Representation of Operators in the Time-Frequency Domain and Generalized Gabor Multipliers

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Abstract Starting from a general operator representation in the time-frequency domain, this paper addresses the problem of approximating linear operators by operators that are diagonal or band-diagonal with respect to Gabor frames. A characterization of operators that can be realized as Gabor multipliers is given and necessary conditions for the existence of (Hilbert-Schmidt) optimal Gabor multiplier approximations are discussed and an efficient method for the calculation of an operator's best approximation by a Gabor multiplier is derived. The spreading function of Gabor multipliers yields new error estimates for these approximations. Generalizations (multiple Gabor multipliers) are introduced for better approximation of overspread operators. The Riesz property of the projection operators involved in generalized Gabor multipliers is characterized, and a method for obtaining an operator's best approximation by a multiple Gabor multiplier is suggested. Finally, it is shown that in certain situations, generalized Gabor multipliers reduce to a finite sum of regular Gabor multipliers with adapted windows.

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1 Introduction

The goal of time-frequency analysis is to provide efficient representations for functions or distributions in terms of decompositions such as

$$f = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle h_\lambda.$$

Here, f is expanded as a weighted sum of *atoms* h_λ well localized in both time and frequency domains. The time-frequency coefficients $\langle f, g_\lambda \rangle$ characterize the function under investigation, and a *synthesis map* usually allows the reconstruction of the original function f.

Concrete applications can be found mostly in signal analysis and processing (see [8, 9, 36] and references therein), but recent works in different areas such as numerical analysis may also be mentioned (see for example [19, 20] and references therein).

Time-frequency analysis of operators, originating in the work on communication channels of Bello [5], Kailath [30] and Zadeh [41], has enjoyed increasing interest during the last few years, [2, 35, 38, 39]. Efficient time-frequency operator representation is a challenging task, and often the intuitively appealing approach of operator approximation by modification of the time-frequency coefficients before reconstruction is the method of choice. If the modification of the coefficients is confined to be multiplicative, this approach leads to the model of time-frequency multipliers, as discussed in Sect. 2.4. The class of operators that may be well represented by time-frequency multipliers depends on the choice of the parameters involved and is restricted to operators performing only small time-shifts or modulations.

The work in this paper is inspired by a general operator representation in the time-frequency domain via a *twisted convolution*. It turns out, that this representation, respecting the underlying structure of the Heisenberg group, has an interesting connection to the so-called spreading function representation of operators. An operator's spreading function comprises the amount of time-shifts and modulations, i.e. of time-frequency-shifts, effected by the operator. Its investigation is hence decisive in the study of time-frequency multipliers and their generalizations. Although no direct discretization of the continuous representation by an operator's spreading function is possible, the twisted convolution turns out to play an important role in the generalizations of time-frequency multipliers. In the main section of this article, we introduce a general model for *multiple Gabor multipliers (MGM)*, which uses several synthesis windows simultaneously. Thus, by jointly adapting the respective masks, more general operators may be well-represented than by regular Gabor multipliers. Specifying to a separable mask in the modification of time-frequency coefficients within MGM, as well as a specific sampling lattice for the synthesis windows, it turns out, that the MGM reduces to one or the sum of a finite number of regular Gabor multipliers with adapted synthesis windows.

For the sake of generality, most statements are given in a *Gelfand-triple*, rather than a pure Hilbert space setting. This choice bears several advantages. First of all, many important operators and signals may not be described in a Hilbert-space setting, starting from simple operators as the identity. Furthermore, by using distributions,

continuous and discrete concepts may be considered in a unified framework. Finally, the Gelfand-triple setting often allows for short-cut proofs of statements formulated in a general context.

This paper is organized as follows. The next section gives a review of the time-frequency plane and the corresponding continuous and discrete transforms. We then introduce the concept of Gelfand triples, which will allow us to consider operators beyond the Hilbert-Schmidt framework. The section closes with the important statement on operator-representation in the time-frequency domain via twisted convolution with an operator's spreading function. Section 2.4 introduces time-frequency multipliers and gives a criterion for their ability to approximate linear operators. A fast method for the calculation of an operator's best approximation by a Gabor multiplier in Hilbert-Schmidt sense is suggested. Section 3 introduces generalizations of Gabor multipliers. The operators in the construction of MGM are investigated and a criterion for their Riesz basis property in the space of Hilbert-Schmidt operators is given. We mention some connections to classical Gabor frames. A numerical example concludes the discussion of general MGM. In the final section, TST (twisted spline type) spreading functions are introduced. It is shown, that under certain conditions, a MGM reduces to a regular Gabor multiplier with an adapted window or a finite sum of regular multipliers with the same mask and adapted windows.

2 Operators from the Time-Frequency Point of View

Whenever one is interested in time-localized frequency information in a signal or operator, one is naturally led to the notion of the time-frequency plane, which, in turn, is closely related to the Weyl-Heisenberg group.

2.1 Preliminaries: The Time-Frequency Plane

The starting point of our operator analysis is the so-called *spreading function operator representation*. This operator representation expresses linear operators as a sum (in a sense to be specified below) of time-frequency shifts $\pi(b, \nu) = M_\nu T_b$. Here, the translation and modulation operators are defined as

$$T_b f(t) = f(t - b), \qquad M_\nu f(t) = e^{2i\pi\nu t} f(t), \quad f \in \mathbf{L}^2(\mathbb{R}).$$

These (unitary) operators generate a group, called the Weyl-Heisenberg group

$$\mathbb{H} = \{(b, \nu, \varphi) \in \mathbb{R} \times \mathbb{R} \times [0, 1[\} , \tag{1}$$

with group multiplication

$$(b, \nu, \varphi)(b', \nu', \varphi') = (b + b', \nu + \nu', \varphi + \varphi' - \nu'b). \tag{2}$$

The specific quotient space $\mathbb{P} = \mathbb{H}/[0, 1]$ of the Weyl-Heisenberg group is called *phase space*, or *time-frequency plane*, which plays a central role in the subsequent analysis. Details on the Weyl-Heisenberg group and the time-frequency plane may

be found in [22, 40]. In the current article, we shall limit ourselves to the basic irreducible unitary representation of $\mathbb{H}$ on $\mathbf{L}^2(\mathbb{R})$, denoted by π^o, and defined by

$$\pi^o(b, v, \varphi) = e^{2i\pi\varphi} M_v T_b. \tag{3}$$

By $\pi(b, v) = \pi^o(b, v, 0)$ we denote the restriction to the phase space. We refer to [11] or [25, Chap. 9] for a more detailed analysis of this quotient operation.

The left-regular (and right-regular) representation generally plays a central role in group representation theory. By unimodularity of the Weyl-Heisenberg group, its left and right regular representations coincide. We thus focus on the left-regular one, acting on $\mathbf{L}^2(\mathbb{H})$ and defined by

$$[L(b', v', \varphi')F](b, v, \varphi) = F(b - b', v - v', \varphi - \varphi' + b'(v - v')). \tag{4}$$

Denote by μ the Haar measure. Given $F, G \in \mathbf{L}^2(\mathbb{H}, d\mu)$, the associated (left) convolution product is the bounded function $F * G$, given by

$$(F * G)(b, v, \varphi) = \int_{\mathbb{H}} F(h)\big[L(h)G\big](b, v, \varphi)\, d\mu(h).$$

After quotienting out the phase term, this yields the *twisted convolution* on $\mathbf{L}^2(\mathbb{P})$:

$$(F \natural G)(b, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(b', v')G(b - b', v - v')e^{-2i\pi b'(v - v')}\, db'\, dv'. \tag{5}$$

The twisted convolution, which admits a nice interpretation in terms of group Plancherel theory [11] is non-commutative (which reflects the non-Abelianess of $\mathbb{H}$) but associative. It satisfies the usual Young inequalities, but is in some sense nicer than the usual convolution, since $\mathbf{L}^2(\mathbb{R}^2)\natural\mathbf{L}^2(\mathbb{R}^2) \subset \mathbf{L}^2(\mathbb{R}^2)$ (see [22] for details).

As explained in [27, 28] (see also [23] for a review), the representation π^o is unitarily equivalent to a subrepresentation of the left regular representation. The representation coefficient is given by a variant of the *short time Fourier transform* (STFT), which we define next.

Definition 1 Let $g \in \mathbf{L}^2(\mathbb{R})$, $g \neq 0$. The STFT of any $f \in \mathbf{L}^2(\mathbb{R})$ is the function on the phase space $\mathbb{P}$ defined by

$$\mathcal{V}_g f(b, v) = \langle f, \pi(b, v)g \rangle = \int_{-\infty}^{\infty} f(t)\overline{g}(t - b)e^{-2i\pi vt}\, dt. \tag{6}$$

This STFT is obtained by quotienting out $[0, 1]$ in the group transform

$$\mathcal{V}_g^o f(b, v, \varphi) = \langle f, \pi^o(b, v, \varphi)g \rangle. \tag{7}$$

The integral transform $\mathcal{V}_g^o$ intertwines L and π^o, i.e. $L(h)\mathcal{V}_g^o = \mathcal{V}_g^o\pi^o(h)$ for all $h \in \mathbb{H}$. The latter relation still holds true (up to a phase factor) when π^o and $\mathcal{V}_g^o$ are replaced with π and $\mathcal{V}_g$ respectively.

It follows from the general theory of square-integrable representations that for any $g \in \mathbf{L}^2(\mathbb{R})$, $g \neq 0$, the transform $\mathcal{V}_g^o$ is (a multiple of) an isometry $\mathbf{L}^2(\mathbb{R}) \rightarrow$

$\mathbf{L}^2(\mathbb{P})$, and thus left invertible by the adjoint transform (up to a constant factor). More precisely, given $h \in \mathbf{L}^2(\mathbb{R})$ such that $\langle g, h \rangle \neq 0$, one has for all $f \in \mathbf{L}^2(\mathbb{R})$

$$f = \frac{1}{\langle h, g \rangle} \int_{\mathbb{P}} \mathcal{V}_g f(b, \nu) \pi(b, \nu) h \, db d\nu. \tag{8}$$

We refer to [8, 25] for more details on the STFT and signal processing applications.

The STFT, being a continuous transform, is not well adapted for numerical calculations, and, for practical issues, is replaced by the Gabor transform, which is a sampled version of it. To fix notation, we outline some steps of the Gabor frame theory and refer to [9, 25] for a detailed account.

Definition 2 (Gabor transform) Given $g \in \mathbf{L}^2(\mathbb{R})$ and two constants $b_0, \nu_0 \in \mathbb{R}^+$, the corresponding Gabor transform associates with any $f \in \mathbf{L}^2(\mathbb{R})$ the sequence of Gabor coefficients

$$\mathcal{V}_g f(mb_0, n\nu_0) = \langle f, M_{n\nu_0} T_{mb_0} g \rangle = \langle f, g_{mn} \rangle, \tag{9}$$

where the functions $g_{mn} = M_{n\nu_0} T_{mb_0} g$ are the Gabor atoms associated to g and the lattice constants b_0, ν_0.

Whenever the Gabor atoms associated to g and the given lattice $\Lambda_{b_0, \nu_0} = b_0 \mathbb{Z} \times \nu_0 \mathbb{Z}$ form a frame,[1] the Gabor transform is left invertible, and there exists $h \in \mathbf{L}^2(\mathbb{R})$ such that any $f \in \mathbf{L}^2(\mathbb{R})$ may be expanded as

$$f = \sum_{m,n} \mathcal{V}_g f(mb_0, n\nu_0) h_{mn}. \tag{10}$$

In addition to the notion of a lattice we need to define the dual concept of adjoint lattices.

Definition 3 (Adjoint lattice) For a given lattice $\Lambda = b_0 \mathbb{Z} \times \nu_0 \mathbb{Z}$ the adjoint lattice is given by $\Lambda^\circ = \frac{1}{\nu_0} \mathbb{Z} \times \frac{1}{b_0} \mathbb{Z}$.

Note that the adjoint lattice is the dual lattice $\Lambda^\perp$ with respect to the symplectic character.

2.2 The Gelfand Triple $(S_0, \mathbf{L}^2, S_0')$

We next set up a framework for the exact description of operators we are interested in. In fact, by their property of being compact operators, the Hilbert space of Hilbert-Schmidt operators turns out to be far too restrictive to contain most operators of

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[1]The operator

$$S_g f = \sum_{m,n \in \mathbb{Z}} \langle f, M_{mb_0} T_{n\nu_0} g \rangle M_{mb_0} T_{n\nu_0} g$$

is the *frame operator* corresponding to g and the lattice defined by (b_0, ν_0). If S_g is invertible on $L^2(\mathbb{R})$, the family of time-frequency shifted atoms $M_{mb_0} T_{n\nu_0} g$, $m, n \in \mathbb{Z}$, is a *Gabor frame* for $L^2(\mathbb{R})$.

practical interest, starting from the identity. Although the classical triple $(\mathcal{S}, \mathbf{L}^2, \mathcal{S}')$ might seem to be the appropriate choice of generalization, we prefer to resort to the Gelfand triple $(S_0, \mathbf{L}^2, S_0')$, which has proved to be more adapted to a time-frequency environment. Additionally, the Banach space property of S_0 guarantees a technically less elaborate account.

Definition 4 (S_0) Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz class. Fix a non-zero "window" function $\varphi \in \mathcal{S}(\mathbb{R})$. The space $S_0(\mathbb{R})$ is given by

$$S_0(\mathbb{R}) = \{f \in \mathbf{L}^2(\mathbb{R}) : \|f\|_{S_0} := \|\mathcal{V}_\varphi f\|_{L^1(\mathbb{R}^2)} < \infty\}.$$

Note that S_0 is a special case of a modulation space [25], namely $M^{1,1}$, and is also called Feichtinger's algebra in the literature. The following proposition summarizes some properties of $S_0(\mathbb{R})$ and its dual, the distribution space $S_0'(\mathbb{R})$.

