J Fourier Anal Appl (2009) 15: 739–752
DOI 10.1007/s00041-009-9091-z

Construction of Interpolating Scaling Vectors with Hermite Interpolation Property

Jianing Sun

Received: 12 November 2008 / Revised: 14 July 2009 / Published online: 19 September 2009

Abstract In this paper, we construct a new family of Hermite-type interpolating scaling vectors with compact support, of which the Hermite interpolation property generalizes the existing results of interpolating scaling vectors and Hermite interpolants. In terms of the Hermite interpolatory mask, we characterize the Hermite interpolation property, approximation property and symmetry property in detail. To illustrate these results, several examples with compact support and high smoothness are exhibited at the end of this paper.

Keywords Hermite-type interpolating scaling vector · Hermite interpolatory mask · Sum rule · Symmetry

Mathematics Subject Classification (2000) 42C15 · 42C40 · 65T60 · 30E05

1 Introduction and Motivation

Wavelet analysis is a very powerful tool in applied mathematics, and wavelet algorithms have been successfully applied in signal processing, computer graphics and many other fields as well. In general, a wavelet is derived from a scaling vector via a multiresolution analysis. We say that a function vector $\phi := [\phi_1, \dots, \phi_r]^T \in$

Communicated by Stephan Dahlke.

This work was supported by Natural Science Foundation for Youths of Northeast Normal University (No. 20090103).

J. Sun (✉)
Mathematics & Statistics School, Northeast Normal University, Changchun 130012, Jilin Province, China
e-mail: math.sjn@gmail.com

$(L_2(\mathbb{R}))^r$ is a scaling vector with multiplicity r and a dyadic factor if

$$\phi(x) := 2\sum_{k\in\mathbb{Z}} a(k)\phi(2x-k), \quad \text{a.e. } x \in \mathbb{R}, \tag{1.1}$$

where $a : \mathbb{Z} \mapsto \mathbb{C}^{r\times r}$ is called the (matrix) mask. When the multiplicity $r = 1$, ϕ is called a scaling function. In the frequency domain, the matrix refinement equation in (1.1) can be rewritten as

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \quad \xi \in \mathbb{R}, \tag{1.2}$$

where $\widehat{a}$ is the Fourier series of the mask a given by

$$\widehat{a}(\xi) := \sum_{k\in\mathbb{Z}} a(k)e^{-ik\xi}, \quad \xi \in \mathbb{R}, \tag{1.3}$$

where i denotes the imaginary unit such that $i^2 = -1$. The Fourier transform $\widehat{f}$ of $f \in L_1(\mathbb{R})$ is defined to be $\widehat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$ and can be extended to square integrable functions and tempered distribution. Throughout the paper, for a smooth function f, $f^{(j)}$ denotes the jth derivative of f.

In the recent years, motivated by the practical applications, interpolating scaling vectors with various additional properties, such as short support, smoothness and symmetry, become of great and increasing interest. Several examples of interpolating scaling functions and their wavelets were studied by Dahlke in [1, 2]. However, the classical wavelet setting is somewhat restricted, re. [3]. To bypass the lack of flexibility, the concept of interpolating scaling vectors was introduced by [16]. There, some interesting examples were also presented. Following the result in [16], some similar approaches were derived in [4, 17]. As a generalization of the work in [1], Koch in [15] presented a systematic construction method and obtained a family of interpolating scaling vectors possessing orthogonality property. On the other hand, as an important subdivision scheme, Hermite interpolants are always of interest in the computer graphics, re. [5, 7, 8, 11]. By linking with Hermite interpolation property, several interesting results of scaling function have been presented by [7]. A complete mathematical characterization of Hermite interpolants in terms of their interpolatory masks is given in [8].

This paper is large motivated by [8] on Hermite interpolants and by [15] on dyadic interpolating scaling vectors. In this paper, we would like to consider the general case of interpolating scaling vectors and investigate their properties. That is, we are interested in a new family of compactly supported interpolating scaling vectors with multiplicity 4 possessing the following Hermite interpolation property.

Definition 1.1 A scaling vector $\phi = [\phi_1, \dots, \phi_4]^T \in (L_2(\mathbb{R}))^4$ is Hermite-type interpolating if ϕ is continuously differentiable and for $p \in \mathbb{Z}$, $j, \rho \in \{0, 1\}$ and $\lambda \in \{1, 2\}$,

$$\left[\phi_{2\rho+\lambda}(\cdot)\right]^{(j)}\left(\frac{p}{2}\right) = \delta_{\lambda-j-1}\delta_{p-\rho}, \tag{1.4}$$

where δ is a Dirac sequence such that $\delta_0 = 1$ and $\delta_k = 0$ for all $k \neq 0$.

For $1 \leqslant \ell \leqslant 4$, let E_ℓ denote the ℓ-th unit coordinate column vector in $\mathbb{R}$, that is, E_ℓ is the 4×1 column vector whose only nonzero entry is located at the ℓ-th component with value 1. Obviously, for all $p \in \mathbb{Z}$, (1.4) may be rewritten equivalently as follows,

$$[\phi(\cdot)]^{(j)}\left(\frac{p}{2}\right) = \delta_{p-\rho} E_{2\rho+j+1}, \tag{1.5}$$

where $\rho, j \in \{0, 1\}$. One advantage of Hermite-type interpolating scaling vector with compact support satisfies a Shannon-like sampling theorem, i.e., for any compactly supported function $f : \mathbb{R}^s \mapsto \mathbb{C}$, the function f can be interpolated and approximated by,

$$\widetilde{f}(x) = \sum_{k\in\mathbb{Z}} \left[f(k), f^{(1)}(k), f\left(k + \frac{1}{2}\right), f^{(1)}\left(k + \frac{1}{2}\right)\right] \phi(x - k).$$

Since ϕ is Hermite-type interpolating and compactly supported, $\widetilde{f}^{(j)}(k/2) = f^{(j)}(k/2)$ for all $k \in \mathbb{Z}$, that is, $\widetilde{f}^{(j)}$ agrees with $f^{(j)}$ on $2^{-1}\mathbb{Z}$ for $j = 0, 1$.

This paper is organized as follows. In the following section, we shall present a complete mathematical characterization for the Hermite-type interpolating scaling vectors in terms of their masks. In Sect. 3, we shall study approximation property and symmetry property of Hermite-type interpolating scaling vectors by constructing a family of Hermite-type interpolatory masks with high orders of sum rule and symmetry structure. Finally, in Sect. 4, some examples of Hermite-type interpolating scaling vectors with their smoothness exponents will be presented.

2 Characterization of Hermite-type Interpolating Scaling Vectors

In this section, we shall provide a complete mathematical characterization for Hermite-type interpolating scaling vectors with compact support in terms of their masks.

