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On Dual Gabor Frame Pairs Generated by Polynomials

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Abstract We provide explicit constructions of particularly convenient dual pairs of Gabor frames. We prove that arbitrary polynomials restricted to sufficiently large intervals will generate Gabor frames, at least for small modulation parameters. Unfortunately, no similar function can generate a dual Gabor frame, but we prove that almost any such frame has a dual generated by a B-spline. Finally, for frames generated by any compactly supported function ϕ whose integer-translates form a partition of unity, e.g., a B-spline, we construct a class of dual frame generators, formed by linear combinations of translates of ϕ. This allows us to chose a dual generator with special properties, for example, the one with shortest support, or a symmetric one in case the frame itself is generated by a symmetric function. One of these dual generators has the property of being constant on the support of the frame generator.

Keywords Gabor frames · Dual frame · Dual generator · Polynomial · B-spline · Strang-Fix conditions

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1 Introduction

The purpose of this paper is to provide simple and concrete constructions of pairs of dual Gabor frames. As generators for the frame we will consider polynomials restricted to compact intervals. Such a function can very well generate a Gabor frame; however, we prove that none of its dual frame generators can have a similar form. We provide sufficient conditions for other types of functions with compact support to generate a dual frame, e.g., the Strang-Fix conditions. In particular, it turns out that almost any Gabor frame generated by a compactly supported polynomial has a dual frame generated by a B-spline.

Finally, for frames generated by any compactly supported function ϕ whose integer-translates form a partition of unity, we construct a class of dual frame generators, formed by linear combinations of translates of ϕ. This allows us to chose a dual generator with special properties, for example, the one with shortest support or a symmetric one in case ϕ itself is symmetric. When applied to one of the B-spline B_N, defined by

$$B_1 = \chi_{[0,1]}, \qquad B_{N+1} = B_N * B_1,$$

one of these duals has the property of being constant on the support of B_N.

In the rest of this section we introduce some notation and state a few results from the literature that will be used in the proofs.

For $a, b \in \mathbb{R}$, consider the translation operator $(T_a f)(x) = f(x-a)$ and the modulation operator $(E_b f)(x) = e^{2\pi i b x} f(x)$, both acting on $L^2(\mathbb{R})$. A *Gabor frame* is a frame for $L^2(\mathbb{R})$ of the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for some $g \in L^2(\mathbb{R})$ and $a, b > 0$; recall that this means that there exist constants $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_{na}g\rangle|^2 \le B\|f\|^2, \quad \forall f \in L^2(\mathbb{R}). \tag{1}$$

If at least the upper condition in (1) is satisfied, $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a *Bessel sequence*.

It is well known that if $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame, then there exists a function $h \in L^2(\mathbb{R})$ such that $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ is a frame and

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g, \quad \forall f \in L^2(\mathbb{R}). \tag{2}$$

Any such function h is called a *dual generator*, and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ is called a *dual frame* of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. We will call (g, h) *a pair of dual frame generators*.

The starting point is the duality conditions for two Gabor systems; see [5] and [7].

Lemma 1.1 *Two Bessel sequences $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R})$ if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g(x-n/b-ka)}h(x-ka) = b\delta_{n,0}, \quad a.e.\ x \in [0,a]. \tag{3}$$

It is well known that any Gabor system generated by a bounded and compactly supported function forms a Bessel sequence. We will use the following elementary fact (see, e.g. [2] or [4]):

Lemma 1.2 *Let $a, b > 0$ be given. For $g \in L^2(\mathbb{R})$, consider the function*

$$G(x) = \sum_{k\in\mathbb{Z}} |g(x-ka)|^2.$$

Then the following hold:

(i) *If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame with bounds A, B, then*

$$bA \le G(x) \le bB, \quad a.e.\ x \in \mathbb{R};$$

(ii) *If g is bounded and has support in an interval I with $|I| < 1/b$, then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame with lower bound A if and only if*

$$bA \le G(x), \quad a.e.\ x \in [0, a].$$

2 Gabor Frames Generated by Polynomials

Via a scaling, we can always restrict our attention to the case where the translation parameter is $a = 1$. We obtain the following consequence of Lemma 1.1:

Lemma 2.1 *Let g and h be real-valued, bounded, and compactly supported functions in $L^2(\mathbb{R})$. Then, for $b > 0$ chosen small enough so that*

$$\operatorname{supp} T_\alpha g \cap \operatorname{supp} h = \emptyset \quad \text{for } |\alpha| \ge 1/b,$$

the following two conditions are equivalent:

