J Fourier Anal Appl (2010) 16: 76–96
DOI 10.1007/s00041-009-9087-8

Criteria for Spectral Gaps of Laplacians on Fractals

Denglin Zhou

Received: 11 April 2008 / Published online: 30 July 2009

Abstract Surprisingly, Fourier series on certain fractals can have better convergence properties than classical Fourier series. This is a result of the existence of gaps in the spectrum of the Laplacian. In this work we prove general criteria for the existence of gaps when the Laplacian admits spectral decimation. The known examples, including the Sierpinski gasket and the level-3 Sierpinski gasket, and the new examples including the fractal-3 tree, the Hexagasket and the infinite family of tree-like fractals satisfy the criteria.

Keywords Analysis on fractals · Spectral analysis · Laplace operator

Mathematics Subject Classification (2000) 28A80 · 42C99 · 31C25

1 Introduction

Laplacians on post critically fractals have been constructed both as a renormalized limit of difference operators and a weak formulation using the theory of Dirichlet forms [11] and [12]. The spectra of Laplacians on a number of fractals have been analyzed both numerically [1] and using the spectral decimation method [5, 14, 16], and [19].

One of the most striking results is that there can be gaps in the spectrum of the Laplacian. (For a given infinite sequence $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_k \le \cdots$, we say that there exist *gaps* in the sequence if $\limsup_{k\ge 1} \frac{\alpha_{k+1}}{\alpha_k} > 1$.) This result was proved for the Laplacian on the Sierpinski gasket by Gibbons, Raj and Strichartz in [9] using results obtained by Fukushima and Shima [8].

Communicated by Robert Strichartz.

D. Zhou (✉)
Department of Pure Mathematics, University of Waterloo, Waterloo, ON N2L 3G1, Canada
e-mail: dzhou@math.uwaterloo.ca

The existence of gaps is an interesting phenomenon in itself, but it also has important applications to analysis on fractals. For instance, Kigami and Lapidus [13] proved that a Weyl-type limit, $\rho(x)/x^{d/2}$ with $\rho(x)$ being the eigenvalue counting function, can not exist for any choice of d. This result also follows from the existence of gaps as the Weyl ratio must drop by a constant factor when x passes through a gap. In addition, one can use the existence of gaps and the heat kernel estimates to derive a version of the Riesz theorem for Fourier series on fractals and even obtain the stronger conclusion that the Fourier series converges for $p=1$ and uniformly when the function is continuous. (See Sect. 3 or [17] for details.)

The main purpose of this paper is to give criteria (Theorems 13, 15 and 16) for the existence of gaps when the Laplacian admits spectral decimation. We show that the criteria apply to various examples, including the Sierpinski gasket, the level-3 Sierpinski gasket, the fractal-3 tree, the Hexagasket, and the infinite family of the n-branch tree-like fractals.

2 Laplacians on Fractals and Spectral Decimation Method

In this section, we review Kigami's method to define Laplacians on p.c.f. fractals [11, 12] and the spectral decimation method to analyze their spectra developed by Shima [16].

2.1 Laplacians on Graphs

We mainly follow the notations and terminologies in [12]. Let $\mathbf{K}$ be a compact metrizable topological space and $\mathcal{L}=\{\mathbf{K}, S, \{F_s\}_{s\in S}\}$ a self-similar structure, where S is a finite set and F_s is a continuous injection from $\mathbf{K}$ to itself for every $s\in S$. We denote $W_n(S)=S^n$ and $W_*(S)=\cup_{n\geq 0}W_n(S)$. For $w=w_1w_2\cdots w_n\in W_n(S)$, let $F_w=F_{w_1}\circ F_{w_2}\circ\cdots\circ F_{w_n}$, and $\mathbf{K}_w=F_w\mathbf{K}$. Assume that there exists a continuous surjection $\pi: S^{\mathbb{N}}\to\mathbf{K}$ satisfying $\pi\circ s=F_s\circ\pi$ for every $s\in S$, where s denotes the map from $S^{\mathbb{N}}$ to $S^{\mathbb{N}}$ defined by $s(w_1w_2\cdots)=sw_1w_2\cdots$. The *critical set* $\mathcal{C}$ and the *post critical set* $\mathcal{P}$ are defined respectively by

$$\mathcal{C}=\pi^{-1}\Bigg(\bigcup_{s,t\in S, s\neq t}(\mathbf{K}_s\cap\mathbf{K}_t)\Bigg),\qquad \mathcal{P}=\bigcup_{n\geq 1}\sigma^n(\mathcal{C}),$$

where $\sigma: S^{\mathbb{N}}\to S^{\mathbb{N}}$ is the left-shift map. A self-similar set is called *post critically finite* (abbreviation p.c.f.) iff the post critical set $\mathcal{P}$ is finite.

We take G_0 to be the complete graph on V_0, where $V_0=\pi(\mathcal{P})$ and is thought of as the boundary of $\mathbf{K}$. Then define the *set of vertices at step m*, V_m, recursively by

$$V_m=\bigcup_s F_sV_{m-1}$$

and define the edge relation $(x,y)\in E_m$ (or $x\sim_m y$) to hold if there exist a word w of length $|w|=m$ such that $x,y\in F_wV_0$. It is not hard to see that the V_m's build up an ascending chain:

$$V_0\subseteq V_1\subseteq V_2\subseteq\cdots.$$

We set

$$V_m^0 = V_m \setminus V_0, \qquad V_* = \bigcup_m V_m$$

and call the elements in V_* *vertices*. We denote $G_m = (V_m, E_m)$ the step-m graph with vertices V_m and edges E_m.

For any set U, we shall use $\ell(U)$ to denote the set of real valued functions on U and

$$\ell_0(V_m) = \{f \in \ell(V_m) : f(p) = 0 \text{ for } p \in V_0\}.$$

For two sets U and V, we define

$$L(U, V) = \{A : \ell(U) \to \ell(V) \text{ and A is linear}\}.$$

In particular, $L(V)$ means $L(V, V)$.

Kigami first defines a Laplacian operator on the vertices V_m as a difference operator. Take D to be a symmetric (Laplacian) matrix in $L(V_0)$ with row sum zero, non-negative off-diagonal entries and negative diagonal entries. Choose $\mathbf{r} = (r_1^{-1}, r_2^{-1}, \ldots, r_{|S|}^{-1}) \in \ell(S)$ and let r_0 be the number such that $r_0^{-1} := \sum_{s\in S} r_s^{-1}$. Define $H_m \in L(V_m)$ by

$$H_m = \sum_{w\in W_m} r_w^{-1} R_w^t D R_w, \tag{2.1}$$

where $R_w \in L(V_m, F_w(V_0))$ is the restriction map defined by $R_w f = f|_{F_w(V_0)}$, and $r_w = r_{w_1} r_{w_2} \ldots r_{w_m}$ for $w = w_1 w_2 \ldots w_m \in W_m$. We call $(H_m, \mathbf{r})$ the *generalized/combinatorial Laplacian with weight* $\mathbf{r}$ on V_m. The special case when all off-diagonal entries of D are 1 and all $r_i = 1$ is called the *Standard Laplacian*. Decompose H_m into

$$H_m = \begin{bmatrix} T_m & J_m^t \\ J_m & X_m \end{bmatrix}, \tag{2.2}$$

where $T_m \in L(V_0)$, $J_m \in L(V_0, V_m^0)$ and $X_m \in L(V_m^0)$. In particular, write $T = T_1$, $J = J_1$ and $X = X_1$.

