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Boundedness of Schrödinger Type Propagators on Modulation Spaces

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Abstract It is known that Fourier integral operators arising when solving Schrödinger-type operators are bounded on the modulation spaces $\mathcal{M}^{p,q}$, for $1 \leq p = q \leq \infty$, provided their symbols belong to the Sjöstrand class $M^{\infty,1}$. However, they generally fail to be bounded on $\mathcal{M}^{p,q}$ for $p \neq q$. In this paper we study several additional conditions, to be imposed on the phase or on the symbol, which guarantee the boundedness on $\mathcal{M}^{p,q}$ for $p \neq q$, and between $\mathcal{M}^{p,q} \to \mathcal{M}^{q,p}$, $1 \leq q < p \leq \infty$. We also study similar problems for operators acting on Wiener amalgam spaces, recapturing, in particular, some recent results for metaplectic operators. Our arguments make heavily use of the uncertainty principle.

Keywords Fourier integral operators · Modulation spaces · Wiener amalgam spaces · Short-time Fourier transform · Sjöstrand's algebra

Mathematics Subject Classification (2000) 25S30 · 47G30

1 Introduction

The paper is concerned with the study of Fourier integral operators (FIOs) defined by

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi(x,\eta)} \sigma(x,\eta) \hat{f}(\eta)\, d\eta, \tag{1.1}$$

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for, say, $f \in \mathcal{S}(\mathbb{R}^d)$. The functions σ and Φ are called symbol and phase, respectively. Here the Fourier transform of f is normalized to be $\hat{f}(\eta) = \int f(x)e^{-2\pi i x\eta}dx$. If $\sigma \in L^\infty$ and the phase Φ is real, the integral converges absolutely and defines a function in L^∞.

The phase function $\Phi(x,\eta)$ fulfills the following properties:

(i) $\Phi \in \mathcal{C}^\infty(\mathbb{R}^{2d})$;
(ii) there exist constants $C_\alpha > 0$ such that

$$|\partial^\alpha \Phi(x,\eta)| \le C_\alpha, \quad \forall \alpha \in \mathbb{Z}_+^{2d},\ |\alpha| \ge 2; \tag{1.2}$$

(iii) there exists $\delta > 0$ such that

$$\left|\det\left(\frac{\partial^2\Phi}{\partial x_i \partial \eta_l}\Big|_{(x,\eta)}\right)\right| \ge \delta \quad \forall (x,\eta) \in \mathbb{R}^{2d}. \tag{1.3}$$

Note that our phases differ from those (positively homogeneous of degree 1 in η) of FIOs arising in the solution of hyperbolic equations (see, e.g., [9, 19, 22, 23]). Indeed, FIOs are a mathematical tool to study a variety of problems in partial differential equations, and our FIOs arise naturally in the study of the Cauchy problem for Schrödinger-type operators (see, e.g., [5, 7, 8, 14, 17, 18]). Basic examples of phase functions within the class under consideration are quadratic forms in the variables x, η (see Example 5.3 below).

Continuing the study pursued in [7], we focus on boundedness results for these operators, when acting on two classes of Banach spaces, widely used in time-frequency analysis, known as *modulation spaces* and *Wiener amalgam spaces*, denoted by $M^{p,q}$ and $W(\mathcal{F}L^p, L^q)$, respectively, with $1 \le p, q \le \infty$. To be definite, we recall the definition of these spaces, introduced by H. Feichtinger (see [10, 16] and Sect. 2 below for details).

In short, given a positive weight function m on $\mathbb{R}^{2d}$, with $m \in \mathcal{S}'(\mathbb{R}^{2d})$, we say that a temperate distribution f belongs to $M_m^{p,q}(\mathbb{R}^d)$, $1 \le p, q \le \infty$, if its short-time Fourier transform (STFT) $V_g f(x,\eta)$, defined in (2.3) below, fulfills $V_g f(x,\eta)m(x,\eta) \in L^{p,q}(\mathbb{R}^{2d}) = L^q(\mathbb{R}^d_\eta, L^p(\mathbb{R}^d_x))$, with the norm

$$\|f\|_{M_m^{p,q}} := \|(V_g f)m\|_{L^q_\eta L^p_x} < \infty. \tag{1.4}$$

Here g is a non-zero (so-called window) function in $\mathcal{S}(\mathbb{R}^d)$, which in (2.3) is first translated and then multiplied by f to localize f near any point x. Changing $g \in \mathcal{S}(\mathbb{R}^d)$ produces equivalent norms. Taking the two norms above in the converse order yields the norm in $W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$:

$$\|f\|_{W(\mathcal{F}L^p, L^q)} := \|V_g f\|_{L^p_x L^q_\eta} < \infty. \tag{1.5}$$

The spaces $\mathcal{M}_m^{p,q}(\mathbb{R}^d)$ and $\mathcal{W}(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$ are defined as the closure of $\mathcal{S}(\mathbb{R}^d)$ in the $M_m^{p,q}(\mathbb{R}^d)$ and $W(\mathcal{F}L^p, L^q)$ norm, respectively. For heuristic purposes, distributions in $M^{p,q}$, as well as in $W(\mathcal{F}L^q, L^p)$, may be regarded as functions which are locally in $\mathcal{F}L^q$ and decay at infinity like functions in L^p. Among their properties, we highlight the important relation $W(\mathcal{F}L^p, L^q) = \mathcal{F}(M^{p,q})$.

The action on the spaces $\mathcal{M}^p := \mathcal{M}^{p,p}$ of FIOs as above already appeared in [4, 7] (see also [1–3, 22]). It is a basic result that FIOs with symbols in the Sjöstrand class $M^{\infty,1}(\mathbb{R}^{2d})$ extend to bounded operators on $\mathcal{M}^p(\mathbb{R}^d)$, $1 \le p \le \infty$. Applications to issues of classical analysis were also given in [9]. Moreover, it was observed in [7] that boundedness generally fails on the spaces $\mathcal{M}^{p,q}(\mathbb{R}^d)$, with $p \neq q$, although it can hold under an additional condition on the phase function. The present paper is devoted to a more systematic study of the conditions which guarantee the boundedness on $\mathcal{M}^{p,q}(\mathbb{R}^d)$, for $p \neq q$.

Our first result is in fact a generalization of [2, Theorem 11] and [7, Theorem 5.2] to the case of rougher symbols.

Theorem 1.1 *Consider a phase function* Φ *satisfying* (i), (ii), *and* (iii), *and a symbol* $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$. *Suppose, in addition, that*

$$\sup_{x,x',\eta \in \mathbb{R}^d} \left| \nabla_x \Phi(x,\eta) - \nabla_x \Phi(x',\eta) \right| < \infty. \tag{1.6}$$

Then, the corresponding Fourier integral operator T *extends to a bounded operator on* $\mathcal{M}^{p,q}(\mathbb{R}^d)$, *for every* $1 \le p, q \le \infty$.

The condition (1.6) is seen to be essential for the conclusion to hold. In fact it was shown in [7, Proposition 7.1] that the pointwise multiplication operator by $e^{-\pi|x|^2}$ (which has phase $\Phi(x,\eta) = x\eta - \frac{|x|^2}{2}$ and symbol $\sigma \equiv 1$) is not bounded on any $\mathcal{M}^{p,q}$, with $p \neq q$ (see also Theorem 6.1 below).

If we drop the condition (1.6), we need some further decay condition on the symbol, as explained by the next result.

For $s_1, s_2 \in \mathbb{R}$, we define the weight function $v_{s_1,s_2}(x,\eta) := \langle x\rangle^{s_1}\langle \eta\rangle^{s_2}$, $(x,\eta) \in \mathbb{R}^{2d}$.

Theorem 1.2 *Consider a phase* Φ *satisfying* (i), (ii) *and* (iii), *and a symbol* $\sigma \in M^{\infty,1}_{v_{s_1,s_2}\otimes 1}(\mathbb{R}^{2d})$, $s_1, s_2 \in \mathbb{R}$.

(i) *Let* $1 \le p \le \infty$. *If* $s_1, s_2 \ge 0$, T *extends to a bounded operator on* $\mathcal{M}^p(\mathbb{R}^d)$.
(ii) *Let* $1 \le q < p \le \infty$. *If* $s_1 > d(\frac{1}{q} - \frac{1}{p})$, $s_2 \ge 0$, T *extends to a bounded operator on* $\mathcal{M}^{p,q}(\mathbb{R}^d)$.
(iii) *Let* $1 \le p < q \le \infty$. *If* $s_1 \ge 0$, $s_2 > d(\frac{1}{p} - \frac{1}{q})$, T *extends to a bounded operator on* $\mathcal{M}^{p,q}(\mathbb{R}^d)$.

In all cases,

$$\|Tf\|_{\mathcal{M}^{p,q}} \lesssim \|\sigma\|_{M^{\infty,1}_{v_{s_1,s_2}\otimes 1}} \|f\|_{\mathcal{M}^{p,q}}. \tag{1.7}$$

Although we do not exhibit a complete set of counterexamples for the thresholds arising in Theorem 1.2, some examples are given in Sect. 6, and show that the thresholds are in fact the expected ones (see Remark 6.5).

Results for boundedness of FIOs between weighted modulation spaces are attained as well (see Sect. 4).

We also turn our attention to the boundedness of FIOs as above from the modulation space $\mathcal{M}^{p,q}(\mathbb{R}^d)$ into $\mathcal{M}^{q,p}(\mathbb{R}^d)$. This study is mostly suggested by the special case of metaplectic operators (corresponding to a quadratic phase and symbol $\sigma \equiv 1$), which was investigated in detail in [5]; see also Example 5.3 below.

Theorem 1.3 *Let* $1 \le q < p \le \infty$. *Consider a phase* $\Phi(x,\eta)$ *satisfying* (i), (ii) *and* (iii). *Moreover, assume one of the following conditions*:

(a) *the symbol* $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ *and, for some* $\delta > 0$,

$$\left|\det\left(\frac{\partial^2\Phi}{\partial x_i \partial x_l}\Big|_{(x,\eta)}\right)\right| \ge \delta \quad \forall (x,\eta) \in \mathbb{R}^{2d}, \tag{1.8}$$

(b) *the symbol* $\sigma \in M^{\infty,1}_{v_{s,0}\otimes 1}(\mathbb{R}^{2d})$, *with* $s > d(\frac{1}{q} - \frac{1}{p})$.

Then, the corresponding FIO T *extends to a bounded operator* $\mathcal{M}^{p,q}(\mathbb{R}^d) \to \mathcal{M}^{q,p}(\mathbb{R}^d)$.

The additional assumptions (a) or (b) are essential to guarantee the boundedness. A counterexample in this connection is given in Proposition 6.7 below. Moreover, even under those conditions, T generally fails to be bounded between $\mathcal{M}^{p,q} \to \mathcal{M}^{q,p}$ if $q > p$; see Proposition 6.6 below.

For the sake of brevity, in this Introduction we only established our results for FIOs acting on modulation spaces. In the subsequent sections we shall provide corresponding results for Wiener amalgam spaces (Corollaries 3.9 and 5.2).

As in [7], the proof of our results relies on a formula expressing the Gabor matrix of the FIO T in terms of the STFT of its symbol σ (see (3.3) below). Here the novelty is provided by the combination of this formula with Schur-type tests and the uncertainty principle, in the form of Bernstein's inequality and some generalizations.

The paper is organized as follows. In Sect. 2 we prove some preliminary results of classical analysis and we also collect the basic definitions and properties of modulation and Wiener amalgam spaces. In Sect. 3 we recall from [7] a useful formula for the Gabor matrix of the operator T, and use it to prove Theorems 1.1 and 1.2. In Sect. 4 we study the action of a FIO T on weighted modulation spaces. In Sect. 5 we prove Theorem 1.3. Finally, in Sect. 6 some examples related to the Schrödinger operators are exhibited: they reveal to be useful tests for the sharpness of the above results.

