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Multipliers, Phases and Connectivity of MRA Wavelets in $L^2(\mathbb{R}^2)$

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Abstract Let A be any 2×2 real expansive matrix. For any A-dilation wavelet ψ, let $\widehat{\psi}$ be its Fourier transform. A measurable function f is called an A-dilation wavelet multiplier if the inverse Fourier transform of $(f\widehat{\psi})$ is an A-dilation wavelet for any A-dilation wavelet ψ. In this paper, we give a complete characterization of all A-dilation wavelet multipliers under the condition that A is a 2×2 matrix with integer entries and $|\det(A)| = 2$. Using this result, we are able to characterize the phases of A-dilation wavelets and prove that the set of all A-dilation MRA wavelets is path-connected under the $L^2(\mathbb{R}^2)$ norm topology for any such matrix A.

Keywords Wavelet multipliers · Phase of wavelet · A-dilation MRA wavelets · Connectivity of wavelets

Mathematics Subject Classification (2000) 42C15 · 42C40

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1 Introduction

One natural problem in wavelet theory concerns the construction of different wavelets. Naturally, one may attempt to construct new wavelets from an existing one. This approach leads to the concept of wavelet multipliers [6]. In the one-dimensional case, wavelet multipliers have been studied extensively and characterized completely [16, 20]. Another area of study in wavelet theory concerns the topological properties of various classes of wavelets. One well known problem in this area asks whether the collection of all or some orthonormal wavelets is path-connected under the $L^2(\mathbb{R})$ norm [6, 20]. In fact, it is still an open question whether the set of all orthonormal wavelets is path-connected under the $L^2(\mathbb{R})$ norm. However, it is proved in [20] that the set of all MRA wavelets is path-connected under the $L^2(\mathbb{R})$ norm. Furthermore, the use of wavelet multipliers played a key role in the establishment of this important result.

The main purpose of this paper is to extend the above mentioned results to the two-dimensional case.

Let A be a 2×2 real expansive matrix, i.e., a matrix with real entries whose eigenvalues are all of modules greater than one. Let $L^2(\mathbb{R}^2)$ be the set of all square Lebesgue integrable functions in $\mathbb{R}^2$. An *A-dilation wavelet* is a function $\psi\in L^2(\mathbb{R}^2)$ such that the set

$$\{|\det A|^{\frac{n}{2}}\psi(A^n\mathbf{t}-\ell): n\in\mathbb{Z}, \ell\in\mathbb{Z}^2\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^2)$. For any function $f(\mathbf{t})\in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$, its Fourier transform is defined by

$$(\mathcal{F}f)(\mathbf{s})=\widehat{f}(\mathbf{s})=\frac{1}{2\pi}\int_{\mathbb{R}^2} f(\mathbf{t})e^{-i\mathbf{t}\circ\mathbf{s}}d\mu, \tag{1.1}$$

where μ denotes the Lebesgue measure in $\mathbb{R}^2$ and $\mathbf{t}\circ\mathbf{s}$ is the standard inner product of the vectors $\mathbf{s},\ \mathbf{t}\in\mathbb{R}^2$. The inverse Fourier transform will be denoted by $\mathcal{F}^{-1}$.

A measurable function f is called an *A-dilation wavelet multiplier* if the inverse Fourier transform of $(f\widehat{\psi})$ is an A-dilation wavelet for any A-dilation wavelet ψ.

A matrix is called an integral matrix if its entries are all integers. In this paper, we will only consider 2×2 expansive integral matrices A such that $|\det(A)|=2$. Although it is possible for the dilation matrix A to be non-integral, such a matrix must be accompanied by a full rank lattice Γ that is compatible with it (namely that $A\Gamma\subset\Gamma$ must hold). (A,Γ) is called an "admissible pair" in [14]. For an admissible pair (A,Γ), one can simplify the problem by a suitable linear transformation $x\mapsto Px$ which takes (A,Γ) to $(PAP^{-1}, P\Gamma)$. If one chooses P such that $P\Gamma=\mathbb{Z}^2$, then PAP^{-1} is an integral matrix. In other words, we can always simplify the problem to the case where A is integral. Furthermore, in this paper we are only interested in MRA systems generated by a single wavelet function. It is known that in the higher-dimensional case, such system exists only when $|\det(A)|=2$ [12, 17]. From now on, all matrices will be 2×2 matrices with such properties unless otherwise stated.

There have been some attempts to characterize A-dilation wavelet multipliers in the two-dimensional case. For example, in [15], a characterization of A-dilation wavelet multipliers is given for the following two specific 2×2 matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

Moreover, it is proven there that for any given A-dilation wavelet ψ_0 (under the above choices of A), the set $\mathcal{M}_{\psi_0} = \{\psi : \widehat{\psi} = v\hat{\psi}_0\}$ where v is an A-dilation wavelet multiplier is path-connected.

In this paper, we generalize the above result to all 2×2 expansive matrices with integer entries such that $|\det(A)| = 2$. We will derive an explicit formula that can be used to construct all A-dilation wavelet multipliers for such matrices A. We then prove that the set of all A-dilation MRA wavelets is path-connected under the $L^2(\mathbb{R}^2)$ norm. We also obtain a characterization of the phases of A-dilation MRA wavelets as an application of the wavelet multipliers.

The rest of the paper is organized as follows. In the next section, we introduce the notations and terms needed for this paper, with some preliminary results needed in later sections. In Sect. 3 we discuss the relationship between wavelets with integrally similar dilation matrices and show that we need only to consider six dilation matrices. Section 4 gives two special MRA wavelets which will be used in Sect. 7. In Sect. 5 we characterize wavelet multipliers in the two-dimensional case. Section 6 is devoted to the phases of A-dilation MRA wavelets. Finally, in Sect. 7 we prove that the set of all A-dilation MRA wavelets is path-connected.

2 Notations, Definitions and Preliminary Results

For a given expansive integral matrix A (such that $|\det(A)| = 2$), we will use T^ℓ, D_A as the translation and dilation unitary operators acting on $L^2(\mathbb{R}^2)$ defined by $(T^\ell f)(\mathbf{t}) = f(\mathbf{t} - \ell)$, $(D_A f)(\mathbf{t}) = |\det(A)|^{\frac{1}{2}} f(A\mathbf{t})$ $\forall f \in L^2(\mathbb{R}^2)$, $\mathbf{t} \in \mathbb{R}^2$ and $\ell \in \mathbb{Z}^2$.

Definition 2.1 A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^2)$ is called an A-dilation multi-resolution analysis (or A-dilation MRA for short) if the following hold:

(i) $V_j \subset V_{j+1}, \forall j \in \mathbb{Z}$;
(ii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^2)$;
(iii) $f(\mathbf{t}) \in V_j$ if and only if $f(A^{-j}\mathbf{t}) \in V_0$ for $j \in \mathbb{Z}$; and
(iv) There exists $\phi(\mathbf{t})$ in V_0 such that $\{\phi(\mathbf{t} - \ell) : \ell \in \mathbb{Z}^2\}$ is an orthonormal basis for V_0.

The function $\phi(\mathbf{t})$ defined in (iv) above is called an A-dilation scaling function for the MRA. In our case, it is known that a single A-dilation wavelet can be derived from the above A-dilation MRA [17]. An A-dilation wavelet ψ so obtained is called an MRA wavelet (and $\psi \in V_1 \cap V_0^\perp$). For any $f \in V_1$, $f(A^{-1}\mathbf{t}) \in V_0$ hence we have

$$f(\mathbf{t}) = |\det(A)| \sum_{\ell\in\mathbb{Z}^2} c_\ell \phi(A\mathbf{t} - \ell). \tag{2.1}$$

If we define $m_f(\mathbf{s}) = \sum_{\ell\in\mathbb{Z}^2} c_\ell e^{-i\ell\circ\mathbf{s}}$, then by taking Fourier transform on both sides of (2.1) we obtain $\widehat{f}(A^\tau\mathbf{s}) = m_f(\mathbf{s})\widehat{\phi}(\mathbf{s})$, where A^τ is the transpose of A. In particular, we have

$$\widehat{\phi}(A^\tau\mathbf{s}) = m(\mathbf{s})\widehat{\phi}(\mathbf{s}) \tag{2.2}$$

for some function $m(s)$ of the form similar to (2.1). The function $m(s)$ is called the *low pass A-dilation filter* of the corresponding A-dilation MRA.

Recall that a measurable function f is called an A-dilation wavelet multiplier if the inverse Fourier transform of $(f\hat{\psi})$ is an A-dilation wavelet whenever ψ is an A-dilation wavelet. A measurable function $f(\mathbf{t}) \in L^2(\mathbb{R}^2)$ is called a $2\pi\mathbb{Z}^2$-translation periodic if $f(\mathbf{t}+2\pi\ell) = f(\mathbf{t})$ a.e. on $\mathbb{R}^2$ for any $\ell\in\mathbb{Z}^2$, and f is called A-dilation periodic if $f(A\mathbf{t}) = f(\mathbf{t})$ a.e. on $\mathbb{R}^2$. Furthermore, f is called A-dilation-translation compatible if there exists a $2\pi\mathbb{Z}^2$-translation periodic function $k(\mathbf{t})$ such that $f(A\mathbf{t}) = k(\mathbf{t})f(\mathbf{t})$. Apparently, the function $m_f(\mathbf{s})$ and the low pass A-dilation filter defined above are $2\pi\mathbb{Z}^2$-translation periodic functions.

The following lemmas are well known results and can be easily obtained by standard arguments [1, 9, 13].

Lemma 2.1 *ψ is an A-dilation wavelet iff the following conditions hold*

(i) $\|\psi\|_2 = 1$;
(ii) $\sum_{j\in\mathbb{Z}} |\widehat{\psi}((A^\tau)^j\mathbf{s})|^2 = 1/(2\pi)^2$ *a.e. and*
(iii) $\sum_{j=0}^{\infty} \widehat{\psi}((A^\tau)^j\mathbf{s})\overline{\widehat{\psi}((A^\tau)^j(\mathbf{s}+2\pi\ell))} = 0$ *a.e.* $\forall\ell\in\mathbb{Z}^2\setminus A^\tau\mathbb{Z}^2$.