(i) $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ *form dual frames for* $L^2(\mathbb{R})$;

(ii)

$$\sum_{k\in\mathbb{Z}} g(x-k)h(x-k) = b, \quad a.e.\ x \in [0, 1]. \tag{4}$$

Our goal is to provide convenient constructions of a Gabor frame and a dual frame. We first notice that the restriction of any polynomial to a sufficiently large interval will generate a Gabor frame for small modulation parameters:

Proposition 2.2 *Let $N \in \mathbb{N}$, and consider any bounded interval $I \subset \mathbb{R}$ with $|I| \ge N$. Then any (nontrivial) polynomial*

$$g(x) = \left(\sum_{k=0}^{N-1} c_k x^k\right) \chi_I(x) \tag{5}$$

generates a Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ for $b \in]0, \frac{1}{|I|}]$.

Proof Choose $c \geq 0$ such that $[c, c+N] \subseteq I$. For $x \in [c, c+1]$, and with $\varphi(x) = \sum_{k=0}^{N-1} c_k x^k$,

$$G(x) = \sum_{k \in \mathbb{Z}} |g(x+k)|^2 \geq \sum_{k=0}^{N-1} |g(x+k)|^2 = \sum_{k=0}^{N-1} |\varphi(x+k)|^2.$$

It is easy to check that the function $\sum_{k=0}^{N-1} |\varphi(\cdot + k)|^2$ is bounded below, so the conclusion follows from Lemma 1.2. □

Note that the condition on the relationship between the support size and the degree of the polynomial is necessary in Proposition 2.2:

Example 2.3 Let

$$g(x) = (x - 1/2)(x - 3/2)\chi_{[0,2]}(x) = (x^2 - 2x + 3/4)\chi_{[0,2]}(x).$$

Then

$$\sum_{k \in \mathbb{Z}} |g(1/2 - k)|^2 = 0,$$

so $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$ is not a Gabor frame for any $b > 0$.

In applications of frames, it is crucial that the frame generator and the dual frame generator have a convenient form. The polynomials in (5) are very convenient, but unfortunately no dual generator of a similar form exist:

Proposition 2.4 *Let I and J be bounded intervals. Two compactly supported polynomials*

$$g(x) = \left(\sum_{k=0}^{N} a_k x^k \right) \chi_I(x)$$

and

$$h(x) = \left(\sum_{k=0}^{M} b_k x^k \right) \chi_J(x)$$

of degree ≥ 1 can not generate dual Gabor frames $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n \in \mathbb{Z}}$ for any $b > 0$.

Proof Assume that $a_N \neq 0$ and $b_M \neq 0$. If $I \cap J = \emptyset$ it is clear from Lemma 1.1 that $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n \in \mathbb{Z}}$ do not form dual Gabor frames, so we will assume that $I \cap J \neq \emptyset$. In that case $I \cap J$ contains an interval. It is easy to check that for $x \in I \cap J$, the expression $\sum_{n\in\mathbb{Z}} g(x+n)h(x+n)$ is a polynomial of degree $N+M$. Thus, if (4) in Lemma 2.1 is satisfied, then necessarily $N = M = 0$. □

Remark A similar proof shows that even finite linear combinations of the type

$$\tilde{g}(x) = \sum c_k T_k g, \qquad \tilde{h}(x) = \sum d_k T_k h$$

with g and h as in Proposition 2.4 can not generate dual Gabor frames $\{E_{mb}T_n\tilde{g}\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n\tilde{h}\}_{m,n\in\mathbb{Z}}$.

We will now analyze how we can find compactly generated dual frames associated to a Gabor frame generated by a compactly supported polynomial. In our first result, we will search for a dual generator supported on the same interval as the frame generator itself. For $k \in \mathbb{N}$, let $D^k f$ denote the kth derivative of a function $f : \mathbb{R} \to \mathbb{C}$.