Proposition 5 $S_0(\mathbb{R})$ *is a Banach space and densely embedded in* $\mathbf{L}^2(\mathbb{R})$. *The definition of* $S_0(\mathbb{R})$ *is independent of the window* $\varphi \in \mathcal{S}(\mathbb{R})$, *and different choices of* $\varphi \in \mathcal{S}(\mathbb{R})$ *yield equivalent norms on* $S_0(\mathbb{R})$.

By duality, $\mathbf{L}^2(\mathbb{R})$ *is densely and weak*-continuously embedded in* $S_0'(\mathbb{R})$ *and can also be characterized by the norm* $\|f\|_{S_0'} = \|\mathcal{V}_\varphi f\|_{\mathbf{L}^\infty}$.

A triple of spaces $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$ consisting of a Banach space $\mathbf{B}$ which is densely embedded in the Hilbert space $\mathcal{H}$, which, in turn is densely embedded in $\mathbf{B}'$, is called (Banach) Gelfand triple. Hence, $(S_0, \mathbf{L}^2, S_0')$ represents a special case of a *Gelfand triple* [24] or Rigged Hilbert space. The prototype of a Gelfand triple is the triple of sequence spaces $(\ell^1, \ell^2, \ell^\infty)$. For a proof of Proposition 5, equivalent characterizations, and more results on S_0 we refer to [13, 14, 17].

Via an isomorphism between integral kernels in the Banach spaces S_0, S_0' and the operator spaces of bounded operators $S_0' \mapsto S_0$ and $S_0 \mapsto S_0'$, we obtain, together with the Hilbert space of Hilbert-Schmidt operators, a Gelfand triple of operator spaces, as follows. We denote by $\mathcal{B}$ the family of operators that are bounded $S_0' \to S_0$ and by $\mathcal{B}'$ the family of operators that are bounded $S_0 \to S_0'$. We have the following correspondence between these operator classes and their integral kernels κ:

$$H \in (\mathcal{B}, \mathcal{H}, \mathcal{B}') \quad \longleftrightarrow \quad \kappa_H \in (S_0(\mathbb{R}^2), \mathbf{L}^2(\mathbb{R}^2), S_0'(\mathbb{R}^2)).$$

We will make use of the principle of *unitary Gelfand triple isomorphisms*, described for the Gelfand triples just introduced in [17]. A linear mapping between two Gelfand triples $(\mathbf{B}_1, \mathcal{H}_1, \mathbf{B}_1')$, $(\mathbf{B}_2, \mathcal{H}_2, \mathbf{B}_2')$ is called unitary Gelfand triple isomorphism if it is an isomorphism between $\mathbf{B}_1, \mathbf{B}_2$, a unitary isomorphism on the Hilbert space level and extends to a weak*-continuous bijection from $\mathbf{B}_1'$ to $\mathbf{B}_2'$. In fact, it may be shown, that it suffices to verify unitarity of a given isomorphic operator on the (dense) subspace S_0 in order to obtain a unitary Gelfand triple isomorphism, see [17, Corollary 7.3.4]. The most prominent examples for a unitary Gelfand triple isomorphism are the Fourier transform and the partial Fourier transform. For all further details on the Gelfand triples just introduced, we again refer to [17], only mentioning here,

that one important reason for investigating operator representations on the level of Gelfand triples instead of just a Hilbert space framework is the fact, that S_0' contains distributions such as the Dirac functionals, Shah distributions or just pure frequencies and $\mathcal{B}'$ contains operators of great importance in signal processing, e.g. convolution, the identity or just time-frequency shifts.

Subsequently, we will usually assume that the analysis and synthesis windows g, h are in S_0. This is a rather mild condition, which has almost become the canonical choice in Gabor analysis, for many good reasons. Among others, this choice guarantees a beautiful correspondence between the ℓ^p-spaces and corresponding modulation spaces [25]. In the ℓ^2-case this means, that the sequence of Gabor atoms generated from time frequency translates of an S_0 window on an arbitrary lattice Λ is automatically a Bessel sequence (in such a case, the window is termed "Bessel atom"), which is not true for general $\mathbf{L}^2$-windows.

The Banach spaces S_0 and S_0' may also be interpreted as Wiener amalgam spaces [13, Sect. 3.2.2]. These time-frequency homogeneous spaces are defined as follows. Let $\mathcal{F}\mathbf{L}^1$ denote the Fourier image of integrable functions and let a compactly supported function $\phi \in \mathcal{F}\mathbf{L}^1(\mathbb{R})$ with $\sum_{n\in\mathbb{Z}} \phi(x - n) \equiv 1$ be given. Then, for $\mathbf{X}(\mathbb{R}) = \mathcal{F}\mathbf{L}^1(\mathbb{R})$ or $\mathbf{X}(\mathbb{R}) = \mathbf{C}(\mathbb{R})$, i.e. the space of continuous functions on $\mathbb{R}$, or any of the Lebesgue spaces, we define, for $p \in [1, \infty)$, with the usual modification for $p = \infty$:

$$\mathbf{W}(\mathbf{X}, \ell^p) = \left\{ f \in \mathbf{X}_{loc} : \|f\|_{\mathbf{W}(\mathbf{X},\ell^p)} = \left(\sum_{n\in\mathbb{Z}} \|f T_n \phi\|_{\mathbf{X}}^p \right)^{1/p} < \infty \right\}. \qquad (11)$$

Now, $S_0 = \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)$ and $S_0' = \mathbf{W}(\mathcal{F}\mathbf{L}^\infty, \ell^\infty)$, see [13, Sect. 3.2.2].

2.3 The Spreading Function Representation and Its Connections to the STFT

The so-called *spreading function* representation, closely related to the integrated Schrödinger representation [25, Sect. 9.2], expresses operators in $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ as a sum of time-frequency shifts. More precisely, one has (see [25, Chap. 9]):

Theorem 6 *Let $H \in (\mathcal{B}, \mathcal{H}, \mathcal{B}')$; then there exists a spreading function η_H in $(S_0(\mathbb{R}^2), \mathbf{L}^2(\mathbb{R}^2), S_0'(\mathbb{R}^2))$ such that*

$$H = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta_H(b, \nu)\pi(b, \nu)\, db d\nu. \qquad (12)$$

For $H \in \mathcal{H}$, the correspondence $H \leftrightarrow \eta_H$ is isometric, i.e. $\|H\|_{\mathcal{H}} = \|\eta_H\|_{\mathbf{L}^2(\mathbb{P})}$.

Remark 7 For $H \in \mathcal{B}$, the decomposition given in (12) is absolutely convergent, whereas, for $H \in \mathcal{B}'$, it holds in the weak sense of bilinear forms on S_0.

When $\eta_H \in \mathbf{L}^2(\mathbb{P})$, H is a Hilbert-Schmidt operator, and the above integral is defined as a Bochner integral.

The spreading function is intimately related to the integral kernel $\kappa = \kappa_H$ of H via

$$\eta_H(b, v) = \int_{-\infty}^{\infty} \kappa_H(t, t - b)e^{-2i\pi vt} \, dt \quad \text{and}$$

$$\kappa_H(t, s) = \int_{-\infty}^{\infty} \eta_H(t - s, v)e^{2i\pi vt} \, dv.$$

As a consequence, for $\kappa_H \in (S_0, \mathbf{L}^2, S_0')$, we also have $\eta_H \in (S_0, \mathbf{L}^2, S_0')$. In particular, this leads to the following expression for a weak evaluation of Gelfand triple operators:

$$\langle K, L \rangle_{(\mathcal{B}, \mathcal{H}, \mathcal{B}')} = \langle \kappa(K), \kappa(L) \rangle_{(S_0, \mathbf{L}^2, S_0')} = \langle \eta(K), \eta(L) \rangle_{(S_0, \mathbf{L}^2, S_0')}. \tag{13}$$

For $L = g \otimes f^*$, i.e. the tensor product with kernel $\kappa(s, t) = g(s)\overline{f}(t)$ and spreading function $\eta(b, v) = \mathcal{V}_f g(b, v)$, we thus have:

$$\langle K, g \otimes f^* \rangle_{(\mathcal{B}, \mathcal{H}, \mathcal{B}')} = \langle \kappa(K), \kappa(g \otimes f^*) \rangle_{(S_0, \mathbf{L}^2, S_0')} = \langle \eta(K), \mathcal{V}_f g \rangle_{(S_0, \mathbf{L}^2, S_0')}. \tag{14}$$

Let us also mention that the spreading function is related to the operator's Kohn-Nirenberg symbol via a *symplectic Fourier transform*, which we define for later reference.

Definition 8 The symplectic Fourier transform is formally defined by

$$\mathcal{F}_s F(t, \xi) = \int_{\mathbb{P}} F(b, v)e^{-2\pi i(b\xi - tv)} db \, dv.$$

The symplectic Fourier transform is a self-inverse unitary automorphism of the Gelfand triple $(S_0, \mathbf{L}^2, S_0')$. We will make use of the following relation.

Lemma 9 *Assume that* $f_1, f_2, g_1, g_2 \in \mathbf{L}^2(\mathbb{R})$. *Then*

$$\mathcal{F}_s(\mathcal{V}_{g_1} f_1 \overline{\mathcal{V}_{g_2} f_2})(x, \omega) = (\mathcal{V}_{f_2} f_1 \overline{\mathcal{V}_{g_2} g_1})(x, \omega).$$

Proof The analogous statement for the conventional (Cartesian) Fourier transform reads $\mathcal{F}(\mathcal{V}_{g_1} f_1 \overline{\mathcal{V}_{g_2} f_2})(x, \omega) = (\mathcal{V}_{f_2} f_1 \overline{\mathcal{V}_{g_2} g_1})(-\omega, x)$ and has been shown in [26, Lemma 2.3.2]. The fact, that

$$[\mathcal{F}_s F](x, \omega) = \hat{F}(-\omega, x)$$

completes the argument. $\square$

Recall that the spreading function of the product of operators corresponds to the twisted convolution of the operators' spreading function. Assume K_1 in $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ and K_2 in $(\mathcal{B}', \mathcal{H}, \mathcal{B})$ then

$$\eta(K_2 \cdot K_1) = \eta(K_2) \natural \eta(K_1). \tag{15}$$

The spreading function representation of operators provides an interesting time-frequency implementation for operators, stated in the following proposition. It turns out to be closely connected to the tools described in the previous section, in particular twisted convolution and STFT.

Proposition 10 *Let H be in $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$, and let $\eta = \eta_H$ be its spreading function in $(S_0(\mathbb{R}^2), \mathbf{L}^2(\mathbb{R}^2), S_0'(\mathbb{R}^2))$. Let $g \in S_0(\mathbb{R})$, then the STFT of Hf is given by a twisted convolution of η_H and $\mathcal{V}_g f$:*

$$\mathcal{V}_g Hf(z) = (\eta_H \natural \mathcal{V}_g f)(z).$$

Proof By (12), we may write

$$
\begin{aligned}
\mathcal{V}_g Hf(z') &= \langle Hf, \pi(z')g \rangle \\
&= \int \langle \eta_H(z)\pi(z)f, \pi(z')g \rangle dz \\
&= \int \eta_H(z)\langle f, \pi(z)^*\pi(z')g \rangle dz \\
&= \int \eta_H(z)e^{-2\pi i b(\nu'-\nu)}\langle f, \pi(z'-z)g \rangle dz \\
&= \int \eta_H(z)\mathcal{V}_g f(z'-z)e^{-2\pi i b(\nu'-\nu)}dz = (\eta_H \natural \mathcal{V}_g f)(z'). \quad (16)
\end{aligned}
$$

Note that S_0 is time-frequency shift-invariant, so $\pi(z)g$ is in S_0 for all z. Hence, the expression in (16) is well-defined.

If $f \in \mathbf{L}^2(\mathbb{R})$ and $H \in \mathcal{H}$, then $\mathcal{V}_g f$, $\mathcal{V}_g Hf$ and $\eta_H \in \mathbf{L}^2(\mathbb{R}^2)$, which is in accordance with the fact that $\mathbf{L}^2 \natural \mathbf{L}^2 \subseteq \mathbf{L}^2$.

If $f \in S_0(\mathbb{R})$, then H may be in $\mathcal{B}'$, such that $\eta_H \in S_0'(\mathbb{R}^2)$, hence $Hf \in S_0'(\mathbb{R})$. Hence, we have $\mathcal{V}_g f \in S_0(\mathbb{R}^2)$ and $\mathcal{V}_g Hf \in S_0'(\mathbb{R}^2)$. This leads to the inclusion $S_0' \natural \mathbf{W}(\mathbf{C}, \ell^1) \subseteq \mathbf{L}^\infty$, which may easily be verified directly.

On the other hand, if $f \in S_0'(\mathbb{R})$ and H in $\mathcal{B}$, such that $\eta_H \in S_0(\mathbb{R}^2)$, then Hf is in $S_0(\mathbb{R})$. Hence, $\mathcal{V}_g f \in S_0'(\mathbb{R}^2)$ and $\mathcal{V}_g Hf \in S_0(\mathbb{R}^2)$. Here, this leads to the conclusion that we have, for $f \in S_0'(\mathbb{R})$:

$$S_0 \natural \mathcal{V}_g f \subseteq S_0. \quad (17)$$

$\square$

Although it is known that $\mathcal{V}_g f$ is not only $\mathbf{L}^\infty(\mathbb{R}^2)$, but also in the Amalgam space $\mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^\infty)$ for $f \in S_0'(\mathbb{R})$ and $g \in S_0(\mathbb{R})$, [13], it is not clear, whether (17) also holds for functions $F \in \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^\infty)$, which are not in the range of $S_0'(\mathbb{R})$ under $\mathcal{V}_g$. This and other interesting open questions concerning the twisted convolution of function spaces are currently under investigation.[2]

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[2]H. Feichtinger and F. Luef. Twisted convolution properties for Wiener amalgam spaces. *In preparation*, 2009.