Lemma 2.2 *An A-dilation wavelet ψ is an A-dilation MRA wavelet iff*

$$D_\psi(\mathbf{s}) = \sum_{n=1}^{\infty}\sum_{\ell\in\mathbb{Z}^2} |\widehat{\psi}((A^\tau)^n(\mathbf{s}+2\pi\ell))|^2 = \frac{1}{(2\pi)^2} \quad a.e. \tag{2.3}$$

Lemma 2.3 *ϕ is an A-dilation scaling function for an MRA iff the following conditions hold*

(i) $\sum_{\ell\in\mathbb{Z}^2} |\widehat{\phi}(\mathbf{s}+2\pi\ell)|^2 = 1/(2\pi)^2$ *a.e.*;
(ii) $\lim_{j\to\infty} |\widehat{\phi}((A^\tau)^{-j}\mathbf{s})| = 1/2\pi$ *a.e. and*
(iii) *there exists a $2\pi\mathbb{Z}^2$-translation periodic function $m(\mathbf{s}) \in L^2([-\pi,\pi)^2)$ such that $\widehat{\phi}(A^\tau\mathbf{s}) = m(\mathbf{s})\widehat{\phi}(\mathbf{s})$.*

Lemma 2.4 *Suppose that ψ is an A-dilation MRA wavelet with scaling function ϕ, then*

$$|\widehat{\phi}(\mathbf{s})|^2 = \sum_{j=1}^{\infty} |\widehat{\psi}((A^\tau)^j\mathbf{s})|^2 \quad a.e. \tag{2.4}$$

Since $|\det(A)| = 2$, the quotient group $\mathbb{Z}^2/A^\tau\mathbb{Z}^2$ has only 2 elements. Let $\ell + A^\tau\mathbb{Z}^2$ be the non-zero element in $\mathbb{Z}^2/A^\tau\mathbb{Z}^2$, where $\ell\in\mathbb{Z}^2$ is a representative of the corresponding coset. Then we have $(A^\tau)^{-1}\ell\notin\mathbb{Z}^2$. Since $|\det((A^\tau)^{-1})| = \frac{1}{2}$ and

$2(A^\tau)^{-1}$ is an integral matrix, there is a unique element $\mathbf{h}_0 \in \{(1/2, 0)^\tau, (0, 1/2)^\tau, (1/2, 1/2)^\tau\}$ such that $(A^\tau)^{-1}\ell \in \mathbf{h}_0 + \mathbb{Z}^2$. Let $\mathbf{u}$ be a constant vector such that $\mathbf{h}_0 \circ \mathbf{u} = 1/2$. We have the following two propositions.

Proposition 2.1 *Let $\phi \in L^2(\mathbb{R}^2)$ be an A-dilation scaling function for an A-dilation MRA $\{V_j\}$ and let m be its associated low pass filter. Let $\psi \in W_0 = V_1 \cap V_0^\perp$, then $\{\psi(\mathbf{t} - \ell) : \ell \in \mathbb{Z}^2\}$ is an orthonormal basis for W_0 iff*

$$\widehat{\psi}(A^\tau \mathbf{s}) = e^{i\mathbf{s}\circ\mathbf{u}} v(A^\tau \mathbf{s}) \overline{m(\mathbf{s} + 2\pi \mathbf{h}_0)} \widehat{\phi}(\mathbf{s}) \quad a.e., \tag{2.5}$$

where v is a $2\pi\mathbb{Z}^2$-translation periodic measurable function with $|v(\mathbf{s})| = 1$ a.e. on $\mathbb{R}^2$.

Let us give an outline of the proof for Proposition 2.1. From the discussion following (2.1), we have $\widehat{\psi}(A^\tau \mathbf{s}) = m_\psi(\mathbf{s})\widehat{\phi}(\mathbf{s})$ for some $2\pi\mathbb{Z}^2$-translation periodic function m_ψ. Again, standard arguments show that $\{\psi(\mathbf{t} - \ell) : \ell \in \mathbb{Z}^2\}$ is an orthonormal basis for W_0 iff equations $|m(\mathbf{s})|^2 + |m(\mathbf{s} + 2\pi \mathbf{h}_0)|^2 = 1$, $|m_\psi(\mathbf{s})|^2 + |m_\psi(\mathbf{s} + 2\pi \mathbf{h}_0)|^2 = 1$ and $m(\mathbf{s})\overline{m_\psi(\mathbf{s})} + m(\mathbf{s} + 2\pi \mathbf{h}_0)\overline{m_\psi(\mathbf{s} + 2\pi \mathbf{h}_0)} = 0$ hold. The reader can verify that the solution for $m_\psi(\mathbf{s})$ (in terms of $m(\mathbf{s})$) is of the form given in the proposition.

Proposition 2.2 *Let ψ be an A-dilation MRA wavelet. Then $e^{i\mathbf{s}\circ\mathbf{u}_1}|\widehat{\psi}(\mathbf{s})|$ is the Fourier transform of an A-dilation MRA wavelet, where $\mathbf{u}_1 = A^{-1}\mathbf{u}$ and $\mathbf{u}$ is the constant vector defined before Proposition 2.1.*

Proof Let ϕ be the corresponding scaling function with low pass filter m, then $\mathcal{F}^{-1}(|\widehat{\phi}|)$ is also an A-dilation scaling function whose associated low pass filter is $|m|$ by Lemma 2.3. Thus, the function ψ_1 defined by

$$\widehat{\psi_1}(A^\tau \mathbf{s}) = e^{i\mathbf{s}\circ\mathbf{u}} |\overline{m(\mathbf{s} + 2\pi \mathbf{h}_0)} \widehat{\phi}(\mathbf{s})| = e^{i\mathbf{s}\circ\mathbf{u}} |\psi(A^\tau \mathbf{s})|$$

is an A-dilation MRA wavelet. The result follows after a simple substitution $\mathbf{t} = A^\tau \mathbf{s}$. □

3 Systems with Integrally Similar Dilation Matrices

Two $d \times d$ integral matrices B and C are said to be *integrally similar* if there exists an integral $d \times d$ matrix P such that $|\det(P)| = 1$ and $P^{-1}BP = C$. In such cases we write $B \sim C$. The main result of this section is the following theorem which reveals the relation between wavelets under integrally similar dilation matrices.

Theorem 3.1 *For any 2×2 integral matrix P with $|\det(P)| = 1$, let $\Phi_P : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}^2)$ be the operator defined by $\Phi_P(g(\mathbf{t})) = g(P\mathbf{t})$. If B and C are two 2×2 integral, expansive matrices such that $P^{-1}BP = C$, then the following statements hold*

(i) *ψ is a B-dilation wavelet iff $\Phi_P(\psi)$ is a C-dilation wavelet*;

(ii) *A function* $f \in L^2(\mathbb{R}^2)$ *is a* B*-dilation wavelet multiplier iff the function* $\Phi_{(P^\tau)^{-1}}(f)$ *is a* C*-dilation wavelet multiplier.*

Proof (i) It suffices to show that $\{D_B^n T^\ell \psi(\mathbf{t})\}$ ($n \in \mathbb{Z}$ and $\ell \in \mathbb{Z}^2$) is an orthonormal basis of $L^2(\mathbb{R}^2)$ iff $\{D_C^n T^\ell \psi_C\}$ is an orthonormal basis of $L^2(\mathbb{R}^2)$, where $\psi_C = \Phi_P(\psi)$. Since $|\det(P)| = 1$, $P\mathbb{Z}^2 = \mathbb{Z}^2$, a simply variable substitution $P\mathbf{t} = \mathbf{s}$ shows that $\{|\det(B)|^{n/2}\psi(B^n P\mathbf{t} - P\ell)\}$ is an orthonormal basis of $L^2(\mathbb{R}^2)$ iff $\{|\det(B)|^{n/2}\psi(B^n\mathbf{t} - \ell)\} = \{D_B^n T^\ell \psi(\mathbf{t})\}$ is an orthonormal basis of $L^2(\mathbb{R}^2)$. But a direct computation shows that

$$\begin{aligned} D_C^n T^\ell \psi_C(\mathbf{t}) &= D_C^n T^\ell \psi(P\mathbf{t}) = D_C^n \psi(P(\mathbf{t} - \ell)) \\ &= |\det(C)|^{n/2}\psi(P(C^n\mathbf{t} - \ell)) = |\det(B)|^{n/2}\psi((PC^nP^{-1})P\mathbf{t} - P\ell) \\ &= |\det(B)|^{n/2}\psi(B^nP\mathbf{t} - P\ell). \end{aligned}$$

(ii) Let f be a B-dilation wavelet multiplier and let ψ_C be a C-dilation wavelet. By (i) above, there exists a B-dilation wavelet ψ such that $\psi_C(\mathbf{t}) = \psi(P\mathbf{t})$. We have

$$\begin{aligned} \widehat{\psi_C}(\mathbf{s}) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \psi_C(\mathbf{t})e^{-i\mathbf{t}\circ\mathbf{s}}d\mathbf{t} = \frac{1}{2\pi}\int_{\mathbb{R}^2} \psi(P\mathbf{t})e^{-i\mathbf{t}\circ\mathbf{s}}d\mathbf{t} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \psi(\mathbf{t})e^{-iP^{-1}\mathbf{t}\circ\mathbf{s}}d\mathbf{t} = \frac{1}{2\pi}\int_{\mathbb{R}^2} \psi(\mathbf{t})e^{-i\mathbf{t}\circ(P^\tau)^{-1}\mathbf{s}}d\mathbf{t} \\ &= \widehat{\psi}((P^\tau)^{-1}\mathbf{s}). \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{F}^{-1}(f_C\widehat{\psi_C})(\mathbf{t}) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} f((P^\tau)^{-1}\mathbf{s})\widehat{\psi}((P^\tau)^{-1}\mathbf{s})e^{i\mathbf{s}\circ\mathbf{t}}d\mathbf{s} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} f(\mathbf{s})\widehat{\psi}(\mathbf{s})e^{iP^\tau\mathbf{s}\circ\mathbf{t}}d\mathbf{s} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} f(\mathbf{s})\widehat{\psi}(\mathbf{s})e^{i\mathbf{s}\circ P\mathbf{t}}d\mathbf{s} \\ &= \mathcal{F}^{-1}(f\widehat{\psi})(P\mathbf{t}). \end{aligned}$$

By the definition of f, $\mathcal{F}^{-1}(f\widehat{\psi})(\mathbf{t})$ is a B-dilation wavelet. Thus by (i) again, $\mathcal{F}^{-1}(f\widehat{\psi})(P\mathbf{t})$ (hence $\mathcal{F}^{-1}(f_C\widehat{\psi_C})(\mathbf{t})$) is a C-dilation wavelet. This proves that f_C is a C-dilation wavelet multiplier. On the other hand, if f_C is a C-dilation wavelet multiplier, reversing the above argument shows that f is a B-dilation wavelet multiplier. □

Remark 3.1 The linear operator $\Phi_P : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}^2)$ defined above is obviously continuous and unitary (since $|\det(P)| = 1$). In the case that P is also integral and $P^{-1}BP = C$, then Theorem 3.1 asserts that $\Phi_P\psi : \mathcal{W}_B \longrightarrow \mathcal{W}_C$ is a continuous and bijective mapping, where $\mathcal{W}_B$ is the set of all B-dilation wavelets and $\mathcal{W}_C$ is the set of all C-dilation wavelets.

Remark 3.2 Using (2.3) and $\widehat{\psi_C}(\mathbf{s}) = \widehat{\psi}((P^\tau)^{-1}\mathbf{s})$ as shown in the proof of Theorem 3.1(ii), it is easy to see that in the case $B \sim C$ by the relation $P^{-1}BP = C$, the operator Φ_P is also a bijection between the set of all B-dilation MRA wavelets and the set of all C-dilation MRA wavelets.

We will now turn our focus on 2×2 integral expansive matrices A with the property $|\det(A)| = 2$. It turns out that there are exactly six integrally similar classes of such integral matrices [14]. A representative from each of these classes is listed below.

$$A_1 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix},$$

and $A_5 = -A_3$, $A_6 = -A_4$.