Theorem 2.5 *Let $N \in \mathbb{N}$. Given a compact interval $I \subset \mathbb{R}$, let*

$$g(x) := \left(\sum_{k=0}^{N-1} c_k x^k\right) \chi_I(x).$$

Let $h \in L^2(\mathbb{R})$ be such that $\operatorname{supp} h \subseteq I$, *and let $b \in]0, \frac{1}{|I|}]$. Then the following are equivalent*:

(i) *There exists $\beta \in \mathbb{R}$ such that $g(x)$ and $\tilde{h}(x) := \beta h(x)$ generate dual frames $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n\tilde{h}\}_{m,n\in\mathbb{Z}}$;*
(ii) *The equations*

$$\begin{cases} \sum_{k=0}^{N-1} c_k \dfrac{D^k\hat{h}(n)}{(-2\pi i)^k} = 0, \quad n \in \mathbb{Z}\setminus\{0\}, \\ \sum_{k=0}^{N-1} c_k \dfrac{D^k\hat{h}(0)}{(-2\pi i)^k} \neq 0, \end{cases} \tag{6}$$

hold.

Proof Let $I := [a, a+|I|]$ and let $N_1 := \lceil |I| \rceil$, where $\lceil\cdot\rceil$ denotes the ceiling function. If (i) holds, then by Lemma 2.1, for some $\beta \neq 0$,

$$\sum_{k=-\infty}^{\infty} g(x+k)\beta h(x+k) = b, \quad x \in \mathbb{R};$$

due to the compact support of g, this is equivalent to

$$F(x) := \sum_{k=0}^{N_1-1} g(x+k)\beta h(x+k) = b, \quad x \in [a, a+1].$$

Assuming that this condition is satisfied, it follows that

$$\begin{aligned}
b\delta_{0,n} &= \int_a^{a+1} F(x)e^{-2\pi inx}dx \\
&= \sum_{k=0}^{N_1-1} \int_a^{a+1} g(x+k)\beta h(x+k)e^{-2\pi inx}dx \\
&= \sum_{k=0}^{N_1-1} \int_{a+k}^{a+k+1} g(x)\beta h(x)e^{-2\pi inx}dx \\
&= \int_a^{a+N_1} g(x)\beta h(x)e^{-2\pi inx}dx \\
&= \beta \int_{-\infty}^{\infty} g(x)h(x)e^{-2\pi inx}dx \\
&= \beta \int_{-\infty}^{\infty} \sum_{k=0}^{N-1} c_k x^k h(x)e^{-2\pi inx}dx \\
&= \beta \sum_{k=0}^{N-1} c_k \frac{D^k\hat{h}(n)}{(-2\pi i)^k},
\end{aligned}$$

where we used that

$$D^k\hat{h}(n) = (-2\pi i)^k \int_{-\infty}^{\infty} x^k h(x)e^{-2\pi inx}dx$$

in the last equality (see [8, Theorem 9.2]). Therefore condition (6) is satisfied. On the other hand, if (ii) holds, choose

$$\beta := \frac{b}{\sum_{k=0}^{N-1} c_k \frac{D^k\hat{h}(0)}{(-2\pi i)^k}}.$$

Then, reversing the above argument, $g(x)$ and $\tilde{h}(x) := \beta h(x)$ generate dual frames $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n\tilde{h}\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$. □

With the aim of obtaining explicit constructions of pairs of dual frame generators we will now restrict our attention to (potential) dual generators satisfying extra requirements. Recall that a function h is said to satisfy the Strang-Fix conditions of order N if

$$\begin{cases} \hat{h}(0) \neq 0; \\ D^k\hat{h}(n) = 0, \quad n \in \mathbb{N}\setminus\{0\},\ k = 0, 1, \dots, N-1. \end{cases}$$

See [9] for more information about the role of that condition. We will now consider a frame generator g of the type in Theorem 2.5, and search for a compactly supported dual generator h satisfying the Strang-Fix conditions. We split into two cases, depending on the support of h:

Theorem 2.6 *Let $N \in \mathbb{N}$. Given a compact interval $I \subset \mathbb{R}$, let*

$$g(x) := \left(\sum_{k=0}^{N-1} c_k x^k \right) \chi_I(x) \tag{7}$$

be a nontrivial polynomial. Suppose that $h \in L^2(\mathbb{R})$ satisfies the Strang-Fix conditions of order N. Let $b \in]0, \frac{1}{|I|}]$. Then the following hold:

(a) *If $I = \operatorname{supp} h$ and*

$$\kappa := \sum_{k=0}^{N-1} c_k \frac{D^k \hat{h}(0)}{(-2\pi i)^k} \neq 0,$$

then $g(x)$ and $\tilde{h}(x) := \frac{b}{\kappa} h(x)$ generate dual frames $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n \tilde{h}\}_{m,n\in\mathbb{Z}}$.