Before proceeding further, let us recall some necessary notions and auxiliary results. For $0 < \alpha \leqslant 1$ and $1 \leqslant p \leqslant \infty$, we say that $f \in \mathrm{Lip}(\alpha, L_p(\mathbb{R}))$ if for all $h > 0$, there is a constant C_f such that $\|f - f(\cdot - h)\|_{L_p(\mathbb{R})} \leqslant C_f h^\alpha$. The L_p smoothness of a function $f \in L_p(\mathbb{R})$ is measured by

$$\nu_p(f) := \sup\left\{n + \alpha : n \in \mathbb{N} \cup \{0\}, 0 < \alpha \leqslant 1, f^{(n)} \in \mathrm{Lip}(\alpha, L_p(\mathbb{R}))\right\}. \tag{2.1}$$

For a function vector $\phi = [\phi_1, \ldots, \phi_r]^T$, we denote $\nu_p(\phi) := \min_{1\leqslant\ell\leqslant r} \nu_p(\phi_\ell)$.

By $(\ell_0(\mathbb{Z}))^{r\times s}$ we denote the linear space of all finitely supported sequences of $r \times s$ matrices on $\mathbb{Z}$, and for $1 \leqslant p \leqslant \infty$, $(\ell_p(\mathbb{Z}^s))^{r\times s}$ denotes the linear space of all sequences v of $r \times s$ matrices on $\mathbb{Z}^s$ such that $\|v\|_{(\ell_p(\mathbb{Z}^s))^{r\times s}} := (\sum_{\beta\in\mathbb{Z}^s} \|v\|^p)^{1/p} < \infty$ for $1 \leqslant p < \infty$ and $\|v\|_{(\ell_\infty(\mathbb{Z}^s))^{r\times s}} := \sup_{\beta\in\mathbb{Z}^s} \|v(\beta)\|$ where $\|\cdot\|$ denotes a matrix norm on $r \times s$ matrices.

In the following, let us recall an important quantity $\nu_p(a; 2)$ from [8], which will play an important role in our investigation of Hermite-type interpolating scaling vectors. The convolution of two sequences v_1 and v_2 as

$$[v_1 * v_2](\alpha) := \sum_{k\in\mathbb{Z}} v_1(k)v_2(\alpha-k), \quad v_1 \in (\ell_0(\mathbb{Z}))^{\ell\times r}, v_2 \in (\ell_0(\mathbb{Z}))^{r\times s}.$$

Here, $\widehat{v_1 * v_2} = \widehat{v}_1\widehat{v}_2$. For a matrix mask a with multiplicity 4 and a dyadic factor, we say that a satisfies the sum rule of order $\Bbbk+1$ if there exists a sequence $y \in (\ell_0(\mathbb{Z}))^{1\times 4}$ such that $\widehat{y}(0)\neq 0$ and

$$\left[\widehat{y}(2\cdot)\widehat{a}(\cdot)\right]^{(j)}(k\pi) = \delta_k \widehat{y}^{(j)}(0), \quad k=0,1,\ j=0,\ldots,\Bbbk. \tag{2.2}$$

For $y \in (\ell_0(\mathbb{Z}))^{1\times 4}$ and a non-negative integer $\Bbbk$, we define the space

$$\mathcal{V}_{\Bbbk+1,y} := \left\{ v \in (\ell_0(\mathbb{Z}))^{4\times 1} : \mathcal{D}^j\left[\widehat{y}(\cdot)\widehat{v}(\cdot)\right](0) = 0,\ \forall\ j=0,\ldots,\Bbbk \right\}, \tag{2.3}$$

and by convention, $\mathcal{V}_{0,y} := (\ell_0(\mathbb{Z}))^{4\times 1}$. It is noted that the equations in (2.2) and (2.3) only depend on the values $\widehat{y}^{(j)}(0)$, $j=0,\ldots,\Bbbk$. As in [8], for a matrix mask a with multiplicity 4, a sequence $y \in (\ell_0(\mathbb{Z}))^{1\times 4}$ and a dyadic factor, we define

$$\rho_\Bbbk(a;2,p,y) := \sup\left\{ \lim_{n\to\infty} \|a_n * v\|^{1/n}_{(\ell_p(\mathbb{Z}))^{4\times 1}} : v \in \mathcal{V}_{\Bbbk,y} \right\}, \quad \Bbbk \in \mathbb{N}\cup\{0\}, \tag{2.4}$$

where $\widehat{a}_n(\xi) := \prod_{j=1}^n \widehat{a}(2^{j-1}\xi)$. For $1 \leqslant p \leqslant \infty$, define

$$\rho(a;2,p) := \inf\{\rho_\Bbbk(a;2,p,y) : (2.2) \text{ holds for some } \Bbbk\in\mathbb{N}\cup\{0\} \text{ and } y \in (\ell_0(\mathbb{Z}))^{1\times 4} \text{ with } \widehat{y}(0)\neq 0\}. \tag{2.5}$$

As presented in [8], we define the following important quantity:

$$\nu_p(a;2) = 1/p - 1 - \log_2 \rho(a;2,p), \quad 1 \leqslant p \leqslant \infty. \tag{2.6}$$

The quantity $\nu_p(a;2)$ plays an important role in characterizing the convergence of a vector cascade algorithm in a Sobolev space $W_p^\Bbbk(\mathbb{R}) := \{f \in L_p(\mathbb{R}) : f^{(j)} \in L_p(\mathbb{R}), \forall\, j=0,\ldots,\Bbbk\}$ and in characterizing the L_p smoothness of a dyadic scaling vector. It was shown in [8, Theorem 4.3] that the vector cascade algorithm associated with mask a and a dyadic factor converges in the Sobolev space if and only if $\nu_p(a;2) > \Bbbk$. In general, $\nu_p(a;2) \leqslant \nu_p(\phi)$ always holds. Moreover, if the shifts of a scaling vector ϕ associated with mask a and a dyadic factor are stable in $L_p(\mathbb{R})$, then $\nu_p(a;2) = \nu_p(\phi)$, re. [8, Theorem 4.1], that is, $\nu_p(a;2)$ characterizes the $L_p(\mathbb{R})$ smoothness exponent of a dyadic scaling vector ϕ in this case. Moreover, we always have $\nu_\infty(a,2) \geqslant \nu_2(a,2) - 1/2$. More details can be consulted in [7–9, 13].

