangle \beta_n = g_t - \frac{\theta_t}{\sqrt{J_s}} \|g_t\|_2^2 \sum_{n=1}^{J_s} \beta_n.$$

And, inserting formulas for β_n , we obtain

$$g_{t+1} = g_t(1 - \theta_t^2 \|g_t\|_2^2) - \sum_{n=1}^{J_s} \frac{\theta_t}{\sqrt{J_s}} \|g_t\|_2^2 (\gamma_n + (J_s - n)\sigma_n) \Psi_{n,s} = \lambda_t g_t + \sum_{n=1}^{J_s} \kappa_n \Psi_{n,s},$$

where $\lambda_t = 1 - \theta_t^2 \|g_t\|_2^2 < 1$ and

$$|\kappa_n| = \frac{\theta_t}{\sqrt{J_s}} \|g_t\|_2^2 |\gamma_n + (J_s - n)\sigma_n| < \frac{2\theta_t}{\sqrt{J_s}} \|g_t\|_2^2. \tag{35}$$

We have

$$I(g_{t+1}) = \sum_{m=1}^d \int_{\mathbf{R}^d} |\xi_m|^{2d} |\widehat{g_{t+1}}|^2 d\xi = (2\pi)^{-2d} \sum_{m=1}^d \|\partial_m^d g_{t+1}\|_2^2.$$

According to our choice of s the support of g_t does not intersect the supports of $\Psi_{n,s}$. Then

$$I(g_{t+1}) = \lambda_t^2 I(g_t) + (2\pi)^{-2d} \left\| \partial_m^d \left(\sum_{n=1}^{J_s} \kappa_n \Psi_{n,s} \right) \right\|_2^2.$$

Now Lemma 4 implies

$$\left\| \partial_m^d \left(\sum_{n=1}^{J_s} \kappa_n \Psi_{n,s} \right) \right\|_2^2 \leq A(d, \Psi) \sum_{n=1}^{J_s} |\kappa_n|^2.$$

Combining the last two inequalities with (35), we obtain

$$I(g_{t+1}) \leq \lambda_t^2 I(g_t) + B(d, \Psi) \theta_t^2 \|g_t\|_2^4 \leq I(g_1) + B(d, \Psi) \sum_{u=1}^t \theta_u^2 \|g_u\|_2^4.$$

We note also that

$$\begin{aligned} 1 &\geq \|P_{B_{t+1,k-1}} f_k\|_2^2 = \|f_k - g_{t+1}\|_2^2 = \|f_k - g_t\|_2^2 + \theta_t^2 \|g_t\|_2^4 \\ &= \|f_k - g_1\|_2^2 + \sum_{u=1}^t \theta_u^2 \|g_u\|_2^4. \end{aligned}$$

Thus $I(g_{t+1}) \leq I(g_1) + B(d, \Psi)$ and $\theta_t^2 > (9 + I(g_1) + B(d, \Psi))^{-1}$ for each t . We take $T_k > 9 + I(g_1) + B(d, \Psi)$ and for $B_k = B_{T_k,k-1}$ we get

$$\|P_{B_k} f_k\|_2^2 = \|f_k - g_1\|_2^2 + \sum_{u=1}^{T_k} \theta_u^2 \|g_u\|_2^4.$$

If for each u we have $\|g_u\|_2 > 1/2$, then

$$\|P_{B_k} f_k\|_2^2 \geq \frac{T_k}{16} (9 + I(g_1) + B(d, \Psi))^{-1} > \frac{1}{16}.$$

If $\|g_u\|_2 < 1/2$ for some u , then

$$\|P_{B_k} f_k\|_2^2 \geq \|f_k - g_u\|_2^2 \geq \frac{1}{4}.$$

We let $B_k = \bigcup_{t=1}^{T_k} C_t$ and finish the proof as above.