Notation We define $|x|^2 = x \cdot x$, for $x \in \mathbb{R}^d$, where $x \cdot y = xy$ is the scalar product on $\mathbb{R}^d$. The space of smooth functions with compact support is denoted by $\mathcal{C}_0^\infty(\mathbb{R}^d)$, the Schwartz class is $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\eta) = \mathcal{F}f(\eta) = \int f(t)e^{-2\pi i t\eta}\,dt$. Translation and modulation operators (*time and frequency shifts*) are defined, respectively, by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\eta f(t) = e^{2\pi i\eta t} f(t).$$

We have the formulas $(T_x f)\hat{} = M_{-x}\hat{f}$, $(M_\eta f)\hat{} = T_\eta \hat{f}$, and $M_\eta T_x = e^{2\pi i x\eta} T_x M_\eta$. The inner product of two functions $f, g \in L^2(\mathbb{R}^d)$ is $\langle f, g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t)}\,dt$, and its extension to $\mathcal{S}' \times \mathcal{S}$ will be also denoted by $\langle \cdot, \cdot \rangle$. The notation $A \lesssim B$ means $A \le cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ means $c^{-1}A \le B \le cA$, for some $c \ge 1$. The symbol $B_1 \hookrightarrow B_2$ denotes the continuous embedding of the space B_1 into B_2. The open ball in $\mathbb{R}^d$ of center x and radius R will be denoted by $B(x, R)$.

2 Preliminary Results

2.1 Bernstein Inequalities

The core of our proofs relies on the classical Bernstein's inequality (see, e.g., [26]) and some of its generalizations, described in what follows. Recall that the ball of center $x \in \mathbb{R}^d$ and radius $R > 0$ is denoted by $B(x, R)$.

Lemma 2.1 (Bernstein's inequality) *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\hat{f}$ is supported in $B(0, R)$, and let $1 \le p \le q \le \infty$. Then, there exists a positive constant C (independent of f, R, p, q), such that*

$$\|f\|_q \le C R^{d(\frac{1}{p}-\frac{1}{q})} \|f\|_p. \tag{2.1}$$

We shall use also the following generalizations of the Bernstein's inequalities.

Lemma 2.2 *Consider a mapping $v : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $|v(x)| \lesssim |x|$, $\forall x \in \mathbb{R}^d$. Then, for any $s \ge 0$,*

$$\sup_{x \in \mathbb{R}^d} \{\langle x\rangle^{-s} |f(v(x))|\} \lesssim \int \langle x\rangle^{-s} |f(x)|\, dx,$$

for every function $f \in \mathcal{S}'(\mathbb{R}^d)$, such that $\hat{f}$ is supported in $B(\eta_0, 1)$, for some $\eta_0 \in \mathbb{R}^d$.

Proof Take a Schwartz function g, satisfying $\hat{g}(\eta) = 1$ for $|\eta| \le 1$. If $\hat{f}$ is supported in $B(\eta_0, 1)$ we have $\hat{f} = \hat{f} T_{\eta_0} \hat{g}$. Hence

$$\langle x\rangle^{-s} |f(v(x))| \le \int \langle x\rangle^{-s} |f(y)||g(v(x) - y)|\, dy. \tag{2.2}$$

Now we have

$$\langle y\rangle^{s} \lesssim \langle y - v(x)\rangle^{s} \langle v(x)\rangle^{s} \lesssim \langle y - v(x)\rangle^{s} \langle x\rangle^{s},$$

so that

$$\langle x\rangle^{-s} |g(v(x) - y)| \le \langle x\rangle^{-s} \langle v(x) - y\rangle^{-N} \lesssim \langle y\rangle^{-s} \langle v(x) - y\rangle^{-N+s}.$$

Using this inequality in (2.2), with $N \ge s$, we attain the desired conclusion. □

We recall from [26, Proposition 5.5] the following localized version of Bernstein's inequality.

Lemma 2.3 *Let $N \geq 1$ be an integer and $\varphi(x) = (1+|x|^2)^{-N}$. For $R > 0$, $x \in \mathbb{R}^d$, let $\varphi_{x,R}(v) = \varphi(\frac{v-x}{R})$, $v \in \mathbb{R}^d$. There exists $C_N > 0$ such that*

$$\sup_{y \in B(x,R)} |f(y)| \leq C_N\, \mu(B(x,R))^{-1} \int_{\mathbb{R}^d} \varphi_{x,R}(v)|f(v)|\, dv,$$

for every $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\hat{f}$ is supported in $B(\eta_0, 1/R)$, for some $\eta_0 \in \mathbb{R}^d$, where μ is the Lebesgue measure.

2.2 Schur-Type Tests

The next proposition collects Schur-type tests assuring the boundedness of integral operators on the mixed-norm spaces $L^q_\eta L^p_x := L^q(\mathbb{R}^d_\eta; L^p(\mathbb{R}^d_x))$, $1 \leq p, q \leq \infty$.

Proposition 2.4 *Consider an integral operator A on $\mathbb{R}^{2d}$, given by*

$$(Af)(x', \eta') = \iint K(x', \eta'; x, \eta) f(x, \eta)\, dx\, d\eta.$$

- (i) *If $K \in L^\infty_\eta L^1_{\eta'} L^\infty_{x'} L^1_x$, then A is continuous on $L^1_\eta L^\infty_x$.*
- (ii) *If $K \in L^\infty_{\eta'} L^1_\eta L^\infty_x L^1_{x'}$, then A is continuous on $L^\infty_\eta L^1_x$.*
- (iii) *If $K \in L^\infty_\eta L^1_{\eta'} L^\infty_{x'} L^1_x \cap L^\infty_{\eta'} L^1_\eta L^\infty_x L^1_{x'}$, and, moreover, $K \in L^\infty_{x',\eta'} L^1_{x,\eta} \cap L^\infty_{x,\eta} L^1_{x',\eta'}$, then the operator A is continuous on $L^q_\eta L^p_x$, for every $1 \leq p, q \leq \infty$.*
- (iv) *If $K \in L^\infty_{\eta,\eta'} L^1_{x,x'}$, then A is continuous $L^1_\eta L^\infty_x \to L^\infty_\eta L^1_x$.*

Proof The proof of all items, but (iv), is just a repetition, with obvious changes, of that of [7, Proposition 5.1], where a discrete version was presented.
Let us now prove (iv). We have

$$\begin{aligned}
\|Af\|_{L^\infty_{\eta'} L^1_{x'}} &= \sup_{\eta' \in \mathbb{R}^d} \int_{\mathbb{R}^d_{x'}} \left| \iint K(x', \eta'; x, \eta) f(x, \eta)\, dx\, d\eta \right| dx' \\
&\leq \sup_{\eta' \in \mathbb{R}^d} \iiint_{\mathbb{R}^{3d}_{x,x',\eta}} |K(x', \eta'; x, \eta) f(x, \eta)|\, dx\, dx'\, d\eta \\
&\leq \sup_{\eta' \in \mathbb{R}^d} \int_{R^d_\eta} \sup_{x \in \mathbb{R}^d} |f(x, \eta)| \iint |K(x', \eta'; x, \eta)|\, dx\, dx'\, d\eta \\
&\leq \|K\|_{L^\infty_{\eta,\eta'} L^1_{x,x'}} \|f\|_{L^1_\eta L^\infty_x}.
\end{aligned}$$

This concludes the proof. □

2.3 Modulation Spaces

See [10–13, 16, 25]. For $s \in \mathbb{R}$, we denote by $\langle\cdot\rangle^s = (1+|\cdot|^2)^{s/2}$. In what follows we limit ourselves to the class of weight functions $v_{s_1,s_2}(x, \eta) = \lambda x\rangle^{s_1} \langle\eta\rangle^{s_2}$, $s_i \in \mathbb{R}$, $i =$

$1, 2$, on $\mathbb{R}^{2d}$, or $m(x, \eta, z, \zeta) = \langle x\rangle^{s_1}\langle\eta\rangle^{s_2} = (v_{s_1,s_2}\otimes 1)(x, \eta, z, \zeta)$, on $\mathbb{R}^{4d}$. In order to define such spaces, we make use of the following time-frequency representation: the short-time Fourier transform (STFT) $V_g f$ of a function/tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window $g \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$V_g f(x, \eta) = \langle f, M_\eta T_x g\rangle = \int_{\mathbb{R}^d} e^{-2\pi i\eta y} f(y)\overline{g(y-x)}\, dy, \tag{2.3}$$

i.e., the Fourier transform $\mathcal{F}$ applied to $f\overline{T_x g}$.

Given a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, a weight function m as those quoted above, and $1 \le p, q \le \infty$, the *modulation space* $M^{p,q}_m(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L^{p,q}_m(\mathbb{R}^{2d})$ (weighted mixed-norm spaces). The norm on $M^{p,q}_m$ is

$$\|f\|_{M^{p,q}_m} = \|V_g f\|_{L^{p,q}_m} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x, \eta)|^p m(x, \eta)^p\, dx\right)^{q/p} d\eta\right)^{1/p}$$

(with obvious changes when $p = \infty$ or $q = \infty$). If $p = q$, we write M^p_m instead of $M^{p,p}_m$, and if $m(z) \equiv 1$ on $\mathbb{R}^{2d}$, then we write $M^{p,q}$ and M^p for $M^{p,q}_m$ and $M^{p,p}_m$, respectively. Then $M^{p,q}_m(\mathbb{R}^d)$ is a Banach space whose definition is independent of the choice of the window g. For the properties of these spaces we refer to the literature quoted at the beginning of this subsection.

We define by $\mathcal{M}^{p,q}_m(\mathbb{R}^d)$ the closure of $\mathcal{S}(\mathbb{R}^d)$ in the $M^{p,q}_m$-norm. Observe that $\mathcal{M}^{p,q}_m = M^{p,q}_m$, whenever the indices p and q are finite. They enjoy the duality property: $(\mathcal{M}^{p,q}_m)^* = \mathcal{M}^{p',q'}_{1/m}$, with $1 < p, q < \infty$, and p', q' being the conjugate exponents.

We recall the inversion formula for the STFT (see, e.g., [16, Corollary 3.2.3]: if $\|g\|_{L^2} = 1$ and, for example, $u \in L^2(\mathbb{R}^d)$, it turns out

$$u = \int_{\mathbb{R}^{2d}} V_g u(x, \eta) M_\eta T_x g\, dx\, d\eta. \tag{2.4}$$

The following inequality, proved in [16, Lemma 11.3.3], is useful for changing windows.

Lemma 2.5 *Let $g_0, g_1, \gamma \in \mathcal{S}(\mathbb{R}^d)$ such that $\langle\gamma, g_1\rangle \neq 0$ and let $f \in \mathcal{S}'(\mathbb{R}^d)$. Then,*

$$|V_{g_0} f(x, \eta)| \le \frac{1}{|\langle\gamma, g_1\rangle|}(|V_{g_1} f| * |V_{g_0}\gamma|)(x, \eta),$$

for all $(x, \eta) \in \mathbb{R}^{2d}$.

The complex interpolation theory for these spaces reads as follows (see, e.g., [13]).

Proposition 2.6 *Let $0 < \theta < 1$, $p_j, q_j \in [1, \infty]$ and m_j be v-moderate weight functions (i.e., $m_j(x + y) \le Cv(x)m_j(y)$, v being a submultiplicative weight), $j = 1, 2$.*

Set

$$\frac{1}{p}=\frac{1-\theta}{p_1}+\frac{\theta}{p_2},\qquad \frac{1}{q}=\frac{1-\theta}{q_1}+\frac{\theta}{q_2},\qquad m=m_1^{1-\theta}m_2^{\theta},$$

then

$$(\mathcal{M}_{m_1}^{p_1,q_1}(\mathbb{R}^d),\mathcal{M}_{m_2}^{p_2,q_2}(\mathbb{R}^d))_{[\theta]}=\mathcal{M}_m^{p,q}(\mathbb{R}^d).$$

Remark 2.7 We observe that our results are established as the existence of a bounded extension $\mathcal{M}^{p,q}\to\mathcal{M}^{\tilde{p},\tilde{q}}$ of a class of FIOs T with symbols σ in a weighted space $M_m^{\infty,1}$, $m\geq 1$. It is important to observe that such an extension follows from a uniform estimate of the type

$$\|Tf\|_{M^{\tilde{p},\tilde{q}}}\leq C\|\sigma\|_{M_m^{\infty,1}}\|f\|_{M^{p,q}},\quad \forall f\in\mathcal{S}(\mathbb{R}^d). \tag{2.5}$$

Indeed, this estimate shows that T extends to a bounded operator $\mathcal{M}^{p,q}\to M^{\tilde{p},\tilde{q}}$. In order to prove that this extension takes values in $\mathcal{M}^{\tilde{p},\tilde{q}}$, it suffices to verify that $Tf\in\mathcal{M}^{\tilde{p},\tilde{q}}$ when f is a Schwartz function. This follows from [7, Theorem 6.1]. Indeed, if $f\in M^1$, then $Tf\in M^1$.