Since the stability and linear independence of a scaling vector will be needed in this paper, let us recall their definition here. For a compactly supported function vector $f := [f_1,\ldots,f_r]^T$ in $L_p(\mathbb{R})$ for $1 \leqslant p \leqslant \infty$, we say that the shifts of f are stable in $L_p(\mathbb{R})$ if there exist two positive constants C_1 and C_2 such that

$$C_1 \sum_{\ell=1}^r \sum_{k\in\mathbb{Z}} |c_\ell(k)|^p \leqslant \left\| \sum_{\ell=1}^r \sum_{k\in\mathbb{Z}} c_\ell(k) f_\ell(x-k) \right\|^p_{L_p(\mathbb{R})} \leqslant C_2 \sum_{\ell=1}^r \sum_{k\in\mathbb{Z}} |c_\ell(k)|^p$$

for all finitely supported sequences $c_1, \dots, c_r$ in $\ell_0(\mathbb{Z})$. For a compactly supported function vector $f := [f_1, \dots, f_r]^T$, we say that the shifts of f are linearly independent if for any sequences $c_1, \dots, c_r : \mathbb{Z} \mapsto \mathbb{C}$ such that

$$\sum_{\ell=1}^{r} \sum_{k\in\mathbb{Z}} c_\ell(k) f_\ell(x-k) = 0, \quad \text{a.e. } x \in \mathbb{R}, \tag{2.7}$$

then one must have $c_\ell(k) = 0$ for all $\ell = 1, \dots, r$ and $k \in \mathbb{Z}$. It is known in [14] that if the shifts of a compactly supported function vector f in $L_p(\mathbb{R})$ are linearly independent, then the shifts of f must be stable in $L_p(\mathbb{R})$. Concerning the Hermite interpolation property of ϕ in (1.4), it is easy to see that the shifts of ϕ are linearly independent. More precisely, by setting $x = p/2 + m$ with $p \in \{0, 1\}$ and $m \in \mathbb{Z}$, we have

$$\begin{aligned} 0 &= \sum_{\ell=1}^{4} \sum_{k\in\mathbb{Z}} c_\ell(k) [\phi_\ell(\cdot)]^{(j)}(p/2 + m - k) \\ &= \sum_{\rho=0}^{1} \sum_{\lambda=1}^{2} \sum_{k\in\mathbb{Z}} c_{2\rho+\lambda}(k) \delta_{\lambda-j-1} \delta_{p-\rho} \delta_{m-k} = c_{2p+j}(m), \end{aligned}$$

where $j \in \{0, 1\}$. So the shifts of ϕ in (1.4) are linearly independent, which deduces that the shifts of ϕ are stable and for $1 \leqslant p \leqslant \infty$, $\nu_p(a; 2) = \nu_p(\phi)$ always holds here.

Based on these discussions, let us characterize the compactly supported Hermite-type interpolating scaling vectors as follows.

Theorem 2.1 *Let $a : \mathbb{Z} \mapsto \mathbb{C}^{4\times 4}$ be a finitely supported sequence of 4×4 matrices on $\mathbb{Z}$. Let $\phi = [\phi_1, \dots, \phi_4]^T$ be a compactly supported scaling vector such that $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$. Then ϕ is Hermite-type Interpolating, that is, ϕ is continuously differentiable and (1.4) holds if and only if the following statements holds:*

(1) $\nu_\infty(a; 2) > 1$;
(2) $[1, 0, 1, 0]\widehat{\phi}(0) = 1$;
(3) *a is a Hermite interpolatory mask, that is, a satisfies the sum rule of order 2 with respect to a nonzero sequence $y \in (\ell_0(\mathbb{Z}))^{1\times 4}$ which satisfies*

$$\widehat{y}(0) = [1, 0, 1, 0], \qquad \widehat{y}^{(1)}(0) = \left[0, i, \frac{i}{2}, i\right], \tag{2.8}$$

where

$$a(p) E_{j+1} = 2^{-j-1} \delta_{p-\rho} E_{2\rho+j+1}, \tag{2.9}$$

where $\rho, j \in \{0, 1\}$ and $p \in \mathbb{Z}$.

Proof Necessity: Following (1.1) and (1.5), for $j \in \{0, 1\}$ and $p \in \mathbb{Z}$, we have

$$\left[\phi\left(\frac{p}{2}\right)\right]^{(j)} = 2\sum_{k\in\mathbb{Z}} a(k) \, [\phi(\cdot)]^{(j)} \, (p-k) = 2\sum_{k\in\mathbb{Z}} a(k) \delta_{p-k} E_{j+1} = 2 \cdot a(p) E_{j+1}.$$

Considering (1.5) again, we deduce that

$$a(p)E_{j+1}=2^{-j-1}\,[\phi(\cdot)]^{(j)}\left(\frac{p}{2}\right)=2^{-j-1}\delta_{p-\rho}E_{2\rho+j+1}.$$

Hence, (2.9) holds.

Due to the stability of the shifts of Hermite-type interpolating scaling vector ϕ, note that ϕ is continuously differentiable, by [8, Corollary 5.1], $\nu_\infty(a;2)>1$ must hold, that is (1).

On the other hand, based on [8, Theorem 4.3], there must exist a nonzero sequence $y\in(\ell_0(\mathbb{Z}))^{1\times 4}$ such that $\widehat{y}(0)\widehat{\phi}(0)=1$ and the mask a satisfies the sum rule of order 2 with respect to y, where the values $\widehat{y}(0)$ is determined uniquely by the mask a. By [8, Proposition 3.2], we may have $[\widehat{y}(\cdot)\widehat{\phi}(\cdot)]^{(j)}(0)=\delta_j$ and $[\widehat{y}(\cdot)\widehat{\phi}(\cdot)]^{(j)}(2\pi k)=0$ for $j\in\{0,1\}$ and all $k\in\mathbb{Z}\backslash\{0\}$, which is equivalently to

$$\sum_{k\in\mathbb{Z}}\sum_{\ell=0}^{1}p^{(\ell)}(k)\left[-i\widehat{y}(\cdot)\right]^{(\ell)}(0)\,[\phi(\cdot)]^{(j)}(x-k)=p^{(j)}(x),\qquad p(x)\in\Pi_1,\tag{2.10}$$

where $j\in\{0,1\}$ and $\Pi_{\Bbbk}$ denotes the linear space of all polynomials with total degree no greater than $\Bbbk$, re. [8]. Taking $x=\lambda/2$ with $\lambda\in\{0,1\}$, since ϕ is Hermite-type Interpolating, we deduce from (1.4) that for $\widehat{y}(0)=[\widehat{y}_1(0),\ldots,\widehat{y}_4(0)]$,

$$\sum_{\ell=0}^{1}p^{(\ell)}(0)\left[-i\widehat{y}_{2\lambda+1}(\cdot)\right]^{(\ell)}(0)=p\left(\frac{\lambda}{2}\right),$$

$$\sum_{\ell=0}^{1}p^{(\ell)}(0)\left[-i\widehat{y}_{2\lambda+2}(\cdot)\right]^{(\ell)}(0)=p^{(1)}\left(\frac{\lambda}{2}\right).$$

By taking $p(x)=1$ and $p(x)=x$ respectively, we may have

$$\widehat{y}(0)=[1,0,1,0],\qquad \widehat{y}^{(1)}(0)=\left[0,i,\frac{i}{2},i\right].$$

Hence, (3) holds.

Combining with $\widehat{y}(0)\widehat{\phi}(0)=1$ and (3), we must have (2).