(b) *If $\operatorname{supp} h$ is an interval with $|\operatorname{supp} h| < |I|$, then there always exist $\alpha, \beta \in \mathbb{R}$ such that $g(x)$ and $\tilde{h}(x) := \beta h(x-\alpha)$ generate dual frames $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n \tilde{h}\}_{m,n\in\mathbb{Z}}$.*

Proof (a) follows from Theorem 2.5 and the Strang-Fix conditions.

For (b), let $I := [a, a+|I|]$ for some $a \in \mathbb{R}$. Let $N_1 := \lceil |I| \rceil$. Choose an interval J such that

$$\alpha \in J \Rightarrow \operatorname{supp} h \subset I - \alpha.$$

Letting $g_1(x) := \sum_{k=0}^{N-1} c_k x^k$, we can write $g(x) = g_1(x)\chi_I(x)$. Note that g_1 is a polynomial of degree at most $N-1$. Depending on the coefficients c_k, the degree might be smaller than $N-1$; let us denote the exact degree by N_0. For any choice of $\alpha \in \mathbb{R}$, a Taylor expansion yields that

$$g_1(x) = \sum_{k=0}^{N_0} c_k x^k = \sum_{k=0}^{N_0} \frac{D^k g_1(\alpha)}{k!}(x-\alpha)^k.$$

By an argument similar to the one in the proof of Theorem 2.5, we have for $\alpha \in J$,

$$\begin{aligned}
&\int_a^{a+1} \sum_{k=0}^{N_1-1} g(x+k)h(x+k-\alpha)e^{-2\pi i n x}\,dx \\
&= \int_{-\infty}^{\infty} g_1(x)h(x-\alpha)e^{-2\pi i n x}\,dx \\
&= \int_{-\infty}^{\infty} \sum_{k=0}^{N_0} \frac{D^k g_1(\alpha)}{k!}(x-\alpha)^k h(x-\alpha)e^{-2\pi i n x}\,dx \\
&= e^{-2\pi i n\alpha} \sum_{k=0}^{N_0} \frac{D^k g_1(\alpha)}{k!} \int_{-\infty}^{\infty} x^k h(x) e^{-2\pi i n x}\,dx
\end{aligned}$$

$$= e^{-2\pi i n\alpha} \sum_{k=0}^{N_0} \frac{D^k g_1(\alpha)}{k!} \frac{D^k \hat{h}(n)}{(-2\pi i)^k}$$

$$= \left(\sum_{k=0}^{N_0} \frac{D^k g_1(\alpha)}{k!} \frac{D^k \hat{h}(0)}{(-2\pi i)^k} \right) \delta_{0,n}, \tag{8}$$

where we used the Strang-Fix conditions in the last equality. Define a polynomial H by

$$H(x) := \sum_{k=0}^{N_0} \frac{D^k g_1(x)}{k!} \frac{D^k \hat{h}(0)}{(-2\pi i)^k}. \tag{9}$$

Note that the term corresponding to $k = 0$ in (9) is $\hat{h}(0)g_1(x)$. By assumption, $\hat{h}(0) \neq 0$, so this term is a polynomial of degree N_0. For all other values of k, the degree of the term $\frac{D^k g_1(x)}{k!} \frac{D^k \hat{h}(0)}{(-2\pi i)^k}$ in (9) is smaller than N_0; thus, the exact degree of the polynomial H is N_0. Take $\alpha_0 \in J$ such that $H(\alpha_0) \neq 0$, then, the above calculation shows that the nth Fourier coefficient for the function

$$\sum_{k=0}^{N_1-1} g(x+k)h(x+k-\alpha_0), \quad x \in [a, a+1],$$

is $H(\alpha_0)\delta_{n,0}$. Taking $\beta := \frac{b}{H(\alpha_0)}$ and using the knowledge of the support of g, we arrive at

$$\sum_{k\in\mathbb{Z}} g(x+k)\beta h(x+k-\alpha_0) = b.$$

Hence $g(x)$ and $\tilde{h}(x) := \beta h(x-\alpha_0)$ generate dual frames $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n \tilde{h}\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$. □

Note that the proof of Theorem 2.6 shows how to choose suitable values for α and β in (b).