By constructing a measure $\hat{\mu}_m$ on V_m as

$$\hat{\mu}_m(x) = \left(\sum_{w\in W_m} r_w^{-1} R_w^t (-T) R_w \right)_{x,x},$$

Kigami [11] defines the *normalized Laplacian* $\widehat{\Delta}_m f(x)$ as

$$\widehat{\Delta}_m f(x) := \frac{H_m f(x)}{\hat{\mu}_m(x)},$$

for $f \in \ell(V_m)$.

Assume the p.c.f. fractal $\mathbf{K}$ is connected and

$$\#(F_s(V_0) \cap V_0) \le 1 \quad \text{for every } s \in S. \tag{2.3}$$

Note the latter assumption implies that T is a diagonal matrix. Define diagonal matrices M and W such that $M_{i,i} = -X_{i,i}$ and $W = \begin{bmatrix} -T & 0 \\ 0 & M \end{bmatrix}$. We also denote $G(\lambda) = (X + \lambda M)^{-1}$ if the inverse matrix exists.

Definition 1 [16] The generalized Laplacian $(H_m, \mathbf{r})$ is said to have a *strong harmonic structure* if there exist rational functions $K_D(\lambda)$ and $K_T(\lambda)$ such that when $X + \lambda M$ is invertible, then

$$T - J^t (X + \lambda M)^{-1} J = K_D(\lambda) D + K_T(\lambda) T. \tag{2.4}$$

$K_D(0)^{-1}$ is called the *energy renormalization constant*.

We denote

$$\mathfrak{F} := \{\lambda \in \mathbf{R} : K_D(\lambda) = 0 \text{ or } \det(X + \lambda M) = 0\}$$

and call elements in $\mathfrak{F}$ the *forbidden eigenvalues*. Moreover, we let

$$\mathfrak{F}_k := \{\lambda \in \mathfrak{F} : \lambda \text{ is an eigenvalue of } -\widehat{\Delta}_k\}$$

and call the elements in $\mathfrak{F}_k$ the *forbidden eigenvalues at step k* or *initial eigenvalues at step k*. The rational function

$$R(\lambda) := \frac{\lambda - K_T(\lambda)}{K_D(\lambda)}$$

is called the *spectral decimation function*.

Example 2.1 (The Sierpinski Gasket $\mathcal{SG}$) The fractal and the first step graph are shown in Fig. 1. The boundary points are indicated by solid dots. It can be calculated that for the standard Laplacian,

$$K_D(\lambda) = \frac{3 - 2\lambda}{(4\lambda - 5)(2\lambda - 1)}, \qquad K_T(\lambda) = \frac{2\lambda}{2\lambda - 1},$$

and so the spectral decimation function is

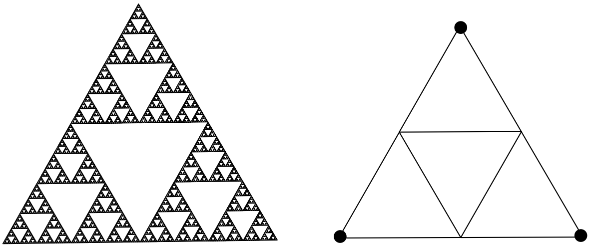
$$R(\lambda) = \lambda(5 - 4\lambda),$$

with forbidden eigenvalues $1/2$, $5/4$, and $3/2$. Notice that since all interior vertices have four neighboring points, the spectral decimation function and the forbidden eigenvalues differ by a constant factor 4 from those calculated in [18].

Suppose we are given a p.c.f. self-similar set (also satisfying our assumption (2.3)) and the generalized Laplacian has a strong harmonic structure. Then the normalized Laplacian has the following spectral decimation property proved by Shima.

Proposition 2 (Shima [16]) *Suppose the generalized Laplacian has a strong harmonic structure. We have the following collective results*:

Fig. 1 The Sierpinski gasket



(1) *If f is an eigenfunction of $-\widehat{\Delta}_{m+1}$ with eigenvalue λ, i.e. $-\widehat{\Delta}_{m+1} f = \lambda f$, and $\lambda \notin \mathfrak{F}$, then $-\widehat{\Delta}_m f|_{V_m} = R(\lambda) f|_{V_m}$,*
(2) *Conversely, if $-\widehat{\Delta}_m f = R(\lambda) f$, and $\lambda \notin \mathfrak{F}$, then there exists a unique extension $\overline{f}$ of f such that $-\widehat{\Delta}_{m+1}\overline{f} = \lambda \overline{f}$.*

The spectral decimation function R has been proved to have the following property.

Proposition 3 (Shima [16]) *The spectral decimation function R satisfies*

$$R(0) = 0, \quad \text{and} \quad R'(0) = \frac{1}{K_D(0) r_0} > 1. \tag{2.5}$$

$\rho = R'(0)$ is called the *Laplacian renormalization constant.*

2.2 Laplacians on Fractals and Spectral Decimation

The (normalized) Laplacian Δ on $\mathbf{K}$ can be defined as a limit of the normalized discrete Laplacians $\widehat{\Delta}_m$ [11] and [12].

Definition 4 (Kigami [11]) Let

$$\mathcal{D} = \Big\{ u \in C(\mathbf{K}) : \text{ there exists a function } f \in C(\mathbf{K}) \text{ and } \lim_{m\to\infty} \rho^m \widehat{\Delta}_m u(x) = f(x) \text{ uniformly for } x \in V_* \setminus V_0 \Big\}.$$

We then define the (*normalized*) *Laplacian on the fractal* $\mathbf{K}$ by

$$\Delta u = f,$$

where f is the function appearing above.

In some cases, spectra of Laplacians can be obtained through an iterative process called spectral decimation.

Definition 5 For a p.c.f. self-similar set $\mathbf{K}$, we say that the Laplacian, $-\Delta$, with Dirichlet boundary conditions, admits *spectral decimation with spectral decimation* *function R* if all eigenvalues of $-\Delta$ are of the form

$$\rho^i \lim_{m\to\infty} \rho^m \phi_v(x), \quad x \in \mathfrak{F}_{i+1} \text{ and } i \in \mathbb{N}\cup\{0\},$$

where $v = v_m \cdots v_1$ with each $v_j \in \{0, \dots, \#(\text{branches of the inverse function of } R) - 1\}$, and $\phi_v = \phi_{v_m} \cdots \phi_{v_1}$ with ϕ_k being the $(k+1)$-th branch of the inverse functions of R from bottom to top; i.e., the ϕ_j are ordered according to their domains, so that if x is in the domain of ϕ_j and y in the domain of ϕ_{j+1}, then $x \le y$. In particular, ϕ_0 is the bottom branch of the inverses.

Remark 6 (1) Note that in the above definition, ϕ_v has to be chosen such that the limit exists. Shima proved if $\phi_0(z) < z$ for all positive real numbers z on its domain, then the existence of the limit is equivalent to the condition that after finitely many steps, we only apply the bottom branch, ϕ_0, of the inverse functions.

In fact, if $\phi_0(z) < z$, then $\phi_0^{(n)}(z)$, the n iterations of ϕ_0, is decreasing in n and so converges. But the limit is a fixed point of ϕ_0 and so it is 0. Hence after applying ϕ_0 sufficiently many times, the resulting value will be close to 0. Proposition 3 then tells us

$$\phi_0(z) = \frac{1}{\rho} z + O(z^2), \quad \text{as } z \to 0.$$

Hence the limit in the above definition exists if we only apply ϕ_0 after finitely many steps to do the extensions. Conversely, if we do not apply ϕ_0 after finitely many steps, then $\phi_v(z)$ with $|v| = m$ does not converge to 0 as $m \to \infty$. Since $\rho > 1$, $\lim_m \rho^m \phi_v(z)$ does not exist.