For the rest of this paper, we will only consider the case where A is one of the above six matrices. By Theorem 3.1 (as well as Remarks 3.1 and 3.2), the discussion of a different 2×2 expansive integral matrix B (with $|\det(B)| = 2$) can be converted to a discussion concerning one of the six matrices listed above by applying the operator Φ_P for some suitable P. For the sake of convenience, let us give the vectors $\mathbf{h}_0$, $\mathbf{u}$ and $A^{-1}\mathbf{u}$ used in Propositions 2.1 and 2.2 here. We can choose $\mathbf{u} = (1,0)^\tau$ for all cases. For $A = A_1$ or $A = A_2$, $\mathbf{h}_0 = (1/2, 0)^\tau$, $\mathbf{u}_1 = A^{-1}\mathbf{u} = (0, 1/2)^\tau$; for $A = \pm A_3$, $\mathbf{h}_0 = (1/2, 1/2)^\tau$, $\mathbf{u}_1 = A^{-1}\mathbf{u} = \pm(1/2, 1/2)^\tau$; for $A = \pm A_4$, $\mathbf{h}_0 = (1/2, 0)^\tau$ and $\mathbf{u}_1 = A^{-1}\mathbf{u} = \pm(1/2, 1/2)^\tau$. Throughout the rest of the paper, $\mathbf{h}_0$, $\mathbf{u}$ and $\mathbf{u}_1$ are so defined with respect to their corresponding dilation matrix A.

4 Examples of Haar and Shannon Type A-dilation Wavelets

Example 4.1 The construction of the Haar-type A-dilation wavelet given here can be found in [4, 11, 14]. The low pass filter m is $m(\mathbf{s}) = \frac{1}{2}(1 + e^{-i\mathbf{s}\circ\mathbf{u}})$, $\widehat{\phi}(\mathbf{s})$ is defined by $\widehat{\phi}(\mathbf{s}) = (1/2\pi)\prod_{j=1}^{\infty} m((A^\tau)^{-j}\mathbf{s})$, and ψ is defined by

$$\widehat{\psi}(\mathbf{s}) = e^{i\mathbf{s}\circ\mathbf{u}_1}\overline{m((A^\tau)^{-1}\mathbf{s} + 2\pi\mathbf{h}_0)}\widehat{\phi}((A^\tau)^{-1}\mathbf{s}). \tag{4.1}$$

Example 4.2 The Shannon type A-dilation MRA wavelet in this example is constructed using the concept of wavelet sets [6, 7, 12]. For each matrix A, we construct a scaling set F such that the set $E = A^\tau F \setminus F$ is an A-dilation wavelet set, i.e., the function $\frac{1}{2\pi}\chi_E$ is the Fourier transform of an A-dilation wavelet. Let Ω be the set $[-\pi, \pi)^2$. The low pass filter, scaling function and wavelet are given by

$$m(\mathbf{s})|_\Omega = \chi_{(A^\tau)^{-1}\Omega}, \ \widehat{\phi}(\mathbf{s}) = \frac{1}{2\pi}\chi_\Omega \text{ and } \widehat{\psi}(\mathbf{s}) = \frac{1}{2\pi}e^{i\mathbf{s}\circ\mathbf{u}_1}\chi_{A^\tau\Omega\setminus\Omega}.$$

Notice that $m(\mathbf{s})$ is a $2\pi\mathbb{Z}^2$-translation periodic and the above formula gives its definition in one complete period (i.e., $\Omega = [-\pi, \pi)^2$). The wavelet set $E = A^\tau\Omega\setminus\Omega$

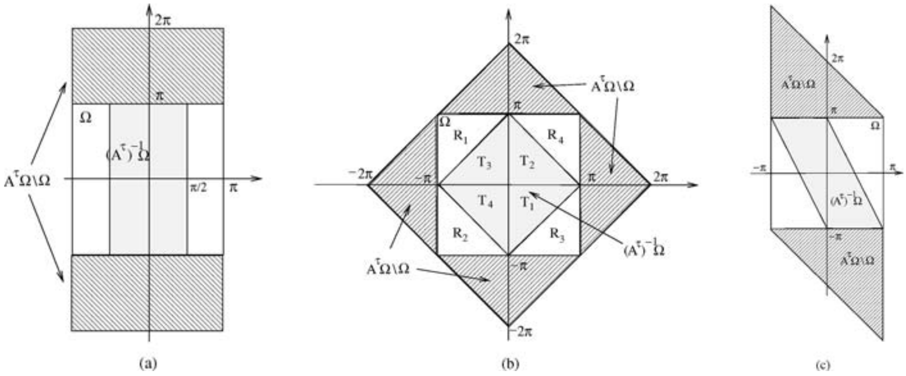

Fig. 1 The supports of $m, \widehat{\phi}$ and $\widehat{\psi}$: (**a**) is for the case of $A = A_1$ or $A = A_2$, (**b**) is for the case of $A = A_3$, A_5 and (**c**) is for the case of $A = A_4$, A_6

(which is the support of $\widehat{\psi}$), the supports of $\widehat{\phi}(\mathbf{s})$ (i.e., Ω) and $m(\mathbf{s})$ (within Ω) are shown in Figs. 1(a) to 1(c) for each case of A.

Remark 4.1 In fact, the function $\widehat{\psi_0}(\mathbf{s}) = \frac{1}{2\pi}\chi_{A^\tau\Omega\backslash\Omega}$ is itself the Fourier transform of an A-dilation MRA wavelet. From this fact, the above results on ψ can also be derived from Proposition 2.2 directly.

5 A-dilation Wavelet Multipliers

In this section, we characterize the A-dilation wavelet multipliers. A necessary condition for a function f to be an A-dilation wavelet multiplier is that $|f| = 1$ [6, 15, 20]. Thus in the following we will limit our discussion to such functions. Instead of trying to characterize the scaling function multiplier or the low pass filter multiplier (which is the approach used in [15]), we will use a different approach. Let us call a function f with the property $|f| = 1$ a *unimodular function*.

Theorem 5.1 *A unimodular function $f \in L^\infty(\mathbb{R}^2)$ is an A-dilation wavelet multiplier iff the function $k(\mathbf{s}) = f(A^\tau \mathbf{s})/f(\mathbf{s})$ is $2\pi\mathbb{Z}^2$-translation periodic.*

Proof "$\Longleftarrow$" Assume that $f \in L^\infty(\mathbb{R}^2)$ is a unimodular function and that $k(\mathbf{s}) = f(A^\tau \mathbf{s})/f(\mathbf{s})$ is $2\pi\mathbb{Z}^2$-translation periodic. To show that f is a wavelet multiplier, we need to show that for any A-dilation wavelet ψ, $\eta = \mathcal{F}^{-1}(f\widehat{\psi})$ is also a wavelet. It suffices to verify that $\widehat{\eta}$ satisfies conditions (ii) and (iii) in Lemma 2.1. It is easy to see that (ii) holds for $\widehat{\eta}$ since $|\widehat{\eta}| = |\widehat{\psi}|$ and (ii) holds for $\widehat{\psi}$. Applying the relation $f(A^\tau \mathbf{s}) = k(\mathbf{s})f(\mathbf{s})$ repeatedly, for any $j \geq 1$ and $\ell \in \mathbb{Z}^2$, we obtain

$$f((A^\tau)^j \mathbf{s}) = k((A^\tau)^{j-1}\mathbf{s}) \cdots k(A^\tau \mathbf{s})k(\mathbf{s})f(\mathbf{s}), \tag{5.1}$$

and

$$
\begin{aligned}
f((A^\tau)^j(\mathbf{s}+2\pi\ell)) &= k((A^\tau)^{j-1}(\mathbf{s}+2\pi\ell))k((A^\tau)^{j-2}(\mathbf{s}+2\pi\ell)) \\
&\quad \cdots k(A^\tau(\mathbf{s}+2\pi\ell))k(\mathbf{s}+2\pi\ell)f(\mathbf{s}+2\pi\ell) \\
&= k((A^\tau)^{j-1}\mathbf{s})\cdots k(A^\tau\mathbf{s})k(\mathbf{s})f(\mathbf{s}+2\pi\ell).
\end{aligned}
$$

Since $k(\mathbf{s})$ is unimodular, this leads to

$$
\begin{aligned}
&f((A^\tau)^j\mathbf{s})\cdot\overline{f((A^\tau)^j(\mathbf{s}+2\pi\ell))} \\
&\quad = k((A^\tau)^{j-1}\mathbf{s})\cdots k(A^\tau\mathbf{s})k(\mathbf{s})f(\mathbf{s})\cdot\overline{k((A^\tau)^{j-1}\mathbf{s})\cdots k(A^\tau\mathbf{s})k(\mathbf{s})f(\mathbf{s}+2\pi\ell)} \\
&\quad = f(\mathbf{s})\overline{f(\mathbf{s}+2\pi\ell)}
\end{aligned}
$$

for any $j\geq 0$ and $\ell\in\mathbb{Z}^2$. Thus

$$
\begin{aligned}
&\sum_{j=0}^{\infty}\widehat{\eta}((A^\tau)^j\mathbf{s})\overline{\widehat{\eta}((A^\tau)^j(\mathbf{s}+2\pi\ell))} \\
&\quad = \sum_{j=0}^{\infty}[f((A^\tau)^j\mathbf{s})\overline{f((A^\tau)^j(\mathbf{s}+2\pi\ell))}\cdot\widehat{\psi}((A^\tau)^j\mathbf{s})\overline{\widehat{\psi}((A^\tau)^j(\mathbf{s}+2\pi\ell))}] \\
&\quad = \sum_{j=0}^{\infty} f(\mathbf{s})\overline{f(\mathbf{s}+2\pi\ell)}\widehat{\psi}((A^\tau)^j\mathbf{s})\overline{\widehat{\psi}((A^\tau)^j(\mathbf{s}+2\pi\ell))} \\
&\quad = f(\mathbf{s})\overline{f(\mathbf{s}+2\pi\ell)}\sum_{j=0}^{\infty}\widehat{\psi}((A^\tau)^j\mathbf{s})\overline{\widehat{\psi}((A^\tau)^j(\mathbf{s}+2\pi\ell))}=0
\end{aligned}
$$

for any $\ell\in\mathbb{Z}^2\backslash A^\tau\mathbb{Z}^2$. So condition (iii) of Lemma 2.1 holds for $\widehat{\eta}$ as well.