Sufficiency: Recalling [8, Corollary 5.2], there exists a Hermite interpolant $d=[d_1,d_2]^T$ with dyadic factor, of which the mask satisfies the sum rule of order 2 with respect to a nonzero sequence $y_1\in(\ell_0(\mathbb{Z}))^{1\times 2}$ such that $\widehat{y}_1(0)=[1,0]$ and $[-i\widehat{y}_1(\cdot)]^{(1)}(0)=[0,1]$. Therefore, the following formula always hold for $j\in\{0,1\}$.

$$\left[\widehat{y}_1(\cdot)\widehat{d}(\cdot)\right]^{(j)}(0)=\delta_j,\qquad \left[\widehat{y}_1(\cdot)\widehat{d}(\cdot)\right]^{(j)}(2\pi k)=0,\quad \forall k\in\mathbb{Z}\backslash\{0\}.\tag{2.11}$$

Define a function vector as follows $f(x):=[d_1(2x),\ 2^{-1}d_2(2x),\ d_1(2x-1),\ 2^{-1}d_2(2x-1)]^T$, of which the Fourier transform is $\widehat{f}(\xi)=2^{-1}\cdot[\widehat{d}_1(\xi/2),2^{-1}\widehat{d}_2(\xi/2),\ e^{-i\xi/2}\widehat{d}_1(\xi/2),\ 2^{-1}e^{-i\xi/2}\widehat{d}_2(\xi/2)]^T$. Note that from the first formula of (2.11), we may have $\widehat{d}_1(0)=1$. Hence linking with (2.8) and (2.11), we have

$$[1,0,1,0]\,\widehat{f}(0)=2^{-1}\,[1,0,1,0]\left[\widehat{d}_1(0),2^{-1}\widehat{d}_2(0),\widehat{d}_1(0),2^{-1}\widehat{d}_2(0)\right]^T=1.$$

In addition, by substituting ϕ in (2.10) with f and setting $\widehat{y}=[\widehat{y}_1, e^{i\xi/2}\widehat{y}_1]$, a simple computation may obtain that for all $k\in\mathbb{Z}\backslash\{0\}$ and $j=0,1$, f satisfies $[\widehat{y}(\cdot)\widehat{f}(\cdot)]^{(j)}(2\pi k)=0$ with respect to y. By [8, Theorem 4.3], it follows from (1) that the cascade algorithm associated with mask a, dyadic factor and the initial function vector f converges to ϕ in $(C^1(\mathbb{R}))^{4\times 1}$. More precisely, let us define the cascade sequence f_n, $n\in\mathbb{N}\cup\{0\}$ to be $f_0:=f$ and

$$f_n := Q_{a,2} f_{n-1} := 2\sum_{k\in\mathbb{Z}} a(k)\phi(2x-k), \quad n\in\mathbb{N},$$

then $\lim_{n\to\infty}\|f_n-\phi\|_{(C^1(\mathbb{R}))^{4\times 1}}=0$.

Now we show by induction on n that

$$\left[f_n(\cdot)\right]^{(j)}\left(\frac{p}{2}\right)=\delta_{p-\rho}E_{2\rho+j+1}, \quad \forall\, p\in\mathbb{Z}, \tag{2.12}$$

where $\rho, j\in\{0,1\}$. Apparently, with the definition of the initial function vector $f_0=f$, (2.12) holds for $n=0$. By the definition of f and (2.9), we deduce that for $p\in\mathbb{Z}$ and the induction hypothesis for $n-1$,

$$\begin{aligned}\left[f_n(\cdot)\right]^{(j)}\left(\frac{p}{2}\right) &= 2^j\left[f_n\left(\frac{p}{2}\right)\right]^{(j)} = 2^{j+1}\sum_{k\in\mathbb{Z}} a(k)\left[f_{n-1}(\cdot)\right]^{(j)}(p-k)\\ &= 2^{j+1}a(p)E_{j+1}\\ &= \delta_{p-\rho}E_{2\rho+j+1},\end{aligned}$$

where $\rho\in\{0,1\}$. Now by induction, (2.12) holds for all $n\in\mathbb{N}\cup\{0\}$. From the discussions before, it is concluded that ϕ is continuously differentiable and (1.4) holds. □

In fact, the Hermite interpolatory condition given by (2.9) can be rewritten as follows

$$\widehat{a}(\xi)=\begin{pmatrix}\frac{1}{2} & 0 & * & *\\ 0 & \frac{1}{4} & * & *\\ \frac{1}{2}e^{-i\xi} & 0 & * & *\\ 0 & \frac{1}{4}e^{-i\xi} & * & *\end{pmatrix}, \tag{2.13}$$

where $*$ denotes some 2π-periodic trigonometric polynomials.

3 Additional Properties

3.1 Approximation Property

In the following, we shall investigate the structure of the vector $\widehat{y}$ in the definition of the sum rules. Note that the sum rule condition in (2.2) can be rewritten equivalently as

$$\widehat{y}(2\xi)\widehat{a}(\xi+\pi\ell)=\delta_\ell\,\widehat{y}(\xi)+O(|\xi|^{\Bbbk+1}), \quad \xi\to 0,\ \ell\in\{0,1\}. \tag{3.1}$$

Now we shall express the sum rule condition above in terms of the cosets of $\widehat{a}(\xi)$. Define the cosets $\widehat{a}_\ell(\xi)$ of $\widehat{a}(\xi)$ by

$$\widehat{a}_\ell(\xi) := \sum_{k\in\mathbb{Z}} a(2k+\ell)e^{-i\xi(2k+\ell)}, \quad \ell \in \{0,1\}. \tag{3.2}$$

Noting that $\widehat{a}(\xi) = \widehat{a}_0(\xi) + \widehat{a}_1(\xi)$ and $\widehat{a}_\ell(\xi+\pi p) = e^{-i\pi p\,\ell}\widehat{a}(\xi)$ for $p \in \mathbb{Z}$, we can rewrite (3.1) as follows

$$\widehat{y}(2\xi)\widehat{a}_0(\xi) + e^{-ip\pi}\widehat{y}(2\xi)\widehat{a}_1(\xi) = \delta_p\,\widehat{y}(\xi) + O(|\xi|^{\Bbbk+1}), \quad \xi \to 0,\ p \in \{0,1\},$$

which equals that,

$$\widehat{y}(2\xi)\widehat{a}_\ell(\xi) = \frac{1}{2}\widehat{y}(\xi) + O(|\xi|^{\Bbbk+1}), \quad \xi \to 0,\ \ell \in \{0,1\}. \tag{3.3}$$

The following result will present the vector $\widehat{y}$ in the definition of sum rules for Hermite interpolatory mask in this paper, which plays an important role for construction of Hermite interpolatory mask with a given order of sum rules later.