Therefore, we conclude that if $\phi_0(z) < z$ for all positive z on its domain, all eigenvalues of $-\Delta$ must be of the form

$$\rho^i \lim_{m\to\infty} \rho^m \phi_0^{(m-j)} \phi_{v'}(z), \tag{2.6}$$

where $z \in \mathfrak{F}_{i+1}$, $|v'| = j$, and $i \in \mathbb{N}\cup\{0\}$.

(2) This definition can also be applied to Laplacians with Neumann boundary condition, where all boundary points satisfy the same type of eigenvalue equations as other interior points. All we need to do is to replace $\mathfrak{F}$ and $\mathfrak{F}_i$ by $\mathfrak{F}\cup\{0\}$ and $\mathfrak{F}_i \cup \{0\}$ respectively since constant functions are always Neumann eigenfunctions corresponding to eigenvalue zero.

Shima has proved the following theorem about the relationship between strong harmonic structure and spectral decimation.

Theorem 7 (Shima [16]) *Suppose the Laplacian $-\Delta$ has a strong harmonic structure. If the number of contraction maps is less than the Laplacian renormalization constant, i.e. $|S| < \frac{1}{K_D(0) r_0}$, then $-\Delta$ admits spectral decimation with a rational function R.*

If we take all r_i to be 1, then $1/r_0 = |S|$. By Theorem 4.10 in [11], it is known that for some $s \in S$, $r_s < K_D(0)^{-1}$. Therefore $K_D(0) < 1$ and $|S| < \frac{1}{K_D(0) r_0}$. Hence we have the following corollary.

Corollary 8 *If a Laplacian has the strong harmonic structure and all $r_i = 1$, then it admits spectral decimation.*

3 Criteria for Spectral Gaps

Under some technical conditions on the self-similar structure $\mathcal{L}$, one can find a complete orthonormal basis of $L^2(\mathbf{K}, \mu)$, formed by Dirichlet or Neumann eigenfunctions of the Laplacians, where μ is a Radon measure on $\mathbf{K}$. (For the existence of such an orthonormal basis, see Theorem 3.4.6 in [12].) Thus we can decompose any function $f \in L^2$ as

$$f = \sum_{j=1}^{\infty} c_j u_j \quad \text{with } c_j = \int_{\mathbf{K}} f u_j d\mu,$$

where u_j are the eigenfunctions of the Laplacian on $\mathbf{K}$, analogous to the sine and cosine functions on the interval. This sum is the *Fourier series* of f and its partial sums converge to f in L^2 norm.

Strichartz proved the following theorems concerning convergence of the Fourier series on $\mathcal{SG}$ and the Littlewood-Paley theorem.

Theorem 9 (Theorem 1 in [17]) *Let $\{N_m\}$ be a sequence of integers such that $\frac{\lambda_{N_m+1}}{\lambda_{N_m}} - 1$ is bounded away from zero. Then the partial sums of the Fourier series $S_{N_m} f$ converge to f as $m \to \infty$ in L^p for $f \in L^p$ $(1 \le p < \infty)$ and uniformly if f is continuous.*

Theorem 10 (Theorem 2 in [17]) *Let $1 < p < \infty$. Let*

$$Sf(x) = \left(\sum_{m=1}^{\infty} \left| \sum_{j=N_{m-1}+1}^{N_m} c_j u_j(x) \right|^2 \right)^{1/2},$$

where $\{N_m\}$ is the same sequence as in the above theorem. Then there exist constants A_p and B_p such that

$$A_p \parallel f \parallel_p \le \parallel Sf \parallel_p \le B_p \parallel f \parallel_p .$$

The proof of the two theorems uses a generic argument developed by Duong, Ouhabaz and Sikora ([7], Theorem 3.1 and 6.2), with two key ingredients. One is the existence of gaps [9] and the other is a suitable heat kernel estimate, which was originally proved on $\mathcal{SG}$ by Barlow and Perkins [4], and have been extended to other fractals. (See [3, 10, 12] and references therein.) Here the existence of gaps for a given sequence is defined as follows.

Definition 11 Let $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_k \le \cdots$ be an infinite sequence. We say that there exist *gaps* in the sequence if $\limsup_{k\to\infty} \frac{\alpha_{k+1}}{\alpha_k} > 1$.

Essentially, the same argument can be applied to any other fractals where there exist gaps in the spectrum of the Laplacian and the heat kernel estimates specified in [7] are satisfied.

In this section we shall give three criteria for the existence of gaps in the spectrum of Laplacians on fractals which admit spectral decimation. The proofs of the theorems we give are similar. In all cases, we show there are gaps in the spectrum between numbers A_k and B_k, where A_k, B_k are of the form

$$A_k = \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)}(x),$$

$$B_k = \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)}(y)$$

with x, y consecutive elements of the set $R^{-1}(\mathfrak{F})$.

Following the proofs of the theorems we will give examples of fractals to which the theorems can be applied. The first criterion applies to the level-2 and level-3 Sierpinski gaskets, $\mathcal{SG}$ and $\mathcal{SG}_3$, the 3-tree fractal, and the Hexagasket. It is shown in [9] that there are two sequences where we can find gaps for $\mathcal{SG}$ and our first criterion detects one of them. Slightly modifying the conditions of the first criterion, we obtain a second theorem which can be used to find the other sequence for $\mathcal{SG}$ where gaps are already known to occur. The third theorem can be used to prove the existence of gaps for the tree-like fractals.

3.1 Gap Theorems for the Sierpinski Gasket and Other Fractals

As before, we let $\mathfrak{F}_k = \{\lambda \in \mathfrak{F} : \lambda \text{ is an eigenvalue of } -\widehat{\Delta}_k\}$, be the set of forbidden/initial eigenvalues appearing at step k. We restrict the domain of R to the non-negative real line.

We denote the functions

$$y_0(z) = \lim_{m\to\infty} \rho^m \phi_0^{(m)}(z), \quad \text{if } v = 0, \tag{3.1}$$

$$y_v(z) = \lim_{m\to\infty} \rho^m \phi_0^{(m-j)} \phi_v(z), \quad \text{if } |v| = j,\ v_j \neq 0, \tag{3.2}$$

for all z such that the limits on the right-hand side exist. We are particularly interested about properties of the function y_0, which will be used in the proof of our criteria for finding spectral gaps.

Lemma 12 *If ϕ_0 is strictly convex on $[0, b]$, where b is the largest forbidden eigenvalue, and $\phi_0(b) < b$, then y_0 exists. Moreover, it is convex on its domain, strictly increasing and continuous.*

Proof We first note that $y_0(z)$ is well-defined for all $0 \le z \le b$ because the strict convexity of ϕ_0 and $\phi_0(0) = 0$ gives

$$\frac{\phi_0(z)}{z} < \frac{\phi_0(b)}{b} < 1,$$

for all z in $[0, b]$ and hence by Remark 6, y_0 exists. (The existence of the limit in a neighborhood of zero also follows from Koenig's Linearization theorem. See [15].)