Theorem 2.6 implies the rather surprising fact that functions of the type in (7) usually have B-splines as dual generators:

Corollary 2.7 *Let $N \in \mathbb{N}$. Given a compact interval $I \subset \mathbb{R}$, let*

$$g(x) := \left(\sum_{k=0}^{N-1} c_k x^k \right) \chi_I(x)$$

be a nontrivial polynomial. Then the following hold:

(a) *Assume that $I = [0, N]$. If $b \leq 1/N$ and*

$$\kappa := \sum_{k=0}^{N-1} c_k \frac{D^k \widehat{B_N}(0)}{(-2\pi i)^k} \neq 0, \tag{10}$$

then $g(x)$ *and* $h(x) := \frac{b}{\kappa} B_N(x)$ *generate dual frames* $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$.

(b) *If* $I \supsetneq [0, N]$ *and* $b \in]0, \frac{1}{|I|}]$, *then there exist* $\alpha, \beta \in \mathbb{R}$ *such that* $g(x)$ *and* $h(x) := \beta B_N(x-\alpha)$ *generate dual frames* $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$.

For the proof, we just need to notice that the B-spline B_N satisfies the Strang-Fix condition of order N, see, e.g., [3, p. 101].

We will now give some concrete applications of Corollary 2.7. We first give an example of a frame generated by a compactly supported polynomial, which does not satisfy the condition (10). However, extending the support and using Corollary 2.7(ii) we are able to obtain a pair of dual frame generators:

Example 2.8 Let

$$g(x) = (x-1)\chi_I(x) := \left(\sum_{k=0}^{1} c_k x^k\right)\chi_I(x).$$

First, let $I = [0, 2]$. A direct calculation shows that $\widehat{B_2}(0) = 1$ and $D\widehat{B_2}(0) = -2\pi i$. Thus we have

$$\kappa = \sum_{k=0}^{1} c_k \frac{D^k\widehat{B_2}(0)}{(-2\pi i)^k} = 0.$$

An immediate direct computation shows that

$$\sum_{k\in\mathbb{Z}} g(x-k)B_2(x-k) = 0.$$

Hence, by Lemma 2.1 the B-spline B_2 is not a dual generator of g for any $b > 0$.

Now, let $I = [-1, 3]$ and let $g_1(x) = x - 1$. As in (9), define

$$H(x) := \sum_{k=0}^{1} \frac{D^k g_1(x)}{k!}\frac{D^k\widehat{B_2}(0)}{(-2\pi i)^k} = (x-1) + 1 = x.$$

Choose $\alpha_0 \in [-1, 1] \setminus \{0\}$. Then we have $\operatorname{supp} B_2 \subset I - \alpha_0$ and $H(\alpha_0) \neq 0$. Let $\beta = b/\alpha_0$. By Theorem 2.6(b), $\beta B_2(x-\alpha_0)$ is a dual generator of g for any choice of $b \in]0, \frac{1}{|I|}]$.

Example 2.9 Let

$$g(x) := x^2(5-x)^2\chi_{[0,5]}(x) = \left(25x^2 - 10x^3 + x^4\right)\chi_{[0,5]}(x)$$
$$=: \left(\sum_{k=0}^{4} c_k x^k\right)\chi_{[0,5]}(x).$$

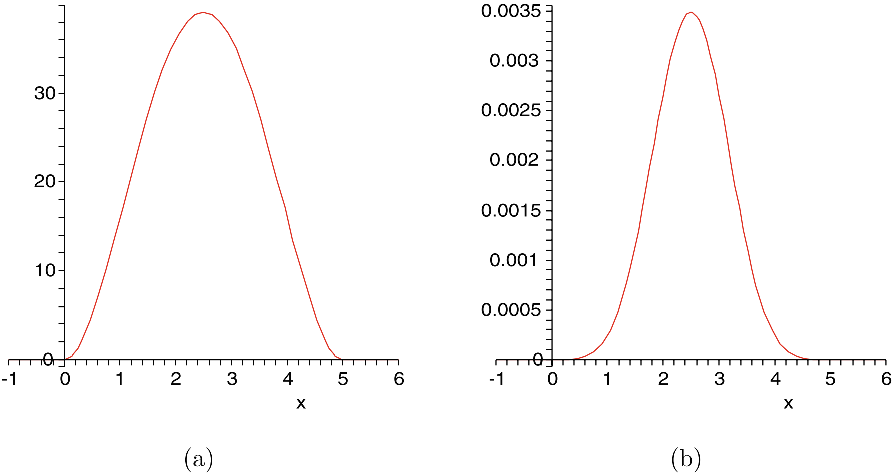


Fig. 1 Plots of the generators for $b = 1/5$ in Example 2.9: (**a**) g; (**b**) h