Lemma 3.1 *Let d be a positive integer such that $d > 1$. Let a be a supported sequence of 4×4 matrices on $\mathbb{Z}$. If for a positive integer $\Bbbk \geqslant 1$, $a \in (\ell_0(\mathbb{Z}))^{4\times 4}$ is a Hermite interpolatory mask, which satisfies the sum rules of order $\Bbbk+1$ with respect to a nonzero sequence $y \in (\ell_0(\mathbb{Z}))^{1\times 4}$ and (2.8) holds, then*

$$\widehat{y}(\xi) = \left[1,\ i\xi,\ e^{i\xi/2},\ i\xi\cdot e^{i\xi/2}\right] + O(|\xi|^{\Bbbk+1}), \quad \xi \to 0. \tag{3.4}$$

Proof Note that the first two columns in the right side of (2.13) can be rewritten as

$$\widehat{a}_\ell(\xi)\,[E_1, E_2] = e^{-i\ell\xi}\left[2^{-1}E_{2\ell+1}, 2^{-2}E_{2\ell+2}\right], \quad \ell \in \{0,1\}. \tag{3.5}$$

Since a satisfies the sum rule of order $\Bbbk+1$ with respect to the vector $\widehat{y}$, it can be obtained from (3.3) that as $\xi \to 0$ and $\ell \in \{0,1\}$,

$$\begin{aligned}\widehat{y}(\xi)\,[E_1, E_2] =& 2\,\widehat{y}(2\xi)\widehat{a}_\ell(\xi)\,[E_1, E_2] + O(|\xi|^{\Bbbk+1})\\ =& \widehat{y}(2\xi)\,e^{-i\ell\xi}\left[E_{2\ell+1}, 2^{-1}E_{2\ell+2}\right] + O(|\xi|^{\Bbbk+1}).\end{aligned}$$

Following $\widehat{y}(\xi) = [\widehat{y}_1(\xi), \dots, \widehat{y}_4(\xi)]$, the formula above can be expressed detailedly as follows,

$$\widehat{y}_{2\ell+1}(2\xi) = e^{i\ell\xi}\widehat{y}_1(\xi) + O(|\xi|^{\Bbbk+1}), \qquad \widehat{y}_{2\ell+2}(2\xi) = 2e^{i\ell\xi}\widehat{y}_2(\xi) + O(|\xi|^{\Bbbk+1}), \tag{3.6}$$

where $\ell \in \{0,1\}$. Thereupon by taking $\ell = 0$, we could have

$$\widehat{y}_1(\xi) = \widehat{y}_1(2^{-1}\xi) + O(|\xi|^{\Bbbk+1}), \qquad \widehat{y}_2(\xi) = 2\,\widehat{y}_2(2^{-1}\xi) + O(|\xi|^{\Bbbk+1}). \tag{3.7}$$

Combining with (2.8), by taking $\xi \to 0$, it can be recognized that for $\ell = 1, 2$ and $j = 2, \dots, \Bbbk$, the values $\widehat{y}_\ell^{(j)}(0)$ are completely and uniquely determined by the system of linear equations in (3.7). Therefore for $\ell = 1, \dots, 4$ and $j = 2, \dots, \Bbbk$, the values $\widehat{y}_\ell^{(j)}(0)$ are uniquely determined by the relation (3.6), that is, if there is a solution to (3.6), then the solution must be unique.

In the following, we shall show that the system of linear equation in (3.6) indeed has a solution. Let $\zeta(\xi) = [1, i\xi, e^{i\xi/2}, i\xi \cdot e^{i\xi/2}]$. Then we have $\zeta(0) = [1, 0, 1, 0]$ and $\zeta^{(1)}(0) = [0, i, i/2, i]$. In addition, a straightforward calculation can verify that $\zeta(\xi)$ satisfies (3.6). Therefore, if setting $\widehat{y}^{(j)}(0) = \widehat{\zeta}^{(j)}(0)$, it is a solution to the system of linear equations in (3.6). By the uniqueness of the solution, we must have (3.4), which completes the proof. □

For discussion conveniently, let $[A]_{k,j}$ denote the (k, j)-entry of the matrix A. We have the following result with regard to Hermite interpolatory masks.

Theorem 3.1 *Let μ, $\Bbbk$ be two positive integers and $a \in (\ell_0(\mathbb{Z}))^{4\times 4}$ be a Hermite interpolatory mask. Then a satisfies the sum rule of order $\Bbbk + 1$ with $\Bbbk \geqslant 1$ if and only if*

$$\sum_{k\in\mathbb{Z}} \sum_{t=0}^{1} \sum_{m=0}^{1} [a(2k+\ell)]_{2t+m+1,3+p}\, 2^m (t-\ell-2k)^{\mu-m} \varrho_\mu(m) = 2^{p-\mu-1} \varrho_\mu(p), \tag{3.8}$$

holds for $0 \leqslant \mu \leqslant \Bbbk$ with $\Bbbk \geqslant 1$, $\ell, p \in \{0, 1\}$ and $k \in \mathbb{Z}$, where $\varrho_\mu(k)$ is a piecewise function such that $\varrho_\mu(k) = 1/(\mu-k)!$ for $\mu \geqslant k$ and $\varrho_\mu(k) = 0$ for $\mu < k$.

Proof Following the remarks mentioned before, by introducing (3.4) and (3.5) into the left side of (3.3), it can be obtained that for $\ell \in \{0, 1\}$,

$$\widehat{y}(2\xi)\,\widehat{a}_\ell(\xi)\,[E_1, E_2] = 2^{-1}[1, i\xi] = 2^{-1}\big[\widehat{y}_1(\xi), \widehat{y}_2(\xi)\big], \quad \xi \to 0.$$

Therefore by (3.3), it is sufficient and necessary to require

$$\widehat{y}(2\xi)\,\widehat{a}_\ell(\xi)\,[E_3, E_4] = 2^{-1}\big[\widehat{y}_3(\xi), \widehat{y}_4(\xi)\big] + O(|\xi|^{\Bbbk+1}), \quad \xi \to 0.$$

Substituting (3.2) into the formula above and applying Lemma 3.1, we have that for $\ell, p \in \{0, 1\}$,

$$\sum_{k\in\mathbb{Z}} \sum_{t=0}^{1} \sum_{m=0}^{1} [a(2k+\ell)]_{2t+m+1,3+p}\, (i2\xi)^m e^{i\xi(t-\ell-2k)} = 2^{-1}(i\xi)^p e^{i\xi/2} + O(|\xi|^{\Bbbk+1}), \tag{3.9}$$

where $\xi \to 0$. By taking μ-th derivative on both sides of (3.9) at $\xi = 0$ and applying the Leibniz differentiation formula, we see that as $\mu = 0$,

$$\sum_{k\in\mathbb{Z}} \sum_{t=0}^{1} [a(2k+\ell)]_{2t+1,3} = 2^{-1}, \qquad \sum_{k\in\mathbb{Z}} \sum_{t=0}^{1} [a(2k+\ell)]_{2t+1,4} = 0, \tag{3.10}$$

and as $1 \leqslant \mu \leqslant \Bbbk$,

$$\sum_{k\in\mathbb{Z}}\sum_{t=0}^{1}\sum_{m=0}^{1}[a(2k+\ell)]_{2t+m+1,3+p}\,\frac{2^m(t-\ell-2k)^{\mu-m}}{(\mu-m)!}=\frac{2^{p-\mu-1}}{(\mu-p)!}, \tag{3.11}$$

where $\ell, p \in \{0,1\}$ and $k \in \mathbb{Z}$. By introducing the piecewise function $\varrho_\mu(k)$ as defined in Theorem 3.1, we have (3.8) by (3.10) and (3.11), which completes the proof. □