We note that continuity follows from convexity. To prove the convexity, we choose z_1 and z_2 from the domain of ϕ_0 with $0 \le z_1 < z_2$ and let $0 < t < 1$. Since ϕ_0 is (strictly) convex, so is $\phi_0^{(m)}$ for all m. Hence,

$$\frac{t y_0(z_1) + (1-t) y_0(z_2)}{y_0(t z_1 + (1-t) z_2)} = \lim_m \frac{t \phi_0^{(m)}(z_1) + (1-t)\phi_0^{(m)}(z_2)}{\phi_0^{(m)}(t z_1 + (1-t) z_2)} \ge 1.$$

Next we prove that y_0 is strictly increasing. Since $y_0(0) = 0$, there is no loss of generality in taking $0 < z_1 < z_2$. By the strict convexity of ϕ_0,

$$\frac{\phi_0(z_1)}{z_1} < \frac{\phi_0(z_2)}{z_2},$$

which we rewrite as

$$\frac{\phi_0(z_2)}{\phi_0(z_1)} > \frac{z_2}{z_1}.$$

Repeated applications of this inequality gives

$$\frac{y_0(z_2)}{y_0(z_1)} = \lim_m \frac{\phi_0^{(m)}(z_2)}{\phi_0^{(m)}(z_1)} \ge \frac{z_2}{z_1} > 1.$$

□

Theorem 13 *Let b be the largest forbidden eigenvalue. There exist gaps in the spectrum of the generalized Laplacian on the fractal if the following conditions are satisfied*:

(1) $R^{-1}([0, b]) \subseteq [0, b]$;
(2) $\phi_1(x)$ *is defined and decreasing on* $[0, b]$;
(3) $\phi_0(x)$ *is strictly convex and* $\phi_0(b) < \phi_1(b)$;
(4) *there exists k_0 such that for all $k \ge k_0$ and all $x \in \mathfrak{F}_k$, $\phi_1(b) \le x$.*

Proof Let

$$A_k = \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m)}(b),$$

$$B_k = \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(b).$$

The first condition tells us that we can further iterate $\phi_0(b)$ and $\phi_1(b)$ by inverse functions of R. By Remark 6, Lemma 12 and the assumptions on ϕ_0 and ϕ_1, the two limits defining A_k and B_k exist and so A_k and B_k are well-defined for any k.

Since ϕ_0 is strictly convex by Lemma 12, we know that A_0 and B_0 are different and $B_0 > A_0$. Since $\frac{B_k}{A_k} = \frac{B_0}{A_0} > 1$, it is sufficient to show that there is no eigenvalue between A_k and B_k for all $k > k_0$ given in condition (4).

As the Laplacian admits spectral decimation, by Definition 5 and Remark 6 all eigenvalues must be of the form

$$\rho^i \lim_{m\to\infty} \rho^m \phi_0^{(m-j)} \phi_v(x),$$

where $i, j \in \mathbb{N} \cup \{0\}$, $|v| = j$, and $x \in \mathfrak{F}_{i+1}$. Hence it suffices to prove the following two claims:

(i) For $i \geq 0$ and $x \in \mathfrak{F}_{i+1}$,

$$B_k \leq \rho^i \lim_{m\to\infty} \rho^m \phi_0^{(m-j)} \phi_v(x) \leq A_{k+1},$$

where $\phi_v = \phi_{v_j} \circ \cdots \circ \phi_{v_1}$ for $v = v_j \cdots v_1$ with $v_j \neq 0$, $|v| = j$, and $i + j = k + 1$;

(ii) For all $k \geq k_0 - 1$ and $x \in \mathfrak{F}_{k+2} \subseteq \mathfrak{F}$, we have

$$B_k \leq \rho^{k+1} \lim_{m\to\infty} \rho^m \phi_0^{(m)}(x) \leq A_{k+1}.$$

Once our claims are proved, only the eigenvalues $\lim_{m\to\infty} \rho^m \phi_0^{(m)}(x)$ with $x \in \mathfrak{F}_1$, and the eigenvalues $\rho^{k+1} \lim_{m\to\infty} \rho^m \phi_0^{(m)}(x)$ with $x \in \mathfrak{F}_{k+2}$ and $k < k_0 - 1$ could lie in $\cup[A_j, B_j]$. As there are only finitely many such eigenvalues, this will not affect the existence of gaps in the sequence.

It is easy to see that the second inequality of (ii) follows directly from the monotonicity of ϕ_0. The first inequality of (ii) is equivalent to

$$\lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(b) \leq \rho \lim_{m\to\infty} \rho^m \phi_0^{(m)}(x).$$

If we replace m by $m' = m - 1$ on the right-hand side, we have

$$\lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(b) \leq \lim_{m\to\infty} \rho^m \phi_0^{(m-1)}(x).$$

The last inequality is true because condition (4) implies that $\phi_1(b) \leq x$ for all those forbidden eigenvalues which can appear at step k_0 or later.

To show (i), note that $\mathfrak{F}_{i+1} \subseteq \mathfrak{F}$ and

$$\rho^i \lim_{m\to\infty} \rho^m \phi_0^{(m-j)} \phi_v(x) = \lim_{m\to\infty} \rho^m \phi_0^{(m-i-j)} \phi_v(x).$$

It is sufficient to prove the following stronger inequalities:

(i′) For $x \in \mathfrak{F}$,

$$B_k \leq \lim_{m\to\infty} \rho^m \phi_0^{(m-k-1)} \phi_v(x) \leq A_{k+1},$$

where $\phi_v = \phi_{v_j} \circ \cdots \circ \phi_{v_1}$ for $v = v_j \cdots v_1$ with $v_j \neq 0$. To show the second inequality of (i′), note that $\phi_v(x) \leq b$ for any v and x by condition (1). Hence

$$\begin{aligned}\lim_{m\to\infty} \rho^m \phi_0^{(m-k-1)} \phi_v(x) &\leq \lim_{m\to\infty} \rho^m \phi_0^{(m-k-1)}(b)\\ &= \lim_{m\to\infty} \rho^{m+k+1} \phi_0^{(m)}(b) = A_{k+1}.\end{aligned}$$

Now we are only left with the first inequality of (i′), which is

$$\rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(b) \le \lim_{m\to\infty} \rho^m \phi_0^{(m-k-1)} \phi_v(x),$$

where $|v| = j$, $v_j \neq 0$, $x \in \mathfrak{F}$. Because of the monotonicity of ϕ_0, it is sufficient to show $\phi_1(b) \le \phi_v(x) = \phi_{v_j} \circ \phi_{v'}(x)$, where $|v'| = |v| - 1$, $v_j \neq 0$, and $x \in \mathfrak{F}$. Note that to make $\phi_v(x)$ the smallest, v_j has to be 1. Since $x \le b$ and ϕ_1 is decreasing, we have $\phi_1(b) \le \phi_v(x)$. □

Remark 14 (1) It is possible that A_k and B_k are not true eigenvalues in the spectrum, but this will not affect the existence of gaps. What is important is that there is no eigenvalue between A_k and B_k for sufficiently large k. Indeed, if A_k and B_k are not true eigenvalues, then there are even larger gaps between the greatest (true) eigenvalue less than A_k and the least eigenvalues greater than B_k.

(2) Since

$$\phi_0''(\lambda) = -\frac{R''(\phi_0(\lambda))}{[R'(\phi_0(\lambda))]^3},$$

the strict convexity of ϕ_0 can be verified by showing that R is strictly concave on the image of ϕ_0.

We further remark that the strict convexity of ϕ_0 of the theorem is used in our proof to verify that A_0 and B_0 exist and are distinct.

(3) Recall that $\phi_0(0) = 0$ and $\phi_0'(0) = \frac{1}{\rho} > 0$, where ρ is the Laplacian renormalization constant, so ϕ_0 is always increasing on its domain. Because we divide the branches of the inverse functions according to where the function turns, as long as R is continuous on $[0, a]$, where a is the least positive root of R, ϕ_1 will be decreasing.

Next we use the above theorem to prove the existence of gaps for the Sierpinski gaskets, $\mathcal{SG}$ and $\mathcal{SG}_3$, the fractal-3 tree and the Hexagasket. Theorem 13 can also be applied to the infinite family of Vicsek sets. We refer the readers to [20] for more details.