Remark 11 As a consequence of the last proposition, H may be realized as a twisted convolution in the time-frequency domain:

$$Hf = \frac{1}{\langle g, h \rangle} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\eta_H \natural \mathcal{V}_g f \right)(b, \nu) M_\nu T_b h \, db d\nu \quad \text{for all } f \in (S_0, \mathbf{L}^2, S_0').$$

(18)

Notice that Proposition 10 implies that the range of $\mathcal{V}_g$ is invariant under left twisted convolution. Notice also that this is no longer true if the left twisted convolution is replaced with the right twisted convolution. Indeed, in such a case, one has

$$\mathcal{V}_g f \natural \eta_H = \mathcal{V}_{H^* g} f.$$

Hence, one has the following simple rule: left twisted convolution on the STFT amounts to acting on the analyzed function f, while right twisted convolution on the STFT amounts to acting on the analysis window g. It is worth noticing that in such a case, applying $\mathcal{V}_g^*$ to $\mathcal{V}_g f \natural \eta_H$ yields the analyzed function f, up to some (possibly vanishing) constant factor.

Example 12 As an illustrative example, let $g, h \in S_0$ be such that $\langle g, h \rangle = 1$ and consider the oblique projection $P : f \mapsto \langle f, g \rangle h$. The spreading function of this operator is given by $\mathcal{V}_g h$, and we have $\mathcal{V}_\varphi P f(z) = \langle f, g \rangle \langle h, \pi(z) \varphi \rangle$. By virtue of the inversion formula for the STFT, which may be written as $\langle f, g \rangle h = \int \langle h, \pi(z) g \rangle \pi(z) f \, dz$, we obtain:

$$\mathcal{V}_\varphi P f(z') = \int \langle h, \pi(z) g \rangle \langle \pi(z) f, \pi(z') \varphi \rangle dz = \mathcal{V}_g h \natural \mathcal{V}_\varphi f.$$

By completely analogous reasoning, we obtain the converse formula, if the operator is applied to the analysing window:

$$\mathcal{V}_{P\varphi} f = \langle f, \pi(z) P \varphi \rangle = \mathcal{V}_\varphi f \natural \mathcal{V}_g h.$$

Remark 13 Notice also that twisted convolution in the phase space is associated with the true translation structure. Indeed, time-frequency shifts take the form of twisted convolutions with a Dirac distribution on $\mathbb{P}$:

$$\delta_{b_0, \nu_0} \natural \mathcal{V}_g f = \mathcal{V}_g M_{\nu_0} T_{b_0} f.$$

This corresponds to the usage of engineers, who "adjust the phases" after shifting STFT coefficients [10, 21].

2.4 Time-Frequency Multipliers

Section 2.3 has shown the close connection between the spreading function representation of Hilbert-Schmidt operators and the short time Fourier transform. However, the twisted convolution representation is generally of poor practical interest in the continuous case, because it does not discretize well. Even in the finite case, it relies on the full STFT on $\mathbb{C}^N$, which represents vectors with N^2 STFT coefficients, which

may be far too large in practice, and sub-sampling is not possible in a straightforward way.

Time-frequency (in particular Gabor) multipliers represent a valuable alternative for time-frequency operator representation (see [18, 29] and references therein for reviews). We analyze below the connections between these representations and the spreading function, and point out some limitations, before turning to generalizations.

2.4.1 Definitions and Main Properties

Let $g, h \in S_0(\mathbb{R})$ be such that $\langle g, h \rangle = 1$, let $\mathbf{m} \in \mathbf{L}^\infty(\mathbb{R}^2)$, and define the STFT multiplier $\mathbb{M}_{\mathbf{m};g,h}$ by

$$\mathbb{M}_{\mathbf{m};g,h} f = \int_\mathbb{P} \mathbf{m}(b, v) \mathcal{V}_g f(b, v) \pi(b, v) h \, db \, dv. \tag{19}$$

This defines a bounded operator on $(S_0(\mathbb{R}), \mathbf{L}^2(\mathbb{R}), S_0'(\mathbb{R}))$.

Similarly, given lattice constants $b_0, v_0 \in \mathbb{R}^+$, set $\pi_{mn} = \pi(mb_0, nv_0) = M_{nv_0} T_{mb_0}$. Then, for $\mathbf{m} \in \ell^\infty(\mathbb{Z}^2)$, the corresponding Gabor multiplier is defined as

$$\mathbb{M}_{\mathbf{m};g,h}^G f = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathbf{m}(m, n) \mathcal{V}_g f(mb_0, nv_0) \pi_{mn} h. \tag{20}$$

Note that Gabor multipliers may be interpreted as STFT multipliers with multiplier $\mathbf{m}$ in S_0'. In fact, in this case, $\mathbf{m}$ is simply a sum of weighted Dirac impulses on the sampling lattice.

The definition of time-frequency multipliers can of course be given for $g, h \in \mathbf{L}^2(\mathbb{R})$, many nice properties only apply with additional assumptions on the windows. Abstract properties of such multipliers have been studied extensively, and we refer to [18] for a review. One may show for example that, whenever the windows g and h are at least in S_0, if $\mathbf{m}$ belongs to $\mathbf{L}^2(\mathbb{P})$ (or $\ell^2(\mathbb{Z}^2)$) then the corresponding multiplier is a Hilbert-Schmidt operator and maps $S_0'(\mathbb{R})$ to $\mathbf{L}^2(\mathbb{R})$.

The spreading function of time-frequency multipliers may be computed explicitly.

Lemma 14 *The spreading function of the STFT multiplier* $\mathbb{M}_{\mathbf{m};g,h}$ *is given by*

$$\eta_{\mathbb{M}_{\mathbf{m};g,h}}(b, v) = \mathcal{M}(b, v) \mathcal{V}_g h(b, v), \tag{21}$$

where $\mathcal{M}$ *is the symplectic Fourier transform of the transfer function* $\mathbf{m}$

$$\mathcal{M}(t, \xi) = \int_\mathbb{P} \mathbf{m}(b, v) e^{2i\pi(vt - \xi b)} \, db \, dv.$$

Specifying to the Gabor multiplier $\mathbb{M}_{\mathbf{m};g,h}^G$, *we see the same expression for the spreading function, however, in this case,* $\mathcal{M} = \mathcal{M}^{(d)}$ *is the* (v_0^{-1}, b_0^{-1})-*periodic symplectic Fourier transform of the discrete transfer function* $\mathbf{m}$

$$\mathcal{M}^{(d)}(t, \xi) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathbf{m}(m, n) e^{2i\pi(nv_0 t - mb_0 \xi)}. \tag{22}$$

Proof For $f \in S_0'$ and $\varphi \in S_0$, we may write

$$\langle \mathbb{M}_{\mathbf{m};g,h} f, \varphi \rangle = \langle \mathbf{m}, \overline{\mathcal{V}_g f} \cdot \mathcal{V}_h \varphi \rangle,$$

where the right-hand side inner product has to be interpreted as an integral or infinite sum, respectively. By Lemma 9, applying the symplectic Fourier transform, we obtain

$$\langle \mathbb{M}_{\mathbf{m};g,h} f, \varphi \rangle = \langle \mathcal{M}, \mathcal{V}_\varphi f \cdot \overline{\mathcal{V}_g h} \rangle = \langle \mathcal{M} \cdot \mathcal{V}_g h, \mathcal{V}_f \varphi \rangle. \qquad (23)$$

By calling on (14), this proves (21). By virtue of that fact that for $g, h \in S_0$, $\mathcal{V}_g h$ is certainly in $\mathbf{L}^1(\mathbb{P})$ and even in the Wiener Amalgam Space $W(C, \mathbf{L}^1)$, hence in particular continuous, the expressions for the spreading function given in the lemma are always well-defined. The symplectic Fourier transform is a Gelfand triple isomorphism of $(S_0, \mathbf{L}^2, S_0')$, i.e., $\mathbf{m} \in (S_0, \mathbf{L}^2, S_0') \Longleftrightarrow \mathcal{M} \in (S_0, \mathbf{L}^2, S_0')$. Hence, $\mathcal{M} \cdot \mathcal{V}_g h \in (S_0, \mathbf{L}^2, S_0')$, which is in accordance with the fact, that for $\mathbf{m}$ in $(\ell^1, \ell^2, \ell^\infty)$, i.e., $(S_0, \mathbf{L}^2, S_0')(\mathbb{Z}^2)$, the kernel of the resulting operator (and hence its spreading function), is in $(S_0, \mathbf{L}^2, S_0')$, while $\mathcal{V}_f \varphi$ is in $(S_0', \mathbf{L}^2, S_0)$, respectively, see [18] for details. Hence, the expression obtained in (23) is well-defined by duality. $\qquad \square$

Remark 15 All expressions derived so far are easily generalized to Gabor frames for $\mathbb{R}^d$ associated to arbitrary lattices $\Lambda \subset \mathbb{R}^{2d}$. In such situations, the spreading function takes a similar form, and involves some discrete symplectic Fourier transform of the transfer function $\mathbf{m}$, which is in that case a Λ°-periodic function, Λ° being the adjoint lattice of Λ.

Notice that as a consequence of Proposition 10, one has the following "intertwining property"

$$\mathcal{V}_g \mathbb{M} f = (\mathcal{M}_{\mathbf{m};g,h} \mathcal{V}_g h) \natural \mathcal{V}_g f.$$

Remark 16 It is clear from the above calculations that a general Hilbert-Schmidt operator may not be well represented by a TF-multiplier. For example, let us assume that the analysis and synthesis windows have been chosen, and let η be the spreading function of the operator under consideration.

- In the STFT case, if the analysis and synthesis windows are fixed, the decay of the spreading function has to be fast enough (at least as fast as the decay of $\mathcal{V}_g h$) to ensure the boundedness of the quotient $\mathcal{M} = \eta / \mathcal{V}_g h$. Such considerations have led to the introduction of the notion of *underspread* operators [32] whose spreading function is compactly supported in a domain of small enough area. A more precise definition of underspread operators will be given below.
- In the Gabor case, the periodicity of $\mathcal{M}^{(d)}$ imposes extra constraints on the spreading function η. In particular, the shape of the support of the spreading function must influence the choice of optimal parameters for the approximation by a Gabor multiplier, i.e. the shape of the window as well as the lattice parameters. The following numerical example indicates the direction for the choice of parameters in the approximation of operators by Gabor multipliers.

Example 17 Consider two operators *OP*1 and *OP*2 with spreading functions as shown in Fig. 1. The values of the spreading functions are random and real, uniformly distributed in $[-0.5, 0.5]$.

Operator 1 has a spreading function with smaller support on the time-axis, which means that the corresponding operator exhibits time-shifts across smaller intervals than Operator 2, whose spreading function is, on the other hand, less extended in frequency. The effect in the opposite direction is, obviously, reverse. These characteristics are illustrated by applying the operators to a sinusoid with frequency 1 and a Dirac impulse at -1, respectively.

Next, we realize approximation[3] by Gabor multipliers with two fixed pairs of lattice constants: $b_1 = 2$, $\nu_1 = 8$ and $b_2 = 8$, $\nu_2 = 2$. Furthermore, the windows are Gaussian windows varying from wide ($j = 0$) to narrow ($j = 100$). Thus, j corresponds to the concentration of the window, in other words, j is the reciprocal of the standard deviation. Now the approximation quality is investigated. The results are shown in the lower plots of Fig. 1, where the left subplot shows the approximation quality for operator *OP*1 for b_1, ν_1 (solid) and b_2, ν_2 (dashed), while the right hand subplot gives the corresponding results for operator 2. The error is measured by $err = \|OP - APP\|_{\mathcal{H}} / \|OP + APP\|_{\mathcal{H}}$. Here, *APP* denotes the approximation operator and the norm is the operator norm. The results show that, as expected, the "adapted" choice of time-frequency parameters leads to more favorable approximation quality. Here, the adapted choice of b and ν mimics the shape of the support of the spreading function according to formula (21) and the periodicity of $\mathcal{M}^{(d)}$. In brief, if the operator realizes frequency-shifts in a wider range, we will need more sampling-points in frequency and vice-versa. It is also visible, that the shape of the window has considerable influence on the approximation quality.

The previous example shows, that the parameters in the approximation by Gabor multipliers must be carefully chosen. Let us point out that the approximation quality achieved in the experiment described in Example 17 is not satisfactory, especially when the time- and frequency shift parameters are not well adapted. Operators with a spreading function that is not well-concentrated around 0, i.e. "overspread operators", don't seem to be well-represented by a Gabor multiplier even with high redundancy (the redundancy used in the example is 8). Moreover, a realistic operator will have a spreading function with a much more complex shape. The next section will give some more details on approximation by Gabor multipliers before generalizations, which allow for approximation of more complex operators, are suggested.

2.4.2 Approximation by Gabor Multipliers

The possibility of approximating operators by Gabor multipliers in Hilbert-Schmidt sense depends on the properties of the rank one operators associated with time-frequency shifted copies of the analysis and synthesis windows.