"$\Longrightarrow$" We need to show that $k(\mathbf{s})=f(A^\tau\mathbf{s})/f(\mathbf{s})$ is $2\pi\mathbb{Z}^2$-translation periodic. Let ψ be any A-dilation MRA wavelet such that $\mathrm{supp}(\widehat{\psi})=\mathbb{R}^2$. Such ψ exists. For example the A-dilation wavelet constructed in Example 4.1 has such a property. By Proposition 2.2, the function $\psi_1(t)$ defined by

$$
\widehat{\psi_1}=e^{i\mathbf{s}\circ\mathbf{u}_1}|\widehat{\psi}(\mathbf{s})|=e^{i\mathbf{s}\circ\mathbf{u}_1}|\widehat{\psi_1}(\mathbf{s})| \tag{5.2}
$$

is an A-dilation wavelet. Since $\mathcal{F}^{-1}(f\widehat{\psi_1})$ is also an A-dilation wavelet, $\widehat{\psi_1}$ and $f\widehat{\psi_1}$ both satisfy condition (iii) of Lemma 2.1, i.e.,

$$
\sum_{j=0}^{\infty}\widehat{\psi_1}((A^\tau)^j\mathbf{s})\cdot\overline{\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))}=0\quad a.e.\quad\text{and} \tag{5.3}
$$

$$
\sum_{j=0}^{\infty} f((A^\tau)^j\mathbf{s})\widehat{\psi_1}((A^\tau)^j\mathbf{s})\cdot\overline{f((A^\tau)^j(\mathbf{s}+2\pi\ell))\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))}=0\quad a.e. \tag{5.4}
$$

for any $\ell\in\mathbb{Z}^2\setminus A^\tau\mathbb{Z}^2$. Since $\ell\in\mathbb{Z}^2\setminus A^\tau\mathbb{Z}^2$, there exists $\ell_1\in\mathbb{Z}^2$ such that $\ell=\ell_0+A^\tau\ell_1=A^\tau(\mathbf{h}_0+\ell_1)$. It follows that $\ell\circ\mathbf{u}_1=A^\tau(\mathbf{h}_0+\ell_1)\circ A^{-1}\mathbf{u}=(\mathbf{h}_0+\ell_1)\circ\mathbf{u}=$

$1/2+m$, where $\mathbf{h}_0 \circ \mathbf{u}$ by the definition of $\mathbf{h}_0$ and $\mathbf{u}$, and $m = \ell_1 \circ \mathbf{u}$ is an integer. Thus

$$\begin{aligned}\widehat{\psi_1}(\mathbf{s})\overline{\widehat{\psi_1}(\mathbf{s}+2\pi\ell)} &= e^{i\mathbf{s}\circ\mathbf{u}_1}|\widehat{\psi_1}(\mathbf{s})| \cdot e^{-i(\mathbf{s}+2\pi\ell)\circ\mathbf{u}_1}|\widehat{\psi_1}(\mathbf{s}+2\pi\ell)| \\ &= e^{i(-\pi-2m\pi)}|\widehat{\psi_1}(\mathbf{s})| \cdot |\widehat{\psi_1}(\mathbf{s}+2\pi\ell)| = -|\widehat{\psi_1}(\mathbf{s})| \cdot |\widehat{\psi_1}(\mathbf{s}+2\pi\ell)|.\end{aligned}$$

On the other hand, for any $j>0$, $(A^\tau)^j\ell \circ \mathbf{u}_1 = \ell \circ A^{-j-1}\mathbf{u} \in \mathbb{Z}$ and hence

$$\begin{aligned}&\widehat{\psi_1}((A^\tau)^j\mathbf{s})\overline{\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))} \\ &\quad = e^{i(A^\tau)^j\mathbf{s}\circ\mathbf{u}_1}|\widehat{\psi_1}((A^\tau)^j\mathbf{s})| \cdot e^{-i((A^\tau)^j(\mathbf{s}+2\pi\ell)\circ\mathbf{u}_1}|\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))| \\ &\quad = |\widehat{\psi_1}((A^\tau)^j\mathbf{s})| \cdot |\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))|.\end{aligned}$$

Thus, (5.3) and (5.4) can be rewritten as

$$\begin{aligned}&|\widehat{\psi_1}(\mathbf{s})| \cdot |\widehat{\psi_1}(\mathbf{s}+2\pi\ell)| \\ &\quad = \sum_{j=1}^{\infty} |\widehat{\psi_1}((A^\tau)^j\mathbf{s})| \cdot |\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))| \quad \text{and} \qquad (5.5)\end{aligned}$$

$$\begin{aligned}&f(\mathbf{s})\overline{f(\mathbf{s}+2\pi\ell)} \cdot |\widehat{\psi_1}(\mathbf{s})| \cdot |\widehat{\psi_1}(\mathbf{s}+2\pi\ell)| \\ &\quad = \sum_{j=1}^{\infty} f((A^\tau)^j\mathbf{s})\overline{f((A^\tau)^j(\mathbf{s}+2\pi\ell))}|\widehat{\psi_1}((A^\tau)^j\mathbf{s})| \cdot |\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))|. \qquad (5.6)\end{aligned}$$

Since f is unimodular, $\overline{f} = 1/f$. Hence (5.6) can be rewritten as

$$\begin{aligned}&\frac{f(\mathbf{s})}{f(\mathbf{s}+2\pi\ell)}|\widehat{\psi_1}(\mathbf{s})| \cdot |\widehat{\psi_1}(\mathbf{s}+2\pi\ell)| \\ &\quad = \sum_{j=1}^{\infty} \frac{f((A^\tau)^j\mathbf{s})}{f((A^\tau)^j(\mathbf{s}+2\pi\ell))}|\widehat{\psi_1}((A^\tau)^j\mathbf{s})| \cdot |\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))|. \qquad (5.7)\end{aligned}$$

Combining this with (5.5) then leads to

$$\begin{aligned}&\sum_{j=1}^{\infty} |\widehat{\psi_1}((A^\tau)^j\mathbf{s})| \cdot |\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))| \\ &\quad = \sum_{j=1}^{\infty} \frac{f(\mathbf{s}+2\pi\ell)}{f(\mathbf{s})}\frac{f((A^\tau)^j\mathbf{s})}{f((A^\tau)^j(\mathbf{s}+2\pi\ell))}|\widehat{\psi_1}((A^\tau)^j\mathbf{s})| \cdot |\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))|. \\ &\qquad (5.8)\end{aligned}$$

Let $\beta_j(\mathbf{s}) = \frac{f(\mathbf{s}+2\pi\ell)}{f(\mathbf{s})}\frac{f((A^\tau)^j\mathbf{s})}{f((A^\tau)^j(\mathbf{s}+2\pi\ell))}$, $\mathrm{Re}\beta_j(\mathbf{s}) = a_j(\mathbf{s})$, $\mathrm{Im}\beta_j(\mathbf{s}) = b_j(\mathbf{s})$. Then (5.8) can be rewritten as

$$\sum_{j=1}^{\infty}(1-a_j(\mathbf{s}))|\widehat{\psi_1}((A^\tau)^j\mathbf{s})| \cdot |\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))|$$

$$= i\sum_{j=1}^{\infty} b_j(\mathbf{s})|\widehat{\psi_1}((A^\tau)^j\mathbf{s})|\cdot|\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))|, \tag{5.9}$$

and hence we have

$$\sum_{j=1}^{\infty}(1-a_j(\mathbf{s}))|\widehat{\psi_1}((A^\tau)^j\mathbf{s})|\cdot|\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))| = 0 \quad \text{and} \tag{5.10}$$

$$\sum_{j=1}^{\infty} b_j(\mathbf{s})|\widehat{\psi_1}((A^\tau)^j\mathbf{s})|\cdot|\widehat{\psi_1}((A^\tau)^j(\mathbf{s}+2\pi\ell))| = 0. \tag{5.11}$$

Since β_j is unimodular by its definition, we have $a_j(\mathbf{s}) \le 1$. So we must have $a_j(\mathbf{s}) = 1$ a.e. in order for (5.10) to hold. Of course this would then imply that $b_j(\mathbf{s}) = 0$ a.e. as well since $a_j^2(\mathbf{s}) + b_j^2(\mathbf{s}) = 1$. Thus,

$$\beta_j(\mathbf{s}) = \frac{f(\mathbf{s}+2\pi\ell)}{f(\mathbf{s})}\,\frac{f((A^\tau)^j\mathbf{s})}{f((A^\tau)^j(\mathbf{s}+2\pi\ell))} = 1 \quad a.e.$$

For $j = 1$, the above is equivalent to

$$\frac{f(A^\tau\mathbf{s})}{f(\mathbf{s})} = \frac{f(A^\tau(\mathbf{s}+2\pi\ell))}{f(\mathbf{s}+2\pi\ell)} \quad a.e. \quad \forall \ell \in \mathbb{Z}^2 \setminus A^\tau\mathbb{Z}^2.$$

If $\ell \in A^\tau\mathbb{Z}^2$, then $\ell - \ell_0 \notin A^\tau\mathbb{Z}^2$ since $\ell_0 \notin A^\tau\mathbb{Z}^2$. We have

$$k(\mathbf{s}+2\pi\ell) = k(\mathbf{s}+2\pi\ell_0+2\pi(\ell-\ell_0)) = k(\mathbf{s}+2\pi\ell_0) = k(\mathbf{s}).$$

Therefore, $k(\mathbf{s})$ is $2\pi\mathbb{Z}^2$-translation periodic. □

Next, we show that all A-dilation wavelet multipliers can be constructed in the way described in the following theorem. Recall that an A-dilation wavelet set E in $\mathbb{R}^2$ is a measurable set such that $\mathcal{F}^{-1}(\frac{1}{2\pi}\chi_E)$ is an A-dilation wavelet. It is known that E is an A-dilation wavelet set iff both the sets $\{A^nE : n \in \mathbb{Z}\}$ and $\{E + 2\pi\ell : \ell \in \mathbb{Z}^2\}$ are partitions of $\mathbb{R}^2$ modulo a null set [7].

Theorem 5.2 *Let E be an A-dilation wavelet set, and let $k(\mathbf{s})$ be a measurable unimodular $2\pi\mathbb{Z}^2$-translation periodic function and $g(\mathbf{s})$ be a measurable unimodular function defined on E. Define*

$$f(\mathbf{s}) = \begin{cases} g(\mathbf{s}), & \mathbf{s} \in E, \\ k((A^\tau)^{-1}\mathbf{s})\cdots k((A^\tau)^{-n}\mathbf{s})\cdot g((A^\tau)^{-n}\mathbf{s}), & \mathbf{s} \in (A^\tau)^nE, n \ge 1, \\ \overline{k(\mathbf{s})k(A^\tau\mathbf{s})\cdots k((A^\tau)^{n-1}\mathbf{s})}\cdot g((A^\tau)^n\mathbf{s}), & \mathbf{s} \in (A^\tau)^{-n}E, n \ge 1, \\ 1, & \mathbf{0}. \end{cases}$$

Then f is an A-dilation wavelet multiplier. Moreover, any A-dilation wavelet multiplier can be constructed this way.

Proof Since $k(\mathbf{s})$ is $2\pi\mathbb{Z}^2$-translation periodic, it suffices (by Theorem 5.1) to show that $f(A^\tau\mathbf{s}) = k(\mathbf{s})f(\mathbf{s})$ in order to show that f is an A-dilation wavelet multiplier.

Case 1. $\mathbf{s} \in E$. Then $A^\tau\mathbf{s} \in A^\tau E$ and

$$f(A^\tau\mathbf{s}) = k((A^\tau)^{-1}A^\tau\mathbf{s})g((A^\tau)^{-1}A^\tau\mathbf{s}) = k(\mathbf{s})g(\mathbf{s}) = k(\mathbf{s})f(\mathbf{s}).$$

Case 2. $\mathbf{s} \in (A^\tau)^n E$ where $n \geq 1$. Then $A^\tau\mathbf{s} \in (A^\tau)^{n+1}E$ and

$$\begin{aligned} f(A^\tau\mathbf{s}) &= k((A^\tau)^{-1}A^\tau\mathbf{s})\cdots k((A^\tau)^{-(n+1)}A^\tau\mathbf{s})g((A^\tau)^{-(n+1)}A^\tau\mathbf{s}) \\ &= k(\mathbf{s})k((A^\tau)^{-1}\mathbf{s})\cdots k((A^\tau)^{-n}\mathbf{s})g((A^\tau)^{-n}\mathbf{s}) \\ &= k(\mathbf{s})f(\mathbf{s}). \end{aligned}$$

Case 3. $\mathbf{s} \in (A^\tau)^{-1}E$. Then $A^\tau\mathbf{s} \in E$ and $f(\mathbf{s}) = \overline{k(\mathbf{s})}g(A^\tau\mathbf{s})$, so $f(A^\tau\mathbf{s}) = g(A^\tau\mathbf{s}) = k(\mathbf{s})f(\mathbf{s})$.