Hence in the subsequent proofs we will prove estimates of the type (2.5).

Boundedness results dealing with FIOs having symbols in weighted modulation spaces and acting on unweighted modulation spaces could be rephrased as boundedness results for FIOs with symbols in unweighted spaces and acting on weighted spaces, as explained below.

Proposition 2.8 *Let T be a FIO with symbol σ and $\tilde{T}$ a FIO with the same phase as T and symbol*

$$\tilde{\sigma}(x,\eta):=\langle x\rangle^{s_1}\sigma(x,\eta)\langle\eta\rangle^{s_2},\quad x,\eta\in\mathbb{R}^d,\ s_i\in\mathbb{R},\ i=1,2.$$

Then,

(i) *the operator T is bounded from $\mathcal{M}^{p,q}$ into $\mathcal{M}^{\tilde{p},\tilde{q}}$ if and only if the operator $\tilde{T}$ is bounded from $\mathcal{M}_{v_{0,s_2}}^{p,q}$ into $\mathcal{M}_{v_{-s_1,0}}^{\tilde{p},\tilde{q}}$.*

(ii) *It holds true*

$$\sigma\in M_{v_{s_1,s_2}\otimes 1}^{\infty,1}(\mathbb{R}^{2d})\quad\Longleftrightarrow\quad\tilde{\sigma}\in M^{\infty,1}(\mathbb{R}^{2d}). \tag{2.6}$$

Proof The proof is an immediate consequence of [24, Theorem 2.2, Corollary 2.3]. Indeed, they guarantee that the vertical arrows of the following commutative diagram define isomorphisms:

$$\begin{array}{ccc} \mathcal{M}^{p,q} & \xrightarrow{T} & \mathcal{M}^{\tilde{p},\tilde{q}} \\ \uparrow \scriptstyle{\langle D\rangle^{s_2}} & & \downarrow \scriptstyle{\text{pointwise product by } \langle x\rangle^{s_1}} \\ \mathcal{M}^{p,q}_{v_{0,s_2}} & \xrightarrow{\tilde{T}} & \mathcal{M}^{\tilde{p},\tilde{q}}_{v_{-s_1,0}} \end{array} \tag{2.7}$$

Moreover, the product by the weight function $\eta_0(x,\eta,\zeta_1,\zeta_2) := \langle x\rangle^{s_1}\langle\eta\rangle^{s_2}$, $x,\eta,\zeta_1,\zeta_2 \in \mathbb{R}^d$, defined on $\mathbb{R}^{4d}$, is a isomorphism from $M^{\infty,1}_{v_{s_1,s_2}\otimes 1}(\mathbb{R}^{2d})$ to $M^{\infty,1}(\mathbb{R}^{2d})$, that is (ii). □

2.4 Wiener Amalgam Spaces

See [10, 12, 15]. For $1 \le p \le \infty$, $s \in \mathbb{R}$, we denote by $L^p_s(\mathbb{R}^d) := L^p_{\langle\cdot\rangle^s}(\mathbb{R}^d)$. Let $\mathcal{F}L^p_s(\mathbb{R}^d)$ be the space of $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\hat{f} \in L^p_s(\mathbb{R}^d)$. Let $g \in \mathcal{S}(\mathbb{R}^d)$ be a non-zero window function. For $1 \le p,q, \le \infty$, the *Wiener amalgam space* $W(\mathcal{F}L^p_{s_1}, L^q_{s_2})(\mathbb{R}^d)$ with local component $\mathcal{F}L^p_{s_1}$ and global component $L^q_{s_2}$, $s_i \in \mathbb{R}$, $i = 1,2$, is defined as the space of all functions or distributions for which the norm

$$\|f\|_{W(\mathcal{F}L^p_{s_1}, L^q_{s_2})} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |\mathcal{F}(f\cdot T_x g)(\eta)|^p \langle\eta\rangle^{ps_1}\,d\eta\right)^{q/p}\langle x\rangle^{qs_2}\,dx\right)^{1/q}$$

is finite. Analogously to modulation spaces, we define by $\mathcal{W}(\mathcal{F}L^p_{s_1}, L^q_{s_2})(\mathbb{R}^d)$ the closure of $\mathcal{S}(\mathbb{R}^d)$ with respect to the $W(\mathcal{F}L^p_{s_1}, L^q_{s_2})$-norm.

The properties of Wiener amalgam spaces are similar to those of modulation spaces, since we have the following relation:

$$\mathcal{F}(\mathcal{M}^{p,q}_{v_{s_1,s_2}}) = \mathcal{W}(\mathcal{F}L^p_{s_1}, L^q_{s_2}), \quad s_i \in \mathbb{R},\ i = 1,2. \tag{2.8}$$

We recall from [6, Lemma 5.3] the following auxiliary result.

Lemma 2.9 *For $a,b \in \mathbb{R}$, $a > 0$, set $G_{a+ib}(x) = (a+ib)^{-d/2}e^{-\frac{\pi|x|^2}{a+ib}}$, and choose the Gaussian $g(y) = e^{-\pi|y|^2}$ as window function. Then, for every $1 \le q,r \le \infty$, it follows that*

$$|\widehat{G_{a+ib}T_x}g(\eta)| = ((a+1)^2+b^2)^{-d/4}e^{-\frac{\pi}{(a+1)^2+b^2}[(a(a+1)+b^2)|\eta|^2+2bx\eta+(a+1)|x|^2]}, \tag{2.9}$$

$$\|G_{a+ib}T_x g\|_{\mathcal{F}L^q} = \frac{((a+1)^2+b^2)^{\frac{d}{2}(\frac{1}{q}-\frac{1}{2})}}{q^{\frac{d}{2q}}(a(a+1)+b^2)^{\frac{d}{2q}}}e^{-\frac{\pi a|x|^2}{a(a+1)+b^2}}, \tag{2.10}$$

and

$$\|G_{a+ib}\|_{W(\mathcal{F}L^q,L^r)} \asymp \frac{((a+1)^2+b^2)^{\frac{d}{2}(\frac{1}{q}-\frac{1}{2})}}{a^{\frac{d}{2r}}(a(a+1)+b^2)^{\frac{d}{2}(\frac{1}{q}-\frac{1}{r})}}. \tag{2.11}$$

Lemma 2.10 *Let* $\sigma \in M^{p,q}_{v_{s_1,s_2}\otimes 1}(\mathbb{R}^{2d})$, $s_i \in \mathbb{R}$, $i=1,2$, *and consider its adjoint* σ^* *defined by*

$$\sigma^*(x,\eta) = \overline{\sigma(\eta,x)}, \quad (x,\eta)\in\mathbb{R}^{2d}. \tag{2.12}$$

Then, $\sigma^* \in M^{p,q}_{v_{s_2,s_1}\otimes 1}(\mathbb{R}^{2d})$.

Proof Fix a window function $g\in\mathcal{S}(\mathbb{R}^d)$. By a direct computation,

$$\begin{aligned}
V_g\sigma^*(z_1,z_2;\zeta_1,\zeta_2) &= \int e^{-2\pi i(y_1\zeta_1+y_2\zeta_2)}\overline{\sigma(y_2,y_1)}\,\overline{g(y_1-z_1,y_2-z_2)}\,dy_1\,dy_2\\
&= \int e^{-2\pi i(v_2\zeta_1+v_1\zeta_2)}\overline{\sigma(v_1,v_2)}\,\overline{g(v_2-z_1,v_1-\zeta_2)}\,dv_1\,dv_2\\
&= V_{{}^tg}\overline{\sigma}(z_2,z_1;\zeta_2,\zeta_1)
\end{aligned}$$

where we used the notation ${}^tg(y_1,y_2) := g(y_2,y_1)$. Since $\overline{\sigma}\in M^{p,q}_{v_{s_1,s_2}\otimes 1}(\mathbb{R}^{2d})$, the result immediately follows by the independence of the window function for the computation of the modulation space norm. □

3 Boundedness of FIOs on $\mathcal{M}^{p,q}$

In the following we assume that the phase function Φ satisfies the assumptions (i), (ii) and (iii) in the Introduction, so that we shall repeatedly use the following property.

Remark 3.1 It follows from the assumptions (i), (ii) and (iii) and Hadamard's global inversion function theorem (see, e.g., [21]) that the mappings

$$x \longmapsto \nabla_\eta \Phi(x,\eta)$$

and

$$\eta \longmapsto \nabla_x \Phi(x,\eta)$$

are global diffeomorphisms of $\mathbb{R}^d$, and their inverse Jacobian determinant are uniformly bounded with respect to all variables.

The proof of our results relies on a formula, obtained in [7, Sect. 6], which expresses the Gabor matrix of the FIO T in terms of the STFT of its symbol σ. Namely, choose a non-zero window function $g\in\mathcal{S}(\mathbb{R}^d)$, and define

$$\Psi_{(x',\eta)}(y,\zeta) := e^{2\pi i\Phi_{2,(x',\eta)}(y,\zeta)}(\bar{g}\otimes\hat{g})(y,\zeta), \quad (y,\zeta)\in\mathbb{R}^{2d} \tag{3.1}$$

with

$$\Phi_{2,(x',\eta)}(y,\zeta) = 2\sum_{|\alpha|=2}\int_0^1 (1-t)\partial^\alpha\Phi((x',\eta)+t(y,\zeta))\,dt\,\frac{(y,\zeta)^\alpha}{\alpha!}. \tag{3.2}$$

Moreover, let

$$g_{x,\eta}(t) = \big(M_\eta T_x g\big)(t), \quad t, x, \eta \in \mathbb{R}^d.$$

Proposition 3.2 *It turns out*

$$|\langle T g_{x,\eta}, g_{x',\eta'}\rangle| = |V_{\Psi_{(x',\eta)}}\sigma((x',\eta),(\eta'-\nabla_x\Phi(x',\eta), x-\nabla_\eta\Phi(x',\eta)))|. \tag{3.3}$$

Remark 3.3 Observe that the window $\Psi_{(x',\eta)}$ of the STFT above depends on the pair (x',η). However, the assumptions (1.2) imply that these windows belong to a bounded subset of the Schwartz space, i.e. the corresponding seminorms are uniformly bounded with respect to (x',η).

In this section we focus on the proofs of Theorems 1.1 and 1.2. We first prove Theorem 1.1. We need the following auxiliary results.

Lemma 3.4 *Let* $\Psi_0 \in \mathcal{S}(\mathbb{R}^{2d})$ *with* $\|\Psi_0\|_{L^2} = 1$ *and* $\Psi_{(x',\eta)}$ *be defined by* (3.1), $(x',\eta) \in \mathbb{R}^{2d}$, *and* $g \in \mathcal{S}(\mathbb{R}^d)$. *Then,*

$$\int_{\mathbb{R}^{4d}} \sup_{(x',\eta)\in\mathbb{R}^{2d}} |V_{\Psi_{(x',\eta)}}\Psi_0(w)|\,dw < \infty. \tag{3.4}$$

Proof We shall show that

$$|V_{\Psi_{(x',\eta)}}\Psi_0(w)| \le C\langle w\rangle^{-(4d+1)}, \quad \forall (x',\eta) \in \mathbb{R}^{2d}. \tag{3.5}$$

Using the switching property of the STFT:

$$(V_f g)(x,\eta) = e^{-2\pi i\eta x}\overline{(V_g f)(-x,-\eta)},$$

we observe that $|V_{\Psi_{(x',\eta)}}\Psi_0(w_1,w_2)| = |V_{\Psi_0}\Psi_{(x',\eta)}(-w_1,-w_2)|$, and by the even property of the weight $\langle\cdot\rangle$, relation (3.5) is equivalent to

$$|V_{\Psi_0}\Psi_{(x',\eta)}(w)| \le C\langle w\rangle^{-(4d+1)}, \quad \forall (x',\eta) \in \mathbb{R}^{2d}. \tag{3.6}$$

Now, the mapping V_{Ψ_0} is continuous from $\mathcal{S}(\mathbb{R}^{2d})$ to $\mathcal{S}(\mathbb{R}^{4d})$ (see [16, Chap. 11]), which combined with the Remark (3.3) yields (3.6). □

Lemma 3.5 *With the notation of Lemma* 2.3, *for every* $R > 0$ *there exists* C_R *such that*

$$\iint \varphi_{x,R}(v) f(v)\,dv\,dx = C_R \int f(x)\,dx,$$

for every measurable function $f \ge 0$.