Theorem 3.1 develops the relevant results in [15] and presents an executable computing approach to obtain Hermite interpolatory mask a with a given order of sum rules. More precisely, by applying (3.8) in Theorem 3.1 up to the given order of sum rules with regard to the unknown coefficients of the mask a as depicted in (2.13), a system of linear equations can be obtained. Particularly, as for the system of linear equations given by (3.8), we note that its coefficient matrix is a confluent Vandermonde matrix and the number of the linear equations determines the sum rule order. Therefore for a given support, if the corresponding coefficient matrix derived from (3.8) is square, that is, invertible, then there exists a maximal sum rule order, and the corresponding Hermite interpolatory mask is unique.

3.2 Symmetry Property

As an important complementary, we shall introduce the symmetry property of Hermite-type interpolating scaling vectors in this section. As known, if for a function ϕ on $\mathbb{R}$, $\phi(2c-x) = \pm\overline{\phi(x)}$, then we say that the symmetry center of ϕ is c and ϕ is symmetric or antisymmetric about the point c. It is easy to see that $\phi(2c-x) = \pm\overline{\phi(x)}$ for all $x \in \mathbb{R}$ if and only if $\overline{\widehat{\phi}(\xi)} = \pm e^{i\,2c\xi}\widehat{\phi}(\xi)$ for all $\xi \in \mathbb{R}$. Moreover, ϕ is real-valued, that is, $\overline{\phi(x)} = \phi(x)$ for all $x \in \mathbb{R}$, if and only if $\overline{\widehat{\phi}(\xi)} = \widehat{\phi}(-\xi)$ for all $\xi \in \mathbb{R}$.

Suppose that $\phi = [\phi_1, \dots, \phi_4]^T$ is a Hermite-type interpolating scaling vector, and for $\ell \in \{1,\dots,4\}$, the symmetry center of each component function ϕ_ℓ is c_ℓ. Then the interpolation property of ϕ in (1.4) implies for all $p \in \mathbb{Z}$, $j, \rho \in \{0,1\}$ and $\lambda \in \{1,2\}$,

$$\begin{aligned}\delta_{\lambda-j-1}\delta_{p-\rho} &= \left[\phi_{2\rho+\lambda}(\cdot)\right]^{(j)}\left(\frac{p}{2}\right)\\ &= \left[\phi_{2\rho+\lambda}(\cdot)\right]^{(j)}\left(2\,c_{2\rho+\lambda}-\frac{p}{2}\right) = \delta_{\lambda-j-1}\delta_{4c_{2\rho+\lambda}-p-\rho},\end{aligned}$$

and consequently $2\,c_{2\rho+\lambda} = \rho$. A simple computation may obtain that for $\rho \in \{0,1\}$, $\phi_{2\rho+1}$ is symmetric about the point $c_{2\rho+1} = \rho/2$ and $\phi_{2\rho+2}$ is antisymmetric about the point $c_{2\rho+2} = \rho/2$.

As a consequence, the following theorem can be derived directly from [10, Lemma 2.4].

Theorem 3.2 *Let $\phi = [\phi_1, \dots, \phi_4]^T$ be a Hermite-type interpolating scaling vector with mask $a \in (\ell_0(\mathbb{Z}))^{4\times 4}$. If for all $x \in \mathbb{R}$, $\rho \in \{0,1\}$ and $\lambda \in \{1,2\}$, $\phi_{2\rho+\lambda}(\rho - x) =$*

$(-1)^{\lambda-1}\phi_{2\rho+\lambda}(x)$, *then* $\widehat{\overline{\phi}(\xi)} = S(\xi)\widehat{\phi}(\xi)$ *and* $\overline{\widehat{a}(\xi)} = S(2\xi)\widehat{a}(\xi)S(\xi)^{-1}$, *where*

$$S(\xi) := \operatorname{diag}(1, -1, e^{i\xi}, -e^{i\xi}). \tag{3.12}$$

In fact, if a is real-valued, then for $t, \rho \in \{0, 1\}$ *and* $\lambda \in \{1, 2\}$,

$$[\widehat{a}(-\xi)]_{2\rho+\lambda,3+t} = (-1)^{\lambda+t-1} e^{(-1)^{1-\rho} i\xi} [\widehat{a}(\xi)]_{2\rho+\lambda,3+t}.$$

4 Examples

In this section, we shall present several examples of Hermite interpolatory masks, of which the corresponding Hermite-type interpolating scaling vectors will be shown as well. More precisely, for each given support, we shall present a Hermite interpolatory mask a with the maximal sum rule order and a Hermite interpolatory mask a_{sym} with symmetry satisfying the possible highest order of sum rules respectively.

For discussion conveniently, let $[\widehat{a}]_{:,3:4}$ denote the 3-th and 4-th column of Hermite interpolatory mask $\widehat{a}(\xi)$. By observing (2.13), it is necessary and sufficient to construct $[\widehat{a}]_{:,3:4}$ of the Hermite interpolatory mask a. Therefore for expression concisely, $[\widehat{a}]_{:,3:4}$ will be presented detailedly in the following examples.

Example 4.1 Let a be supported in $[-1, 1]$. Then we can obtain a Hermite interpolatory mask a satisfying the sum rules of order 4 and a symmetric Hermite interpolatory mask a_{sym} satisfying the sum rules of order 4, of which the Fourier series is given by

$$[\widehat{a}]_{:,3:4} = \begin{pmatrix} \frac{243}{1024}e^{i\xi} + \frac{1}{4} + \frac{13}{1024}e^{-i\xi} & \frac{405}{512}e^{i\xi} - \frac{3}{4} - \frac{5}{512}e^{-i\xi} \\ -\frac{81}{2048}e^{i\xi} + \frac{1}{32} + \frac{3}{2048}e^{-i\xi} & -\frac{81}{1024}e^{i\xi} - \frac{1}{16} - \frac{1}{1024}e^{-i\xi} \\ \frac{13}{1024}e^{i\xi} + \frac{1}{4} + \frac{243}{1024}e^{-i\xi} & \frac{5}{512}e^{i\xi} + \frac{3}{4} - \frac{405}{512}e^{-i\xi} \\ -\frac{3}{2048}e^{i\xi} - \frac{1}{32} + \frac{81}{2048}e^{-i\xi} & -\frac{1}{1024}e^{i\xi} - \frac{1}{16} - \frac{81}{1024}e^{-i\xi} \end{pmatrix},$$

$$\left[\widehat{a}_{\text{sym}}\right]_{:,3:4} = \begin{pmatrix} \frac{1}{4}(e^{i\xi}+1) & \frac{3}{4}(e^{i\xi}-1) \\ -\frac{1}{32}(e^{i\xi}-1) & -\frac{1}{16}(e^{i\xi}+1) \\ \frac{1}{4}(1+e^{-i\xi}) & \frac{3}{4}(1-e^{-i\xi}) \\ -\frac{1}{32}(1-e^{-i\xi}) & -\frac{1}{16}(1+e^{-i\xi}) \end{pmatrix}.$$