4.5 Classical Means and Dispersions

We remark that in Theorem 4 we don't claim that q_n and r_n are generalized means of b_n and \widehat{b}_n respectively, for the definition of generalized means we refer the reader to [14]. However for $p = 2$ the construction above yields

Corollary 3 *For $d \geq 2$ there exists a basis $\{b_n\}_{n=1}^\infty$ for $L^2(\mathbf{R}^d)$ such that the sequences $\{\mu(b_n)\}_{n=1}^\infty$, $\{\mu(\widehat{b}_n)\}_{n=1}^\infty$ and $\{\Delta(b_n)\Delta(\widehat{b}_n)\}_{n=1}^\infty$ are bounded.*

Proof We repeat the construction described in the previous sections, but start with a function $\Psi^{(e)} = \frac{1}{\sqrt{2}}(\Psi(x) + \Psi(-x))$. Then $\Psi_{j,s}^{(e)}$ has the same properties as $\Psi_{j,s}$ above with (23) replaced with

$$\text{supp}(\Psi_{j,s}^{(e)}) \subset \{x = (x_1, \dots, x_d) : 2^{s-1} < |x_m| < 2^s, m = 1, \dots, d\}.$$

In addition we get $|\Psi_{j,s}^{(e)}(x)| = |\Psi_{j,s}^{(e)}(-x)|$ and

$$\mu(\Psi_{j,s}^{(e)}) = \int_{\mathbf{R}^d} x |\Psi_{j,s}^{(e)}(x)|^2 dx = 0. \tag{36}$$

Now for $d \geq 2$ and any β_l in (27) we have

$$|\mu(\widehat{\beta}_l)| \leq \int_{\mathbf{R}^d} |\xi| |\widehat{\beta}_l(\xi)|^2 d\xi \leq \int_{\mathbf{R}^d} (2 + |\xi - 2^{-s} j_l|^2) |\widehat{\beta}_l(\xi)|^2 d\xi \leq C_1.$$

Clearly,

$$\Delta(\beta_l)\Delta(\widehat{\beta}_l) \leq \int_{\mathbf{R}^d} |x|^2 |\beta_l(x)|^2 dx \int_{\mathbf{R}^d} |\xi - 2^{-s} j_l|^2 |\widehat{\beta}_l(\xi)|^2 d\xi \leq C_2.$$

Finally, to estimate $\mu(\beta_l)$ we use (36)

$$\begin{aligned} |\mu(\beta_l)| &= |\mu(\beta_l) - \mu(\gamma_l \Psi_{l,s}^{(e)})| \leq \frac{\theta^2}{J_s} \int_{\mathbf{R}^d} |x| |f(x)|^2 dx \\ &\quad + \int_{\mathbf{R}^d} |x| \left| \left| \sum_{n=1}^{l-1} \sigma_n \Psi_{n,s}^{(e)}(x) + \gamma_l \Psi_{l,s}^{(e)}(x) \right|^2 - |\gamma_l \Psi_{l,s}^{(e)}(x)|^2 \right| dx. \end{aligned}$$

The first term is bounden by $J_s^{-1}I_2(f) \leq 1$. (Here g_t should replace f for the case $d = 2$.) We estimate the second term by

$$\begin{aligned} & (1 + 2^s) \int_{\mathbf{R}^d} |x| \left| \sum_{n=1}^{l-1} \sigma_n \Psi_{n,s}^{(e)}(x) \right|^2 dx + 2^{-s} \int_{\mathbf{R}^d} |x| \left| \gamma_l \Psi_{l,s}^{(e)}(x) \right|^2 dx \\ & \leq (1 + 2^s) 2^s \int_{[-1,1]^d} |y| \left| \sum_{n=1}^{l-1} \sigma_n e^{2\pi i j(n,s) \cdot y} \Psi^{(e)}(y) \right|^2 dy \\ & \quad + \int_{[-1,1]^d} |y| \left| \Psi^{(e)}(y) \right|^2 dy \\ & \leq 2^{2s+1} \sqrt{d} \sum_{n=1}^{l-1} |\sigma_n|^2 + \sqrt{d} \leq \sqrt{d} (1 + 2^{2s+1} J_s^{-1}) < C_3. \end{aligned}$$

Thus $\mu(\beta_l)$ are bounded. □

4.6 Degenerate Homogeneous Weights

We now consider uncertainty inequalities with homogeneous weights in \mathbf{R}^d , obtained by B. Demange, see [8]. We describe a simple construction of a bases of functions which have uniformly bounded weighted norms with there Fourier transforms. The construction is once again based on the argument form [4]. This time a simple version of the argument implies

Lemma 5 *Let u, v be non-negative functions on \mathbf{R}^d . Suppose that there exists an orthonormal sequence $\{\phi_j\}_{j=1}^\infty$ in $L^2(\mathbf{R}^d)$ of compactly supported smooth functions such that*

$$\int_{\mathbf{R}^d} u(x) |\phi_j(x)|^2 dx \leq C_1^2 \quad \text{and} \quad \int_{\mathbf{R}^d} v(\xi) |\widehat{\phi}_j(\xi)|^2 d\xi \leq C_2^2,$$

for each j . We assume also that the supports of ϕ_j form a locally finite family of sets; i.e. each compact set intersects only finite number of these supports.