Proof The proof is just an application of Fubini's theorem on the exchange of integrals, since $\int \varphi_{x,R}(v)\,dx = C_R$ is independent of v. □

Proposition 3.6 *Let $\Psi_0 \in \mathcal{S}(\mathbb{R}^{2d})$ supported in the ball $B(0, 1/R)$. Then*

$$\sup_{v\in B(0,R)} |V_{\Psi_0}\sigma(u; z_1 + v, z_2)| \le C_N\, \mu(B(z_1, R))^{-1} \int \varphi_{z_1,R}(v)|V_{\Phi_0}\sigma(u; v, z_2)|\,dv,$$

for every $u \in \mathbb{R}^{2d}$, $z_1, z_2 \in \mathbb{R}^d$.

Proof It suffices to apply Proposition 2.3 to the function $\mathbb{R}^d \ni v \longmapsto V_{\Psi_0}\sigma(u; v, z_2)$. Indeed, setting $u = (u_1, u_2) \in \mathbb{R}^d \times \mathbb{R}^d$, its Fourier transform has support contained in the ball $B(u_1, 1/R)$. □

Proof of Theorem 1.1 Since the boundedness on $\mathcal{M}^p = \mathcal{M}^{p,p}$, $1 \le p \le \infty$, was already proved in [7, Theorem 6.1], see also [4, Theorem 2.1], the desired result will follow by interpolation from the cases $(p, q) = (\infty, 1)$ and $(p, q) = (1, \infty)$. To prove them, we observe that the inversion formula (2.4) for the STFT gives

$$V_g(Tu)(x', \eta') = \int_{\mathbb{R}^{2d}} \langle T g_{x,\eta}, g_{x',\eta'}\rangle V_g u(x, \eta)\,dx\,d\eta.$$

By Remark 2.7, the desired estimate therefore follows if we prove that the map K_T defined by

$$K_T G(x', \eta') = \int_{\mathbb{R}^{2d}} \langle T g_{x,\eta}, g_{x',\eta'}\rangle G(x, \eta)\,dx\,d\eta$$

is continuous on $L^1_\eta L^\infty_x$ and $L^\infty_\eta L^1_x$. By Proposition 2.4 it suffices to prove that its integral kernel

$$K_T(x', \eta'; x, \eta) = \langle T g_{x,\eta}, g_{x',\eta'}\rangle$$

satisfies

$$K_T \in L^\infty_\eta L^1_{\eta'} L^\infty_{x'} L^1_x, \tag{3.7}$$

and

$$K_T \in L^\infty_{\eta'} L^1_\eta L^\infty_x L^1_{x'}. \tag{3.8}$$

Let us verify (3.7). By (3.3) we have

$$|K_T(x', \eta'; x, \eta)| = |V_{\Psi_{(x',\eta)}}\sigma(z(x', \eta', x, \eta))|,$$

where

$$z(x', \eta', x, \eta) = (x', \eta, \eta' - \nabla_x\Phi(x', \eta), x - \nabla_\eta\Phi(x', \eta)).$$

By Lemma 2.5 for $g_1 = \gamma = \Psi_0$, $\Psi_0 \in \mathcal{S}(\mathbb{R}^{2d})$, $\|\Psi_0\|_2 = 1$ and $\sup \Psi_0 \subset B(0, 1/R)$, where $R > 0$ will be chosen later, we have

$$|V_{\Psi_{(x',\eta)}}\sigma(z)| \le (|V_{\Psi_0}\sigma| * |V_{\Psi_{(x',\eta)}}\Psi_0|)(z), \quad z \in \mathbb{R}^{4d},$$

so that

$$\begin{aligned}
&\|V_{\Psi_{(x',\eta)}}\sigma(z(x',\eta',x,\eta))\|_{L^\infty_\eta L^1_{\eta'} L^\infty_{x'} L^1_x}\\
&\quad\le \int \|V_{\Psi_0}\sigma(z(x',\eta',x,\eta)-w)\|_{L^\infty_\eta L^1_{\eta'} L^\infty_{x'} L^1_x} \sup_{(x',\eta)\in\mathbb{R}^{2d}} |V_{\Psi_{(x',\eta)}}\Psi_0(w)|\,dw\\
&\quad\le \sup_{w\in\mathbb{R}^{4d}} \|V_{\Psi_0}\sigma(z(x',\eta',x,\eta)-w)\|_{L^\infty_\eta L^1_{\eta'} L^\infty_{x'} L^1_x} \int \sup_{(x',\eta)\in\mathbb{R}^{2d}} |V_{\Psi_{(x',\eta)}}\Psi_0(w)|\,dw.
\end{aligned}$$

In view of Lemma 3.4 we are therefore reduced to proving the estimate

$$\|V_{\Psi_0}\sigma(z(x',\eta',x,\eta)-w)\|_{L^\infty_\eta L^1_{\eta'} L^\infty_{x'} L^1_x} \le C\|\sigma\|_{M^{\infty,1}},$$

uniformly with respect to $w\in\mathbb{R}^{4d}$. Since,

$$|V_{\Psi_0}\sigma(z(x',\eta',x,\eta)-w)| \le \sup_{u\in\mathbb{R}^{2d}} |V_{\Psi_{(x',\eta)}}\sigma(u,\tilde z(x',\eta',x,\eta,w_2))|,$$

with

$$\tilde z(x',\eta',x,\eta,w_2) = (\eta'-\nabla_x\Phi(x',\eta), x-\nabla_\eta\Phi(x',\eta))-w_2,\quad w=(w_1,w_2)\in\mathbb{R}^{4d},$$

we shall prove that

$$\Big\|\sup_{u\in\mathbb{R}^{2d}} |V_{\Psi_0}\sigma(u,\tilde z(x',\eta',x,\eta,w_2))|\Big\|_{L^\infty_\eta L^1_{\eta'} L^\infty_{x'} L^1_x} \le C\|\sigma\|_{M^{\infty,1}}, \tag{3.9}$$

uniformly with respect to $w_2\in\mathbb{R}^{2d}$. A translation shows that the left-hand side in (3.9) is indeed independent of w_2, and coincides with

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d_{\eta'}} \sup_{x'\in\mathbb{R}^d}\int_{\mathbb{R}^d_x} \sup_{u\in\mathbb{R}^{2d}} |V_{\Psi_0}\sigma(u;\eta'-\nabla_x\Phi(x',\eta), x-\nabla_\eta\Phi(x',\eta))|\,dx\,d\eta'.$$

Here we perform the change of variables $x\longmapsto x-\nabla_\eta\phi(x',\eta)$. The last expression will be

$$\le \sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d_{\eta'}}\int_{\mathbb{R}^d_x} \sup_{u\in\mathbb{R}^{2d}}\sup_{x'\in\mathbb{R}^d} |V_{\Psi_0}\sigma(u;\eta'-\nabla_x\Phi(x',\eta),x)|\,dx\,d\eta'.$$

Now we observe that

$$\nabla_x\Phi(x',\eta)=\nabla_x\Phi(0,\eta)+A(x',\eta),$$

where, by the assumption (1.6), $|A(x',\eta)|\le R$ for some $R>0$. By Proposition 3.6 we can continue the majorization as

$$\lesssim \sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d_{\eta'}}\int_{\mathbb{R}^d_x} \sup_{u\in\mathbb{R}^{2d}}\int_{R^d_v} \varphi_{\eta'-\nabla_x\Phi(0,\eta),R}(v)|V_{\Psi_0}\sigma(u;v,x)|\,dv\,dx\,d\eta'.$$

Now we bring the supremum with respect to u inside the more interior integral and perform the change of variables $\eta' \longmapsto \eta' - \nabla_x \phi(0,\eta)$, obtaining

$$\leq \int_{\mathbb{R}^d_{\eta'}} \int_{\mathbb{R}^d_x} \int_{R^d_v} \varphi_{\eta',R}(v) \sup_{u\in\mathbb{R}^{2d}} |V_{\Psi_0}\sigma(u;v,x)|\,dv\,dx\,d\eta'.$$

Finally we exchange the two more interior integral and apply Lemma 3.5. The last expression is seen to be

$$= C_R \int_{\mathbb{R}^d_{\eta'}} \int_{\mathbb{R}^d_x} \sup_{u\in\mathbb{R}^{2d}} |V_{\Psi_0}\sigma(u;\eta',x)|\,dx\,d\eta' = C_R\|\sigma\|_{M^{\infty,1}}.$$

We now prove (3.8). Here we will use repeatedly the Remark 3.1. By arguing as above we are reduced to proving that

$$\sup_{\eta'\in\mathbb{R}^d} \int_{\mathbb{R}^d_\eta} \sup_{x\in\mathbb{R}^d} \int_{R^d_{x'}} \sup_{u\in\mathbb{R}^{2d}} |V_{\Psi_0}\sigma(u;\eta' - \nabla_x\Phi(x',\eta), x - \nabla_\eta\Phi(x',\eta))|\,dx'\,d\eta \\ \leq C\|\sigma\|_{M^{\infty,1}}. \tag{3.10}$$

By performing the change of variables $x' \longmapsto \nabla_\eta\Phi(x',\eta)$ and a subsequent translation, the left-hand side in (3.10) is seen to be

$$\lesssim \sup_{\eta'\in\mathbb{R}^d} \int_{\mathbb{R}^d_\eta} \sup_{x\in\mathbb{R}^d} \int_{R^d_{x'}} \sup_{u\in\mathbb{R}^{2d}} |V_{\Psi_0}\sigma(u;\eta' - \nabla_x\Phi(B(x'+x,\eta),\eta), x')|\,dx'\,d\eta,$$

for a suitable function B coming from the inverse change of variables. Now, by the assumption (1.6) we have $\nabla_x\Phi(B(x'+x,\eta),\eta) = \nabla_x\Phi(0,\eta) + A'(x,x',\eta)$, with $|A'(x,x',\eta)| \leq R$ for a suitable $R > 0$. It turns out that the last expression is

$$\leq \sup_{\eta'\in\mathbb{R}^d} \int_{\mathbb{R}^d_\eta} \int_{R^d_{x'}} \sup_{u\in\mathbb{R}^{2d}} \sup_{v\in B(0,R)} |V_{\Psi_0}\sigma(u;\eta' - \nabla_x\Phi(0,\eta) + v, x')|\,dx'\,d\eta.$$

By the Proposition 3.6 this is

$$\lesssim \sup_{\eta'\in\mathbb{R}^d} \int_{\mathbb{R}^d_\eta} \int_{R^d_{x'}} \sup_{u\in\mathbb{R}^{2d}} \int_{\mathbb{R}^d_v} \varphi_{\eta'-\nabla_x\Phi(0,\eta),R}(v)|V_{\Psi_0}\sigma(u;v,x')|\,dv\,dx'\,d\eta.$$

Now we perform the change of variable $\eta \longmapsto \nabla_x\Phi(0,\eta)$, and a subsequent translation, obtaining

$$\lesssim \int_{\mathbb{R}^d_\eta} \int_{R^d_{x'}} \sup_{u\in\mathbb{R}^{2d}} \int_{\mathbb{R}^d_v} \varphi_{\eta,R}(v)|V_{\Psi_0}\sigma(u;v,x')|\,dv\,dx'\,d\eta.$$

Finally we can bring the supremum with respect to u inside the more interior integral, exchange the integrals with respect to v and η' and apply Lemma 3.5, obtaining

$$\le C_R \int_{\mathbb{R}^d_\eta} \int_{R^d_{x'}} \sup_{u\in\mathbb{R}^{2d}} |V_{\Psi_0}\sigma(u;\eta,x')|\,dx'\,d\eta = C_R\|\sigma\|_{M^{\infty,1}},$$

as desired. □

We now prove Theorem 1.2. To this end, we first consider the cases $(p,q) = (\infty,1)$ (Theorem 3.7) and $(p,q) = (1,\infty)$ (Theorem 3.8).