Example 3.1 (Sierpinski Gasket $\mathcal{SG}$, Fig. 1) For the Sierpinski gasket, the spectral decimation function is

$$R(\lambda) = \lambda(5 - 4\lambda).$$

(See [16, 19] or Sect. 1.) Hence the inverse functions are

$$\phi_{0,1}(\lambda) = \frac{5 \mp \sqrt{25 - 16\lambda}}{8}.$$

The set of forbidden eigenvalues, $\mathfrak{F}$, is $\{1/2, 5/4, 3/2\}$ and the set of forbidden eigenvalues at step k, $\mathfrak{F}_k$, is $\{5/4, 3/2\}$ for all $k \ge 2$. Figure 2 shows ϕ_0 and ϕ_1.

It is clear that 3/2 is in the domain of ϕ_0 and ϕ_1, $R^{-1}([0, 3/2]) = [0, 5/4] \subseteq [0, 3/2]$, and ϕ_1 is decreasing. The strict convexity of ϕ_0 can be easily obtained either by checking $\phi_0'' > 0$ or verifying $R'' < 0$ and ϕ_0 is increasing on its domain, as mentioned in Remark 14. The last condition is also satisfied since $\phi_1(3/2) = 3/4 < 5/4$,

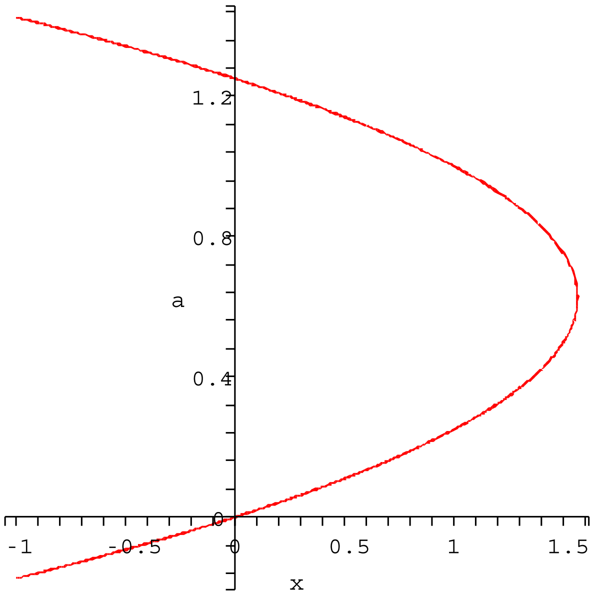


Fig. 2 Inverse functions of R for the Sierpinski gasket

the smallest forbidden eigenvalue in $\mathfrak{F}_k$ for $k \geq 2$. Hence all conditions of Theorem 13 are satisfied and so there are gaps in the spectrum.

Note that $\phi_0(3/2) = 1/2$ and $1/2$ does not appear as a forbidden eigenvalue at any step $k \geq 2$, so $A_k = 5^k \lim_{m\to\infty} \rho^m \phi_0^{(m)}(3/2)$ for $k \geq 2$ is not a true eigenvalue in the spectrum. But this does not affect the existence of gaps, as we pointed out in Remark 14. Indeed, we can replace A_k by the greatest eigenvalue less than A_k, say A'_k, and obtain larger gaps. We claim

$$A'_k = 5^k \lim_{m\to\infty} 5^m \phi_0^{(m)}(5/4) \quad \text{for } k \geq 2.$$

To see this, we first note that since $\phi_l(x) \leq 5/4$ for all $x \in [0, 3/2]$ and $l = 0, 1$, it follows that if $i + j = k + 1$ and $x \in \mathfrak{F}_{i+1}$,

$$5^i \lim_{m\to\infty} 5^m \phi_0^{(m-j)} \phi_v(x) \leq A'_{k+1}, \quad |v| = j, \text{ and } v_j \neq 0.$$

Furthermore, as $5/4$ is the second largest forbidden eigenvalue, we have

$$5^{k+1} \lim_{m\to\infty} 5^m \phi_0^{(m)}(x) \leq A'_{k+1}$$

for $x \in \mathfrak{F}_{k+2} \setminus \{3/2\}$. These observations establish the claim and show there are gaps in (A'_k, B_k) for $B_k = 5^k \lim_{m\to\infty} 5^m \phi_0^{(m-1)} \phi_1(3/2)$, as previously proven in [9].

Example 3.2 (The level-3 Sierpinski Gasket $\mathcal{SG}_3$, Fig. 3) The spectrum of the standard Laplacian on this fractal has also been obtained independently in [2] and [6].

The spectral decimation function is

$$R(\lambda) = \frac{6\lambda(\lambda - 1)(4\lambda - 3)(4\lambda - 5)}{6\lambda - 7}$$

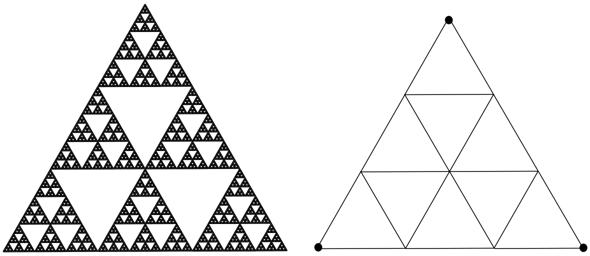


Fig. 3 Level-3 Sierpinski Gasket $\mathcal{SG}_3$

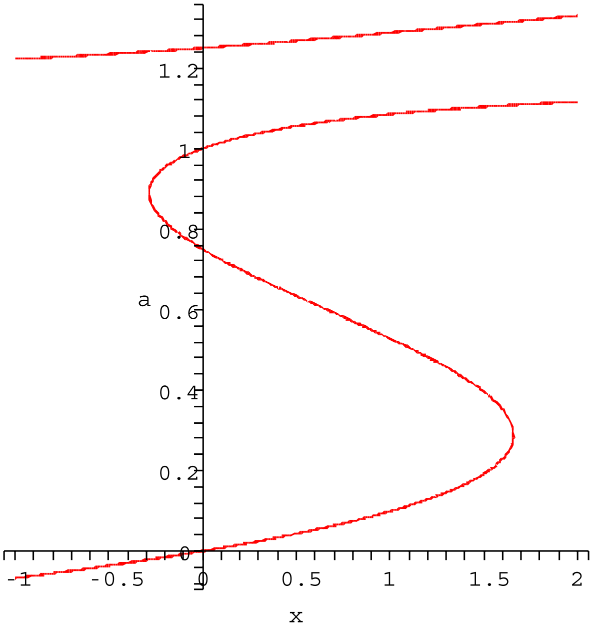


Fig. 4 Inverse functions of R for the level-3 Sierpinski gasket

and the forbidden eigenvalues are $\frac{3\pm\sqrt{5}}{4}$, $3/4, 7/6, 5/4,$ and $3/2$. The set of forbidden eigenvalues at step k, $\mathfrak{F}_k$, is $\{3/4, 5/4, 3/2\}$ for $k \geq 2$. Numerical data in [6] suggested that there are gaps in the spectrum. Now we can apply Theorem 13 to verify this result.

As R is continuous on $[0, 7/6)$, ϕ_1 is decreasing. Note that $R(\lambda) = 0$ has four real roots $0, 3/4, 1,$ and $5/4$. Furthermore,

$$R(\lambda) - 3/2 = \frac{3(4\lambda^2 - 6\lambda + 1)(16\lambda^2 - 24\lambda + 7)}{12\lambda - 14},$$

and it has four real roots $\frac{3\pm\sqrt{2}}{4}$, $\frac{3\pm\sqrt{5}}{4}$. Since

$$\begin{aligned} R^{-1}([0, 3/2]) &= [0, \max\{\text{largest root of } R(\lambda),\ \text{largest root of } R(\lambda) - 3/2\}] \\ &= \left[0, \frac{3+\sqrt{5}}{4}\right], \end{aligned}$$

which is contained in $[0, 3/2]$, the first condition of Theorem 13 is satisfied.