For the B-spline B_5, we have

$$D^k \widehat{B_5}(0)/(-2\pi i)^k = \begin{cases} 20/3, & k = 2; \\ 75/4, & k = 3; \\ 331/6, & k = 4. \end{cases}$$

Thus $\kappa := \sum_{k=0}^{4} c_k \frac{D^k \widehat{B_5}(0)}{(-2\pi i)^k} = 103/3 \neq 0$. Hence $h(x) := \frac{b}{\kappa} B_5$ is a dual generator of $g(x)$ for any choice of $b \leq 1/5$ by Corollary 2.7(a). See Fig. 1. Note that the frame generator g and the dual generator are symmetric.

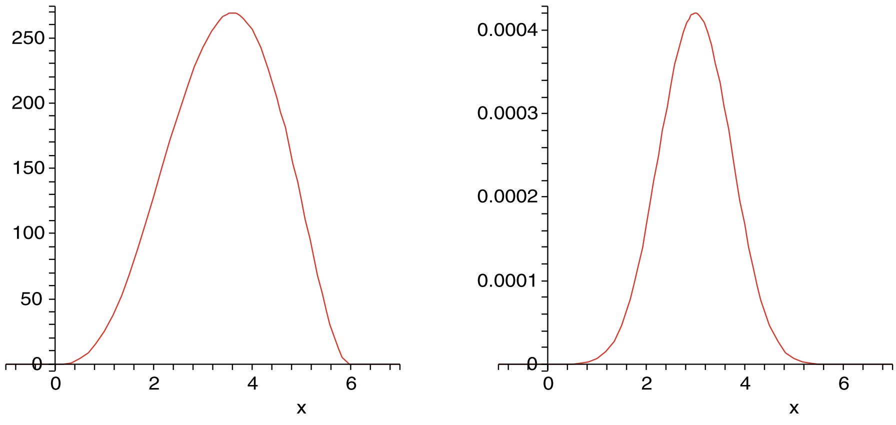
Example 2.10 Let

$$g(x) := x^3(6-x)^2 \chi_{[0,6]}(x) = \left(36x^3 - 12x^4 + x^5\right) \chi_{[0,6]}(x)$$
$$=: \left(\sum_{k=0}^{5} c_k x^k\right) \chi_{[0,6]}(x).$$

For the B-spline B_6, we have

$$D^k \widehat{B_6}(0)/(-2\pi i)^k = \begin{cases} 63/2, & k = 3; \\ 1087/10, & k = 4; \\ 777/2, & k = 5. \end{cases}$$

Thus $\kappa := \sum_{k=0}^{5} c_k \frac{D^k \widehat{B_6}(0)}{(-2\pi i)^k} = 2181/10 \neq 0$. Hence $h(x) := \frac{b}{\kappa} B_6$ is a dual generator of $g(x)$ for any choice of $b \leq 1/6$ by Corollary 2.7(a). See Fig. 2. Note that the dual frame generator is symmetric, but not the frame generator itself.



(a) (b)

Fig. 2 Plots of the generators for $b=1/6$ in Example 2.10: (**a**) g; (**b**) h

We notice that the results can be applied to other types of refinable functions than the B-splines:

Corollary 2.11 *Let $N\in\mathbb{N}$. Given a compact interval $I\subset\mathbb{R}$, let*

$$g(x):=\left(\sum_{k=0}^{N-1}c_kx^k\right)\chi_I(x).$$

Suppose that m_0 is a trigonometric polynomial that can be factorized as

$$m_0(\gamma)=\left(\frac{1+e^{-2\pi i\gamma}}{2}\right)^N F(\gamma),$$

where F is a trigonometric polynomial with $F(0)=1$, and that $\varphi\in L^2(\mathbb{R})$ is the refinable function associated with m_0, that is,

$$\hat{\varphi}(2\gamma)=m_0(\gamma)\hat{\varphi}(\gamma).$$

Then the following hold:

(a) *Assume that $I=\operatorname{supp}\varphi$. If $b\le\frac{1}{|I|}$ and*

$$\kappa:=\sum_{k=0}^{N-1}c_k\frac{D^k\hat{\varphi}(0)}{(-2\pi i)^k}\neq 0,$$

then $g(x)$ and $h(x):=\frac{b}{\kappa}\varphi(x)$ generate dual frames $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_nh\}_{m,n\in\mathbb{Z}}$.