Theorem 3.7 *Consider a phase* Φ *satisfying* (i), (ii) *and* (iii), *and a symbol* $\sigma \in M^{\infty,1}_{v_{s,0}\otimes 1}(\mathbb{R}^{2d})$, *with* $s > d$. *Then the corresponding FIO* T *extends to a bounded operator* $\mathcal{M}^{\infty,1}(\mathbb{R}^d) \to \mathcal{M}^{\infty,1}(\mathbb{R}^d)$.

Proof By arguing as at the beginning of the proof of Theorem 1.1, we see that it suffices to prove the estimate

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d_{\eta'}} \sup_{x'\in\mathbb{R}^d} \int_{\mathbb{R}^d_x} |V_{\Psi_0}\sigma(x',\eta;\eta' - \nabla_x\Phi(x',\eta), x - \nabla_\eta\Phi(x',\eta))|\,dx\,d\eta'$$
$$\le C\|\sigma\|_{M^{\infty,1}_{v_{s,0}\otimes 1}}.$$

The left-hand side of this estimate is seen to be

$$\le \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d_{\eta'}} \sup_{x'\in\mathbb{R}^d} \int_{\mathbb{R}^d_x} \sup_{(u_1,u_2)\in\mathbb{R}^{2d}} \langle x'\rangle^{-s}$$
$$\times |\langle u_1\rangle^s V_{\Psi_0}\sigma(u_1,u_2;\eta' - \nabla_x\Phi(x',\eta), x - \nabla_\eta\Phi(x',\eta))|\,dx\,d\eta',$$

which coincides with

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d_{\eta'}} \sup_{x'\in\mathbb{R}^d} \int_{\mathbb{R}^d_x} \sup_{(u_1,u_2)\in\mathbb{R}^{2d}} \langle x'\rangle^{-s}$$
$$\times |\langle u_1\rangle^s V_{\Psi_0}\sigma(u_1,u_2;\eta' - \nabla_x\Phi(x',\eta), x)|\,dx\,d\eta'$$
$$\le \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d_{\eta'}} \int_{\mathbb{R}^d_x} \sup_{(u_1,u_2)\in\mathbb{R}^{2d}} \langle u_1\rangle^s \sup_{x'\in\mathbb{R}^d} \langle x'\rangle^{-s}$$
$$\times |V_{\Psi_0}\sigma(u_1,u_2;\eta' - \nabla_x\Phi(x',\eta), x)|\,dx\,d\eta'.$$

Now we apply Lemma 2.2 to the function

$$f(\zeta) := V_{\Psi_0}\sigma(u_1,u_2;\eta' - \nabla_x\Phi(0,\eta) - \zeta, x)$$

with $v(x') = \nabla_x\Phi(x',\eta) - \nabla_x\Phi(0,\eta)$ (so that $f(v(x')) = V_{\Psi_0}\sigma(u_1,u_2;\eta' - \nabla_x\Phi(x',\eta), x)$). The assumptions are indeed satisfied uniformly with respect to all parameters; in particular

$$|v(x')| = |\nabla_x\Phi(x',\eta) - \nabla_x\Phi(0,\eta)| \le C|x'|$$

for every $(x',\eta) \in \mathbb{R}^{2d}$ by (ii).

It follows that we can continue our majorization as

$$\begin{aligned}
&\lesssim \sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d_{\eta'}}\int_{\mathbb{R}^d_x}\sup_{(u_1,u_2)\in\mathbb{R}^{2d}}\langle u_1\rangle^s\int_{\mathbb{R}^d_{x'}}\langle x'\rangle^{-s}\\
&\qquad\times|V_{\Psi_0}\sigma(u_1,u_2;\eta'-\nabla_x\Phi(0,\eta)-x',x)|\,dx'\,dx\,d\eta'\\
&\le \sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d_{x'}}\langle x'\rangle^{-s}\int_{\mathbb{R}^d_{\eta'}}\int_{\mathbb{R}^d_x}\sup_{(u_1,u_2)\in\mathbb{R}^{2d}}\langle u_1\rangle^s\\
&\qquad\times|V_{\Psi_0}\sigma(u_1,u_2;\eta'-\nabla_x\Phi(0,\eta)-x',x)|\,dx\,d\eta'\,dx'.
\end{aligned}$$

By performing the translation $\eta'\longmapsto\eta'-\nabla_x\Phi(0,\eta)-x'$ and using the integrability condition $s>d$ one sees that this last expression is

$$=C\int_{\mathbb{R}^d_{\eta'}}\int_{\mathbb{R}^d_x}\sup_{(u_1,u_2)\in\mathbb{R}^{2d}}\langle u_1\rangle^s|V_{\Psi_0}\sigma(u_1,u_2;\eta',x)|\,dx\,d\eta'=C\|\sigma\|_{M^{\infty,1}_{v_{s,0}\otimes1}}.$$

This concludes the proof. □

Theorem 3.8 *Consider a phase* Φ *satisfying* (i), (ii), *and* (iii) *and a symbol* $\sigma\in M^{\infty,1}_{v_{0,s}\otimes1}(\mathbb{R}^{2d})$, *with* $s>d$. *Then, the corresponding FIO* T *extends to a bounded operator* $\mathcal{M}^{1,\infty}(\mathbb{R}^d)\to\mathcal{M}^{1,\infty}(\mathbb{R}^d)$.

Proof By arguing as at the beginning of the proof of Theorem 3.7, we see that it suffices to prove the estimate

$$\begin{aligned}
&\sup_{\eta'\in\mathbb{R}^d}\int_{\mathbb{R}^d_\eta}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d_{x'}}|V_{\Psi_0}\sigma(x',\eta;\eta'-\nabla_x\Phi(x',\eta),x-\nabla_\eta\Phi(x',\eta))|\,dx'\,d\eta\\
&\quad\le C\|\sigma\|_{M^{\infty,1}_{v_{0,s}\otimes1}}.
\end{aligned}$$

The left-hand side of this estimate is seen to be

$$\begin{aligned}
&\le \sup_{\eta'\in\mathbb{R}^d}\int_{\mathbb{R}^d_\eta}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d_{x'}}\sup_{(u_1,u_2)\in\mathbb{R}^{2d}}\langle\eta\rangle^{-s}\\
&\qquad\times|\langle u_2\rangle^s V_{\Psi_0}\sigma(u_1,u_2;\eta'-\nabla_x\Phi(x',\eta),x-\nabla_\eta\Phi(x',\eta))|\,dx'\,d\eta.
\end{aligned}$$

By performing the change of variables $x'\longmapsto\nabla_\eta\Phi(x',\eta)$ (see Remark 3.1) and a subsequent translation, we obtain

$$\begin{aligned}
&\le \sup_{\eta'\in\mathbb{R}^d}\int_{\mathbb{R}^d_\eta}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d_{x'}}\sup_{(u_1,u_2)\in\mathbb{R}^{2d}}\langle\eta\rangle^{-s}\\
&\qquad\times|\langle u_2\rangle^s V_{\Psi_0}\sigma(u_1,u_2;\eta'-\nabla_x\Phi(B(x+x',\eta),\eta),x')|\,dx'\,d\eta,
\end{aligned}$$

for a suitable function $B(x', \eta)$ coming from the inverse change of variables. Bringing the supremum over η' inside,

$$\leq \int_{\mathbb{R}^d_\eta} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d_{x'}} \sup_{(u_1,u_2)\in\mathbb{R}^{2d}} \langle\eta\rangle^{-s} \\ \times \langle u_2\rangle^s \sup_{\eta'\in\mathbb{R}^d} |V_{\Psi_0}\sigma(u_1, u_2; \eta' - \nabla_x\Phi(B(x + x', \eta), \eta), x')|\, dx'\, d\eta. \quad (3.11)$$

Using Bernstein's inequality (2.1):

$$\sup_{\eta'\in\mathbb{R}^d} |f(\eta')| \lesssim \int_{\mathbb{R}^d} |f(\eta')|\, d\eta',$$

for $f(\eta') = V_{\Psi_0}\sigma(u_1, u_2; \eta' - \nabla_x\Phi(B(x + x', \eta), \eta), x')$, and the translation invariance of the Lebesgue measure $d\eta'$, the expression (3.11) is less than

$$\int_{\mathbb{R}^d_\eta} \langle\eta\rangle^{-s}\, d\eta \int_{\mathbb{R}^d_{x'}} \int_{\mathbb{R}^d_{\eta'}} \sup_{(u_1,u_2)\in\mathbb{R}^{2d}} \langle u_2\rangle^s |V_{\Psi_0}\sigma(u_1, u_2; \eta', x')|\, dx'\, d\eta'.$$

Using the integrability condition $s > d$, this last expression is equal to

$$C \int_{\mathbb{R}^d_{x'}} \int_{\mathbb{R}^d_{\eta'}} \sup_{(u_1,u_2)\in\mathbb{R}^{2d}} \langle u_2\rangle^s |V_{\Psi_0}\sigma(u_1, u_2; \eta', x')|\, dx'\, d\eta = C\|\sigma\|_{M^{\infty,1}_{v_{0,s}\otimes 1}},$$

as desired. □

Proof of Theorem 1.2 Item (i) was already proved in [7, Theorem 6.1], see also [4, Theorem 2.1].

Items (ii)–(iii). By Proposition 2.8, the conclusion in Theorem 1.2 is equivalent to saying that any FIO $\tilde{T}$, with phase Φ satisfying (i), (ii) and (iii) and symbol $\tilde{\sigma} \in M^{\infty,1}(\mathbb{R}^{2d})$, extends to a bounded operator $\mathcal{M}^{p,q}_{v_{0,s_2}}(\mathbb{R}^d) \to \mathcal{M}^{p,q}_{v_{-s_1,0}}(\mathbb{R}^d)$, for s_1, s_2 as in the statement, and moreover

$$\|\tilde{T} f\|_{\mathcal{M}^{p,q}_{v_{-s_1,0}}} \leq C\|\tilde{\sigma}\|_{M^{\infty,1}}\|f\|_{\mathcal{M}^{p,q}_{v_{0,s_2}}}.$$

Since this was already proved for $(p, q) = (\infty, 1)$, for all $1 \leq p = q \leq \infty$, and for $(p, q) = (1, \infty)$, the desired result follows from Proposition 2.6. □

Theorem 1.2 has the following counterpart in the framework of Wiener amalgam spaces. Of course, we are interested in $\mathcal{W}(\mathcal{F}L^p, L^q)$, with $p \neq q$, since $\mathcal{W}(\mathcal{F}L^p, L^p) = \mathcal{M}^p$.

Corollary 3.9 *Consider a phase Φ and a symbol σ as in Theorem 1.2.*

(i) *Let $1 \leq q < p \leq \infty$. If $s_1 > d(\frac{1}{q} - \frac{1}{p})$, $s_2 \geq 0$, T extends to a bounded operator on $\mathcal{W}(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$.*

(ii) *Let* $1 \le p < q \le \infty$. *If* $s_1 \ge 0$, $s_2 > d(\frac{1}{p} - \frac{1}{q})$, T *extends to a bounded operator on* $\mathcal{W}(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$.

In all cases,

$$\|Tf\|_{\mathcal{W}(\mathcal{F}L^p,L^q)} \lesssim \|\sigma\|_{M^{\infty,1}_{v_{s_1,s_2}\otimes 1}} \|f\|_{\mathcal{W}(\mathcal{F}L^p,L^q)}. \tag{3.12}$$

Proof The result easily follows from Theorem 1.2. Indeed, consider an operator T, with symbol σ and phase Φ, satisfying the assumptions of Corollary 3.9. Conjugating with the Fourier transform yields the operator

$$\tilde{T} f(x) = \mathcal{F} \circ T \circ \mathcal{F}^{-1} f(x).$$

Since $\mathcal{M}^{p,q} = \mathcal{F}^{-1}\mathcal{W}(\mathcal{F}L^p, L^q)$, it suffices to prove that $\tilde{T}$ extends to a bounded operator on $\mathcal{M}^{p,q}$. By duality and an explicit computation this is equivalent to verifying that the operator

$$\tilde{T}^* f(x) = \int_{\mathbb{R}^d} e^{-2\pi i \Phi(\eta,x)} \overline{\sigma(\eta,x)} \hat{f}(\eta)\, d\eta$$

extends to a bounded operator on $\mathcal{M}^{p',q'}$. Since $\sigma \in M^{\infty,1}_{v_{s_1,s_2}\otimes 1}(\mathbb{R}^{2d})$ implies that $\sigma^*(x,\eta) = \overline{\sigma(\eta,x)} \in M^{\infty,1}_{v_{s_2,s_1}\otimes 1}(\mathbb{R}^{2d})$ by Lemma 2.10, the desired result is attained from Theorem 1.2. □

4 Boundedness of FIOs on Weighted $\mathcal{M}^{p,q}$

Thanks to the commutativity of the diagram (2.7), the results of Theorem 1.2 may be equivalently stated as the action of a FIO T on weighted modulation spaces. Namely, we have the following result.

