J Fourier Anal Appl (2010) 16: 34–51
DOI 10.1007/s00041-009-9073-1

Fourier Inversion of Distributions Supported by a Hypersurface

F.J. González Vieli · E. Seifert

Received: 16 February 2008 / Published online: 27 March 2009

Abstract Let μ_Σ be the natural measure on $\mathbf{R}^N$ ($N \geq 3$) supported by a compact oriented analytic hypersurface Σ, ψ a smooth function on $\mathbf{R}^N$ and $P(D)$ a differential operator in N variables of order m. We determine a sufficient condition on the number λ such that the Fourier integral of the distribution $P(D)\psi\mu_\Sigma$ be summable by Cesàro means of order λ to zero in a point outside the hypersurface. This condition depends on m and on the position of the point with respect to the caustic of the hypersurface.

Keywords Fourier transform · Distribution · Hypersurface · Cesàro means

Mathematics Subject Classification (2000) Primary 42B10 · Secondary 46F12

1 Introduction

Although the Fourier inversion problem—how to reconstruct an integrable function f from its Fourier transform $\mathcal{F}f$—has been thoroughly investigated, some interesting phenomena remained long overlooked. For example, in 1953 Hewitt observed that, in the apparently simple case of f being the indicator function χ_B of the unit ball $B = B(0, 1)$ in $\mathbf{R}^N$, the Fourier integral of χ_B,

$$\int_{\|t\|\leq R} \mathcal{F}\chi_B(t)\mathrm{e}^{2\pi i(x|t)}dt,$$

Communicated by Tom Körner.

F.J. González Vieli (✉)
Montoie 45, 1007 Lausanne, Switzerland
e-mail: Francisco-Javier.Gonzalez@gmx.ch

E. Seifert
Case postale 7387, 1002 Lausanne, Switzerland
e-mail: eseifert@olympe.ch

converges, for R tending to $+\infty$, everywhere except at $x = 0$ if $N \geq 3$ [9, Theorem 3.10, p. 468]. This phenomenon surfaced again in 1993 when Pinsky established necessary and sufficient conditions for the pointwise Fourier inversion of piecewise smooth functions on $\mathbf{R}^N$ [10]. But it was only in 1997 that this difference in behaviour was better understood, when Pinsky and Taylor, and Popov separately, studied, instead of the unit ball, an open set U whose boundary Σ is a regular hypersurface. They showed that the rate of convergence of the Fourier integral of χ_U at $x \notin \Sigma$ depends on the position of x with respect to the caustic of Σ [11, 12].

In 2000, the first author showed that a pointwise Fourier inversion is also partially possible for distributions on $\mathbf{R}^N$ when summation methods are used: the Fourier integral of a distribution with compact support is summable to zero by Cesàro means outside the support of the distribution [6]. He then studied in more detail the case of derivatives of the natural measure on $\mathbf{R}^2$ supported by a regular closed curve γ [7] and showed that the order λ of the Cesàro means which permit the Fourier inversion of these distributions at a point $x \notin \gamma$ depends on the position of x with respect to the caustic of γ in a way perfectly analogous to the dependencies established by [11] and [12] for χ_U.

Here we extend that work to the case of the natural measure on $\mathbf{R}^N$ ($N \geq 3$) supported by a compact oriented analytic hypersurface and some distributions constructed from it. Before establishing our main result in Sect. 6, we prove in Sect. 3 two auxiliary results about Cesàro summability, recall in Sect. 4 useful facts about oscillatory integrals and study in Sect. 5 two ways of defining the caustic of a hypersurface. But first we introduce some notations in Sect. 2. In Sect. 7 finally, we consider the example of an ellipsoid in $\mathbf{R}^3$ with axes of different lengths.

2 Preliminaries

We let $\mathbf{N} := \{1, 2, 3, \ldots\}$ and $\mathbf{N}_0 := \{0\} \cup \mathbf{N}$. Given a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}_0^n$, we put

$$|\alpha| := \alpha_1 + \cdots + \alpha_n, \qquad \alpha! := \alpha_1! \cdots \alpha_n!, \qquad D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

and, if $x \in \mathbf{R}^n$,

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

For $1 \leq k \leq n$, ϵ_k is the multiindex given by $(\epsilon_k)_l = \delta_{kl}$, and we put

$$\partial_k := D^{\epsilon_k}, \qquad \partial_{kl}^2 := \partial_l \partial_k.$$

If F is a subset of $\mathbf{R}^n$, we write χ_F the indicator function of F. If f is a real valued function, we define $f_+ := \max(f, 0)$. We write $(x|y)$ the usual scalar product of x, $y \in \mathbf{R}^n$.