Then we have the critical Sobolev exponent $\nu_2(a, 2) \approx 2.7723$ and $\nu_2(a_{\text{sym}}, 2) = 2.5000$, that is, $\nu_\infty(a, 2) \geqslant \nu_2(a, 2) - 1/2 \approx 2.2723 > 1$ and $\nu_\infty(a_{\text{sym}}, 2) \geqslant 2.0 > 1$. Therefore, the scaling vector ϕ associated with a is C^2 Hermite-type interpolating and the symmetric scaling vector ϕ_{sym} associated with a_{sym} is C^1 Hermite-type interpolating.

Example 4.2 Let the support of a be $[-1, 2]$. Then we can obtain a Hermite-type interpolatory mask a with the sum rule of order 8 and a symmetric Hermite-type

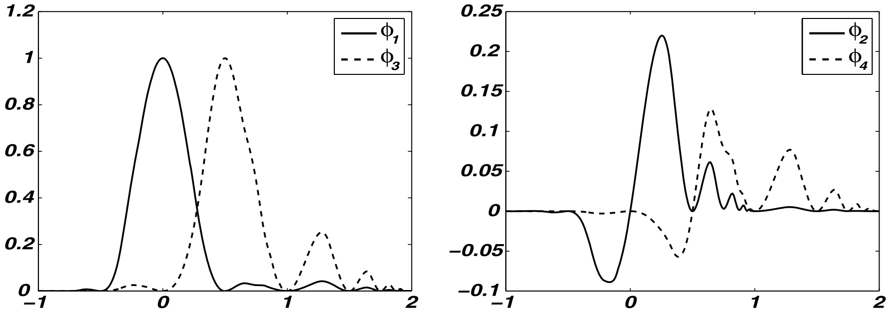

Fig. 1 Hermite-type interpolating scaling vector $\phi = [\phi_1, \dots, \phi_4]^T$ associated with mask a in $[-1, 2]$ of Example 4.2, of which the mask a satisfies the sum rule of order 8 and possesses $\nu_2(a, 2) \approx 2.0243$

interpolatory mask a_{sym} with the sum rules of order 6. Their Fourier series are given respectively as

$$[\widehat{a}]_{:,3:4} = \begin{pmatrix} \frac{243}{1024}e^{i\xi} + \frac{225}{1024} + \frac{13}{1024}e^{-i\xi} + \frac{61}{3072}e^{-i2\xi} & \frac{405}{512}e^{i\xi} - \frac{435}{512} - \frac{5}{512}e^{-i\xi} + \frac{155}{4608}e^{-i2\xi} \\ -\frac{81}{2048}e^{i\xi} + \frac{225}{2048} + \frac{3}{2048}e^{-i\xi} + \frac{5}{2048}e^{-i2\xi} & -\frac{81}{1024}e^{i\xi} + \frac{15}{1024} - \frac{1}{1024}e^{-i\xi} + \frac{13}{3072}e^{-i2\xi} \\ \frac{13}{1024}e^{i\xi} + \frac{425}{3072} + \frac{243}{1024}e^{-i\xi} + \frac{125}{1024}e^{-i2\xi} & \frac{5}{512}e^{i\xi} + \frac{3085}{4608} - \frac{405}{512}e^{-i\xi} + \frac{75}{512}e^{-i2\xi} \\ -\frac{3}{2048}e^{i\xi} - \frac{25}{2048} + \frac{81}{2048}e^{-i\xi} + \frac{75}{2048}e^{-i2\xi} & -\frac{1}{1024}e^{i\xi} - \frac{155}{3072} - \frac{81}{1024}e^{-i\xi} + \frac{55}{1024}e^{-i2\xi} \end{pmatrix},$$

$$\left[\widehat{a}_{\mathrm{sym}}\right]_{:,3:4} = \begin{pmatrix} \frac{9}{32}(e^{i\xi}+1) & \frac{3}{4}(e^{i\xi}-1) \\ -\frac{9}{128}(e^{i\xi}-1) & -\frac{3}{64}(e^{i\xi}+1) \\ \frac{11}{256}(e^{i\xi}+e^{-i2\xi}) + \frac{45}{256}(1+e^{-i\xi}) & -\frac{3}{128}(e^{i\xi}-e^{-i2\xi}) + \frac{93}{128}(1-e^{-i\xi}) \\ -\frac{3}{512}(e^{i\xi}-e^{-i2\xi}) - \frac{9}{512}(1-e^{-i\xi}) & \frac{1}{256}(e^{i\xi}+e^{-i2\xi}) - \frac{15}{256}(1+e^{-i\xi}) \end{pmatrix},$$

with the critical Sobolev exponents $\nu_2(a, 2) \approx 2.0243$ and $\nu_2(a_{\mathrm{sym}}, 2) \approx 2.4945$, that is, $\nu_\infty(a, 2) \geqslant \nu_2(a, 2) - 1/2 \approx 1.5243$ and $\nu_\infty(a_{\mathrm{sym}}, 2) \geqslant \nu_2(a_{\mathrm{sym}}, 2) - 1/2 \approx 1.9945$. The corresponding scaling vectors, that is, ϕ and ϕ_{sym} are graphed in Fig. 1 and 2 respectively.

Note that $\nu_2(\phi) = \nu_2(a; 2)$ here. In Table 1, we shall summarize the L_2 smoothness exponents of Hermite-type interpolating scaling vectors derived from this paper and interpolating vectors constructed by the methods in [7, 15] for comparison. It can be concluded from Table 1 that for a given support, on the assumption that both possess their own maximal accuracy orders, besides paying for with a vector of length four instead of two, compactly supported Hermite-type interpolating scaling vector constructed in this paper usually provides slightly better smoothness than the result in [7, 15]. In addition, it also shows that the smoothness exponent of scaling vector can not increase with respect to the sum rule order, or equivalently, the support size. Recalling the results in [7, 15], based on optimality of conditions, we can get a smoother scaling vector with less sum rule order than the relevant result in this paper.