(b) *If* $I \supsetneq \operatorname{supp}\varphi$ *and* $b \in]0, \frac{1}{|I|}]$, *then there exist* $\alpha, \beta \in \mathbb{R}$ *such that* $g(x)$ *and* $h(x) := \beta\varphi(x-\alpha)$ *generate dual frames* $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$.

Proof It is known that φ satisfies the Strang-Fix conditions of order N, see [6, Theorem 1.32]. Hence (a) and (b) follow from Theorem 2.6. □

3 Gabor Frames Generated by B-Splines

The principles discussed so far can be used to search for simple dual generators for many other types of frame generators. For frames generated by a function g whose integer-translates form a partition of unity, we can provide an explicit description of a class of dual frame generators, all of which are formed by linear combinations of translates of the given function g:

Theorem 3.1 *Let* $N \in \mathbb{N}$. *Let* $g \in L^2(\mathbb{R})$ *be a real-valued bounded function with* $\operatorname{supp} g \subset [0, N]$, *for which*

$$\sum_{n\in\mathbb{Z}} g(x-n) = 1.$$

Let $b \in]0, \frac{1}{2N-1}]$. *Consider any scalar sequence* $\{a_n\}_{n=-N+1}^{N-1}$ *for which*

$$a_0 = b \quad \text{and} \quad a_n + a_{-n} = 2b, \quad n = 1, 2, \ldots, N-1, \tag{11}$$

and define $h \in L^2(\mathbb{R})$ *by*

$$h(x) = \sum_{n=-N+1}^{N-1} a_n g(x+n). \tag{12}$$

Then g *and* h *generate dual frames* $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ *for* $L^2(\mathbb{R})$.

Proof Note that with the definition (12), we have

$$\operatorname{supp} h \subset [-N+1, 2N-1],$$

thus Lemma 2.1 applies if $b \in]0, \frac{1}{2N-1}]$. Thus we only need to check that

$$b = \sum_{k\in\mathbb{Z}} g(x+k)h(x+k), \quad x \in [0,1];$$

due to the compact support of g, this is equivalent to

$$b = \sum_{k=0}^{N-1} g(x+k)h(x+k), \quad x \in [0,1]. \tag{13}$$

To check that (13) holds, let

$$g_n(x) := \sum_{k=0}^{N-1} g(x+k)g(x+k+n).$$

Note that for $x \in [0,1]$ and $n = 1, 2, \ldots, N-1$,

$$\begin{aligned} g_{-n}(x) &= \sum_{k=0}^{N-1} g(x+k)g(x+k-n) \\ &= \sum_{k=n}^{N-1} g(x+k)g(x+k-n) \\ &= \sum_{\ell=0}^{N-1-n} g(x+\ell+n)g(x+\ell) \\ &= \sum_{\ell=0}^{N-1} g(x+\ell)g(x+\ell+n) \\ &= g_n(x). \end{aligned}$$

Putting this and (12) into the right-hand side of (13), we have that for $x \in [0,1]$,

$$\begin{aligned} \sum_{k=0}^{N-1} g(x+k)h(x+k) &= \sum_{k=0}^{N-1} g(x+k) \sum_{n=-N+1}^{N-1} a_n g(x+k+n) \\ &= \sum_{n=-N+1}^{N-1} a_n \sum_{k=0}^{N-1} g(x+k)g(x+k+n) \\ &= \sum_{n=-N+1}^{N-1} a_n g_n(x) \\ &= a_0 g_0(x) + \sum_{n=1}^{N-1} (a_n + a_{-n}) g_n(x) \\ &= b\left[g_0(x) + 2\sum_{n=1}^{N-1} g_n(x) \right]. \end{aligned} \tag{14}$$

On the other hand, for $x \in [0,1]$ the partition of unity property implies that

$$\begin{aligned} \sum_{n=-N+1}^{N-1} g_n(x) &= \sum_{n=-N+1}^{N-1} \sum_{k=0}^{N-1} g(x+k)g(x+k+n) \\ &= \sum_{k=0}^{N-1} g(x+k) \sum_{n=-N+1}^{N-1} g(x+k+n) \end{aligned}$$

$$= \sum_{k=0}^{N-1} g(x+k)$$
$$= 1.$$

Since $g_{-n}(x) = g_n(x)$ for $x \in [0, 1]$ and $n = 1, 2, \ldots, N-1$, it follows that

$$g_0(x) + 2\sum_{n=1}^{N-1} g_n(x) = 1 \quad \text{for } x \in [0, 1].$$

This together with (14) implies that

$$\sum_{k=0}^{N-1} g(x+k)h(x+k) = b \quad \text{for } x \in [0, 1].$$

This completes the proof. □

A special choice of the coefficients a_n in (11) leads to a dual generator with very nice properties:

Corollary 3.2 *Under the assumptions in Theorem 3.1, the function*

$$h(x) = b \sum_{n=-N+1}^{N-1} g(x+n) \tag{15}$$

generates a dual frame of $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$. *The function h satisfies that $h = b$ on the support of g. Furthermore, if g is symmetric, then h is symmetric.*

Proof The function in (15) appears by the choice $a_n = b$ in (11). For $x \in [0, N]$, the partition of unity property together with the compact support of g implies that $h(x) = b$. It is clear that h is symmetric in case g is symmetric. □

Another choice of the coefficients a_n leads to a dual generator considered already in [1]. Among the dual frame generators in Theorem 3.1, it has the shortest support:

Corollary 3.3 *Under the assumptions in Theorem 3.1, the function*

$$h(x) = 2bg(x) + b\sum_{n=1}^{N-1} g(x+n)$$

generates a dual frame of $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$.

Example 3.4 For the B-spline

$$B_2(x) = \begin{cases} x, & x \in [0, 1[, \\ 2-x, & x \in [1, 2[, \\ 0, & x \notin [0, 2[, \end{cases}$$

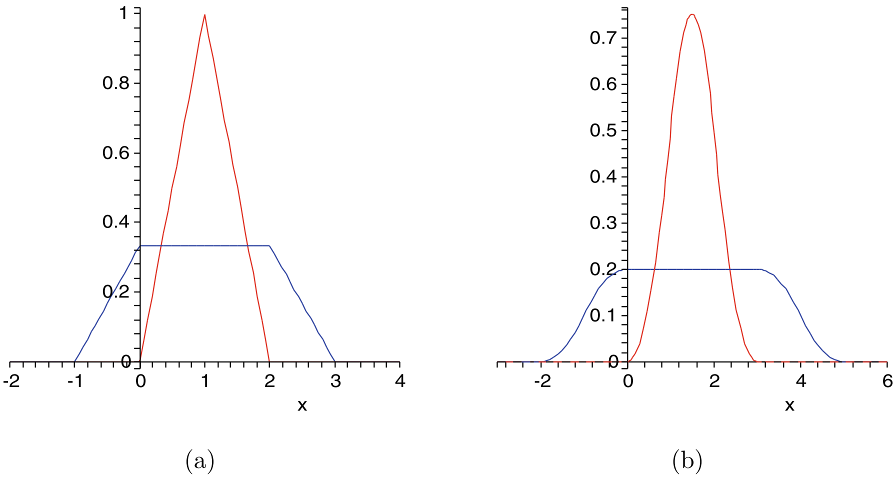


Fig. 3 (**a**) The B-spline B_2 and the dual generator h_2 in (16). (**b**) The B-spline B_3 and the dual generator h_3 in (17)

we can use Theorem 3.1 for $b \in]0, 1/3]$. For $b = 1/3$ we obtain the symmetric dual

$$h_2(x) = \begin{cases} 1/3x + 1/3, & x \in [-1, 0[, \\ 1/3, & x \in [0, 2[, \\ 1 - 1/3x, & x \in [2, 3[, \\ 0, & x \notin [-1, 3[. \end{cases} \tag{16}$$

See Fig. 3(a). For the B-spline

$$B_3(x) = \begin{cases} 1/2\, x^2, & x \in [0, 1[, \\ -3/2 + 3x - x^2, & x \in [1, 2[, \\ 9/2 - 3x + 1/2\, x^2, & x \in [2, 3[, \\ 0, & x \notin [0, 3[, \end{cases}$$

and $b = 1/5$, we obtain the symmetric dual

$$h_3(x) = \begin{cases} 1/10\, x^2 + 2/5\, x + 2/5, & x \in [-2, -1[, \\ -1/10\, x^2 + 1/5, & x \in [-1, 0[, \\ 1/5, & x \in [0, 3[, \\ -1/10\, x^2 + 3/5\, x - 7/10, & x \in [3, 4[, \\ 1/10\, x^2 - x + 5/2, & x \in [4, 5[, \\ 0, & x \notin [0, 5[. \end{cases} \tag{17}$$

See Fig. 3(b).

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