Case 4. $\mathbf{s} \in (A^\tau)^{-n}E$ where $n > 1$. Then $A^\tau\mathbf{s} \in (A^\tau)^{-(n-1)}E$ and

$$\begin{aligned} f(A^\tau\mathbf{s}) &= \overline{k(A^\tau\mathbf{s})\cdots k((A^\tau)^{n-2}A^\tau\mathbf{s})}g((A^\tau)^{n-1}A^\tau\mathbf{s}) \\ &= k(\mathbf{s})\overline{k(\mathbf{s})k(A^\tau\mathbf{s})\cdots k((A^\tau)^{n-1}\mathbf{s})}g((A^\tau)^n\mathbf{s}) \\ &= k(\mathbf{s})f(\mathbf{s}). \end{aligned}$$

Since $\{(A^\tau)^n E : n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}^2$ modulo a null set, the above four cases have exhausted all possibilities for a.e. $\mathbf{s} \in \mathbb{R}^2$.

Now suppose that $f(\mathbf{s})$ is an A-dilation wavelet multiplier. Let $g(\mathbf{s}) = f(\mathbf{s})$ for $\mathbf{s} \in E$, and $k(\mathbf{s}) = f(A^\tau\mathbf{s})/f(\mathbf{s})$. Then $k(\mathbf{s})$ is $2\pi\mathbb{Z}^2$-translation periodic and is unimodular. We leave it to our reader to verify that $f(\mathbf{s})$ has the form given in the theorem. □

6 Phases of A-dilation MRA Wavelets

The linear phase filtering problem is considered in signal processing where wavelets and scaling functions are considered as filter functions. For more detailed discussions on the linear-phase problems concerning wavelet and scaling functions, interested reader may refer to [3, Sect. 5.5].

A function $f(t) \in L^2(\mathbb{R}^2)$ is said to have a *linear phase* if its Fourier transform has the form

$$\widehat{f}(\mathbf{s}) = \pm|\widehat{f}(\mathbf{s})| \cdot e^{-i\mathbf{s}\circ\mathbf{a}} \quad a.e.$$

for some constant vector $\mathbf{a} \in \mathbb{R}^2$, which is the *phase* of $\widehat{f}(\mathbf{s})$.

The following theorem concerning the phase of an A-dilation MRA wavelet in $L^2(\mathbb{R}^2)$ is an application of the results obtained in Sect. 5.

Theorem 6.1 *Let $\psi(t) \in L^2(\mathbb{R}^2)$ be an A-dilation MRA wavelet. Then*

$$\widehat{\psi}(\mathbf{s}) = e^{i\mathbf{s}\circ\mathbf{u}_1} f(\mathbf{s})|\widehat{\psi}(\mathbf{s})|$$

for some A-dilation wavelet multiplier $f(\mathbf{s})$.

Proof By Proposition 2.1, the Fourier transform of an A-dilation MRA wavelet $\psi(t)$ has the form

$$\widehat{\psi}(\mathbf{s}) = e^{i\mathbf{s}\circ\mathbf{u}_1} v(\mathbf{s})\overline{m((A^\tau)^{-1}\mathbf{s} + 2\pi\mathbf{h}_0)}\widehat{\phi}((A^\tau)^{-1}\mathbf{s}), \tag{6.1}$$

where v is some unimodular and $2\pi\mathbb{Z}^2$-translation periodic function. Recall from (2.2) that $\widehat{\phi}(A^\tau\mathbf{s}) = m(\mathbf{s})\widehat{\phi}(\mathbf{s})$ and that the low pass filter $m(\mathbf{s})$ is $2\pi\mathbb{Z}^2$-translation periodic. Let $\widehat{\phi}(\mathbf{s}) = g(\mathbf{s})|\widehat{\phi}(\mathbf{s})|$. Then $\widehat{\phi}(A^\tau\mathbf{s}) = g(A^\tau\mathbf{s})|\widehat{\phi}(A^\tau\mathbf{s})| = g(A^\tau\mathbf{s})|m(\mathbf{s})\widehat{\phi}(\mathbf{s})|$ and $\widehat{\phi}(A^\tau\mathbf{s}) = m(\mathbf{s})\widehat{\phi}(\mathbf{s}) = m(\mathbf{s})g(\mathbf{s})|\widehat{\phi}(\mathbf{s})|$. Thus $g(A^\tau\mathbf{s})/g(\mathbf{s}) = m(\mathbf{s})/|m(\mathbf{s})|$. Now let $E = \mathrm{Supp}(\widehat{\phi})$. For any $\mathbf{s} \in (A^\tau)^{-1}E$, $A^\tau\mathbf{s} \in E$ so $0 \neq \widehat{\phi}(A^\tau\mathbf{s}) = m(\mathbf{s})\widehat{\phi}(\mathbf{s})$. It follows that $\widehat{\phi}(\mathbf{s}) \neq 0$ so $\mathbf{s} \in E$. This shows that $(A^\tau)^{-1}E \subset E$ (which then implies that $(A^\tau)^n E \subset (A^\tau)^{n+1}E$ for any $n \in \mathbb{Z}$). Since $m(\mathbf{s})$ is $2\pi\mathbb{Z}^2$-translation periodic and the support of $m(\mathbf{s})$ contains $(A^\tau)^{-1}E$, $g(A^\tau\mathbf{s})/g(\mathbf{s})$ is $2\pi\mathbb{Z}^2$-translation periodic on $(A^\tau)^{-1}E$ as well. Thus the restriction of $g(A^\tau\mathbf{s})/g(\mathbf{s})$ on $(A^\tau)^{-1}E$ can be extended to a $2\pi\mathbb{Z}^2$-translation periodic function $k_0(\mathbf{s})$ over the set $\bigcup_{\ell\in\mathbb{Z}^2}((A^\tau)^{-1}E + 2\pi\ell)$. We then define a unimodular and $2\pi\mathbb{Z}^2$-translation periodic function $k(\mathbf{s})$ by

$$k(\mathbf{s}) = \begin{cases} k_0(\mathbf{s}), & \mathbf{s} \in \bigcup_{\ell\in\mathbb{Z}^2}((A^\tau)^{-1}E + 2\pi\ell), \\ 1, & \text{otherwise.} \end{cases}$$

We will now use $k(\mathbf{s})$ to extend the domain of g to $\mathbb{R}^2$. If $\mathbf{s} \in E$, $g(\mathbf{s})$ is already defined by its definition $\widehat{\phi}(\mathbf{s}) = g(\mathbf{s})|\widehat{\phi}(\mathbf{s})|$. If $\mathbf{s} \in A^\tau E\backslash E$, then $g((A^\tau)^{-1}\mathbf{s})$ is defined since $(A^\tau)^{-1}\mathbf{s} \in E$. Thus we can define $g(\mathbf{s}) = k((A^\tau)^{-1}\mathbf{s}) \cdot g((A^\tau)^{-1}\mathbf{s})$. In general, assume that $g(\mathbf{s})$ has been defined on $(A^\tau)^n E$, then for any $\mathbf{s} \in (A^\tau)^{n+1}E\backslash(A^\tau)^n E$, define $g(\mathbf{s}) = k((A^\tau)^{-1}\mathbf{s}) \cdot g((A^\tau)^{-1}\mathbf{s})$. The support of $\widehat{\psi}$ is contained in $A^\tau E$ by (6.1). By Lemma 2.1(ii), $\bigcup_{n\in\mathbb{Z}}(A^\tau)^n E = \mathbb{R}^2$ modulo a null set. Thus the extended g has been defined on the entire $\mathbb{R}^2$. The function g is an A-dilation wavelet multiplier since $k(\mathbf{s}) = g(A^\tau\mathbf{s})/g(\mathbf{s})$ is unimodular and $2\pi\mathbb{Z}^2$-translation periodic. For $\mathbf{s} \in (A^\tau)^{-1}E$, we have $m(\mathbf{s}) = \widehat{\phi}(A^\tau\mathbf{s})/\widehat{\phi}(\mathbf{s}) = k(\mathbf{s})|m(\mathbf{s})|$. For $\mathbf{s} \in E \setminus (A^\tau)^{-1}E$, $\widehat{\phi}(A^\tau\mathbf{s}) = m(\mathbf{s})\widehat{\phi}(\mathbf{s}) = 0$ while $\widehat{\phi}(\mathbf{s}) \neq 0$. So $m(\mathbf{s}) = 0$. Thus $m(\mathbf{s}) = k(\mathbf{s})|m(\mathbf{s})|$ also holds. This means $m(\mathbf{s}) = k(\mathbf{s})|m(\mathbf{s})|$ holds for all $\mathbf{s} \in \mathbb{R}^2$ since $m(\mathbf{s})$ is $2\pi\mathbb{Z}^2$-translation periodic and $\bigcup_{k\in\mathbb{Z}^2}(E + 2\pi\ell) = \mathbb{R}^2$ modulo a null set by Lemma 2.3(i). Finally, (6.1) becomes

$$\begin{aligned}\widehat{\psi}(\mathbf{s}) &= e^{i\mathbf{s}\circ\mathbf{u}_1} v(\mathbf{s})\overline{k((A^\tau)^{-1}\mathbf{s} + 2\pi\mathbf{h}_0)}|m((A^\tau)^{-1}\mathbf{s} + 2\pi\mathbf{h}_0)| \\ &\quad \times g((A^\tau)^{-1}\mathbf{s})|\widehat{\phi}((A^\tau)^{-1}\mathbf{s})| \\ &= e^{i\mathbf{s}\circ\mathbf{u}_1} v(\mathbf{s})\overline{k((A^\tau)^{-1}\mathbf{s} + 2\pi\mathbf{h}_0)}g((A^\tau)^{-1}\mathbf{s})|\widehat{\psi}(\mathbf{s})|.\end{aligned}$$

Let $f(\mathbf{s}) = v(\mathbf{s})\overline{k((A^\tau)^{-1}\mathbf{s} + 2\pi\mathbf{h}_0)}g((A^\tau)^{-1}\mathbf{s})$. Since the support for each of $v(\mathbf{s})$, $k(\mathbf{s})$ and $g(\mathbf{s})$ is $\mathbb{R}^2$, the support for $f(\mathbf{s})$ is $\mathbb{R}^2$. Furthermore,

$$\begin{aligned}\frac{f(A^\tau\mathbf{s})}{f(\mathbf{s})} &= \frac{v(A^\tau\mathbf{s})\overline{k(\mathbf{s} + 2\pi\mathbf{h}_0)}g(\mathbf{s})}{v(\mathbf{s})\overline{k((A^\tau)^{-1}\mathbf{s} + 2\pi\mathbf{h}_0)}g((A^\tau)^{-1}\mathbf{s})} \\ &= (v(A^\tau\mathbf{s})/v(\mathbf{s}))\overline{k(\mathbf{s} + 2\pi\mathbf{h}_0)}k((A^\tau)^{-1}\mathbf{s} + 2\pi\mathbf{h}_0)k((A^\tau)^{-1}\mathbf{s}).\end{aligned} \tag{6.2}$$