At the end of this section, we shall explicitly present the L_2 smoothness exponents of some function vectors generated from the masks constructed by Theorem 2.1, Theorem 3.1 and Theorem 3.2 in Table 2 and Table 3. Here, if $\nu_2(\phi) > 1.5$, then the

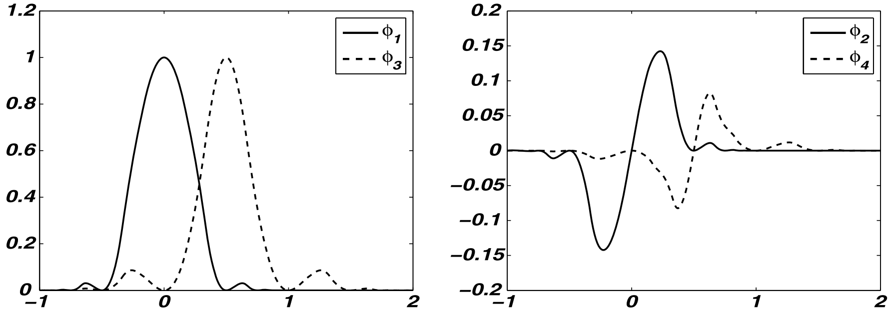

Fig. 2 Symmetric Hermite-type interpolating scaling vector $\phi = [\phi_1, \ldots, \phi_4]^T$ associated with mask a_{sym} in $[-1, 2]$ of Example 4.2, of which the mask a_{sym} satisfies the sum rule of order 6 and possesses $\nu_2(a_{\text{sym}}, 2) \approx 2.4945$

Table 1 For some given supports, the L_2 smoothness exponents of some types of scaling vectors, including $\phi^{\text{H-Iv}}$, $\phi^{\text{H-I}}$ and ϕ^{Iv} with and without symmetry are presented, where $\phi^{\text{H-Iv}}$ denotes Hermite-type interpolating scaling vector constructed in this paper, $\phi^{\text{H-I}}$ denotes the function vector constructed in [7] and ϕ^{Iv} denotes the function vector constructed in [15]

Support	$[-1, 1]$	$[-1, 2]$	$[-2, 2]$	$[-2, 3]$	$[-3, 3]$	$[-3, 4]$
$\nu_2(\phi^{\text{H-Iv}})$	2.7723	2.0243	3.5751	3.1271	3.8641	3.4343
$\nu_2(\phi^{\text{H-Iv}}_{\text{sym}})$	2.5000	2.4945	3.3950	3.0518	3.7620	3.6465
$\nu_2(\phi^{\text{H-I}})$	2.5000	2.5000	2.5000	2.7006	3.3950	3.3950
$\nu_2(\phi^{\text{Iv}})$	1.7173	1.8916	2.7581	2.3109	3.4059	2.6654
$\nu_2(\phi^{\text{Iv}}_{\text{sym}})$	1.5000	1.8390	2.4408	2.1598	3.1751	2.6762

Table 2 The L_2 smoothness exponents $\nu_2(\phi)$ of dyadic Hermite-type interpolating scaling vectors associated with Hermite interpolatory masks a supported in $[K_1, K_2]$

$K_1 \backslash K_2$	1	2	3	4	5	6
0	0.5000	0.4486	−3.7962	−3.8059	−7.4539	−7.4732
−1	2.7723	2.0243	2.0245	0.1473	−0.0808	−2.8494
−2	2.5940	3.5751	3.1271	2.9977	1.4181	1.1810
−3	2.4422	2.9327	3.8641	3.4343	3.3565	2.3126
−4	−0.0724	2.9075	3.5033	3.9999	3.7780	2.8254

corresponding function vector is Hermite-type interpolating. Specifically, the masks in Table 2 don't possess symmetry and satisfy the sum rule of order $4\lfloor \frac{K_2 - K_1 + 1}{2} \rfloor$, where $\lfloor \cdot \rfloor$ denotes a round estimate towards zero, and in Table 3, the masks possess symmetry and satisfy the sum rule of order $4K - 2c$.

Acknowledgements The author would like to thank reviewers for their many valuable suggestions that lead to a significant improvement for my manuscript. In addition, the author would also like to thank Stephan Dahlke for his sharing of Matlab routine, [12].

Table 3 The L_2 smoothness exponents $\nu_2(\phi_{\mathrm{sym}})$ of symmetric dyadic Hermite-type interpolating scaling vectors associated with Hermite interpolatory masks a_{sym} supported in $[c-K, K]$

$c\backslash K$	1	2	3	4	5
0	2.5000	3.3950	3.7620	3.9732	4.0000
1	0.5000	2.4945	3.0518	3.6465	3.7912

References

1. Dahlke, S., Maass, P.: Interpolating refinable functions and wavelets for general scaling matrices. Numer. Funct. Anal. Optim. **18**(5–6), 521–539 (1997)
2. Dahlke, S., Grochenig, K., Maass, P.: A new approach to interpolating scaling functions. Appl. Anal. **72**, 485–500 (1999)
3. Daubechies, I.: Ten Lectures on Wavelets, CBMS-NSF Series in Applied Mathematics. SIAM, Philadelphia (1992)
4. Davidson, T.N., Luo, Z.-Q., Wong, K.M., Zhang, J.-K.: Design of interpolating biorthogonal multiwavelet systems with compact support. Appl. Comput. Harmon. Anal. **11**, 420–438 (2001)
5. Dubuc, S., Merrien, J.L.: Dyadic Hermite interpolation on a rectangular mesh. Adv. Comput. Math. **10**, 343–365 (1999)
6. Han, B.: Analysis and construction of optimal multivariate biorthogonal wavelets with compact support. SIAM J. Math. Anal. **31**(2), 274–304 (1999)
7. Han, B.: Approximation properties and construction of Hermite interpolants and biorthogonal multiwavelets. J. Approx. Theory **110**(1), 18–53 (2001)
8. Han, B.: Vector cascade algorithms and refinable function vectors in Sobolev spaces. J. Approx. Theory **124**, 44–88 (2003)
9. Han, B., Jia, R.-Q.: Multivariate refinement equations and convergence of subdivision scheme. SIAM J. Math. Anal. **29**(5), 1177–1199 (1998)
10. Han, B., Mo, Q.: Multiwavelet frames from refinable function vectors. Adv. Comput. Math. **18**, 211–245 (2003)
11. Han, B., Piper, B., Yu, P.-Y.: Multivariate refinable Hermite interpolants. Math. Comput. **73**(248), 1913–1935 (2004)
12. http://www.mathematik.uni-marburg.de/~dahlke/ag-numerik/research/software
13. Jia, R.-Q., Jiang, Q.-T.: Spectral analysis of the transition operator and its applications to smoothness analysis of wavelets. SIAM J. Matrix Anal. Appl. **24**(4), 1071–1109 (2003)
14. Jia, R.-Q., Micchelli, C.A.: On linear independence of integer translates of a finite number of functions. Proc. Edinb. Math. Soc. (1992), 69–85
15. Koch, K.: Interpolating scaling vectors. Int. J. Wavelets Multiresolut. Inf. Process. **3**, 389–416 (2005)
16. Selesnick, I.W.: Interpolating multiwavelet bases and the sampling theorem. IEEE Trans. Signal Process. **47**(6), 1615–1621 (1999)
17. Zhou, D.-X.: Interpolatory orthogonal multiwavelets and refinable functions. IEEE Trans. Signal Process. **50**, 520–527 (2002)