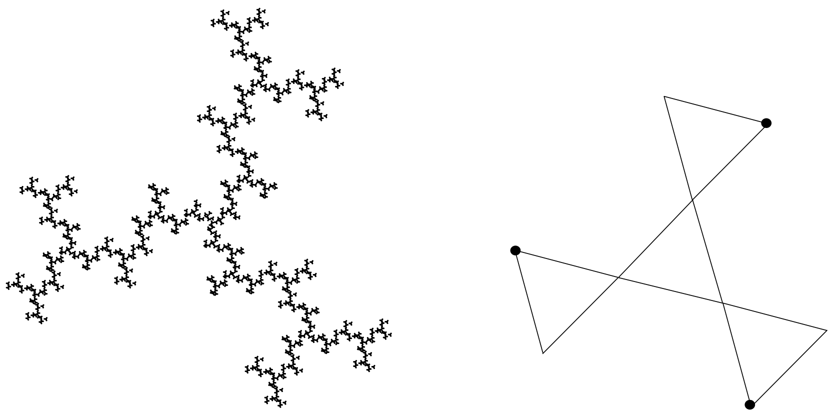


Fig. 5 A fractal-3 tree[1]

To check the third condition of the theorem, we note that R is differentiable on $(-\infty, 7/6)$ and

$$R'(x) = \frac{6(288x^4 - 1024x^3 + 1290x^2 - 658x + 105)}{(6x-7)^2}.$$

Using Maple, we find that R' are has only two real roots, 0.2880979998 and 0.8900943083, correct to six decimal places. The first turning point of R is 0.2880979998 and so the range of ϕ_0 is contained in $[0, 0.3]$. Using Maple again, we find

$$R''(x) = \frac{12(1728x^4 - 7104x^3 + 10752x^2 - 7056 + 1673)}{(6x-7)^3}$$

and the only two real roots of R'' are 0.5988200688 and 1.314052020. Particularly, this tells us that the sign of R'' remains unchanged on $[0, 0.3]$. We can easily check that it is negative, so R is strictly concave from 0 to 0.3. Therefore condition (3) is satisfied.

Note that $\phi_1(3/2)$ is equal to the second root of $R(\lambda) - 3/2 = 0$. Thus $\phi_1(3/2) = \frac{3-\sqrt{2}}{4}$, which is less than $3/4$, the smallest element in $\mathfrak{F}_k$ for $k \geq 2$, so the last condition of the theorem is satisfied. Therefore, by Theorem 13, there exist gaps in the spectrum.

Example 3.3 (A fractal-3 tree, Fig. 5) This fractal can be approximated by triangles with 3 boundary points and appeared as the limit set of the Gupta-Sidki group. (See [2] and references therein.)

The spectral decimation function R and the forbidden eigenvalues have been shown in [2] to be $R(\lambda) = 4\lambda(\lambda - 1)(4\lambda - 3)$, $\mathfrak{F} = \{\frac{3-\sqrt{3}}{4}, \frac{3}{4}, \frac{3+\sqrt{3}}{4}, \frac{3}{2}\}$. The inverse functions are shown in Fig. 6.

[1] The figure of this fractal is taken from Teplyaev with his permission.

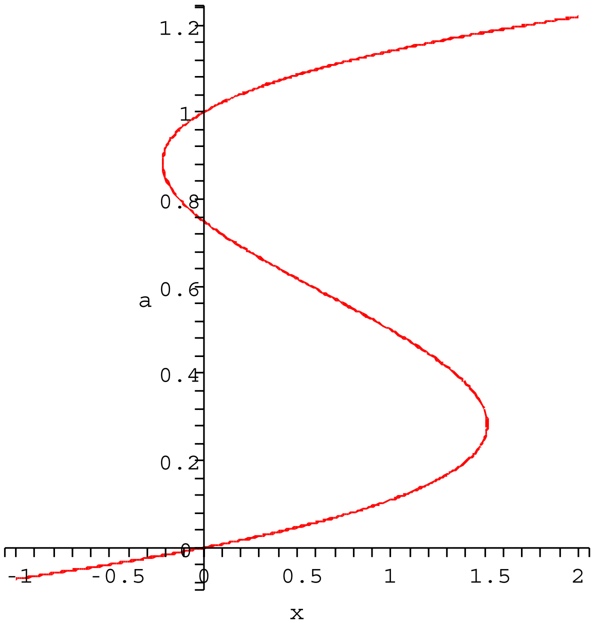


Fig. 6 Inverse functions of R for the fractal-3 tree

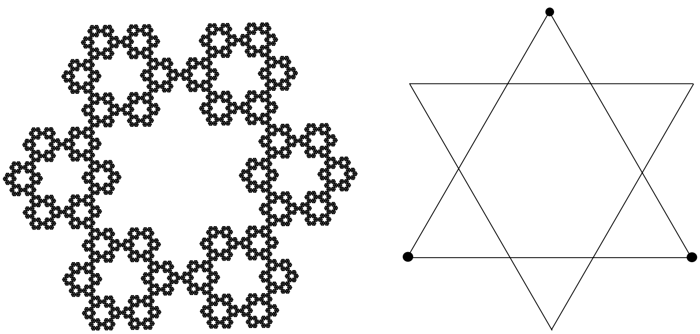


Fig. 7 Hexagasket

We note $R(\lambda) = 3/2$ has solutions $\frac{1}{4}, \frac{3\pm\sqrt{3}}{4}$ and so $\phi_1(3/2) = \frac{3-\sqrt{3}}{4}$, which is the smallest element in $\mathfrak{F}$. The rest of conditions in Theorem 13 can also be easily verified.

Example 3.4 (The Hexagasket, Fig. 7) The Hexagasket, or the Hexakun, can be obtained by starting with a regular hexagon and six contraction maps $\{F_j\}_{j=1}^6$ with ratio $1/3$ and fixed points equal to the vertices of the hexagon. The boundary points in this case are the six vertices of the hexagon. But this fractal can also be generated by a slightly different i.f.s, with smaller boundary. For each F_j, we compose it with a rotation of angle $\frac{j\pi}{3}$. Then there are only three boundary points, every other one of

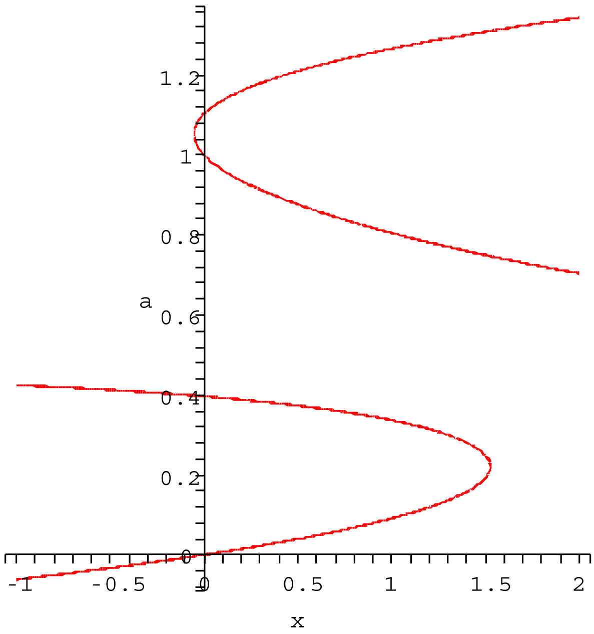


Fig. 8 Inverse functions of R for the hexagasket

the six vertices of the hexagon, indicated as solid dots in Fig. 7. This fractal has been studied in [2, 12, 18].