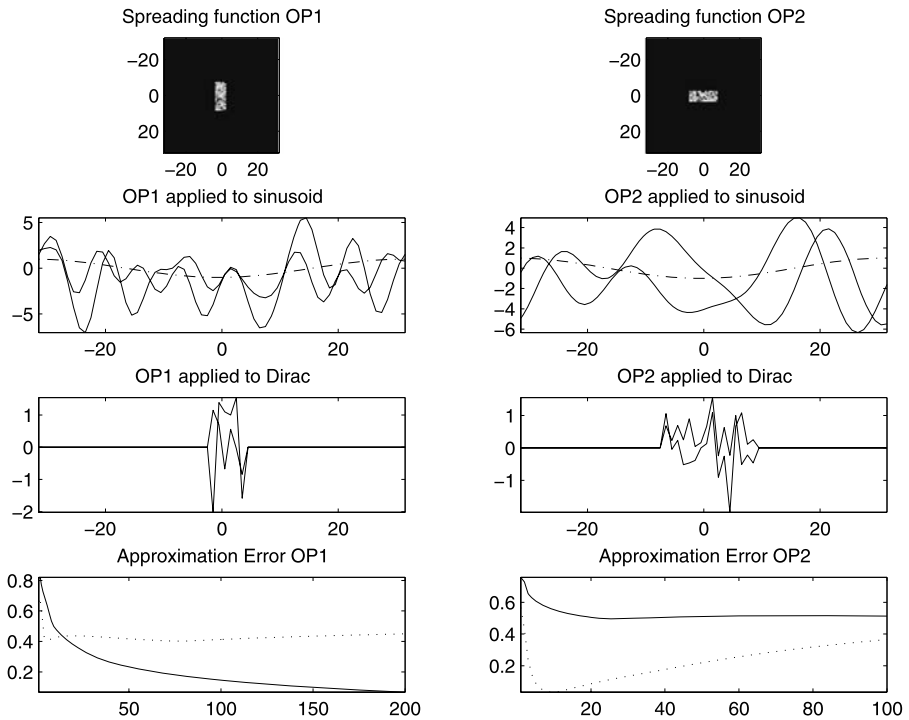[3]Best approximation is realized in Hilbert-Schmidt sense, see the next section for details.

Fig. 1 Approximation by Gabor Multipliers with parameters b_1, ν_1 (*solid*) and b_2, ν_2 (*dashed*)

Let $g, h \in S_0(\mathbb{R})$ be such that $\langle g, h \rangle = 1$. Let $\lambda = (b_1, \nu_1) \in \mathbb{P}$, and consider the rank one operator (oblique projection) P_λ defined by

$$P_\lambda f = (g_\lambda^* \otimes h_\lambda) f = \langle f, g_\lambda \rangle h_\lambda, \quad f \in (S_0(\mathbb{R}), \mathbf{L}^2(\mathbb{R}), S_0'(\mathbb{R})). \tag{24}$$

Direct calculations show that the kernel of P_λ is given by

$$\kappa_{P_\lambda}(t, s) = \overline{g}_\lambda(s) h_\lambda(t), \tag{25}$$

and its spreading function reads

$$\eta_{P_\lambda}(b, \nu) = e^{2i\pi(\nu_1 b - b_1 \nu)} \mathcal{V}_g h(b, \nu). \tag{26}$$

The following result characterizes the situations for which time-frequency rank one operators form a Riesz sequence, in which case the best approximation by a Hilbert-Schmidt operator is well-defined. This result first appeared in [15]. Here, we give a slightly different version, which is obtained from the original statement by applying Poisson summation formula. This result was also given in [6] for general full-rank lattices in $\mathbb{R}^d$.

We will subsequently denote by $\Lambda_{b_0, \nu_0}^\circ = \frac{1}{\nu_0} \mathbb{Z} \times \frac{1}{b_0} \mathbb{Z}$ the adjoint lattice of $\Lambda_{b_0, \nu_0}^\circ$, and by $\square^\circ = [-\frac{1}{2\nu_0}, \frac{1}{2\nu_0}[\times [-\frac{1}{2b_0}, \frac{1}{2b_0}[$ its fundamental domain.

Proposition 18 *Let $g, h \in \mathbf{L}^2(\mathbb{R})$, with $\langle g, h \rangle \neq 0$, let $b_0, v_0 \in \mathbb{R}^+$, and set*

$$\mathcal{U}(t, \xi) = \sum_{k, \ell = -\infty}^{\infty} \left| \mathcal{V}_g h \left(t + \frac{k}{v_0}, \xi + \frac{\ell}{b_0} \right) \right|^2. \tag{27}$$

The family $\{P_{mb_0, nv_0}, m, n \in \mathbb{Z}\}$ is a Riesz sequence in $\mathcal{H}$ if and only if there exist real constants $0 < A \leq B < \infty$ such that

$$0 < A \leq \mathcal{U}(t, \xi) \leq B < \infty \quad a.e. \text{ on } \square^\circ. \tag{28}$$

We call this condition the $\mathcal{U}$ condition.

Note that validity of the $\mathcal{U}$ condition, via the fact that the subspace spanned by the corresponding Riesz basis of projection operators is closed, guarantees the existence of a unique solution to the problem of best approximation of an arbitrary Hilbert-Schmidt operator in a given Gabor multiplier setting. Accordingly it turns out, that the approximation of a given operator via a standard minimization process yields an expression, which is only well-defined if the $\mathcal{U}$ condition (28) holds.

Theorem 19 *Assume that $\mathcal{V}_g h$ and $b_0, v_0 \in \mathbb{R}^+$ are such that the $\mathcal{U}$ condition (28) is fulfilled. Then the best Gabor multiplier approximation (in Hilbert-Schmidt sense) of $H \in \mathcal{H}$ is defined by the time-frequency transfer function $\mathbf{m}$ whose discrete symplectic Fourier transform reads*

$$\mathcal{M}(b, v) = \frac{\sum_{k, \ell = -\infty}^{\infty} \overline{\mathcal{V}_g h}(b + k/v_0, v + \ell/b_0) \eta_H (b + k/v_0, v + \ell/b_0)}{\sum_{k, \ell = -\infty}^{\infty} |\mathcal{V}_g h(b + k/v_0, v + \ell/b_0)|^2}. \tag{29}$$

Proof Let us set $\mathcal{V} = \mathcal{V}_g h$ for simplicity of notation. First, notice that if $\eta_H \in L^2(\mathbb{P})$, then the function $(b, v) \in \square^\circ \to \sum_{k, \ell} |\eta_H (b + k/v_0, v + \ell/b_0)|^2$ is in $L^2(\square^\circ)$, and is therefore well defined almost everywhere in $\square^\circ$. Thus, by Cauchy-Schwarz inequality, the numerator in (29) is well-defined a.e.

The Hilbert-Schmidt optimization is equivalent to the problem

$$\min_{\mathcal{M} \in \mathbf{L}^2(\square^\circ)} \|\eta_H - \mathcal{M}\mathcal{V}\|^2.$$

The latter squared norm may be written as

$$\|\eta_H - \mathcal{M}\mathcal{V}\|^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\eta_H(b, v) - \mathcal{M}(b, v)\mathcal{V}(b, v)|^2 \, db \, dv$$

$$= \sum_{k, \ell = -\infty}^{\infty} \int \int_{\square^\circ} |\eta_H(b + k/v_0, v + \ell/b_0)$$

$$- \mathcal{M}(b, v)\mathcal{V}(b + k/v_0, v + \ell/b_0)|^2 \, db \, dv$$

$$= \int \int_{\square^\circ} \left[\sum_{k, \ell} |\eta_H(b + k/v_0, v + \ell/b_0)|^2 \right.$$

$$-2\Re\Bigg(\overline{\mathcal{M}}(b,v)\sum_{k,\ell}\eta_H(b+k/v_0,v+\ell/b_0)$$

$$\times\,\overline{\mathcal{V}}(b+k/v_0,v+\ell/b_0)\Bigg)$$

$$+\,|\mathcal{M}(b,v)|^2\sum_{k,\ell}|\mathcal{V}(b+k/v_0,v+\ell/b_0)|^2\Bigg]\,dbdv.$$

From this expression, the Euler-Lagrange equations may be obtained, which read

$$\mathcal{M}(b,v)\sum_{k,\ell}|\mathcal{V}(b+k/v_0,v+\ell/b_0)|^2$$

$$=\sum_{k,\ell}\eta_H(b+k/v_0,v+\ell/b_0)\overline{\mathcal{V}}(b+k/v_0,v+\ell/b_0),$$

and the result follows. $\square$

We next derive an error estimate for the approximation. Let us set, for $(b,v)\in\square^\circ$,

$$\mathcal{E}(b,v)=\frac{|\sum_{k,\ell}\eta_H(b+k/v_0,v+\ell/b_0)\overline{\mathcal{V}}(b+k/v_0,v+\ell/b_0)|^2}{\sum_{k,\ell}|\eta_H(b+k/v_0,v+\ell/b_0)|^2\mathcal{U}(b,v)}.$$

Corollary 20 *With the above notation, we obtain the estimate*

$$\|H-\mathbb{M}_{\mathbf{m}}\|_{\mathcal{H}}^2\leq\|\eta_H\|^2\,\|1-\mathcal{E}\|_\infty$$

for the best approximation of H by a Gabor multiplier $\mathbb{M}_{\mathbf{m}}$ according to (29).

Proof Set $\Gamma_H(b,v)=\sum_{k,\ell}|\eta_H(b+k/v_0,v+\ell/b_0)|^2$. Replacing the expression for $\mathcal{M}$ obtained in (29) into the error term, we have

$$\|H-\mathbb{M}_{\mathbf{m}}\|_{\mathcal{H}}^2=\int\int_{\square^\circ}\Big[\Gamma_H(b,v)-|\mathcal{M}(b,v)|^2\mathcal{U}(b,v)\Big]\,dbdv$$

$$=\int\int_{\square^\circ}\Gamma_H(b,v)[1-\mathcal{E}(b,v)]\,dbdv$$

$$\leq\|\eta_H\|^2\,\|1-\mathcal{E}\|_\infty,$$

where we have used the fact that $\|\Gamma_H\|_{\mathbf{L}^2(\square^\circ)}=\|\eta_H\|_{\mathbf{L}^2(\mathbb{P})}$. Clearly, Cauchy-Schwarz inequality gives $|\mathcal{E}(b,v)|\leq1$ on $\square^\circ$, with equality if and only if there exists a function ϕ such that $\eta=\phi\mathcal{V}$, i.e. if and only if H is a multiplier with the prescribed window functions. Hence we obtain

$$\|H-\mathbb{M}_{\mathbf{m}}\|_{\mathcal{H}}^2\leq\|H\|_{\mathcal{H}}^2\left[1-\underset{(b,v)\in\square^\circ}{ess\inf}\mathcal{E}(b,v)\right].\qquad(30)$$

$\square$

Remark 21 $\mathcal{E}(b, v)$ essentially represents the cosine of the angle between vectors $\{\eta_H(b + k/v_0, v + \ell/b_0), \, k, \ell \in \mathbb{Z}\}$ and $\{\mathcal{V}(b + k/v_0, v + \ell/b_0), \, k, \ell \in \mathbb{Z}\}$. In other words, the closer to colinear these vectors, the better the approximation.

An example for operators which are poorly represented by this class of multipliers are those with a spreading function that is not "well-concentrated". These are, in technical terms, overspread operators. The underspread/overspread terminology seems to originate from the context of time-varying multipath wave propagation channels [31]. However, different definitions exist in the literature. Here, we give the definition used in [33]. Note that underspread operators have recently found renewed interest [7, 33–35, 37, 39].

Definition 22 Consider an operator K with compactly supported spreading function:

$$\text{supp}(\eta_K) \subseteq Q(t_0, \xi_0), \quad \text{where } Q(t_0, \xi_0) := [-t_0, t_0] \times [-\xi_0, \xi_0].$$

Then, K is called underspread, if $vol(Q) < 1$.

Most generally, operators which are *not* underspread, will be called overspread. It is generally known, that an operator must be underspread in order to be well-approximated by Gabor multipliers. Formula (29) enables us to make this statement more precise.

Corollary 23 *Consider an underspread operator H. Choose (b_0, v_0) such that $1/b_0 > 2\xi_0$ and $1/v_0 > 2t_0$ and consider the Gabor frame $\{g_{mn}, m, n \in \mathbb{Z}\}$, with lattice constants (b_0, v_0), and dual window h. Then, the symplectic Fourier transform $\mathcal{M}$ of the time-frequency transfer function of the best Gabor multiplier takes the form*

$$\mathcal{M} = \frac{\eta_H \overline{\mathcal{V}_g h}}{\mathcal{U}},$$

and the approximation error can be bounded by

$$\|\eta_H - \mathcal{M}\mathcal{V}_g h\|^2 \leq \|\eta_H\|^2 \underset{(b,v)\in \square_H^\circ}{\text{ess sup}} \left[1 - \frac{|\mathcal{V}_g h(b, v)|^2}{\mathcal{U}(b, v)} \right]$$

with $\square_H^\circ = \square^\circ \cap \text{supp}(\eta_H)$.

Proof

$$\|\eta_H - \mathcal{M}\mathcal{V}_g h\|^2 = \int \int_{\square^\circ} \left[|\eta(b, v)|^2 - \frac{|\eta(b, v)\mathcal{V}_g h(b, v)|^2}{\mathcal{U}(b, v)} \right] db\, dv$$

$$\leq \|\eta_H\|^2 \underset{(b,v)\in \square^\circ}{\text{ess sup}} \left[1 - \frac{|\mathcal{V}_g h(b, v)|^2}{\mathcal{U}(b, v)} \right]. \qquad \square$$

The estimate in the last corollary shows, that approximation quality is a joint property of window and lattice, which is in accordance with the results of Example 17.

Remark 24 Note that, although technically only defined for Hilbert-Schmidt operators, the approximation by Gabor multipliers can formally be extended to operators from $\mathcal{B}'$, see [18, Sect. 5.8]. Also, the expression given in (29) is well-defined in S_0' whenever η_H is at least in S_0'. However, for non-Hilbert-Schmidt operators, it is not clear, in which sense the resulting Gabor multiplier represents the original operator. The following example shows that at least in some cases, the result is however the intuitively expected one.