Theorem 4.1 *Consider a phase* Φ *satisfying* (i), (ii) *and* (iii), *and a symbol* $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$.

(i) *If* $1 \le q < p \le \infty$, $s_1 < -d(\frac{1}{q} - \frac{1}{p})$, *and* $s_2 \ge 0$, *then* T *extends to a bounded operator from* $\mathcal{M}^{p,q}_{v_{0,s_2}}(\mathbb{R}^d)$ *to* $\mathcal{M}^{p,q}_{v_{s_1,0}}(\mathbb{R}^d)$.

(ii) *If* $1 \le p < q \le \infty$, $s_1 \le 0$, *and* $s_2 > d(\frac{1}{p} - \frac{1}{q})$, T *extends to a bounded operator from* $\mathcal{M}^{p,q}_{v_{0,s_2}}(\mathbb{R}^d)$ *to* $\mathcal{M}^{p,q}_{v_{s_1,0}}(\mathbb{R}^d)$.

In both cases,

$$\|Tf\|_{\mathcal{M}^{p,q}_{v_{s_1,0}}} \lesssim \|\sigma\|_{M^{\infty,1}} \|f\|_{\mathcal{M}^{p,q}_{v_{0,s_2}}}.$$

One can easily rephrase the results of Corollary 3.9 in term of weighted Wiener amalgam spaces.

We now study the weighted cases not contained above.

Theorem 4.2 *Consider a phase* Φ *satisfying* (i), (ii) *and* (iii), *and a symbol* $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$. *Then, the corresponding FIO* T *extends to a bounded operator between the following spaces*:

$$\mathcal{M}^{\infty,1}(\mathbb{R}^d) \to \mathcal{M}^{\infty,1}_{v_{0,s}}(\mathbb{R}^d), \qquad \mathcal{M}^{1,\infty}(\mathbb{R}^d) \to \mathcal{M}^{1,\infty}_{v_{s,0}}(\mathbb{R}^d),$$

with $s < -d$, *and its norm is bounded from above by* $C\|\sigma\|_{M^{\infty,1}}$, *for a suitable* $C > 0$.

Proof To prove the boundedness of T from $\mathcal{M}^{\infty,1}(\mathbb{R}^d)$ to $\mathcal{M}^{\infty,1}_{v_{0,s}}(\mathbb{R}^d)$, we have to show that the integral kernel

$$K_T(x', \eta'; x, \eta) = \langle T g_{x,\eta}, g_{x',\eta'} \rangle \langle \eta' \rangle^s$$

satisfies $K_T \in L^\infty_\eta L^1_{\eta'} L^\infty_{x'} L^1_x$. The arguments are similar to those of Theorem 3.7, we sketch them for sake of clarity. The quantity

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d_{\eta'}} \sup_{x'\in\mathbb{R}^d} \int_{\mathbb{R}^d_x} |V_{\Psi_0}\sigma(x', \eta; \eta' - \nabla_x\Phi(x', \eta), x - \nabla_\eta\Phi(x', \eta))| \langle \eta' \rangle^s \, dx \, d\eta'$$

can be controlled from above by

$$\begin{aligned}
&C \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d_{\eta'}} \sup_{x'\in\mathbb{R}^d} \int_{\mathbb{R}^d_x} \sup_{(u_1,u_2)\in\mathbb{R}^{2d}} |V_{\Psi_0}\sigma(u_1, u_2; \eta' - \nabla_x\Phi(x', \eta), x)| \langle \eta' \rangle^s dx\, d\eta', \\
&\quad \le C \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d_{\eta'}} \sup_{x'\in\mathbb{R}^d} \int_{\mathbb{R}^d_x} \sup_{(u_1,u_2)\in\mathbb{R}^{2d}} \sup_{\zeta\in\mathbb{R}^d} |V_{\Psi_0}\sigma(u_1, u_2; \zeta, x)| \langle \eta' \rangle^s dx\, d\eta' \\
&\quad \le C' \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d_{\eta'}} \sup_{x'\in\mathbb{R}^d} \int_{\mathbb{R}^d_x} \sup_{(u_1,u_2)\in\mathbb{R}^{2d}} \int_{\mathbb{R}^d_\zeta} |V_{\Psi_0}\sigma(u_1, u_2; \zeta, x)| d\zeta \, \langle \eta' \rangle^s dx\, d\eta',
\end{aligned}$$

where in the last estimate we applied Bernstein's inequality (2.1) to the function $f(\zeta) = V_{\Psi_0}\sigma(u_1, u_2; \zeta, x)$. The last estimate, since $s < -d$, is dominated from above by $\|\sigma\|_{M^{\infty,1}}$, as desired.

Similarly, to show $T\colon \mathcal{M}^{1,\infty} \to \mathcal{M}^{1,\infty}_{v_{s,0}}$, is equivalent to proving that the integral kernel

$$K_T(x', \eta'; x, \eta) = \langle T g_{x,\eta}, g_{x',\eta'} \rangle \langle x' \rangle^s$$

satisfies $K_T \in L^\infty_{\eta'} L^1_\eta L^\infty_x L^1_{x'}$. The arguments are quite similar to those of Theorem 3.8. Again, we sketch the proof for sake of clarity. By performing the change of variables $\eta \longmapsto \nabla_x\Phi(x', \eta)$, the quantity

$$\sup_{\eta'\in\mathbb{R}^d} \int_{\mathbb{R}^d_{x'}} \int_{\mathbb{R}^d_\eta} \sup_{x\in\mathbb{R}^d} |V_{\Psi_0}\sigma(x', \eta; \eta' - \nabla_x\Phi(x', \eta), x - \nabla_\eta\Phi(x', \eta))| \langle x' \rangle^s \, d\eta \, dx'$$

is less than

$$\begin{aligned}
&\sup_{\eta'\in\mathbb{R}^d} \int_{\mathbb{R}^d_{x'}} \int_{\mathbb{R}^d_\eta} \sup_{x\in\mathbb{R}^d} \sup_{(u_1,u_2)\in\mathbb{R}^{2d}} |V_{\Psi_0}\sigma(u_1, u_2; \eta, x - \nabla_\eta\Phi(x', B(x', \eta + \eta')))| \\
&\qquad \times \langle x' \rangle^s \, dx' \, d\eta,
\end{aligned}$$

for a suitable function $B(x',\eta)$ coming from the inverse change of variables. The result is attained using Bernstein's inequality (2.1): $\sup_{x\in\mathbb{R}^d}|f(x)|\lesssim\int_{\mathbb{R}^d}|f(x)|\,dx$, for $f(x)=V_{\Psi_0}\sigma(u_1,u_2;\eta,x)$, and the condition $s<-d$. □

Since the boundedness of the FIO T on $\mathcal{M}^p$ is provided by Theorem 1.2, the complex interpolation with the preceding results yields:

Theorem 4.3 *Consider a phase* Φ *satisfying* (i), (ii) *and* (iii), *a symbol* $\sigma\in M^{\infty,1}(\mathbb{R}^d)$.

(i) *If* $1\le q<p\le\infty$, $s_1\ge 0$, *and* $s_2<-d(\frac{1}{q}-\frac{1}{p})$, *then* T *extends to a bounded operator from* $\mathcal{M}^{p,q}_{v_{s_1,0}}(\mathbb{R}^d)$ *to* $\mathcal{M}^{p,q}_{v_{0,s_2}}(\mathbb{R}^d)$.

(ii) *If* $1\le p<q\le\infty$, $s_1<-d(\frac{1}{p}-\frac{1}{q})$, *and* $s_2\ge 0$, *then* T *extends to a bounded operator from* $\mathcal{M}^{p,q}_{v_{0,s_2}}(\mathbb{R}^d)$ *to* $\mathcal{M}^{p,q}_{v_{s_1,0}}(\mathbb{R}^d)$.

In both cases the norm of T *is bounded from above by* $C\|\sigma\|_{M^{\infty,1}}$, *for a suitable* $C>0$.

The results for Wiener amalgam spaces are obtained by similar arguments as those in Corollary 3.9 and left to the reader.

5 Boundedness of FIOs $\mathcal{M}^{p,q}\to\mathcal{M}^{q,p}$, $p\ge q$

In this section we shall prove the boundedness of an operator T between $\mathcal{M}^{p,q}\to\mathcal{M}^{q,p}$, namely Theorem 1.3. As a byproduct, conditions for the boundedness from $W(\mathcal{F}L^p,L^q)$ to $W(\mathcal{F}L^q,L^p)$ are attained as well (Corollary 5.2). By using complex interpolation, as in the proof of Theorem 1.2, we see that it suffices to prove the desired results for $(p,q)=(\infty,1)$:

Theorem 5.1 *Consider a phase* $\Phi(x,\eta)$ *satisfying* (i) *and* (ii). *Moreover, assume one of the following conditions*:

(a) *the symbol* $\sigma\in M^{\infty,1}(\mathbb{R}^{2d})$ *and, for some* $\delta>0$,

$$\left|\det\left(\frac{\partial^2\Phi}{\partial x_i\partial x_l}\bigg|_{(x,\eta)}\right)\right|\ge\delta\quad\forall(x,\eta)\in\mathbb{R}^{2d},\tag{5.1}$$

(b) *the symbol* $\sigma\in M^{\infty,1}_{v_{s,0}\otimes 1}(\mathbb{R}^{2d})$, *with* $s>d$.

Then, the corresponding FIO T *extends to a bounded operator* $\mathcal{M}^{\infty,1}(\mathbb{R}^d)\to\mathcal{M}^{1,\infty}(\mathbb{R}^d)$.

Proof We argue as in the first part of the proof of Theorem 1.1. We see that it suffices to prove that the map K_T defined by

$$K_TG(x',\eta')=\int_{\mathbb{R}^{2d}}\langle Tg_{x,\eta},g_{x',\eta'}\rangle G(x,\eta)\,dx\,d\eta$$

is continuous $L^1_\eta L^\infty_x \to L^\infty_\eta L^1_x$. By Proposition 2.4 it suffices to prove that its integral kernel

$$K_T(x', \eta'; x, \eta) = \langle T g_{x,\eta}, g_{x',\eta'} \rangle$$

satisfies

$$K_T \in L^\infty_{\eta,\eta'} L^1_{x,x'}. \tag{5.2}$$

(a) The same arguments as in the proof of Theorem 1.1 show that it is enough to verify the estimate

$$\begin{aligned}&\sup_{(\eta,\eta')\in\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}_{x,x'}} \sup_{u\in\mathbb{R}^{2d}} |V_{\Psi_0}\sigma(u; \eta' - \nabla_x\Phi(x', \eta), x - \nabla_\eta\Phi(x', \eta))|\, dx\, dx' \\ &\quad\le C\|\sigma\|_{M^{\infty,1}}.\end{aligned} \tag{5.3}$$

To this end, we first perform the translation $x \to x - \nabla_\eta\Phi(x', \eta)$ in the left-hand side of (5.3), obtaining

$$\sup_{(\eta,\eta')\in\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}_{x,x'}} \sup_{u\in\mathbb{R}^{2d}} |V_{\Psi_0}\sigma(u; \eta' - \nabla_x\Phi(x', \eta), x)|\, dx\, dx'.$$

Then we perform the change of variables $x' \longmapsto \nabla_x\Phi(x', \eta)$ (followed by a translation) which by Remark 3.1 is a global diffeomorphism of $\mathbb{R}^d$, and whose inverse Jacobian determinant is uniformly bounded with respect to all variables. Hence the last expression is seen to be

$$\lesssim \int_{\mathbb{R}^d_{x'}} \int_{\mathbb{R}^d_x} \sup_{u\in\mathbb{R}^{2d}} |V_{\Psi_0}\sigma(u; x', x)|\, dx\, dx' = \|\sigma\|_{M^{\infty,1}}.$$