Since $(v(A^\tau \mathbf{s})/v(\mathbf{s}))\overline{k(\mathbf{s}+2\pi\mathbf{h}_0)}$ is $2\pi\mathbb{Z}^2$-translation periodic by the definitions of v and k, we only need to show that $k((A^\tau)^{-1}\mathbf{s}+2\pi\mathbf{h}_0)k((A^\tau)^{-1}\mathbf{s})$ is also $2\pi\mathbb{Z}^2$-translation periodic. If $\ell \in A^\tau\mathbb{Z}^2$ then it is obvious that $k((A^\tau)^{-1}(\mathbf{s}+2\pi\ell)+2\pi\mathbf{h}_0)k((A^\tau)^{-1}(\mathbf{s}+2\pi\ell)) = k((A^\tau)^{-1}\mathbf{s}+2\pi\mathbf{h}_0)k((A^\tau)^{-1}\mathbf{s})$. Otherwise, we have $\ell = \ell_0 + A^\tau\ell_1 = A^\tau(\mathbf{h}_0+\ell_1)$ for some $\ell_1 \in \mathbb{Z}^2$. It follows that $k((A^\tau)^{-1}(\mathbf{s}+2\pi\ell)+2\pi\mathbf{h}_0)k((A^\tau)^{-1}(\mathbf{s}+2\pi\ell)) = k((A^\tau)^{-1}\mathbf{s}+2\pi\ell_1+4\pi\mathbf{h}_0)k((A^\tau)^{-1}\mathbf{s}+2\pi\ell_1+2\pi\mathbf{h}_0) = k((A^\tau)^{-1}\mathbf{s}+2\pi\mathbf{h}_0)k((A^\tau)^{-1}\mathbf{s})$ since $2\mathbf{h}_0 \in \mathbb{Z}^2$. This proves that $f(A^\tau\mathbf{s})/f(\mathbf{s})$ is indeed $2\pi\mathbb{Z}^2$-translation periodic. $f(A^\tau\mathbf{s})/f(\mathbf{s})$ is unimodular since every term in the right side of (6.2) is unimodular. Thus f is an A-dilation wavelet multiplier by Theorem 5.1. □

Corollary 6.1 *For every A-dilation wavelet ψ, there exists an A-dilation wavelet ψ' such that $|\widehat{\psi}| = |\widehat{\psi'}|$ and ψ' has a linear phase $-\mathbf{u}_1 = -A^{-1}\mathbf{u}$.*

Proof By Theorem 6.1, there exists an A-dilation wavelet multiplier f such that

$$\widehat{\psi}(\mathbf{s}) = e^{i\mathbf{s}\circ\mathbf{u}_1} f(\mathbf{s})|\widehat{\psi}(\mathbf{s})|.$$

Since f is unimodular, multiplying $\overline{f}$ on both sides of the above equation yields

$$\overline{f(\mathbf{s})}\widehat{\psi}(\mathbf{s}) = e^{i\mathbf{s}\circ\mathbf{u}_1}|\widehat{\psi}(\mathbf{s})| = e^{i\mathbf{s}\circ\mathbf{u}_1}|\overline{f(\mathbf{s})}\widehat{\psi}(\mathbf{s})|.$$

Since $\overline{f}$ is also an A-dilation wavelet multiplier, ψ' defined by $\widehat{\psi'}(\mathbf{s}) = \overline{f(\mathbf{s})}\widehat{\psi}(\mathbf{s})$ is also an A-dilation wavelet. By definition, $-\mathbf{u}_1$ is a linear phase of ψ'. □

Remark 6.1 If B is a 2×2 integral expansive matrix with $|\det(B)| = 2$ and $P^{-1}AP = B$ for some integral matrix P with $|\det(P)| = 1$, then for any given B-dilation wavelet ψ_B, there exists a B-dilation wavelet ψ'_B such that $|\widehat{\psi_B}| = |\widehat{\psi'_B}|$ and ψ'_B has a linear phase of the form $-P^\tau\mathbf{u}_1$.

7 Path-connectivity of the Set of A-dilation MRA Wavelets

As another application of Theorem 5.1, in this section we prove that the set of A-dilation MRA wavelets is path-connected under the $L^2(\mathbb{R}^2)$ norm topology. In the one-dimensional case, the path-connectedness of the set of all orthonormal wavelets is still an open question, although many results have been obtained for special classes of wavelets and frame wavelets. In [19], Speegle showed that the class of all minimally supported frequency (MSF) wavelets is path-connected. Paluszyński *et al.* showed the connectivity for the class of MRA tight frame wavelets [18]. Garrigós *et al.* showed that the class of all tight frame wavelets satisfying certain mild conditions on their spectrum is also connected [10]. Dai *et al.* showed that the sets of s-elementary tight frame wavelets (for any given frame bound) and s-elementary frame wavelets are all path-connected [5, 8]. These efforts were further extended to the set of all frame wavelets by Bownik [2], where he showed that this much larger set is path-connected under a differently defined norm called $L^2_*(\mathbb{R})$ (he also showed that this result holds in the higher-dimensional case). Despite all these efforts, so far there has been little activity in attacking the path-connectivity problem of MRA wavelets in higher dimensions. While it is generally expected that the set of all MRA wavelets is path-connected in the higher-dimensional case, the establishment of such a result is not a trivial generalization of the one-dimensional case due to the complexity introduced by the dilation matrices.

Our main result of this section is the following theorem.

Theorem 7.1 *For any two A-dilation MRA wavelets ψ_0 and ψ_1, there exists a continuous map $\gamma : [0,1] \longrightarrow L^2(\mathbb{R}^2)$ such that $\gamma(0) = \psi_0$, $\gamma(1) = \psi_1$ and $\gamma(t)$ is an A-dilation MRA wavelet for $\forall\, t \in [0,1]$.*

We will prove the theorem by directly constructing a continuous path connecting the two MRA wavelets. The proof is given for the case where A is one of the matrices A_1, A_2, $\pm A_3$ and $\pm A_4$. In general, if $B \sim A$ for one of the matrices A above, then we can simply apply the unitary operator Φ_P to the set of all A-dilation MRA wavelets (recall Remark 3.2). The proof is of constructive nature and long. So we break it into several lemmas. For a given A-dilation wavelet ψ_0, define $\mathcal{M}_{\psi_0} = \{\psi : \widehat{\psi} = v\widehat{\psi_0}$ for some A-dilation wavelet multiplier $v\}$, and $\mathcal{W}_{\psi_0} = \{\psi :$ ψ is an A-dilation wavelet with $|\widehat{\psi}| = |\widehat{\psi_0}|\}$. Furthermore, in the case that ψ_0 is an A-dilation MRA wavelet with ϕ_0 being the corresponding A-dilation scaling function for the MRA, define $\mathcal{S}_{\psi_0} = \{\psi :\ \psi$ is an A-dilation MRA wavelet with $|\phi| = |\phi_0|\}$.

Lemma 7.1 *For any A-dilation MRA wavelet ψ_0 we have $\mathcal{S}_{\psi_0} = \mathcal{M}_{\psi_0} = \mathcal{W}_{\psi_0}$.*

Proof $\mathcal{W}_{\psi_0} \subseteq \mathcal{S}_{\psi_0}$ follows from equation (2.4) of Lemma 2.4. $\mathcal{M}_{\psi_0} \subseteq \mathcal{W}_{\psi_0}$ by definition. $\mathcal{S}_{\psi_0} \subseteq \mathcal{M}_{\psi_0}$ follows from an argument similar to the one used in the proof of Theorem 1.2 in [15] and Proposition 2.1. □

Lemma 7.2 *Let ψ_0 be an A-dilation MRA wavelet. Then $\mathcal{M}_{\psi_0}$ is path-connected.*

Proof This is proved in [15] for a special case of A. However the proof for the general case is similar and thus omitted. □

By Lemma 7.1 we have $\mathcal{S}_{\psi_0} = \mathcal{M}_{\psi_0}$. Thus, to show that any two A-dilation MRA wavelets are connected by a continuous path, it suffices to show that for any A-dilation MRA wavelet ψ, there exists a $\psi_1 \in \mathcal{S}_\psi$, such that ψ_1 is path-connected to the generalized Shannon wavelet ψ_0 defined by

$$\widehat{\psi_0}(\mathbf{s}) = \frac{1}{2\pi} e^{i\mathbf{s}\circ\mathbf{u}_1} \chi_{A^\tau\Omega\setminus\Omega}(\mathbf{s}). \tag{7.1}$$

We will choose $\psi_1 \in \mathcal{S}_\psi$ so that it is associated with a scaling function ϕ_1 such that $\widehat{\phi_1} \geq 0$ and $m_1 \geq 0$ and

$$\widehat{\psi_1}(\mathbf{s}) = e^{i\mathbf{s}\circ\mathbf{u}_1} m_1((A^\tau)^{-1}\mathbf{s} + 2\pi\mathbf{h}_0)\widehat{\phi_1}((A^\tau)^{-1}\mathbf{s}). \tag{7.2}$$

The existence of such a ψ_1 is guaranteed by Lemma 2.3 and Proposition 2.2. Note that the corresponding scaling function and low pass filter of ψ_0 are given by $\widehat{\phi_0}(\mathbf{s}) = (1/2\pi)\chi_\Omega$ and $m_0(\mathbf{s})|_\Omega = \chi_{(A^\tau)^{-1}\Omega}$, respectively.

We will now build a path that connects the low pass filters first, then use it to construct the path for the scaling functions and the connected path for the wavelet functions. We will describe the construction for the case of $A = A_3$. The other cases can be dealt with similarly. Notice in this case $2\pi\mathbf{h}_0 = (\pi, \pi)^\tau$ and $\mathbf{s} \circ \mathbf{u}_1 = (s_1 + s_2)/2$ where $\mathbf{s} = (s_1, s_2)^\tau$. For $t \in [0, 1]$, $\mathbf{s} \in \Omega = [-\pi, \pi)^2$, define

$$m_t(\mathbf{s}) = \begin{cases} (1-t)m_0(\mathbf{s}) + tm_1(\mathbf{s}), & \mathbf{s} \in (A^\tau)^{-1}\Omega \setminus (1-t)(A^\tau)^{-1}\Omega, \\ 1, & \mathbf{s} \in (1-t)(A^\tau)^{-1}\Omega, \\ \sqrt{1 - |m_t(\mathbf{s} + (\pi, -\pi)^\tau)|^2}, & \mathbf{s} \in R_1, \\ \sqrt{1 - |m_t(\mathbf{s} + (\pi, \pi)^\tau)|^2}, & \mathbf{s} \in R_2, \\ \sqrt{1 - |m_t(\mathbf{s} + (-\pi, \pi)^\tau)|^2}, & \mathbf{s} \in R_3, \\ \sqrt{1 - |m_t(\mathbf{s} + (-\pi, -\pi)^\tau)|^2}, & \mathbf{s} \in R_4, \end{cases}$$

where the regions R_j $(1 \le j \le 4)$ are as marked in Fig. 1(b). The general $m_t(\mathbf{s})$ is then defined by extending the above $2\pi\mathbb{Z}^2$-periodically. Of course, for $t = 0$ and $t = 1$, $m_t(\mathbf{s})$ is just the $m_0(\mathbf{s})$ and $m_1(\mathbf{s})$ given before. Furthermore, it is easy to see that $|m_t(\mathbf{s})| \le 1$ for any t by its definition and that $m_t(\mathbf{s})$ satisfies the equation

$$|m_t(\mathbf{s})|^2 + |m_t(\mathbf{s} + (\pi, \pi)^\tau)|^2 = 1.$$

Define:

$$\widehat{\phi_t}(\mathbf{s}) = \frac{1}{2\pi} \prod_{j=1}^{\infty} m_t((A^\tau)^{-j}\mathbf{s}), \tag{7.3}$$

$$\widehat{\psi_t}(\mathbf{s}) = e^{i\frac{s_1+s_2}{2}} m_t((A^\tau)^{-1}\mathbf{s} + (\pi, \pi)^\tau)\widehat{\phi_t}((A^\tau)^{-1}\mathbf{s}) \tag{7.4}$$

for $\mathbf{s} \in \mathbb{R}^2$. Then $\widehat{\phi_t}$ is well defined since $0 \le m_t(\mathbf{s}) \le 1$, so is $\widehat{\psi_t}$. Furthermore, for $t = 0$ and $t = 1$, ψ_t coincides with the $\widehat{\psi_0}$ and $\widehat{\psi_1}$ defined in (7.1) and (7.2), respectively.