Example 25 Consider the operator $\pi(\lambda)$, i.e. a time-frequency shift. Although this operator is clearly not a Hilbert-Schmidt operator, we may consider its approximation by a Gabor-multiplier according to (29). First note that the spreading function of the time-frequency shift $\pi(\lambda) = \pi(b_1, \nu_1)$ is given by $\eta_\pi = \delta(b - b_1) \cdot \delta(\nu - \nu_1)$. Then, we have

$$
\begin{aligned}
\mathcal{M}(b, \nu) &= \sum_{k,l} \frac{\overline{\mathcal{V}_g h}(b + k/\nu_0, \nu + l/b_0)\delta(b - b_1 + k/\nu_0, \nu - \nu_1 + l/b_0)}{\sum_{k',l'} |\mathcal{V}_g h(b + k'/\nu_0, \nu + l'/b_0)|^2} \\
&= \overline{\mathcal{V}_g h}(b_1, \nu_1) \sum_{k,l} \frac{\delta(b - b_1 + k/\nu_0, \nu - \nu_1 + l/b_0)}{\sum_{k',l'} |\mathcal{V}_g h(b_1 + \frac{k'-k}{\nu_0}, \nu_1 + \frac{l'-l}{b_0})|^2} \\
&= \frac{\overline{\mathcal{V}_g h}(b_1, \nu_1)}{\mathcal{U}(b_1, \nu_1)} \sum_{k,l} \delta(b - b_1 + k/\nu_0, \nu - \nu_1 + l/b_0).
\end{aligned}
$$

Hence, from the inverse (discrete) symplectic Fourier transform we obtain:

$$
\begin{aligned}
\mathbf{m}(m, n) &= \frac{\overline{\mathcal{V}_g h}(b_1, \nu_1)}{\mathcal{U}(b_1, \nu_1)} \int_0^{\frac{1}{\nu_0}} \int_0^{\frac{1}{b_0}} \sum_{k,l} \delta(b - b_1 + k/\nu_0, \nu - \nu_1 + l/b_0) \\
&\quad \times e^{-2\pi i(mb_0\nu - n\nu_0 b)} \, db d\nu \\
&= \frac{\overline{\mathcal{V}_g h}(b_1, \nu_1)}{\mathcal{U}(b_1, \nu_1)} e^{-2\pi i(mb_0\nu_1 - n\nu_0 b_1)}.
\end{aligned}
$$

As expected, the absolute value of the mask is constant and the phase depends on the displacement of λ from the origin. This confirms the key role played by the phase of the mask of a Gabor multiplier. Specializing to $\lambda = 0$, we obtain a constant mask and thus, if h is a dual window of g with respect to $\Lambda = b_0\mathbb{Z} \times \nu_0\mathbb{Z}$, up to a constant factor, the identity.

3 Generalizations: Multiple Gabor Multipliers and TST Spreading Functions

In the last section it has become clear that most operators are not well represented as a STFT or Gabor multiplier.

Guided by the desire to extend the good approximation quality that Gabor multipliers warrant for underspread operators to the class of their overspread counterparts, we introduce generalized TF-multipliers. The basic idea is to allow for an extended

scheme in the synthesis part of the operator: instead of using just one window h, we suggest the use of a set of windows $\{h^{(j)}\}$ in order to obtain the class of *Multiple Gabor Multipliers* (MGM for short).

Definition 26 (Multiple Gabor Multipliers) Let $g \in S_0(\mathbb{R})$ and a family of reconstruction windows $h^{(j)} \in S_0(\mathbb{R})$, $j \in \mathcal{J}$, as well as corresponding masks $\mathbf{m}_j \in \ell^\infty$ be given. Operators of the form

$$\mathbb{M} = \sum_{j \in \mathcal{J}} \mathbb{M}^G_{\mathbf{m}_j; g, h^{(j)}} \tag{31}$$

will be called *Multiple Gabor Multipliers* (MGM for short).

Remark 27

(a) The most natural generalization of Gabor multipliers would allow for both several analysis and synthesis windows. However, we will see below that the generalization introduced in (31) is sufficient in order to achieve perfect representation of most operators of interest.
(b) Note, that we need to impose additional assumptions in order to obtain a well-defined operator. For example, we may assume $\sum_j \sup_\lambda |m_j(\lambda)| = C < \infty$ and $\max_j \|h^j\|_{S_0} = C < \infty$, which guarantees a bounded operator on $(S_0, \mathbf{L}^2, S_0')$. This follows easily from the boundedness of a Gabor multiplier under the condition that $\mathbf{m}$ is ℓ^∞. Conditions for function space membership of MGMs are easily derived in analogy to the Gabor multiplier case. For example, if $\sum_j \sum_\lambda |m_j(\lambda)|^2 < \infty$, we obtain a Hilbert-Schmidt operator, similarly, trace-class membership follows from an analogous ℓ^1-condition.
(c) The spreading function of a MGM is (formally) given by a sum of the spreading functions corresponding to the single Gabor multipliers involved:

$$\eta^G_{\mathbb{M}_{\mathbf{m}_j; g, h^{(j)}}}(b, \nu) = \sum_{j \in \mathcal{J}} \mathcal{M}^{(j)}(b, \nu) \mathcal{V}_g h^{(j)}(b, \nu), \tag{32}$$

where the (ν_0^{-1}, b_0^{-1})-periodic functions $\mathcal{M}^{(j)}$ are the symplectic Fourier transforms of the transfer functions $\mathbf{m}_j$.
(d) Note that in practice $|\mathcal{J}|$ will often be finite. Let us mention that by assuming $\mathbf{m}_j \in \ell^2(\mathbb{Z}^3)$, we have $\|\eta^G_{\mathbb{M}_{\mathbf{m}_j; g, h^{(j)}}}\|_2 = C\|\mathbf{m}\|_2$.

It is immediately obvious that the new model gives much more freedom in generating overspread operators. However, in order to obtain structural results, we will have to impose further specifications.

Before doing so, we will state a generalization of Proposition 18 to the more general situation of the family of projection operators P_λ^j defined by

$$P_\lambda^j f = (g_\lambda^* \otimes h_\lambda^j) f = \langle f, g_\lambda \rangle h_\lambda^j, \quad \text{where } \lambda \in \Lambda, \ j \in \mathcal{J}. \tag{33}$$

Note that these projection operators are the building blocks for the MGM. The following theorem characterizes their Riesz property.

Proposition 28 *Let $g, h^j \in \mathbf{L}^2(\mathbb{R})$, $j \in \mathcal{J}$, with $\langle g, h^j \rangle \neq 0$, let $b_0, v_0 \in \mathbb{R}^+$, and let the matrix $\Gamma(b, v)$ be defined by*

$$\Gamma(b, v)_{jj'} = \sum_{k,\ell} \overline{\mathcal{V}_g h^{(j)}}(b + k/v_0, v + \ell/b_0) \mathcal{V}_g h^{(j')}(b + k/v_0, v + \ell/b_0) \qquad (34)$$

a.e. on $\square^\circ = [0, v_0^{-1}[\times [0, b_0^{-1}[$. Then the family of projection operators $\{P_\lambda^j, j \in \mathbb{Z}, \lambda \in \Lambda\}$ is a Riesz sequence in $\mathcal{H}$ if and only if Γ is invertible a.e. with uniform bounds $0 < A \leq B < \infty$ such that

$$A \leq \Gamma(b, v) \leq B \quad a.e. \text{ on } \square^\circ. \qquad (35)$$

Alternatively, the Riesz basis property is characterized by bounded invertibility of the matrix U defined as

$$\mathcal{U}^{jj'}(t, \xi) = \sum_{k,l} [\overline{\mathcal{V}_g g} \cdot \mathcal{V}_{h^j} h^{j'}](kb_0, lv_0) e^{-2\pi i (lv_0 t - kb_0 \xi)} \qquad (36)$$

a.e. on the fundamental domain of Λ.

Proof Recall that the family $\{P_\lambda^j, j \in \mathbb{Z}, \lambda \in \Lambda\}$ is a Riesz basis for its closed linear span if there exist constants $0 < A, B < \infty$ such that

$$A\|c\|_2^2 \leq \left\| \sum_\lambda \sum_j c_\lambda^j P_\lambda^j \right\|_{\mathcal{HS}}^2 \leq B\|c\|_2^2 \qquad (37)$$

for all finite sequences c defined on $(\Lambda \times \mathcal{J})$. We have

$$\left\| \sum_\lambda \sum_j c_\lambda^j P_\lambda^j \right\|_{\mathcal{HS}}^2 = \left\langle \sum_\lambda \sum_j c_\lambda^j P_\lambda^j, \sum_\mu \sum_{j'} c_\mu^{j'} P_\mu^{j'} \right\rangle$$

$$= \sum_{\lambda\mu} \sum_{jj'} c_\lambda^j \overline{c_\mu^{j'}} \overline{\langle g_\lambda, g_\mu \rangle} \langle h_\lambda^j, h_\mu^{j'} \rangle.$$

Hence, by setting $U^{jj'}(b, v) = [\overline{\mathcal{V}_g g} \cdot \mathcal{V}_{h^j} h^{j'}](b, v)$, we may write

$$\left\| \sum_\lambda \sum_j c_\lambda^j P_\lambda^j \right\|_{\mathcal{HS}}^2 = \sum_{\lambda\mu} \sum_{jj'} c_\lambda^j \overline{c_\mu^{j'}} U^{jj'}(\mu - \lambda)$$

$$= \sum_\mu \sum_{jj'} \overline{c_\mu^{j'}} (c^j * U^{jj'})(\mu)$$

$$= b_0 v_0 \sum_{jj'} \langle \mathcal{U}^{jj'} \cdot \mathcal{C}^j, \mathcal{C}^{j'} \rangle_{\mathbf{L}^2(\square^\circ)},$$

where $\mathcal{U}^{jj'}(t, \xi) = \sum_{k,l} U^{jj'}(kb_0, lv_0) e^{-2\pi i (lv_0 t - kb_0 \xi)}$ is the discrete symplectic Fourier transform of $U^{jj'}$, and, analogously, $\mathcal{C}^j$ is the discrete symplectic Fourier

transform of the sequence c^j, defined on Λ, for each j. Hence, these are $v_0^{-1} \times b_0^{-1}$-periodic functions. The last equation can be rewritten as

$$\left\| \sum_\lambda \sum_j c_\lambda^j P_\lambda^j \right\|_2^2 = b_0 v_0 \sum_{jj'} \int_{\square^\circ} \mathcal{U}^{jj'}(t,\xi) \cdot \mathcal{C}^j(t,\xi)\overline{\mathcal{C}^{j'}}(t,\xi)dt d\xi$$

$$= b_0 v_0 \langle \mathcal{U} \cdot \mathcal{C}, \mathcal{C} \rangle_{\mathbf{L}^2(\square^\circ) \times \ell^2(\mathbb{Z})},$$

where $\mathcal{U}$ is the matrix with entries $\mathcal{U}^{jj'}$. Note that this proves statement (36) by positivity of the operator $\mathcal{U}$.

In order to obtain the condition for Γ given in (34), first note that

$$\mathcal{F}_s(\overline{\mathcal{V}_g g} \cdot \mathcal{V}_{h^j} h^{j'})(\lambda) = (\mathcal{V}_g h^{j'} \cdot \overline{\mathcal{V}_g h^j})(\lambda)$$

by applying Lemma 9. Furthermore, $F := \mathcal{V}_g h^{j'} \cdot \overline{\mathcal{V}_g h^j}$ is always in $\mathbf{L}^1$ for $g, h^j \in \mathbf{L}^2$. We may therefore look at the Fourier coefficients of its Λ°-periodization, with $\lambda = (mb_0, nv_0)$:

$$\mathcal{F}_s^{-1}(P_{\Lambda^\circ}F)(\lambda) = \int_{\square^\circ} \left(\sum_{k,\ell} F\left(b + \frac{k}{v_0}, v + \frac{\ell}{b_0}\right)\right) e^{2\pi i(bnv_0 - vmb_0)} \, db\, dv$$

$$= \int_{\square^\circ} \left(\sum_{k,\ell} F\left(b + \frac{k}{v_0}, v + \frac{\ell}{b_0}\right)\right) e^{2\pi i((b+\frac{k}{v_0})nv_0 - (v+\frac{\ell}{b_0})mb_0)} \, db\, dv$$

$$= \int_{\mathbb{R}^2} F(b, v)e^{2\pi i(bnv_0 - vmb_0)} \, db\, dv = \mathcal{F}_s^{-1}(\mathcal{V}_g h^{j'} \cdot \overline{\mathcal{V}_g h^j})(\lambda).$$

Hence, we may apply the Poisson summation formula, with convergence in $\mathbf{L}^2(\square^\circ)$, to obtain:

$$P_{\Lambda^\circ}(\mathcal{V}_g h^{j'} \cdot \overline{\mathcal{V}_g h^j})(b, v) = b_0 v_0 \sum_{k,l} \mathcal{F}_s^{-1}(\mathcal{V}_g h^{j'} \cdot \overline{\mathcal{V}_g h^j})(kb_0, lv_0)e^{-2\pi i(b\ell v_0 - vkb_0)}$$

$$= b_0 v_0 \sum_{k,l} \overline{\mathcal{V}_g g} \cdot \mathcal{V}_{h^j} h^{j'}(kb_0, lv_0)e^{-2\pi i(b\ell v_0 - vkb_0)}.$$

We conclude that

$$\left\| \sum_j \sum_{\lambda \in \Lambda} c_\lambda^j P_\lambda^j \right\|_2^2 = b_0 v_0 \sum_{jj'} \int \Gamma(b, v)_{jj'} C^j(b, v)\overline{C^{j'}(b, v)} \, db\, dv, \qquad (38)$$

and the Riesz basis property is equivalent to the invertibility of Γ. $\qquad\square$

In the sequel, the discrete symplectic Fourier transforms of $\mathbf{m}_j$ will be denoted by $\mathcal{M}_j$, and the vector with $\mathcal{M}_j$ as coordinates will be denoted by $\mathcal{M}$. We then obtain an expression for the best multiplier in analogy to the Gabor multiplier case discussed in Theorem 19.