The spectral decimation function R and the forbidden eigenvalues have been shown in [2] to be $R(\lambda) = \frac{2\lambda(\lambda-1)(16\lambda^2-24\lambda+7)}{2\lambda-1}$, $\mathfrak{F} = \{\frac{3\pm\sqrt{2}}{4}, \frac{3\pm\sqrt{5}}{4}, \frac{1}{2}, \frac{3}{2}\}$. Moreover, $\mathfrak{F}_k = \{\frac{3\pm\sqrt{2}}{4}, \frac{1}{2}, \frac{3}{2}\}$, for $k \geq 2$. Inverse functions of R are shown in Fig. 8.

Note that $R(\lambda) = 3/2$ has solutions $\frac{1}{4}, \frac{3}{4}, \frac{3\pm\sqrt{5}}{4}$, so $\phi_1(3/2) = 1/4$ and it is less than the smallest element in $\mathfrak{F}_k$. Other conditions in Theorem 13 can also be easily verified and so there exist gaps in the spectrum of the Laplacian.

In [9], it is shown that there exists a second sequence of gaps for $\mathcal{SG}$. We can slightly modify our conditions in Theorem 13 and identify those gaps.

Theorem 15 *Let $a < b$ be the two largest forbidden eigenvalues in $\mathfrak{F}$. There exist gaps in the spectrum of the generalized Laplacian if the following conditions are satisfied*:

(1) $R^{-1}([0, b]) \subseteq [0, a]$;
(2) $\phi_1(x)$ *is defined and decreasing on* $[0, b]$;
(3) $\phi_0(x)$ *is strictly convex*;
(4) *there exists* k_0 *such that for all* $k \geq k_0$ *and all* $x \in \mathfrak{F}_k$, $\phi_1(a) \leq x$.

Proof The proof is quite similar to the above theorem. Let

$$A_k = \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(b),$$

$$B_k = \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(a).$$

We can show that all eigenvalues of the form

$$\rho^i \lim_{m\to\infty} \rho^m \phi_0^{(m-j)} \phi_v(x), \quad \text{with } |v| = j,\ v_j \neq 0,\ i + j = k + 1,$$

are between B_k and A_{k+1}, except the case when $v = 1$ and $x = b$ for A_k. The rest of the proof is almost identical to that of Theorem 13. □

Example 3.5 (Sierpinski Gasket $\mathcal{SG}$) In Example 3.1 we actually verified that the conditions of Theorem 15 were satisfied with $b = 3/2$, $a = 5/4$, and $k_0 = 2$. Therefore there are gaps between

$$A_k = 5^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(3/2)$$

and

$$B_k = 5^k \lim_{m\to\infty} \rho^m \phi_0^{(m-1)} \phi_1(5/4).$$

That is the second sequence of gaps for $\mathcal{SG}$ discovered in [9].

3.2 A Gap Theorem for the n-branch Tree-Like Fractal

We have the following alternative criterion to determine whether there are gaps in the spectrum of Laplacian. We shall use it to prove the existence of gaps for the tree-like fractals which will be described after the proof of the theorem.

Theorem 16 *Suppose $\alpha \leq \beta$ are two consecutive forbidden eigenvalues in $\mathfrak{F}$. Let $c \geq b$ be such that $R^{-1}([0, b]) \subseteq [0, c]$, where b is the largest forbidden eigenvalue. If the following conditions are satisfied, then there must be gaps in the spectrum.*

(1) $\phi_1(x) \geq \beta$, *for all* $x \in [0, c]$;
(2) $\phi_0(c) \leq \alpha$;
(3) $\phi_0(x)$ *is strictly convex.*

Proof The proof of this theorem is also very similar to previous theorems.

For $k \geq 0$, we let

$$A_k = \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m)}(\alpha),$$

$$B_k = \rho^k \lim_{m\to\infty} \rho^m \phi_0^{(m)}(\beta).$$

By the same reasoning as in the previous theorems, we can prove that both A_k and B_k exist and $\frac{B_k}{A_k} = \frac{B_0}{A_0} > 1$ for any k. We claim there is no eigenvalue between A_k and B_k for any k and hence there are gaps in the spectrum.

Let $k > 1$, $i + j = k$, $|v| = j$, and $v_j \neq 0$. We claim that for all $x \in \mathfrak{F}_{i+1} \subseteq \mathfrak{F}$, for which $\phi_0^{(m-j)} \phi_v(x)$ is defined,

$$B_k \leq \rho^i \lim_{m\to\infty} \rho^m \phi_0^{(m-j)} \phi_v(x) \leq A_{k+1}. \tag{3.3}$$

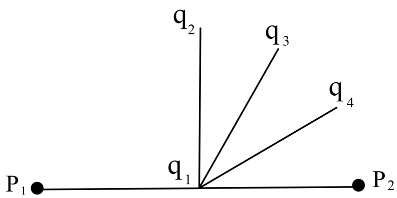


Fig. 9 Step-1 graph of the 5-branch tree-like fractal

Note that the first inequality above is equivalent to

$$\lim_{m\to\infty} \rho^m \phi_0^{(m)}(\beta) \le \lim_{m\to\infty} \rho^{m+i-k} \phi_0^{(m-j)} \phi_v(x).$$

Since $i-k=-j$, if we let $m'=m-j$ on the right-hand side, the above inequality is equivalent to

$$\lim_{m\to\infty} \rho^m \phi_0^{(m)}(\beta) \le \lim_{m\to\infty} \rho^{(m)} \phi_0^m \phi_v(x).$$

As ϕ_0 is increasing on its domain, it suffices to show that

$$\phi_v(x) \ge \beta, \;\; v_j \ne 0, \;\; x \in \mathfrak{F}_{i+1}.$$

By (1), $\phi_i(x) \ge \beta$ for all $x \in [0, c]$, so it is clearly true.

The second inequality of (3.3) is equivalent to

$$\lim_{m\to\infty} \rho^{m+k+1-i} \phi_0^{(m)}(\alpha) \ge \lim_{m\to\infty} \rho^m \phi_0^{(m-j)} \phi_v(x).$$

Again, notice that $k-i=j$ and let $m'=m+j+1$ on the left-hand side. We would have

$$\lim_{m\to\infty} \rho^m \phi_0^{(m-j-1)}(\alpha) \ge \lim_{m\to\infty} \rho^m \phi_0^{(m-j)} \phi_v(x).$$

Hence it suffices to show that

$$\phi_0(\phi_v(x)) \le \alpha$$

for all w. Since $R^{-1}([0,b]) \subseteq [0,c]$ and ϕ_0 is increasing, the largest possible value on the left-hand side is $\phi_0(c)$. Hence the last inequality is true by (2) and we have the desired result. □

Example 3.6 (The n-branch tree-like fractal) This infinite family of fractals can be obtained by sticking $n-2$ intervals to the middle of a unit interval and then repeating this process (sticking $n-2$ branches to each interval obtained in the previous step with half of its length). Figure 9 shows the first step graph for $n=5$.

For Laplacians on this infinite family of fractals, one can choose a weighted vector $\mathbf{r} = (1, 1, \underbrace{r^{-1}, \ldots, r^{-1}}_{n-2})$ with $r > 0$ so that the weights on the two outer branches are 1 and the weights on all $(n-2)$ inner branches are r. (See [16] for the case when $n=5$.)