(b) Similar arguments as above show that the result is attained once we prove the estimate

$$\begin{aligned}&\sup_{(\eta,\eta')\in\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}_{x,x'}} |V_{\Psi_0}\sigma(x', \eta; \eta' - \nabla_x\Phi(x', \eta), x - \nabla_\eta\Phi(x', \eta))|\, dx\, dx' \\ &\quad\le C\|\sigma\|_{M^{\infty,1}_{v_{s,0}\otimes 1}}.\end{aligned} \tag{5.4}$$

For $s \ge 0$, and performing the change of variables $x \longmapsto x - \nabla_\eta\Phi(x', \eta)$, the left-hand side above is controlled by

$$\begin{aligned}&\le \sup_{(\eta,\eta')\in\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}_{x,x'}} \langle x'\rangle^{-s} \sup_{(u_1,u_2)\in\mathbb{R}^{2d}} \langle u_1\rangle^s |V_{\Psi_0}\sigma(u_1, u_2; \eta' - \nabla_x\Phi(x', \eta), x)|\, dx\, dx' \\ &\le \sup_{\eta'\in\mathbb{R}^d} \iint_{\mathbb{R}^{2d}_{x,x'}} \langle x'\rangle^{-s} \sup_{(u_1,u_2)\in\mathbb{R}^{2d}} \langle u_1\rangle^s \\ &\quad\times \sup_{\eta\in\mathbb{R}^d} |V_{\Psi_0}\sigma(u_1, u_2; \eta' - \nabla_x\Phi(x', \eta), x)|\, dx\, dx'.\end{aligned}$$

Now we apply Lemma 2.2 to the function

$$f(\zeta) := V_{\Psi_0}\sigma(u_1, u_2; \eta' - \nabla_x\Phi(x', 0) - \zeta, x)$$

with $v(\eta) = \nabla_x\Phi(x', \eta) - \nabla_x\Phi(x', 0)$ (so that $f(v(\eta)) = V_{\Psi_0}\sigma(u_1, u_2; \eta' - \nabla_x\Phi(x', \eta), x)$). The assumptions are indeed satisfied uniformly with respect to all parameters; in particular $|v(\eta)| \leq C|\eta|$, for every $(x', \eta) \in \mathbb{R}^{2d}$ by (ii).

Continuing our majorizations, we obtain

$$\begin{aligned}
&\lesssim \sup_{\eta'\in\mathbb{R}^d} \int_{\mathbb{R}^d_{\eta'}} \int_{\mathbb{R}^d_x} \sup_{(u_1,u_2)\in\mathbb{R}^{2d}} \langle u_1\rangle^s \int_{\mathbb{R}^d_{x'}} \langle x'\rangle^{-s} \\
&\quad \times |V_{\Psi_0}\sigma(u_1, u_2; \eta' - \nabla_x\Phi(x', 0) - \eta, x)|\, d\eta\, dx'\, dx \\
&\leq \sup_{\eta'\in\mathbb{R}^d} \int_{\mathbb{R}^d_{x'}} \langle x'\rangle^{-s} \int_{\mathbb{R}^d_\eta} \int_{\mathbb{R}^d_x} \sup_{(u_1,u_2)\in\mathbb{R}^{2d}} \langle u_1\rangle^s |V_{\Psi_0}\sigma(u_1, u_2; \eta, x)|\, d\eta\, dx\, dx'
\end{aligned}$$

where in the last row we perform the translation (up to a sign) $\eta \longmapsto \eta' - \nabla_x\Phi(x', 0) - \eta$ and using the integrability condition $s > d$ one sees that this last expression is dominated by $\|\sigma\|_{M^{\infty,1}_{v_{s,0}\otimes 1}}$. □

From Theorem 1.3 and using the same arguments as in Corollary 3.9, one can prove the following result.

Corollary 5.2 *Let $1 \leq q < p \leq \infty$. Consider a phase $\Phi(x, \eta)$ satisfying* (i), (ii) *and* (iii). *Moreover, assume one of the following conditions*:

(a) *the symbol $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ and, for some $\delta > 0$,*

$$\left|\det\left(\frac{\partial^2\Phi}{\partial\eta_i\partial\eta_l}\bigg|_{(x,\eta)}\right)\right| \geq \delta \quad \forall (x, \eta) \in \mathbb{R}^{2d}, \tag{5.5}$$

or

(b) *the symbol $\sigma \in M^{\infty,1}_{v_{0,s}\otimes 1}(\mathbb{R}^{2d})$, with $s > d(\frac{1}{q} - \frac{1}{p})$.*

Then, the corresponding FIO T extends to a bounded operator $\mathcal{W}(\mathcal{F}L^p, L^q)(\mathbb{R}^d) \to \mathcal{W}(\mathcal{F}L^q, L^p)(\mathbb{R}^d)$.

Example 5.3 (Metaplectic operators) Consider the particular case of quadratic phases, namely phases of the type

$$\Phi(x, \eta) = \frac{1}{2}Ax \cdot x + Bx \cdot \eta + \frac{1}{2}C\eta \cdot \eta + \eta_0 \cdot x - x_0 \cdot \eta, \tag{5.6}$$

where $x_0, \eta_0 \in \mathbb{R}^d$, A, C are real symmetric $d \times d$ matrices and B is a real $d \times d$ nondegenerate matrix.

It is easy to see that, if we take the symbol $\sigma \equiv 1$ and the phase (5.6), the corresponding FIO T is (up to a constant factor) a metaplectic operator. This can be seen by means of the easily verified factorization

$$T = M_{\eta_0} U_A D_B \mathcal{F}^{-1} U_C \mathcal{F} T_{x_0}, \tag{5.7}$$

where U_A and U_C are the multiplication operators by $e^{\pi i A x\cdot x}$ and $e^{\pi i C\eta\cdot\eta}$ respectively, and D_B is the dilation operator $f \mapsto f(B\cdot)$. Each of the factors is (up to a constant factor) a metaplectic operator (see e.g. the proof of [20, Theorem 18.5.9]), so T is. We refer to [5, Proposition 2.7(ii)] and [7, Sect. 7] for details about the symplectic matrix which yields such an operator.

We see that the assumptions (i) and (ii) in the Introduction are clearly satisfied, whereas the hypotheses (iii) in the Introduction, (1.8) and (5.5) are equivalent to $\det B \neq 0$, $\det A \neq 0$ and $\det C \neq 0$ respectively. In particular, the first part of Corollary 5.2 generalizes [5, Theorem 4.1].

6 Some Counterexamples Related to the Schrödinger multiplier

In this section, we study the action of the Scrödinger multiplier

$$f \longmapsto \mathcal{F}^{-1}\left(e^{i\pi|\cdot|^2}\hat{f}\right),$$

that is the multiplier with symbol $e^{i\pi|\eta|^2}$, on the Wiener amalgam spaces $W(\mathcal{F}L^p, L^q_s)$. Equivalently, we study the action of the pointwise multiplication operator

$$Af(x) = e^{i\pi|x|^2} f(x), \quad x \in \mathbb{R}^d \tag{6.1}$$

on the modulation spaces $\mathcal{M}^{p,q}_{v_{0,s}}$. This will provide useful examples to test the sharpness of the thresholds arising in our results. Notice that A is the FIO with phase $\Phi(x,\eta) = x\eta + \frac{|\eta|^2}{2}$ and symbol $\sigma \equiv 1$. In particular, it satisfies (1.8).

It was shown in [7, Proposition 7.1] that A is bounded on $\mathcal{M}^{p,q}$ (if and) only if $p = q$ (for a proof of the positive results see [2, 5, 7]). It is natural to wonder whether for suitable negative values of s it is bounded as an operator $\mathcal{M}^{p,q} \to \mathcal{M}^{p,q}_{v_{0,s}}$. The next result deals with the optimal range of s for this to happen.

Proposition 6.1 *If the operator A in (6.1) is bounded as an operator $\mathcal{M}^{p,q}(\mathbb{R}^d) \to \mathcal{M}^{p,q}_{v_{0,s}}(\mathbb{R}^d)$, for some $1 \le p, q \le \infty$, $s \in \mathbb{R}$, then $p \ge q$ and $s \le -d(1/q - 1/p)$. Moreover, if $p = \infty$ and $q < \infty$, then $s < -d/q$.*

Proof We will prove later on that $p \ge q$. Let us verify now the remaining conditions. We test the estimate

$$\|Af\|_{M^{p,q}_{v_{0,s}}} \lesssim \|f\|_{M^{p,q}} \tag{6.2}$$

on the family of functions $f_\lambda(x) = e^{-\pi\lambda|x|^2}$, $\lambda > 0$. Applying Lemma 2.9 with $a = \lambda$, $b = 0$ we see that

$$\|f_\lambda\|_{M^{p,q}} \asymp \|G_{a+ib}\|_{W(\mathcal{F}L^p, L^q)} \asymp \lambda^{-\frac{d}{2p}}, \quad \text{as } \lambda \to 0^+. \tag{6.3}$$

On the other hand, with the notation of Lemma 2.9, we have

$$\|Af_\lambda\|_{M^{p,q}_{v_{0,s}}} \asymp \|G_{a+ib}\|_{W(\mathcal{F}L^p,L^q_s)}$$

with $a=\lambda$, $b=-1$. We now estimate this last expression. We see that, by (2.10),

$$\|G_{a+ib}T_x g\|_{\mathcal{F}L^p} \asymp e^{-\frac{\pi\lambda|x|^2}{1+\lambda+\lambda^2}}, \quad \lambda\leq 1.$$

Let $\mu=\lambda/(1+\lambda+\lambda^2)$ (observe that $\mu\sim\lambda$ and $\log\mu\sim\log\lambda$ as $\lambda\to 0^+$). Then, for $q<\infty$,

$$\begin{aligned}\|G_{a+ib}\|_{W(\mathcal{F}L^p,L^q_s)} &\asymp \left(\int e^{-\pi q\mu|x|^2}(1+|x|)^{qs}\,dx\right)^{1/q}\\ &=\mu^{-\frac{d}{2q}}\left(\int e^{-\pi q|x|^2}(1+\mu^{-\frac{1}{2}}|x|)^{qs}\,dx\right)^{1/q}. \qquad (6.4)\end{aligned}$$

First, assume $s\leq 0$. If $\mu^{\frac{1}{2}}\leq|x|\leq 1$ we have $e^{-\pi q|x|^2}\gtrsim 1$ and $1+\mu^{-\frac{1}{2}}|x|\leq 2\mu^{-\frac{1}{2}}|x|$. Hence the last expression turns out to be

$$\begin{aligned}&\geq \mu^{-\frac{d}{2q}}\left(\int_{\mu^{\frac{1}{2}}\leq|x|\leq 1} e^{-\pi q|x|^2}(1+\mu^{-\frac{1}{2}}|x|)^{qs}\,dx\right)^{1/q}\\ &\gtrsim \mu^{-\frac{d}{2q}-\frac{s}{2}}\left(\int_{\mu^{\frac{1}{2}}\leq|x|\leq 1}|x|^{qs}\,dx\right)^{1/q}.\end{aligned}$$

On the other hand, using polar coordinates one sees that, as $\mu\to 0^+$,

$$\int_{\mu^{\frac{1}{2}}\leq|x|\leq 1}|x|^{qs}\,dx \asymp \begin{cases}1 & \text{if } sq>-d,\\ |\log\mu| & \text{if } sq=-d,\\ \mu^{\frac{d}{2}+\frac{sq}{2}} & \text{if } sq<-d.\end{cases}$$

Secondly, assume $s>0$ in (6.4). Since $1+\mu^{-\frac{1}{2}}|x|\geq\mu^{-\frac{1}{2}}|x|$, we have

$$\|G_{a+ib}\|_{W(\mathcal{F}L^p,L^q_s)}\gtrsim\mu^{-\frac{d}{2q}-\frac{s}{2}}.$$

Hereby, we deduce, as $\lambda\to 0^+$,

$$\|Af_\lambda\|_{M^{p,q}_{v_{0,s}}}\gtrsim\begin{cases}\lambda^{-\frac{d}{2q}-\frac{s}{2}} & \text{if } sq>-d,\\ \lambda^{-\frac{d}{2q}-\frac{s}{2}}|\log\lambda|^{\frac{1}{q}} & \text{if } sq=-d,\\ 1 & \text{if } sq<-d.\end{cases} \qquad (6.5)$$

By combining this estimate with (6.2) and (6.3) and letting $\lambda\to 0^+$ we deduce the desired conclusion, when $q<\infty$. An easy modification of the above arguments yields also the case $p=q=\infty$.