Proposition 29 *Let $g \in S_0(\mathbb{R})$ and $h^{(j)} \in S_0(\mathbb{R})$, $j \in \mathcal{J}$ be such that for almost all b, v, the matrix $\Gamma(b, v)$ defined in (34) is invertible a.e. on $\square^\circ$.*

Let $H \in (\mathcal{B}, \mathcal{H}, \mathcal{B}')$ be an operator with spreading function $\eta \in (S_0, \mathbf{L}^2, S_0')$. Then the functions $\mathcal{M}_j$ yielding approximation of the form (31) may be obtained as

$$\mathcal{M} = \Gamma^{-1} \cdot \mathcal{B}, \tag{39}$$

where $\mathcal{B}$ is the vector whose entries read

$$\mathcal{B}_{j_0}(b, v) = \sum_{k,\ell} \eta(b + k/v_0, v + \ell/b_0)\overline{\mathcal{V}}_g h^{j_0}(b + k/v_0, v + \ell/b_0). \tag{40}$$

For operators in $\mathcal{H}$ the obtained approximation is optimal in Hilbert-Schmidt sense.

Proof The proof follows the lines of the Gabor multiplier case. The optimal approximation of the form (31) is obtained by minimizing

$$\left\| \eta - \sum_j \mathcal{M}_j \mathcal{V}_j \right\|^2 = \sum_{k,\ell} \int_{\square^\circ} \left| \eta(b + k/v_0, v + \ell/b_0) \right. $$
$$\left. - \sum_j \mathcal{M}_j(b, v)\mathcal{V}_j(b + k/v_0, v + \ell/b_0) \right|^2 db\, dv$$

where one has set $\mathcal{V}_j = \mathcal{V}_g h^{(j)}$. Setting to zero the Gâteaux derivative with respect to $\overline{\mathcal{M}}_{j_0}$, we obtain the corresponding variational equation

$$\sum_j \mathcal{M}_j(b, v) \sum_{k,\ell} \mathcal{V}_j(b + k/v_0, v + \ell/b_0)\overline{\mathcal{V}}_{j_0}(b + k/v_0, v + \ell/b_0) = \mathcal{B}_{j_0}(b, v),$$

where $\mathcal{B}_j(b, v)$ are as defined in (40). Provided that the $\Gamma(b, v)$ matrices are invertible for almost all b, v, this implies that the functions $\mathcal{M}_j$ for approximation of the form (31) may indeed be obtained as in (39). $\qquad\square$

In a next step, we are going to discern two basic approaches:

(a) $m_j(\lambda) = m(\mu, \lambda)$, i.e. the synthesis windows are time-frequency shifted versions (on a lattice) of a single synthesis window: $h_j = \pi(\mu_j)h$, $\mu_j \in \Lambda_1$.
(b) $m_j(\lambda) = m_1(\lambda)m_2(j)$, i.e. a separable multiplier function. If we set $h^{(j)}(t) = \pi(b_j, v_j)h(t)$ then this approach leads to what will be called TST spreading functions in Sect. 3.1.

In both cases we will be especially interested in the situation in which the h^j are given as time-frequency shifted versions of a single synthesis window on the adjoint lattice Λ°.

We fix the synthesis windows h_j to be time-frequency translates of a fixed window function, i.e.

$$h^{(j)}(t) = \pi(b_j, v_j)h(t) = e^{2i\pi v_j t}h(t - b_j). \tag{41}$$

We may turn our attention to the projection operators associated with two fixed (Gabor) families (g, Λ_1) and (h, Λ_2). Note that it has been shown by Benedetto and Pfander [6] that the family of projection operators $\{P_\lambda, \lambda \in \Lambda\}$, as discussed in Sect. 2.4 either forms a Riesz basis or not a frame (for its closed linear span). The next corollary shows that, on the other hand, if we use the extended family of projection operators $\{P_{\lambda,\mu}, (\lambda, \mu) \in \Lambda_1 \times \Lambda_2\}$, where $P_{\lambda,\mu} f = \langle f, \pi(\lambda)g\rangle \pi(\mu)h$, we obtain a frame of operators for the space of Hilbert-Schmidt operators, whenever (g, Λ_1) and (h, Λ_2) are Gabor frames. This corollary is a special case of Theorem 4.1 in [3] and Proposition 3.2 in [4].

Corollary 30 *Let two Gabor frames (g, Λ_1) and (h, Λ_2) be given. Then the family of projection operators $\{P_{\lambda,\mu}, (\lambda, \mu) \in \Lambda_1 \times \Lambda_2\}$ form a frame of operators in $\mathcal{H}$ and any Hilbert-Schmidt operator H may be expanded as*

$$H = \sum_{\lambda \in \Lambda_1, \mu \in \Lambda_2} \mathbf{c}(\lambda, \mu) P_{\lambda,\mu}.$$

The coefficients are given by $\mathbf{c}(\lambda, \mu) = \langle H, (P_{\lambda,\mu})^*\rangle = \langle H\pi(\mu)h, \pi(\lambda)g\rangle$.

Note that an analogous statement holds for Riesz sequences. Very recently, it has been shown [1], that the converse of Corollary 30 holds true for both frames and Riesz bases, i.e. the family of projection operators $\{P_{\lambda,\mu}, (\lambda, \mu) \in \Lambda_1 \times \Lambda_2\}$ is a frame (a Riesz basis) for $\mathcal{H}$ if and only if the two generating sequences form a frame (a Riesz basis) for $\mathbf{L}^2(\mathbb{R})$. In particular, this leads to the conclusion, that the characterization of Riesz sequences given in Proposition 28 also yields a characterization of frames for $\mathbf{L}^2(\mathbb{R})$—it is well known, that $(g_\lambda, \lambda \in \Lambda)$ form a Gabor frame if and only if $(g_\mu, \mu \in \Lambda^\circ)$ form a Riesz sequence. We can draw two conclusions.

Corollary 31 *Let $g \in \mathbf{L}^2(\mathbb{R})$ and a lattice $\Lambda = b_0\mathbb{Z} \times v_0\mathbb{Z}$ be given.*

(a) *The Gabor family $\{g_\lambda, \lambda \in \Lambda\}$ forms a frame for $\mathbf{L}^2(\mathbb{R})$ if and only if the matrix*

$$\Gamma_{mn,m'n'}(b, v)$$
$$= \sum_{k,\ell} \exp\left(-2i\pi[m/v_0(v + \ell v_0 - n/b_0) - m'/v_0(v + \ell v_0 - n'/b_0)]\right)$$
$$\times \overline{\mathcal{V}_g g}(b - m/v_0 + kb_0, v - n/b_0 + \ell v_0)$$
$$\times \mathcal{V}_g g(b - m'/v_0 + kb_0, v - n'/b_0 + \ell v_0)$$

is, a.e. on $\square^\circ$, invertible on ℓ^2.

(b) *In addition, we may state the following "Balian-Low Theorem for the tensor products of Gabor frames":*

A family of projection operators given by $\{P_{\lambda,\mu} = g_\lambda^* \otimes g_\mu, (\lambda, \mu) \in \Lambda \times \Lambda^\circ\}$ forms a frame for the space of Hilbert-Schmidt operators on $\mathbf{L}^2(\mathbb{R})$ if and only if it forms a Riesz basis. Hence, in this case, g cannot be in $S_0(\mathbb{R})$.

Proof Statement (a) is easily obtained from (34) by observing that

$$\mathcal{V}_g \pi(mb_0, n\nu_0)g(b, \nu) = e^{-2i\pi mb_0(\nu - n\nu_0)} \mathcal{V}_g h(b - mb_0, \nu - n\nu_0).$$

We then have that $\{P_{\lambda,\mu} = g_\lambda^* \otimes g_\mu, (\lambda, \mu) \in \Lambda^\circ \times \Lambda^\circ\}$ forms a Riesz basis in $\mathcal{H}$ if and only if $\Gamma_{mn,m'n'}(b, \nu)$ is invertible. By the converse of Corollary 30, this is equivalent to the Riesz property of $g_\mu, \mu \in \Lambda^\circ$, which, in turn, is equivalent to the frame property of $\{g_\lambda, \lambda \in \Lambda\}$ by Ron-Shen duality, see, e.g. [25].

To see (b), note that in this case $P_{\lambda,\mu}$ is a frame for $\mathcal{H} \Leftrightarrow (g_\lambda, \Lambda)$ and (g_μ, Λ°) form a frame $\Leftrightarrow (g_\lambda, \Lambda)$ and (g_μ, Λ°) form a Riesz basis $\Leftrightarrow \{P_{\lambda,\mu} = g_\lambda^* \otimes g_\mu, (\lambda, \mu) \in \Lambda \times \Lambda^\circ\}$ is a Riesz basis for $\mathcal{H}$. Furthermore, by the classical Balian-Low theorem, if a Gabor system is an $\mathbf{L}^2$-frame and at the same time a Riesz sequence (hence an $\mathbf{L}^2$-Riesz basis), then the generating window g cannot be in S_0.[4] $\square$

We may next ask, when the projection operators form a Riesz sequence, if the reconstruction windows are TF-shifted versions of a single window h on the adjoint lattice of $\Lambda = b_0\mathbb{Z} \times \nu_0\mathbb{Z}$. In fact, in this case, the matrix Γ turns out to enjoy quite a simple form. To fix some notation, let

$$\mathcal{A}_{mn}(b, \nu) = \sum_{k,\ell} e^{2i\pi m[\nu - \ell/\nu_0]} \overline{\mathcal{V}_g h}(b - k/\nu_0, \nu - \ell/b_0)$$

$$\times \mathcal{V}_g h(b - (k - m)/\nu_0, \nu - (\ell - n)/b_0),$$

and introduce the right twisted convolution operator

$$K_{\mathcal{A}}^\natural(b, \nu): \quad \mathcal{M}(b, \nu) \rightarrow \mathcal{M}(b, \nu) \natural \mathcal{A}(b, \nu).$$

We may derive the form of Γ given in the following corollary by direct calculation:

Corollary 32 *Let $g, h \in \mathbf{L}^2(\mathbb{R})$ as well as b_0, ν_0 be given. Furthermore, let $h^{(j)} = \pi(\frac{m}{\nu_0}, \frac{n}{b_0})h$. Then the variational equations read*

$$\mathcal{M}(b, \nu) \natural \mathcal{A}(b, \nu) = \mathcal{B}(b, \nu). \tag{42}$$

Hence, if for all $b, \nu \in \mathbb{R}^2$, the discrete right twisted convolution operator $K_{\mathcal{A}}^\natural$ is invertible, then the family $P_{\lambda,\mu} = g_\lambda^* \otimes g_\mu, (\lambda, \mu) \in \Lambda \times \Lambda^\circ$ forms a Riesz sequence and the best MGM approximation of an Hilbert-Schmidt operator with spreading function η is given by the family of transfer functions

$$\mathcal{M}_{mn}(b, \nu) = \left[(K_A^\natural(b, \nu))^{-1}\mathcal{B}(b, \nu)\right]_{mn},$$

where $\mathcal{B}$ is given in (40).

We close this section with some results of numerical experiments testing the approximation quality of MGMs for slowly time-varying systems. Note that more extensive numerical results are presented in [12].

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[4]More precisely, g cannot even be in the space of continuous functions in the Wiener space $W(\mathbb{R})$, see [25, Theorem 8.4.1].
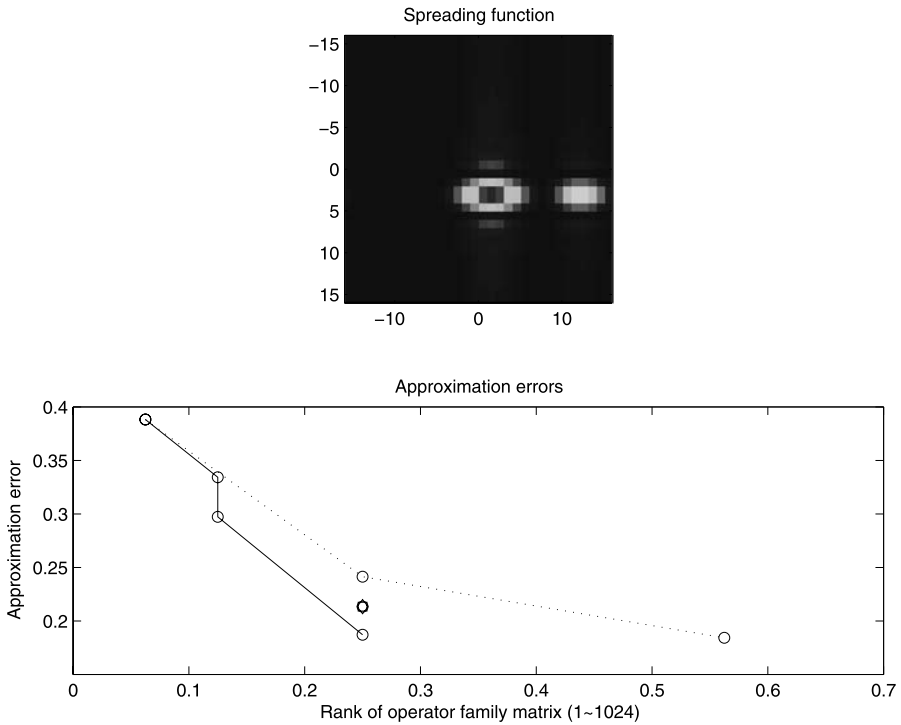
Fig. 2 Approximation by multiple Gabor multipliers

Example 33 We study the approximation of a (slowly) time-varying operator. The operator has been generated by perturbing a time-invariant operator. The spreading function is shown in the upper display of Fig. 2. The signal length is 32, time- and frequency-parameters are $b_0 = 4$ and $v_0 = 4$, such that the redundancy of the Gabor frame used in the MGM approximation is 2. The approximation is then realized in several steps for two different schemes. Scheme 1 adds three synthesis window corresponding to a frequency-shift by 4, a time-shift by 4 and a time-frequency-shift by (4, 4). The first step 1 calculates the regular Gabor multiplier approximation. Step 2 adds one (only frequency-shift) and so on. The rank of the resulting operator families is 64, 128 for both step 2 and step 3 (adding either time- or frequency-shift) and 256 (time-shifted, frequency-shifted and time-frequency-shifted window added). The resulting approximation-errors are given by the solid line in the lower display of Fig. 2.