Let D and H_1 be the matrices representing the graph Laplacians on V_0 and V_1 respectively. Then $D = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ and $H_1 = \begin{bmatrix} T & J^t \\ J & X \end{bmatrix}$, where T is the diagonal matrix with $T_{i,i} = D_{i,i}$,

$$J = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}_{(n-1)\times 2},$$

$$X = \begin{bmatrix} -(2+(n-2)r) & r & r & \cdots & r \\ r & -r & 0 & \cdots & 0 \\ & \vdots & \vdots & & \\ r & 0 & 0 & \cdots & -r \end{bmatrix}_{(n-1)\times(n-1)}.$$

Hence

$$X + \lambda M = \begin{bmatrix} (2+(n-2)r)(\lambda-1) & r & r & \cdots & r \\ r & (\lambda-1)r & 0 & \cdots & 0 \\ & \vdots & \vdots & & \\ r & 0 & 0 & \cdots & (\lambda-1)r \end{bmatrix}_{(n-1)\times(n-1)},$$

and so the normalized Laplacian for any function f on $V_1 \setminus V_0$ satisfies

$$\widehat{\Delta} f(q_i) = f(q_1) - f(q_i), \quad \text{for } 2 \le i \le n-1,$$

and

$$\widehat{\Delta} f(q_1) = \frac{1}{2+(n-2)r}\left[f(p_1) + f(p_2) + r\sum_{i=2}^{n-1} f(q_i) \right] - f(q_1).$$

Recall that the spectral decimation function R is given by

$$R(\lambda) = \frac{\lambda - K_T(\lambda)}{K_D(\lambda)},$$

where K_D and K_T are defined by

$$T - J^t(X+\lambda M)^{-1}J = K_D(\lambda)D + K_T(\lambda)T.$$

We can easily see that $K_D(\lambda) = (-J^t G(\lambda)J)_{1,2}$, where $G(\lambda) = (X+\lambda M)^{-1}$. Since

$$(J^t G(\lambda)J)_{1,2} = \sum_{k,j} J^t_{1,k} G_{k,j} J_{j,2} = G_{1,1},$$

we have that

$$K_D(\lambda) = -G(\lambda)_{1,1} = \frac{-1}{\det(X+\lambda M)}[(\lambda-1)r]^{n-2}.$$

So now the question is to find $\det(X + \lambda M)$. Repeatedly expanding by the last column, we have

$$\det(X+\lambda M) = ((\lambda-1)r)^{n-3}\begin{vmatrix}(2+(n-2)r)(\lambda-1) & r \\ r & (\lambda-1)r\end{vmatrix}$$
$$- (n-3)r^{n-1}(\lambda-1)^{n-3}$$
$$= (\lambda-1)^{n-3}r^{n-2}[(2+(n-2)r)\lambda^2 - 2(2+(n-2)r)\lambda + 2].$$

Therefore,

$$K_D(\lambda) = -\frac{\lambda-1}{(2+(n-2)r)\lambda^2 - 2(2+(n-2)r)\lambda + 2}.$$

Moreover,

$$K_T(\lambda) = \frac{T_{1,1} - (J^t G(\lambda) J)_{1,1} - K_D(\lambda) D_{1,1}}{T_{1,1}}$$
$$= 1 - 2K_D(\lambda).$$

It follows then

$$R(\lambda) = \frac{\lambda - K_T(\lambda)}{K_D(\lambda)} = -(2+(n-2)r)\lambda(\lambda-2).$$

The forbidden eigenvalues, by definition, are zeros of K_D and $\det(X + \lambda M)$, so $\mathfrak{F} = \{1, \alpha_1, \alpha_2\}$, where $\alpha_1 = 1 - \sqrt{\frac{(n-2)r}{2+(n-2)r}}$ and $\alpha_2 = 1 + \sqrt{\frac{(n-2)r}{2+(n-2)r}}$.

We now use Theorem 16 to show that there are gaps in the spectrum of the Laplacian on this infinite family of fractals. Let ϕ_0, ϕ_1 be the 2 branches of the inverse function of $R(\lambda)$ from bottom to top:

$$\phi_{0,1}(x) = 1 \mp \sqrt{\frac{2+(n-2)r-x}{2+(n-2)r}}.$$

Note that $\phi_i(x) \le 2$ for all $x \ge 0$. The domain of ϕ_0 and ϕ_1 (restricted to $[0,\infty]$) contains $[0,2]$ for any n and r. Hence we can take $b = \alpha_2 < 2$ and $c = 2$ in Theorem 16 and the condition $R^{-1}([0,b]) \subseteq [0,c]$ is satisfied. Let $\alpha = \alpha_1$ and $\beta = 1$. Clearly $\phi_1(x) \ge 1 = \beta$ for all $x \in [0,2]$ and so (1) is satisfied. As $\phi_0(2) = \alpha_1$, condition (2) is satisfied. As before, condition (3) of the strict convexity of ϕ_0 can be obtained by checking $R' > 0$ and $R'' < 0$ on $\mathrm{Im}(\phi_0)$.

Acknowledgements The author thanks Huojun Ruan and Alexander Teplyaev for their help on drawing the figures, and the referee for many helpful suggestions. He would like to express his deep appreciation to Kathryn Hare for her supervision on this project and many stimulating and fruitful discussions.

References

1. Adams, B., Smith, S.A., Strichartz, R., Teplyaev, A.: The Spectrum of the Laplacian on the Pentagasket. Trends in Mathematics, Fractals in Graz 2001. Birkhauser, Basel (2003), pp. 1–24

2. Bajorin, N., Chen, T., Dagan, A., Emmons, C., Hussein, M., Khalil, M., Mody, P., Steinhurst, B., Teplyaev, A.: Vibration modes of $3n$-gaskets and other fractals. J. Phys. A: Math. Theor. **41** (2008)
3. Barlow, M.: Diffusion on Fractals. Lecture Notes Math., vol. 1690. Springer, Berlin (1998)
4. Barlow, M., Perkins, E.: Brownian motion on the Sierpinski gasket. Probab. Theory Relat. Fields **79**, 543–623 (1988)
5. Drenning, S.: Fractal analysis: Extending the domain. Cornell Univ. Math. Dept. Senior Thesis (2005)
6. Drenning, S., Strichartz, R.: Spectral decimation on Hambly's homogeneous hierarchical gaskets. Preprint
7. Duong, X.T., Ouhabaz, E.M., Sikora, A.: Plancherel type estimates and sharp spectral multipliers. J. Funct. Anal. Appl. **196**, 443–485 (2002)
8. Fukushima, M., Shima, T.: On a spectral analysis for the Sierpinski gasket. Potential Anal. **1**, 1–35 (1992)
9. Gibbons, M., Raj, A., Strichartz, R.: The finite element method on the Sierpinski gasket. Constr. Approx. **17**, 561–588 (2001)
10. Hambly, B., Kumagai, T.: Transition density estimates for diffusion processes on post critically finite self-similar fractals. Proc. Lond. Math. Soc. **78**, 431–458 (1999)
11. Kigami, J.: Harmonic calculus on p.c.f. self-similar sets. Trans. Am. Math. Soc. **335**, 721–755 (1993)
12. Kigami, J.: Analysis on Fractals. Cambridge Univ. Press, New York (2001)
13. Kigami, J., Lapidus, M.L.: Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals. Commun. Math. Phys. **158**, 93–125 (1993)
14. Malozemov, L., Teplyaev, A.: Self-similarity, operators and dynamics. Math. Phys. Anal. Geom. **6**, 201–218 (2003)
15. Milnor, J.: Dynamics in one Complex Variable, 3rd edn. Princeton Univ. Press, New Jersey (2006)
16. Shima, T.: On eigenvalue problems for Laplacians on p.c.f. self-similar sets. Jpn. J. Ind. Appl. Math. **13**, 1–23 (1996)
17. Strichartz, R.: Laplacians on fractals with spectral gaps have nicer Fourier series. Math. Res. Lett. **12**, 269–274 (2005)
18. Strichartz, R.: Differential Equations on Fractals: A Tutorial. Princeton Univ. Press, New Jersey (2006)
19. Teplyaev, A.: Spectral analysis on infinite Sierpinski gasket. J. Funct. Anal. **159**, 537–567 (1998)
20. Zhou, D.: Spectral analysis of Laplacians on the Vicsek set. Pac. J. Math. **241**(2), 369–398 (2009)