To complete the proof of Theorem 7.1, we need to show

1. ϕ_t is an A-dilation scaling function, so ψ_t is an A-dilation MRA wavelet.
2. The mapping $[0, 1] \to L^2(\mathbb{R}^2)$ defined by $t \mapsto \psi_t$ is continuous.

These two statements will be proved in the next three lemmas.

Lemma 7.3 *For each $t \in [0, 1]$, let ϕ_t and ψ_t be functions as defined in* (7.3) *and* (7.4), *respectively. Then ϕ_t is an A-dilation scaling function and ψ_t is an A-dilation MRA wavelet.*

Proof The statement holds trivially for $t = 0$ and 1, so we only need to consider the case $0 < t < 1$. From the definition of ϕ_t, we have

$$\widehat{\phi_t}(A^\tau\mathbf{s}) = m_t(\mathbf{s})\widehat{\phi_t}(\mathbf{s}), \quad \mathbf{s} \in \mathbb{R}^2, \tag{7.5}$$

$$\widehat{\phi_t}(\mathbf{s}) = \frac{1}{2\pi}, \quad \mathbf{s} \in (1-t)\Omega. \tag{7.6}$$

So $\widehat{\phi_t}(\mathbf{s})$ satisfies conditions (ii) and (iii) of Lemma 2.3. We will prove that ϕ_t satisfies condition (i) of Lemma 2.3 as well, which then implies that ϕ_t is a scaling function.

For $\forall\, \mathbf{s} \in \Omega$, we have $(A^\tau)^{-j}\mathbf{s} \in (A^\tau)^{-1}\Omega\ \forall\ j \geq 1$. So by the definition of $m_t(\mathbf{s})$, we have $m_t((A^\tau)^{-j}\mathbf{s}) \geq 1-t$. Since A is expansive, for any fixed $0 < t < 1$, we can choose k_0 sufficiently large such that $(A^\tau)^{-k}\Omega \subset (1-t)\Omega$ when $k \geq k_0$. Hence, if $\mathbf{s} \in \Omega$ and $k \geq k_0$, then $\widehat{\phi_t}((A^\tau)^{-k}\mathbf{s}) = 1/2\pi$ by (7.6) and

$$\widehat{\phi_t}(\mathbf{s}) = \frac{1}{2\pi}\prod_{j=1}^{\infty} m_t((A^\tau)^{-j}\mathbf{s}) = \frac{1}{2\pi}\prod_{k=1}^{k_0} m_t((A^\tau)^{-k}\mathbf{s}) \prod_{k=k_0+1}^{\infty} m_t((A^\tau)^{-k}\mathbf{s})$$

$$= \widehat{\phi_t}((A^\tau)^{-k_0}\mathbf{s}) \prod_{k=1}^{k_0} m_t((A^\tau)^{-k}\mathbf{s}) = \frac{1}{2\pi}\prod_{k=1}^{k_0} m_t((A^\tau)^{-k}\mathbf{s}) \geq \frac{1}{2\pi}(1-t)^{k_0}.$$

This implies that $\chi_\Omega(\mathbf{s}) \leq 2\pi\widehat{\phi_t}(\mathbf{s})/(1-t)^{k_0}$. Define

$$\mu_{t,k}(\mathbf{s}) = \frac{1}{2\pi}\chi_\Omega((A^\tau)^{-k}\mathbf{s}) \cdot \prod_{j=1}^{k} m_t((A^\tau)^{-j}\mathbf{s}),\ k \geq 1.$$

Then

$$\mu_{t,k}(\mathbf{s}) \leq \frac{\widehat{\phi_t}((A^\tau)^{-k}\mathbf{s})}{(1-t)^{k_0}} \prod_{j=1}^{k} m_t((A^\tau)^{-j}\mathbf{s}) = \frac{\widehat{\phi_t}(\mathbf{s})}{(1-t)^{k_0}}.$$

For $k \geq 2$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\mu_{t,k}(\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}} d\mathbf{s} \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} |\chi_\Omega((A^\tau)^{-k}\mathbf{s})|^2 \cdot \prod_{j=1}^{k} |m_t((A^\tau)^{-j}\mathbf{s})|^2 \cdot e^{-i\mathbf{n}\circ\mathbf{s}} d\mathbf{s} \\
&= \frac{2^k}{4\pi^2}\int_{\Omega} \prod_{j=1}^{k} |m_t((A^\tau)^{k-j}\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})} d\mathbf{s} \\
&= \frac{2^k}{4\pi^2}\int_{\Omega} \prod_{j=0}^{k-1} |m_t((A^\tau)^{j}\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})} d\mathbf{s} \\
&= \frac{2^k}{4\pi^2}\int_{\Omega} |m_t(\mathbf{s})|^2 \prod_{j=1}^{k-1} |m_t((A^\tau)^{j}\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})} d\mathbf{s}.
\end{aligned}$$

Let R_j and T_j $(1 \leq j \leq 4)$ be the regions marked in Fig. 1(b) and let $U_j = R_j \cup T_j$. To compute the last integral in the above equality, we divide Ω into these smaller regions. We have

$$\int_{\mathbb{R}^2} |\mu_{t,k}(\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}} d\mathbf{s}$$

$$= \frac{2^k}{4\pi^2}\left(\int_{\cup_{1\le j'\le 4}U_{j'}} |m_t(\mathbf{s})|^2 \prod_{j=1}^{k-1} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})} d\mathbf{s}\right)$$

$$= \frac{2^k}{4\pi^2}\sum_{j'=1}^{4}\int_{U_{j'}} |m_t(\mathbf{s})|^2 \prod_{j=1}^{k-1} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})} d\mathbf{s}.$$

We have

$$\begin{aligned}
&\int_{U_1} |m_t(\mathbf{s})|^2 \prod_{j=1}^{k-1} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})} d\mathbf{s} \\
&\quad= \int_{R_1} |m_t(\mathbf{s})|^2 \prod_{j=1}^{k-1} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})} d\mathbf{s} \\
&\qquad+ \int_{T_1} |m_t(\mathbf{s})|^2 \prod_{j=1}^{k-1} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})} d\mathbf{s} \\
&\quad= \int_{T_1} \prod_{j=1}^{k-1} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})} d\mathbf{s},
\end{aligned}$$

where the second equality is obtained by substituting $\mathbf{s}$ with $\mathbf{s}-(\pi,-\pi)^\tau$ in the integral over T_1 together with the equality that $|m_t(\mathbf{s})|^2 + |m_t(\mathbf{s}+(\pi,-\pi)^\tau)|^2 = 1$ for any $\mathbf{s}\in R_1$. Similarly, for each $j'=2$, 3 and 4 we also have

$$\begin{aligned}
&\int_{U_{j'}} |m_t(\mathbf{s})|^2 \prod_{j=1}^{k-1} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})} d\mathbf{s} \\
&\quad= \int_{T_{j'}} \prod_{j=1}^{k-1} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})} d\mathbf{s}
\end{aligned}$$

So

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\mu_{t,k}(\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}} d\mathbf{s} \\
&\quad= \frac{2^k}{4\pi^2}\int_{\cup_{1\le j'\le 4}T_{j'}} \prod_{j=1}^{k-1} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})} d\mathbf{s} \\
&\quad= \frac{2^k}{4\pi^2}\int_{(A^\tau)^{-1}\Omega} \prod_{j=1}^{k-1} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^k\mathbf{s})} d\mathbf{s} \\
&\quad= \frac{2^{k-1}}{4\pi^2}\int_{\Omega} \prod_{j=0}^{k-2} |m_t((A^\tau)^j\mathbf{s})|^2 e^{-i\mathbf{n}\circ((A^\tau)^{k-1}\mathbf{s})} d\mathbf{s}
\end{aligned}$$

$$= \int_{\mathbb{R}^2} |\mu_{t,k-1}(\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}} d\mathbf{s}.$$

Repeating the above procedure then leads to

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\mu_{t,k}(\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}} d\mathbf{s} \\
&\quad = \int_{\mathbb{R}^2} |\mu_{t,1}(\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}} d\mathbf{s} \\
&\quad = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} |\chi_\Omega((A^\tau)^{-1}\mathbf{s})|^2 \cdot |m_t((A^\tau)^{-1}\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}} d\mathbf{s} \\
&\quad = \frac{2}{4\pi^2} \int_{\Omega} |m_t(\mathbf{s})|^2 e^{-i\mathbf{n}\circ(A^\tau\mathbf{s})} d\mathbf{s} \\
&\quad = \frac{2}{4\pi^2} \int_{(A^\tau)^{-1}\Omega} e^{-i\mathbf{n}\circ(A^\tau\mathbf{s})} d\mathbf{s} = \frac{1}{4\pi^2} \int_{\Omega} e^{-i\mathbf{n}\circ\mathbf{s}} d\mathbf{s} = \delta_{\mathbf{n},\mathbf{0}}.
\end{aligned}$$

So $\|\mu_{t,k}\|^2 = 1$. Clearly $\lim_{k\to\infty} \mu_{t,k}(\mathbf{s}) = \widehat{\phi_t}(\mathbf{s})$ for all $\mathbf{s} \in \mathbb{R}^2$. Thus $\phi_t \in L^2(\mathbb{R}^2)$ by Fatou's Lemma. Since $\mu_{t,k}(\mathbf{s})$ is dominated by $\frac{\widehat{\phi_t}(\mathbf{s})}{(1-t)^{k_0}}$, we get

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} |\mu_{t,k}(\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}} d\mathbf{s} = \int_{\mathbb{R}^2} |\widehat{\phi_t}(\mathbf{s})|^2 e^{-i\mathbf{n}\circ\mathbf{s}} d\mathbf{s} = \delta_{\mathbf{n},\mathbf{0}}$$

by Lebesgue's dominated convergence theorem. This is equivalent to the condition that $\sum_{\ell\in\mathbb{Z}^2} |\widehat{\phi_t}(\mathbf{s} + 2\pi\ell)|^2 = \frac{1}{4\pi^2}$ *a.e.* By Lemma 2.3, ϕ_t is a scaling function for some MRA. Consequently, ψ_t is an A-dilation MRA wavelet. □

Lemma 7.4 $\lim_{t\to t_0} \widehat{\phi_t}(\mathbf{s}) = \widehat{\phi_{t_0}}(\mathbf{s})$ *a.e. for any* $t_0 \in [0, 1]$.