Scheme 2 considers synthesis windows shifted in time and frequency on the sublattice generated by $a = 8$, $b = 8$, the resulting families having rank 64, 256 and 576. Here, we only plot the results for the case corresponding to three and eight additional synthesis windows, respectively. The results are given by the dotted line.

For comparison, an approximation with a regular Gabor multiplier with redundancy 8, i.e. an approximation family of rank 256, has been performed. The approximation error for this situation is the diamond in the middle of the display. It is easy to see that, depending on the behavior of the spreading function, different schemes per-

form advantageously for a certain redundancy. Note that for scheme 1, the best MGM with the same rank as the regular Gabor multiplier performs better than the latter. In the case of the present operator, scheme 2 performs the "wrong" time-frequency shifts on the synthesis windows in order to capture important characteristics of the operator. However, in a different setting, this scheme might be favorable (e.g. if an echo with a longer delay is present).

The example shows, that the choice of an appropriate sampling scheme for the synthesis windows is extremely important in order to achieve a good and efficient approximation by MGM. An optimal sampling scheme depends on the analysis window's STFT, the lattice used in the analysis and on the behavior of the operator's spreading function, which reflects the amount of delay and Doppler-shift created by the operator. Additionally, structural properties of the family of projections operators used in the approximation, based on the results in this section, have to be exploited to achieve numerical efficiency. An algorithm for optimization of these parameters is currently under development.[5]

3.1 Varying the Synthesis Window: TST Spreading Functions

We next turn to the special case of separable functions $\mathbf{m}_j(\lambda) = m_1(\lambda)m_2(j)$ for the mask in the definition (31) of MGMs. In this case the resulting operator is of the form

$$\mathbb{M}f = \sum_\lambda m_1(\lambda)\rho(\lambda)\left(\sum_j m_2(j)\big(g^* \otimes h^j\big)\right)(f) = \sum_\lambda m_1(\lambda)\rho(\lambda)\mathbb{P}_m f,$$

where $\mathbb{P}_m f = \sum_j m_2(j)\langle f, g\rangle h^j$ and $\rho(\lambda)$ denotes a tensor product of time-frequency shifts:

$$\rho(\lambda)H := \pi(\lambda)H\pi^*(\lambda).$$

Hence, the spreading function of $\mathbb{M}$ is given by

$$\eta_{\mathbb{M}} = \mathcal{M} \cdot \eta_{\mathbb{P}_m},$$

where $\mathcal{M}$ is the discrete symplectic FT of m_1. If the reconstruction windows are given by $h^j = \pi(\mu^j)h$, $\mu_j = (b_j, \nu_j)$, this becomes

$$\eta_{\mathbb{M}} = \mathcal{M} \cdot \sum_j m_2(j)\mathcal{V}_g h(\lambda - \mu_j)e^{-2\pi i(\nu-\nu_j)b_j}.$$

Motivated by this result, we introduce the following definition.

Definition 34 (TST spreading functions) Let ϕ be a given function from the function spaces $(S_0(\mathbb{R}^2), \mathbf{L}^2(\mathbb{R}^2), S_0'(\mathbb{R}^2))$ and let b_1, ν_1 denote positive numbers. Let α be

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[5]P. Balazs, M. Dörfler, F. Jaillet and B. Torrésani. An optimized sampling scheme for generalized Gabor multipliers. *In preparation*, 2009.

in $\ell^1(\mathbb{Z}^2)$. A spreading function $\eta = \eta_H$ of $H \in (\mathcal{B}, \mathcal{H}, \mathcal{B}')$, that may be written as

$$\eta(b, \nu) = \sum_{k,\ell} \alpha_{k\ell} \phi(b - kb_1, \nu - \ell\nu_1) e^{-2i\pi(\nu-\ell\nu_1)kb_1} \tag{43}$$

will be called *Twisted Spline Type* function (TST for short).

Remark 35 By α in ℓ^1, the series defining η is absolutely convergent in $(S_0, \mathbf{L}^2, S_0')$. For ℓ^2-sequences α, we obtain an $\mathbf{L}^2$-function η for $\phi \in \mathbf{L}^2$.

TST functions are nothing but spline type functions (following the terminology introduced in [15]), in which usual (Euclidean) translations are replaced with the natural (i.e. $\mathbb{H}$-covariant) translations on the phase space $\mathbb{P}$. In fact, by writing $\alpha(b, \nu) = \sum_{k,\ell} \alpha_{k\ell} \delta(b - kb_1, \nu - \ell\nu_1)$, the TST spreading function may be written as a twisted convolution: $\eta(b, \nu) = \alpha \natural \phi$. This leads to the following property of operators associated with TST spreading functions.

Lemma 36 *An operator $H \in (\mathcal{B}, \mathcal{H}, \mathcal{B}')$ possesses a TST spreading function $\eta \in (S_0, \mathbf{L}^2, S_0')$ as in* (43) *if and only if it is of the form*

$$H_\eta = \sum_{k,\ell} \alpha_{k\ell} \pi(kb_1, \ell\nu_1) H_\phi, \tag{44}$$

where H_ϕ is the linear operator with spreading function ϕ.

The proof consists of a straight-forward computation which may be spared by noting that we have, by (15):

$$H_\eta = H_{\alpha \natural \phi} = H_\alpha \cdot H_\phi = \sum_{k,\ell} \alpha_{k\ell} \pi(kb_1, \ell\nu_1) H_\phi.$$

As before, we are particularly interested in the situation of the synthesis windows being given by time-frequency shifted versions of a single window: $h^j = \pi(\mu_j)h$. In a next step we note, that under the condition $\pi(\lambda)\pi(\mu_j) = \pi(\mu_j)\pi(\lambda)$, i.e., $\mu_j \in \Lambda^\circ$, the MGM with separable multiplier results in a TST spreading function with a Gabor multiplier as basic operator H_ϕ.

Lemma 37 *Assume that a MGM $\mathbb{M}$ with multiplier $m_j(\lambda) = m_1(\lambda)m_2(j)$ is given. If the synthesis windows h^j are given by $h^j = \pi(\mu_j)h$, with $\mu_j \in \Lambda^\circ$, then*

$$\mathbb{M} = \sum_j m_2(j)\pi(\mu_j)\mathbb{M}_{m_1;g,h}^G,$$

i.e., here, the operator H_ϕ is given by a regular Gabor multiplier with mask m_1 and synthesis window h.

Remark 38 Comparing the expression in the previous lemma to the expression $\mathbb{M} = \sum_\lambda m_1(\lambda)\rho(\lambda)\mathbb{P}_m$ for the same operator, we note that in this situation, the operator may either be interpreted as a (weighted) sum of Gabor multipliers or as a Gabor multiplier with a generalized projection operator $\mathbb{P}_\mathbf{m}$ in the synthesis process. In this situation, we may ask, whether the family of generalized projection operators, $\{\rho(\lambda)\mathbb{P}_\mathbf{m}\}_{\lambda\in\Lambda}$ form a frame or Riesz basis for their linear span. In fact, if $\mathbf{m}$ is in ℓ^1 and $g, h \in S_0$, this question is easily answered by generalizing the result proved in [6, Theorem 3.2]. Here, $\{\rho(\lambda)\mathbb{P}_\mathbf{m}\}_{\lambda\in\Lambda}$ is either a Riesz basis or not a frame for its closed linear span. Furthermore, there exists $r > 0$ such that $\{\rho(\alpha\lambda)\mathbb{P}_\mathbf{m}\}_{\lambda\in\Lambda}$ is a Riesz basis for its closed linear span whenever $\alpha > r$.

In generalizing the result of Lemma 37, it is a natural further step to assume that the basic function ϕ entering in the composition of η is the spreading function of a Gabor multiplier (at least in an approximate sense). According to the discussion of Sect. 2.4, this essentially means that ϕ is sufficiently well concentrated in the time-frequency domain.

(In the sequel we will write π_{mn} for $\pi(mb_0, n\nu_0)$ whenever the applicable lattice constants are sufficiently clear from the context.)

Hence, we assume that a Gabor multiplier H_ϕ, as defined in (20) is given. We may formally compute

$$Hf = \sum_{k,\ell} \alpha_{k\ell} \pi_{k\ell} \sum_{m,n} \mathbf{m}(m, n) \mathcal{V}_g f(mb_0, n\nu_0) \pi(mb_0, n\nu_0) h$$

$$= \sum_{m,n} \mathbf{m}(m, n) \mathcal{V}_g f(mb_0, n\nu_0) \sum_{k,\ell} \alpha_{k\ell} \pi(kb_1, \ell\nu_1) \pi(mb_0, n\nu_0) h. \quad (45)$$

Based on this expression, one may pursue two different choices of the sampling-points $(kb_1, \ell\nu_1)$. First, in extension of the result given in Lemma 37, we assume that the sampling points are associated to the adjoint lattice $\Lambda^\circ = \frac{1}{\nu_0}\mathbb{Z} \times \frac{1}{b_0}\mathbb{Z}$ of $\Lambda = b_0\mathbb{Z} \times \nu_0\mathbb{Z}$. The second choice of sampling points on the original lattice leads to a construction as introduced in [11] as Gabor twisters and will not be further discussed in the present contribution.

The following theorem extends the result given in Lemma 37 to the case in which the sampling points in the TST expansion are chosen from a lattice containing the adjoint lattice. It turns out that the TST spreading function then leads to a representation as a sum of Gabor multipliers.

Theorem 39 *Let $b_0, \nu_0 \in \mathbb{R}^+$ generate the time-frequency lattice Λ, and let Λ° denote the adjoint lattice. Let $g, h \in S_0(\mathbb{R})$ denote respectively Gabor analysis and synthesis windows, such that the $\mathcal{U}$ condition (28) is fulfilled. Let H denote the operator in $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ defined by the twisted spline type spreading function η as in (43), with $b_1, \nu_1 \in \mathbb{R}^+$.*

1. *Assume that b_1 and ν_1 are multiple of the dual lattice constants. Then H is a Gabor multiplier, with analysis window g, synthesis window*

$$\gamma = \sum_{k,\ell} \alpha_{k\ell} \pi(kb_1, \ell\nu_1) h, \quad (46)$$

and transfer function

$$\mathbf{m}(m, n) = b_0 v_0 \int_{\square^\circ} \mathcal{M}(b, v) e^{-2i\pi(nv_0 b - mb_0 v)} \, db dv, \qquad (47)$$

with $\square^\circ$ the fundamental domain of the adjoint lattice Λ°, and

$$\mathcal{M}(b, v) = \frac{\sum_{k,\ell=-\infty}^{\infty} \overline{\mathcal{V}_g h}(b + k/v_0, v + \ell/b_0) \phi(b + k/v_0, v + \ell/b_0)}{\sum_{k,\ell=-\infty}^{\infty} |\mathcal{V}_g h(b + k/v_0, v + \ell/b_0)|^2}. \qquad (48)$$

2. *Assume that the lattice generated by b_1 and v_1 contains the adjoint lattice:*

$$b_1 = \frac{1}{pv_0}, \qquad v_1 = \frac{1}{qb_0}. \qquad (49)$$

Then H may be written as a finite sum of Gabor multipliers

$$Hf = \sum_{i=1}^{p} \sum_{j=1}^{q} \left(\sum_{m \equiv i \,[\mathrm{mod}\, p]} \sum_{n \equiv j \,[\mathrm{mod}\, q]} \mathbf{m}(m, n) \mathcal{V}_g f(mb_0, nv_0) \pi_{mn} \right) \gamma_{ij}, \qquad (50)$$

with at most $p \cdot q$ different synthesis windows γ_{ij} and the transfer function given in (47) *and* (48).

Proof Let us formally compute

$$Hf = \sum_{m,n} \mathbf{m}(m, n) \mathcal{V}_g f(mb_0, nv_0) \sum_{k,\ell} \alpha_{k\ell} \pi(kb_1, \ell v_1) \pi_{mn} h$$

$$= \sum_{m,n} \mathbf{m}(m, n) \mathcal{V}_g f(mb_0, nv_0) \pi_{mn} \gamma_{mn},$$

where

$$\gamma_{mn} = \sum_{k,\ell} \alpha_{k\ell} e^{2i\pi[knb_0 v_1 - \ell m v_0 b_1]} \pi(kb_1, \ell v_1) h.$$

Now observe that if $(b_1, v_1) \in \Lambda^\circ$, one obviously has

$$\gamma_{mn} = \sum_{k,\ell} \alpha_{k\ell} \pi(kb_1, \ell v_1) h = \gamma_{00}, \quad \text{for } (m, n) \in \mathbb{Z}^2,$$

i.e. the above expression for Hf involves a single synthesis window $\gamma = \gamma_{00}$. Therefore, in this case, H takes the form of a standard Gabor multiplier, with fixed time-frequency transfer function, and a synthesis window prescribed by the coefficients in the TST expansion. This proves the first part of the theorem.