Let us now prove the condition $p \geq q$. By duality, it suffices to prove that if the inequality

$$\|Af\|_{M^{p',q'}} \leq C\|f\|_{M^{p',q'}_{v_{0,s}}}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d), \tag{6.6}$$

holds true for some $s \in \mathbb{R}$, then $p \geq q$. To see this, we test the estimates on the Schwartz functions f_λ, $0 < \lambda < 1$, already used before. It follows from (6.5), in particular, that

$$\|Af_\lambda\|_{M^{p',q'}} \gtrsim \lambda^{-\frac{d}{2q'}}, \quad 0 < \lambda \leq 1. \tag{6.7}$$

On the other hand, by (2.9) with $a = \lambda$, $b = 0$,

$$\|G_{a+ib}T_x g\|_{\mathcal{F}L^{p'}} \asymp \lambda^{-\frac{d}{2p'}} e^{-\pi\nu|x|^2}, \quad 0 < \lambda \leq 1,$$

where $\nu = \lambda/(\lambda^2 + \lambda) \asymp 1$ for $0 < \lambda \leq 1$. Hence

$$\|f_\lambda\|_{M^{p',q'}_{v_{0,s}}} \lesssim \lambda^{-\frac{d}{2p'}}, \quad 0 < \lambda \leq 1.$$

This estimate, combined with (6.6) and (6.7) and letting $\lambda \to 0^+$ yields $p' \leq q'$, namely $p \geq q$. □

Here is a related example.

Example 6.2 Let us show the unboundedness of the operator A between $M^{1,\infty}(\mathbb{R}^d)$ and $M^{\infty,q}(\mathbb{R}^d)$, for every $q < \infty$. Consider the tempered distribution δ, defined by $\langle \delta, \varphi\rangle = \bar{\varphi}(0)$, for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then, for a fixed non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, the STFT $V_g\delta(x,\eta) = \langle \delta, M_\eta T_x g\rangle = \overline{g(-x)} \in L^{1,\infty}(\mathbb{R}^{2d})$, that is, $\delta \in M^{1,\infty}(\mathbb{R}^d)$. Now,

$$V_g(A\delta)(x,\eta) = \langle \delta, e^{-\pi i|\cdot|^2} M_\eta T_x g\rangle = \overline{g(-x)} \notin L^{\infty,q}(\mathbb{R}^{2d}), \tag{6.8}$$

for every $q < \infty$.

This also shows the unboundedness of the operator A between $M^\infty(\mathbb{R}^d)$ and $M^\infty_{v_{0,s}}(\mathbb{R}^d)$, for every $s > 0$. Indeed, using the inclusion relations

$$M^\infty_{v_{0,s}} \subset M^{\infty,q}, \quad s > \frac{1}{q},$$

we see that, if A were bounded between M^∞ and $M^\infty_{v_{0,s}}$ for some $s > 0$, then the operator A would be bounded between M^∞ and $M^{\infty,q}$ as well, and this is false, as shown by (6.8).

Proposition 6.3 *If the operator A in (6.1) is bounded as an operator $\mathcal{M}^{p,q}(\mathbb{R}^d) \to \mathcal{M}^{p,q}_{v_{s,0}}(\mathbb{R}^d)$ for some $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, then $s \leq -d(1/q - 1/p)$. Moreover, if $p = \infty$ and $q < \infty$, then $s < -d/q$.*

We need the following elementary result

Lemma 6.4 *Let $a_1, a_2, C_0 \in \mathbb{R}$ with $0 < a_1 \le C_0$, $C_0^{-1} \le |a_2| \le C_0$. Then*

$$\int_{\mathbb{R}^d} e^{-a_1|\eta+a_2x|^2} \langle\eta\rangle^s \, d\eta \gtrsim \langle x\rangle^s,$$

where the constant implicit in the notation $\gtrsim$ only depends on d, C_0 and s.

Proof Let us observe that $e^{-a_1|\eta+a_2x|^2} \ge e^{-1}$ for $\eta \in B(-a_2x, a_1^{-1/2})$. Hence

$$\int_{\mathbb{R}^d} e^{-a_1|\eta+a_2x|^2} \langle\eta\rangle^s \, d\eta \gtrsim \int_{B(0,a_1^{-1/2})} \langle y - a_2x\rangle^s \, dy \gtrsim \langle a_2x\rangle^s \int_{B(0,a_1^{-1/2})} \langle y\rangle^{-|s|} \, dy,$$

which gives the conclusion by the assumptions on a_1 and a_2. □

Proof of Proposition 6.3 In view of what we proved in Proposition 6.1, it suffices to verify that if the estimate

$$\|Af\|_{\mathcal{M}^{p,q}_{v_{s,0}}} \le C\|f\|_{\mathcal{M}^{p,q}}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d),$$

holds true, then (6.2) holds as well, at least for all $f(x) = f_\lambda(x) = e^{-\pi\lambda|x|^2}$, $0 < \lambda < 1$. Hence we are left to prove that

$$\|Af_\lambda\|_{\mathcal{M}^{p,q}_{v_{s,0}}} \gtrsim \|Af_\lambda\|_{\mathcal{M}^{p,q}_{v_{0,s}}},$$

or equivalently that

$$\|G_{a+ib}\|_{W(\mathcal{F}L^p_s, L^q)} \gtrsim \|G_{a+ib}\|_{W(\mathcal{F}L^p, L^q_s)},$$

with $a = \lambda$, $b = -1$ (see the notation of Lemma 2.9). To this end, observe that by (2.9) we have, for $p < \infty$,

$$\begin{aligned}
&\|G_{a+ib}T_xg\|_{\mathcal{F}L^p_s} \\
&= ((a+1)^2+b^2)^{-d/4} e^{-\frac{\pi(a+1)|x|^2}{(a+1)^2+b^2}} \\
&\quad \times \left(\int e^{-\frac{p\pi}{(a+1)^2+b^2}[(a^2+b^2+a)|\eta|^2+2bx\eta]} \langle\eta\rangle^{ps} \, d\eta\right)^{\frac{1}{p}} \\
&= ((a+1)^2+b^2)^{-d/4} e^{-\frac{\pi a|x|^2}{a(a+1)+b^2}} \\
&\quad \times \left(\int e^{-\frac{p\pi}{(a+1)^2+b^2}|\sqrt{a(a+1)+b^2}\eta+\frac{b}{\sqrt{a(a+1)+b^2}}x|^2} \langle\eta\rangle^{ps} \, d\eta\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $a = \lambda \le 1$ and $b = -1$ an application of Lemma 6.4 gives

$$\|G_{a+ib}T_xg\|_{\mathcal{F}L^p_s} \gtrsim e^{-\frac{\pi\lambda|x|^2}{\lambda^2+\lambda+1}} \langle x\rangle^s \asymp \|G_{a+ib}T_xg\|_{\mathcal{F}L^p} \langle x\rangle^s,$$

where for the last estimate we used (2.9).

Similarly one treats the case $p=\infty$. □

Remark 6.5 By duality, the assumption in Theorem 6.1 could be rephrased as the boundedness of A itself between $\mathcal{M}^{p',q'}_{v_{0,-s}}(\mathbb{R}^d)\to\mathcal{M}^{p',q'}(\mathbb{R}^d)$. Hence, as a consequence of Proposition 2.8 and Proposition 6.1, we see that the threshold for s_2 arising in Theorem 1.2(iii) (with $s_1=0$) is essentially sharp; namely, it is sharp for $p=1$, whereas when $p>1$ only the case of equality remains open.

Similarly, Proposition 6.3 shows that the threshold for s_1 arising in Theorem 1.2(ii) (with $s_2=0$) is essentially sharp.

The following result shows that the conclusion in Theorem 1.3 generally fails if $q>p$.

Proposition 6.6 *If the operator A in (6.1) is bounded as an operator $\mathcal{M}^{p,q}(\mathbb{R}^d)\to\mathcal{M}^{q,p}(\mathbb{R}^d)$ for some $1\le p,q\le\infty$, then $q\le p$.*

Proof We test the estimate

$$\|Af\|_{\mathcal{M}^{q,p}}\lesssim\|f\|_{\mathcal{M}^{p,q}},\qquad\forall f\in\mathcal{S}(\mathbb{R}^d),$$

on the family of Schwartz functions $f_\lambda(x)=e^{-\pi\lambda|x|^2}$, $\lambda>0$. Applying Lemma 2.9 with $a=\lambda$, $b=0$ we see that

$$\|f_\lambda\|_{M^{p,q}}\asymp\|G_{a+ib}\|_{W(\mathcal{F}L^p,L^q)}\asymp\lambda^{\frac{d}{2q}-\frac{d}{2}}\quad\text{as }\lambda\to+\infty. \tag{6.9}$$

On the other hand, with the notation of Lemma 2.9 with $a=\lambda$, $b=1$, we have

$$\|Af_\lambda\|_{M^{q,p}_s}\asymp\|G_{a+ib}\|_{W(\mathcal{F}L^q,L^p)}\asymp\lambda^{\frac{d}{2p}-\frac{d}{2}}\quad\text{as }\lambda\to+\infty.$$

Hence it turns out $q\le p$. □

Finally we present a counterexample related to Theorem 1.3.

Proposition 6.7 *The Schrödinger multiplier $f\longmapsto\mathcal{F}^{-1}(e^{i\pi|\cdot|^2}\hat{f})$ is not bounded as an operator $\mathcal{M}^{p,q}(\mathbb{R}^d)\to\mathcal{M}^{q,p}(\mathbb{R}^d)$ if $p\ne q$.*

Proof It suffices to prove that the pointwise multiplication operator A in (6.1) is not bounded as an operator $\mathcal{W}(\mathcal{F}L^p,L^q)(\mathbb{R}^d)\to\mathcal{W}(\mathcal{F}L^q,L^p)(\mathbb{R}^d)$ if $p\ne q$.

We test the estimate

$$\|Af\|_{\mathcal{W}(\mathcal{F}L^q,L^p)}\le C\|f\|_{\mathcal{W}(\mathcal{F}L^p,L^q)},\quad\forall f\in\mathcal{S}(\mathbb{R}^d),$$

on the family of Schwartz functions $f_\lambda(x)=e^{-\pi\lambda|x|^2}$, $\lambda>0$.

By applying Lemma 2.9 with $a=1/\lambda$ and $b=0$ we obtain

$$\|f_\lambda\|_{\mathcal{W}(\mathcal{F}L^p,L^q)}=\lambda^{-d/2}\|G_{a+ib}\|_{\mathcal{W}(\mathcal{F}L^p,L^q)}\asymp\begin{cases}\lambda^{-\frac{d}{2q}} & 0<\lambda\le1,\\ \lambda^{-\frac{d}{2p'}} & \lambda\ge1.\end{cases}$$

Similarly we have, with $a = \frac{\lambda}{\lambda^2+1}$, $b = \frac{1}{\lambda^2+1}$,

$$\|Af_\lambda\|_{\mathcal{W}(\mathcal{F}L^q,L^p)} = (a^2+b^2)^{d/4}\|G_{a+ib}\|_{\mathcal{W}(\mathcal{F}L^q,L^p)} \asymp \begin{cases} \lambda^{-\frac{d}{2p}} & 0<\lambda\le 1, \\ \lambda^{-\frac{d}{2q'}} & \lambda \ge 1. \end{cases}$$

Letting $\lambda \to 0^+$ and $\lambda \to +\infty$ one deduces a contradiction unless $p = q$. $\square$

Notice that this multiplier is the FIO with phase $\Phi(x,\eta) = x\eta + \frac{|\eta|^2}{2}$ and symbol $\sigma \equiv 1$, so that neither (1.8) nor condition (b) is satisfied (whereas (5.5) is). Indeed, for $\sigma \equiv 1$, $g \in \mathcal{S}(\mathbb{R}^{2d})$, we have

$$|V_g\sigma(z,\zeta)| = |\hat{\bar{g}}(\zeta)| \notin L^1(\mathbb{R}^{2d}_\zeta, L^\infty_{v_{s,0}}(\mathbb{R}^{2d}_z)), \quad \forall s>0.$$

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