Proof By the definition of $m_t(\mathbf{s})$, the mapping $t \mapsto m_t(\mathbf{s})$ is continuous with respect to t a.e. for $\mathbf{s} \in \mathbb{R}^2$. Since $\widehat{\phi_1} \geq 0$, $\lim_{j\to\infty} \widehat{\phi_1}((A^\tau)^{-j}\mathbf{s}) = 1/2\pi$ a.e. For any given $\varepsilon > 0$ and $\mathbf{s} \in \mathbb{R}^2$, there exists a positive integer n_0 such that $\widehat{\phi_1}((A^\tau)^{-n}\mathbf{s}) > 1/2\pi - \varepsilon/2$ and $(A^\tau)^{-n}\mathbf{s} \subset (A^\tau)^{-1}\Omega$ for any $n \geq n_0$. It follows that $m_t((A^\tau)^{-n}\mathbf{s})$ is either 1 or $(1-t) + t m_1((A^\tau)^{-n}\mathbf{s})$ for any $t \in [0, 1]$. In either case, $m_t((A^\tau)^{-n}\mathbf{s}) \geq m_1((A^\tau)^{-n}\mathbf{s})$. So the following inequality holds for any $t \in [0, 1]$:

$$\begin{aligned}
\widehat{\phi_t}((A^\tau)^{-n}\mathbf{s}) &= \frac{1}{2\pi} \prod_{j=1}^{\infty} m_t((A^\tau)^{-j}(A^\tau)^{-n}\mathbf{s}) \\
&\geq \frac{1}{2\pi} \prod_{j=1}^{\infty} m_1((A^\tau)^{-j}(A^\tau)^{-n}\mathbf{s}) = \widehat{\phi_1}((A^\tau)^{-n}\mathbf{s}).
\end{aligned}$$

Since $\widehat{\phi_t}(\mathbf{s}') \leq 1/2\pi$ for any $\mathbf{s}' \in \mathbb{R}^2$ by its definition, it follows that for any t_1, $t_2 \in [0, 1]$, we have

$$|\widehat{\phi_{t_1}}((A^\tau)^{-n}\mathbf{s}) - \widehat{\phi_{t_2}}((A^\tau)^{-n}\mathbf{s})| < \varepsilon/2. \tag{7.7}$$

On the other hand, since $t \mapsto m_t((A^\tau)^{-j}\mathbf{s})$ is continuous for each j, we have that the mapping $t \mapsto \prod_{j=1}^{n_0} m_t((A^\tau)^{-j}\mathbf{s})$ is continuous. Hence for each $t_0 \in [0,1]$, there exists $\delta > 0$ such that for $|t - t_0| < \delta$ and $t \in [0,1]$,

$$\left|\prod_{j=1}^{n_0} m_t((A^\tau)^{-j}\mathbf{s}) - \prod_{j=1}^{n_0} m_{t_0}((A^\tau)^{-j}\mathbf{s})\right| < \varepsilon.$$

Now, we obtain

$$\begin{aligned}
&|\widehat{\phi_t}(\mathbf{s}) - \widehat{\phi_{t_0}}(\mathbf{s})| \\
&\quad= \left|\frac{1}{2\pi}\prod_{j=1}^{\infty} m_t((A^\tau)^{-j}\mathbf{s}) - \frac{1}{2\pi}\prod_{j=1}^{\infty} m_{t_0}((A^\tau)^{-j}\mathbf{s})\right| \\
&\quad= \left|\prod_{j=1}^{n_0} m_t((A^\tau)^{-j}\mathbf{s})\widehat{\phi_t}((A^\tau)^{-n_0}\mathbf{s}) - \prod_{j=1}^{n_0} m_{t_0}((A^\tau)^{-j}\mathbf{s})\widehat{\phi_{t_0}}((A^\tau)^{-n_0}\mathbf{s})\right| \\
&\quad= \left|\prod_{j=1}^{n_0} m_t((A^\tau)^{-j}\mathbf{s}) \cdot \widehat{\phi_t}((A^\tau)^{-n_0}\mathbf{s}) - \prod_{j=1}^{n_0} m_{t_0}((A^\tau)^{-j}\mathbf{s})\widehat{\phi_t}((A^\tau)^{-n_0}\mathbf{s})\right. \\
&\qquad\left. + \prod_{j=1}^{n_0} m_{t_0}((A^\tau)^{-j}\mathbf{s})\widehat{\phi_t}((A^\tau)^{-n_0}\mathbf{s}) - \prod_{j=1}^{n_0} m_{t_0}((A^\tau)^{-j}\mathbf{s})\widehat{\phi_{t_0}}((A^\tau)^{-n_0}\mathbf{s})\right| \\
&\quad\le \frac{1}{2\pi}\left|\prod_{j=1}^{n_0} m_t((A^\tau)^{-j}\mathbf{s}) - \prod_{j=1}^{n_0} m_{t_0}((A^\tau)^{-j}\mathbf{s})\right| + |\widehat{\phi_t}((A^\tau)^{-n_0}\mathbf{s}) - \widehat{\phi_{t_0}}((A^\tau)^{-n_0}\mathbf{s})| \\
&\quad< \frac{\varepsilon}{2\pi} + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Therefore we have proved that $\lim_{t\to t_0}\widehat{\phi_t}(\mathbf{s}) = \widehat{\phi_{t_0}}(\mathbf{s})$. □

By the continuity of $m_t(\mathbf{s})$ and $\widehat{\phi_t}$, we now have $\lim_{t\to t_0}\widehat{\psi_t}(\mathbf{s}) = \widehat{\psi_{t_0}}(\mathbf{s})$ a.e.

Lemma 7.5 *For* $t_0, t \in [0,1]$, $\lim_{t\to t_0}\|\widehat{\psi_t} - \widehat{\psi_{t_0}}\|^2 = 0$.

Proof Since $\|\widehat{\psi_t}\|^2 = \|\widehat{\psi_{t_0}}\|^2 = 1$, $\|\widehat{\psi_t} - \widehat{\psi_{t_0}}\|^2 = \langle\widehat{\psi_t} - \widehat{\psi_{t_0}}, \widehat{\psi_t} - \widehat{\psi_{t_0}}\rangle = 2 - \langle\widehat{\psi_t}, \widehat{\psi_{t_0}}\rangle - \langle\widehat{\psi_{t_0}}, \widehat{\psi_t}\rangle$. Thus it suffices to show that $\lim_{t\to t_0}\langle\widehat{\psi_t}, \widehat{\psi_{t_0}}\rangle = 1$.

Since $\widehat{\psi_{t_0}} \in L^2(\mathbb{R}^2)$, for any given $\varepsilon > 0$, there exists a sufficiently large number $r > 0$ such that $(\int_{|\mathbf{s}|>r}|\widehat{\psi_{t_0}}(\mathbf{s})|^2 d\mathbf{s})^{\frac{1}{2}} < \varepsilon/4$. By Hölder Inequality, we then have

$$\begin{aligned}
&\int_{|\mathbf{s}|>r}|\widehat{\psi_t}(\mathbf{s}) - \widehat{\psi_{t_0}}(\mathbf{s})| \cdot |\overline{\widehat{\psi_{t_0}}(\mathbf{s})}| d\mathbf{s} \\
&\qquad\le \|\widehat{\psi_t}(\mathbf{s}) - \widehat{\psi_{t_0}}(\mathbf{s})\| \left(\int_{|\mathbf{s}|>r}|\widehat{\psi_{t_0}}(\mathbf{s})|^2 d\mathbf{s}\right)^{\frac{1}{2}} < \varepsilon/2
\end{aligned}$$

since

$$\|\widehat{\psi_t}(\mathbf{s}) - \widehat{\psi_{t_0}}(\mathbf{s})\| \leq \|\widehat{\psi_t}(\mathbf{s})\| + \|\widehat{\psi_{t_0}}(\mathbf{s})\| = 2.$$

On the other hand, we have

$$|\widehat{\psi_t}(\mathbf{s}) - \widehat{\psi_{t_0}(\mathbf{s})}| \leq 1/\pi$$

since

$$|\widehat{\psi_t}(\mathbf{s})| \leq 1/2\pi \quad \text{and} \quad |\widehat{\psi_{t_0}(\mathbf{s})}| \leq 1/2\pi$$

by (7.3), (7.4) and $|m_t| \leq 1$. Thus by the dominated convergence theorem, we have

$$\lim_{t\to t_0} \int_{|\mathbf{s}|\leq r} |\widehat{\psi_t}(\mathbf{s}) - \widehat{\psi_{t_0}}(\mathbf{s})| d\mathbf{s} = 0.$$

Therefore, there exists a number $\delta > 0$ such that $\int_{|\mathbf{s}|\leq r} |\widehat{\psi_t}(\mathbf{s}) - \widehat{\psi_{t_0}}(\mathbf{s})| d\mathbf{s} < \pi\varepsilon/2$ whenever $|t - t_0| < \delta$. Combining the above leads to

$$|\langle\widehat{\psi_t}, \widehat{\psi_{t_0}}\rangle - 1| = |\langle\widehat{\psi_t}, \widehat{\psi_{t_0}}\rangle - \langle\widehat{\psi_{t_0}}, \widehat{\psi_{t_0}}\rangle| = \left|\int_{\mathbb{R}^2} (\widehat{\psi_t}(\mathbf{s}) - \widehat{\psi_{t_0}}(\mathbf{s})) \cdot \overline{\widehat{\psi_{t_0}}(\mathbf{s})} d\mathbf{s}\right|$$

$$\leq \int_{|\mathbf{s}|\leq r} |(\widehat{\psi_t}(\mathbf{s}) - \widehat{\psi_{t_0}}(\mathbf{s}))\overline{\widehat{\psi_{t_0}}(\mathbf{s})}| d\mathbf{s} + \int_{|\mathbf{s}|>r} |(\widehat{\psi_t}(\mathbf{s}) - \widehat{\psi_{t_0}}(\mathbf{s}))\overline{\widehat{\psi_{t_0}}(\mathbf{s})}| d\mathbf{s} < \varepsilon.$$

So $\lim_{t\to t_0} \|\widehat{\psi_t} - \widehat{\psi_{t_0}}\|^2 = 0$. □

Since the inverse Fourier transform is continuous, we know that the mapping $t \mapsto \psi_t$ is continuous. This completes the proof of Theorem 7.1.

Let us end this paper with the following discussion about the possibility of extending the results of this paper to higher dimensions. One apparent limitation of the approach used here is that the proof depends heavily on the reduction of the number of dilation matrices that need to be considered. It is difficult to find all the equivalent classes of integrally similar dilation matrices for higher dimensions. Even if we have found all these classes, there is no guarantee that the construction we used here will still work since the situation can be much more complicated. Therefore, a general approach that does not depends on the specific structure of a dilation matrix will be more desirable.

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