Let us now assume that the TST expansion of the spreading function is finer than the one prescribed by the lattice Λ°, but nevertheless the lattice $\Lambda_1 = \mathbb{Z}b_1 \times \mathbb{Z}v_1$ contains Λ°. In other words, there exist positive integers p, q such that (49) holds.

We then have

$$\gamma_{mn} = \sum_{k,\ell} \alpha_{k\ell} e^{2i\pi \left[\frac{knp-lmq}{pq}\right]} \pi(kb_1, \ell\nu_1)h \tag{51}$$

and it is readily seen that there are at most pq different synthesis windows γ_{ij},

$$\gamma_{ij} = \gamma_{m\,[\mathrm{mod}\,p],n\,[\mathrm{mod}\,q]}, \quad i = 1, \dots, p; \ j = 1, \dots, q. \tag{52}$$

The operator H may hence be written as a sum of Gabor multipliers, with one prescribed time-frequency transfer function, which is sub-sampled on several sub-lattices of the lattice Λ:

$$\Lambda_{ij} = (pb_0 \cdot \mathbb{Z} + i \cdot b_0) \times (q\nu_0 \cdot \mathbb{Z} + j \cdot \nu_0), \quad i = 0, \dots, p-1; \ j = 0, \dots, q-1,$$

and a single synthesis window per sub-lattice as given in (52). The resulting expression for H is hence as given in (50).

The expression for the transfer function is derived in analogy to the case discussed in Sect. 2.4.   □

Remark 40 Let us observe that in this approximation, the time-frequency transfer function **m** is completely characterized by the function ϕ used in the TST expansion. The choice of ϕ therefore imposes a fixed mask for the multipliers that come into play in (50).

Example 41 We first assume, that for a given primal lattice $\Lambda = b_0\mathbb{Z} \times \nu_0\mathbb{Z}$, the representation of a spreading function η is given by 5 building blocks:

$$\eta(b, \nu) = \sum_{k=-1}^{1} \alpha_{k0}\phi\left(b - \frac{k}{\nu_0}, \nu\right) + \sum_{\ell=-1}^{1} \alpha_{0\ell}\phi\left(b, \nu - \frac{\ell}{b_0}\right).$$

In this case, we obtain a single Gabor multiplier with synthesis window

$$\gamma_{00} = \sum_{k=-1}^{1} \alpha_{k0}\pi\left(\frac{k}{\nu_0}, 0\right)h + \sum_{\ell=-1}^{1} \alpha_{0\ell}\pi\left(0, \frac{\ell}{b_0}\right)h.$$

If we add the windows $\phi(b \pm \frac{1}{2\nu_0}, \nu \pm \frac{1}{2b_0})$ to the representation of η, we are now dealing with the finer lattice $\Lambda = \frac{1}{2\nu_0}\mathbb{Z} \times \frac{1}{2b_0}\mathbb{Z}$ and we obtain the sum of 4 Gabor multipliers with the following synthesis windows:

$$\gamma_{00} = \sum_{k=-1}^{1} \alpha_{k0}\pi\left(\frac{k}{2\nu_0}, 0\right)h + \sum_{\ell=-1}^{1} \alpha_{0\ell}\pi\left(0, \frac{\ell}{2b_0}\right)h,$$

$$\gamma_{01} = \sum_{k=-1}^{1} \alpha_{k0}e^{\pi i k}\pi\left(\frac{k}{2\nu_0}, 0\right)h + \sum_{\ell=-1}^{1} \alpha_{0\ell}\pi\left(0, \frac{\ell}{2b_0}\right)h,$$

$$\gamma_{10} = \sum_{k=-1}^{1} \alpha_{k0} \pi \left(\frac{k}{2\nu_0}, 0 \right) h + \sum_{\ell=-1}^{1} e^{\pi i \ell} \alpha_{0\ell} \pi \left(0, \frac{\ell}{2b_0} \right) h,$$

$$\gamma_{11} = \sum_{k=-1}^{1} \alpha_{k0} e^{\pi i k} \pi \left(\frac{k}{2\nu_0}, 0 \right) h + \sum_{\ell=-1}^{1} \alpha_{0\ell} e^{-\pi i \ell} \pi \left(0, \frac{\ell}{2b_0} \right) h,$$

and corresponding lattices: $\Lambda_{00} = 2\mathbb{Z}b_0 \times 2\mathbb{Z}\nu_0$, $\Lambda_{01} = 2\mathbb{Z}b_0 \times (2\mathbb{Z}+1)\nu_0$, $\Lambda_{10} = (2\mathbb{Z}+1)b_0 \times 2\mathbb{Z}\nu_0$, and $\Lambda_{11} = (2\mathbb{Z}+1)b_0 \times (2\mathbb{Z}+1)\nu_0$.

It is important to note, that in both cases described in Theorem 39 as well as the above example, the transfer function **m** can be calculated as the best approximation by a regular Gabor multiplier—a procedure which may be efficiently realized using (29). Fast algorithms for this task also exist in the literature, see [16], however, the algorithm derived from (29) is faster, since only one two-dimensional Fourier-transform, whose size depends on the fundamental domain of the adjoint lattice, is necessary. Moreover, our method may easily be adapted to the generalized case according to (39).

4 Conclusions and Perspectives

Starting from an operator representation in the continuous time-frequency domain via a twisted convolution, we have introduced generalizations of conventional time-frequency multipliers in order to overcome the restrictions of this model in the approximation of general operators. The model of multiple Gabor multipliers in principle allows the representation of any given linear operator. However, in oder to achieve computational efficiency as well as insight in the operator's characteristics, the parameters used in the model must be carefully chosen. An algorithm choosing the optimal sampling points for the family of synthesis windows, based on the spreading function, is the topic of ongoing research. On the other hand, the model of twisted spline type functions allows the approximation of a given spreading function and results in an adapted window or family of windows. By refining the sampling lattice in the TST approximation, a rather wide class of operators should be well-represented. The practicality of this approach has to be shown in the context of operators of practical relevance. All the results given in this work will also be applied in the context of estimation rather than approximation.

As a further step of generalization, frame types other than Gabor frames may be considered. Surprisingly little is known about wavelet frame multipliers, hence it will be interesting to generalize the achieved results to the affine group.

References

1. Bourouihiya, A.: The tensor product of frames. Sampl. Theory Signal Image Process. (STSIP) **7**(1), 65–76 (2008)

2. Balan, R.M., Rickard, S., Poor, H.V., Verdú, S.: Canonical time-frequency, time-scale, and frequency-scale representations of time-varying channels. J. Commun. Inf. Syst. **5**(2), 197–226 (2005)
3. Balazs, P.: Hilbert-Schmidt operators and frames—classification, best approximation. Int. J. Wavelets Multiresolut. Inf. Process. **6**(2), 315–330 (2008)
4. Balazs, P.: Matrix-representation of operators using frames. Sampl. Theory Signal Image Process. (STSIP) **7**(1), 39–54 (2008)
5. Bello, P.: Characterization of randomly time-variant linear channels. IEEE Trans. Commun. **11**, 360–393 (1963)
6. Benedetto, J.J., Pfander, G.E.: Frame expansions for Gabor multipliers. Appl. Comput. Harmon. Anal. **20**(1), 26–40 (2006)
7. Bölcskei, H., Koetter, R., Mallik, S.: Coding and modulation for underspread fading channels. In: IEEE International Symposium on Information Theory (ISIT) 2002, p. 358. ETH-Zürich, June 2002
8. Carmona, R., Hwang, W., Torrésani, B.: Practical Time-Frequency Analysis: Continuous Wavelet and Gabor Transforms, with an Implementation in *S*. Wavelet Analysis and its Applications, vol. 9. Academic Press, San Diego (1998)
9. Daubechies, I.: Ten Lectures on Wavelets. SIAM, Philadelphia (1992)
10. Dolson, M.: The phase vocoder: a tutorial. Comput. Music. J. **10**(4), 11–27 (1986)
11. Dörfler, M., Torrésani, B.: Spreading function representation of operators and Gabor multiplier approximation. In: Sampling Theory and Applications (SAMPTA'07), Thessaloniki, June 2007
12. Dörfler, M., Torrésani, B.: Representation of operators by sampling in the time-frequency domain. In: Proceedings of SAMPTA09 (2009)
13. Feichtinger, H., Zimmermann, G.: A Banach space of test functions for Gabor analysis. In: Feichtinger, H., Strohmer, T. (eds.) Gabor Analysis and Algorithms: Theory and Applications, pp. 123–170. Birkhäuser, Boston (1998). Chap. 3
14. Feichtinger, H.G.: On a new Segal algebra. Mon. Math. **92**(4), 269–289 (1981)
15. Feichtinger, H.G.: Spline type spaces in Gabor analysis. In: Zhou, D. (ed.) Wavelet Analysis: Twenty Years' Developments. World Scientific, Singapore (2002)
16. Feichtinger, H.G., Hampejs, M., Kracher, G.: Approximation of matrices by Gabor multipliers. IEEE Signal Process. Lett. **11**(11), 883–886 (2004)
17. Feichtinger, H.G., Kozek, W.: Quantization of TF lattice-invariant operators on elementary LCA groups. In: Gabor Analysis and Algorithms, pp. 233–266. Birkhäuser, Boston (1998)
18. Feichtinger, H.G., Nowak, K.: A first survey of Gabor multipliers. In: Feichtinger, H.G., Strohmer, T. (eds.) Advances in Gabor Analysis. Birkhäuser, Boston (2002)
19. Feichtinger, H.G., Strohmer, T.: Gabor Analysis and Algorithms. Theory and Applications. Birkhäuser, Boston (1998)
20. Feichtinger, H.G., Strohmer, T.: Advances in Gabor Analysis. Birkhäuser, Boston (2003)
21. Flanagan, J., Golden, R.M.: Phase vocoder. Bell Syst. Tech. **45**, 1493–1509 (1966)
22. Folland, G.: Harmonic Analysis in Phase Space. Princeton University Press, Princeton (1989)
23. Führ, H.: Abstract Harmonic Analysis of Continuous Wavelet Transforms. Lecture Notes in Mathematics, vol. 1863. Springer, Berlin (2005)
24. Gel'fand, I.M., Vilenkin, N.Y.: Generalized Functions. Applications of Harmonic Analysis, vol. 4. Academic Press, New York (1964). Translated from the Russian by Amiel Feinstein
25. Gröchenig, K.: Foundations of Time-Frequency Analysis. Appl. Numer. Harmon. Anal. Birkhäuser, Boston (2001)
26. Gröchenig, K.: Uncertainty principles for time-frequency representations. In: Feichtinger, H., Strohmer, T. (eds.) Advances in Gabor Analysis, pp. 11–30. Birkhäuser, Boston (2003)
27. Grossmann, A., Morlet, J., Paul, T.: Transforms associated to square integrable group representations I: general results. J. Math. Phys. **26**, 2473–2479 (1985)
28. Grossmann, A., Morlet, J., Paul, T.: Transforms associated to square integrable group representations II: examples. Ann. Inst. Henri Poincaré **45**, 293 (1986)
29. Hlawatsch, F., Matz, G.: Linear time-frequency filters. In: Boashash, B. (ed.) Time-Frequency Signal Analysis and Processing: A Comprehensive Reference, pp. 466–475. Elsevier, Amsterdam (2003)
30. Kailath, T.: Measurements on time-variant communication channels. IEEE Trans. Inf. Theory **8**(5), 229–236 (1962)
31. Kennedy, R.: Fading Dispersive Communication Channels. Wiley, New York (1969)
32. Kozek, W.: Matched Weyl-Heisenberg expansions of nonstationary environments. PhD thesis, NuHAG, University of Vienna (1996)

33. Kozek, W.: Adaptation of Weyl-Heisenberg frames to underspread environments. In: Feichtinger, H., Strohmer, T. (eds.) Gabor Analysis and Algorithms: Theory and Applications, pp. 323–352. Birkhäuser, Boston (1997)

34. Kozek, W.: On the transfer function calculus for underspread LTV channels. ISP **45**(1), 219–223 (1997)

35. Kozek, W., Pfander, G.E.: Identification of operators with bandlimited symbols. SIAM J. Math. Anal. **37**(3), 867–888 (2006)

36. Mallat, S.: A Wavelet Tour of Signal Processing. Academic Press, San Diego (1998)

37. Matz, G., Hlwatsch, F.: Time-frequency transfer function calculus (symbolic calculus) of linear time-varying systems (linear operators) based on a generalized underspread theory. J. Math. Phys. **39**(8), 4041–4070 (1998)

38. Matz, G., Hlawatsch, F.: Time-frequency characterization of random time-varying channels. In: Boashash, B. (ed.) Time Frequency Signal Analysis and Processing: A Comprehensive Reference, pp. 410–419. Elsevier, Amsterdam (2003)

39. Pfander, G.E., Walnut, D.F.: Operator identification and Feichtinger's algebra. Sampl. Theory Signal Image Process. (STSIP) **5**(2), 183–200 (2006)

40. Schempp, W.: Harmonic Analysis on the Heisenberg Nilpotent Lie Group. Pitman Series, vol. 147. Wiley, New York (1986)

41. Zadeh, L.A.: Time-varying networks I. In: Proceedings of the IRE, vol. 49, pp. 1488–1502, October 1961