Suppose also that there is a sequence $\{f_k\}_{k=1}^\infty$ that is dense in the unit sphere of $L^2(\mathbf{R}^d)$ and such that the functions f_k are smooth, have compact supports, and

$$\int_{\mathbf{R}^d} u(x) |f_k(x)|^2 dx < \infty \quad \text{and} \quad \int_{\mathbf{R}^d} v(\xi) |\widehat{f}_k(\xi)|^2 d\xi < \infty, \tag{37}$$

for each k .

Then for each $\epsilon > 0$ there is an orthonormal basis $\{b_n\}_{n=1}^\infty$ for $L^2(\mathbf{R}^d)$ such that

$$\int_{\mathbf{R}^d} u(x) |b_n(x)|^2 dx \leq (C_1 + \epsilon)^2 \quad \text{and} \quad \int_{\mathbf{R}^d} v(\xi) |\widehat{b}_n(\xi)|^2 d\xi \leq (C_2 + \epsilon)^2,$$

for each n .

Using this lemma we prove the following.

Theorem 6 *Let $d > 1$ and $\alpha_1, \dots, \alpha_d > 0$ then there exists an orthonormal basis for $L^2(\mathbf{R}^d)$ such that*

$$\sup_n \left(\int_{\mathbf{R}^d} |x_1|^{\alpha_1} \dots |x_d|^{\alpha_d} |b_n(x)|^2 dx + \int_{\mathbf{R}^d} |\xi_1|^{\alpha_1} \dots |\xi_d|^{\alpha_d} |\widehat{b}_n(\xi)|^2 d\xi \right) < \infty.$$

Proof First, there exists a dense sequence $\{f_k\}_{k=1}^\infty$ in the unit sphere of $L^2(\mathbf{R}^d)$ that consist of smooth functions with compact supports; it satisfies (37) for

$$u(x) = v(x) = |x|^{\alpha_1} \dots |x_d|^{\alpha_d}.$$

To apply the lemma above we need to construct an orthonormal sequence bounded with their Fourier transforms in $L^2(\mathbf{R}^d, u(x))$. Let ϕ be a smooth function such that its support is contained in the unit cube $[-1, 1]^d$ and $\|\phi\|_2 = 1$. We define

$$\phi_j(x) = 2^{\frac{j}{2}(\frac{\alpha_1}{\alpha_2}-1)} \phi(2^{-j}x_1 - 3, 2^{\frac{j\alpha_1}{\alpha_2}}x_2, x_3, \dots, x_d).$$

Then clearly $\|\phi_j\|_2 = 1$ and the support of ϕ_j is contained in the set

$$E_j = \{x \in \mathbf{R}^d : x_1 \in [2^{j+1}, 2^{j+2}], |x_2| \leq 2^{-j\frac{\alpha_1}{\alpha_2}}, |x_s| \leq 1, s = 3, \dots, d\},$$

these sets are disjoint and hence the sequence $\{\phi_j\}$ is orthogonal. Further,

$$\int_{\mathbf{R}^d} u(x)|\phi_j(x)|^2 dx = \int_{E_j} |x_1|^{\alpha_1} \dots |x_d|^{\alpha_d} |\phi_j(x)|^2 dx \leq 2^{(j+2)\alpha_1} 2^{-j\alpha_1} \|\phi_j\|_2^2 \leq 2^{2\alpha_1}.$$

The Fourier transforms satisfy

$$\widehat{\phi}_j(\xi) = e^{-2^{j+1}3\pi i \xi_1} 2^{\frac{j}{2}(1-\frac{\alpha_1}{\alpha_2})} \widehat{\phi}(2^j \xi_1, 2^{-j\frac{\alpha_1}{\alpha_2}} \xi_2, \dots, \xi_d).$$

Thus

$$\int_{\mathbf{R}^d} u(\xi)|\widehat{\phi}_j(\xi)|^2 d\xi = \int_{\mathbf{R}^d} |\xi_1|^{\alpha_1} |\xi_2|^{\alpha_2} \dots |\xi_d|^{\alpha_d} |\widehat{\phi}(\xi)|^2 d\xi = C.$$

The theorem follows. □

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