The Bessel function J_ν of the first kind and order ν has an asymptotic expansion which can be written as follows: given $p \in \mathbf{N}$ and $R > 0$, there exist a constant $C_{\nu p}$

and an analytic function $\Upsilon_{\nu p}$ such that, for all $t \geq R$,

$$J_\nu(t) = \Re\mathrm{e}\left[\mathrm{e}^{it}\sum_{m=0}^{p-1} c_{\nu m} t^{-m-1/2} + \Upsilon_{\nu p}(t)\right] \tag{1}$$

and

$$|\Upsilon_{\nu p}(t)| \leq C_{\nu p} t^{-p-1/2},$$

where the constants $c_{\nu m} \in \mathbf{C} \setminus \{0\}$ [16, pp. 198–199].

3 Cesàro Summability

Let $b \in L^1_{loc}(\mathbf{R}_+)$, $B \in \mathbf{C}$ and $\lambda \geq 0$. The integral $\int_0^{+\infty} b(t)dt$ is said to be *summable in Cesàro means of order* λ (or (C,λ)-*summable*) *to* $B \in \mathbf{C}$ if

$$\lim_{R\to+\infty} \int_0^R (1 - t/R)^\lambda\, b(t)dt = B.$$

Remark 1 1. If b is integrable, $\int_0^{+\infty} b(t)dt$ is $(C,0)$-summable to the integral of b.

2. If $\int_0^{+\infty} b(t)dt$ is (C,λ)-summable to B, it is (C,λ')-summable to B for all $\lambda' \geq \lambda$ [14, p. 27].

3. The Cesàro means are also called Riesz means.

Proposition 1 *Let* $\lambda \geq 0$, $\alpha > -1$, $m \in \mathbf{N}_0$ *and* $c > 0$. *We write for all* $R > 0$

$$I(R) := \int_0^R (1 - t/R)^\lambda\, t^\alpha (\ln t)^m\, \mathrm{e}^{ict} dt.$$

i) *If* $\lambda > \alpha$, *we have*

$$\lim_{R\to+\infty} I(R) = \mathrm{e}^{(\alpha+1)\pi i/2} \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!}\left(\frac{\pi i}{2}\right)^{m_1} \int_0^{+\infty} t^\alpha (\ln t)^{m_2}\, \mathrm{e}^{-ct} dt.$$

ii) *If* $\lambda = \alpha$, *we have, as* $R \to +\infty$,

$$I(R) \approx \mathrm{e}^{(\alpha+1)\pi i/2} \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!}\left(\frac{\pi i}{2}\right)^{m_1} \int_0^{+\infty} t^\alpha (\ln t)^{m_2}\, \mathrm{e}^{-ct} dt$$
$$+ \mathrm{e}^{-(\alpha+1)\pi i/2} (\ln R)^m\, \mathrm{e}^{icR} \int_0^{+\infty} t^\alpha\, \mathrm{e}^{-ct} dt.$$

Proof Let Log be the principal branch of the logarithm on $\mathbf{C}\backslash]-\infty, 0]$. For any $w \in \mathbf{C}$ and $z \in \mathbf{C}\backslash]-\infty, 0]$ put $z^w := \exp(w \operatorname{Log} z)$. For $R > 0$ fixed, we write

$$f_R(z) := (1 - z/R)^\lambda z^\alpha (\operatorname{Log} z)^m\, \mathrm{e}^{icz};$$

the function f_R is holomorphic on $\mathbf{C} \setminus (]-\infty, 0] \cup [R, +\infty[)$. We choose ρ, τ such that $0 < \rho < 1/\mathrm{e}$ and $R < \tau$; we now consider the rectangle D with corners 0, R, $R + i\tau$, $i\tau$ and indented around 0 and R with quarters of circles of radius ρ. Integrating f_R along D we get by Cauchy

$$\oint f_R(\zeta)d\zeta = 0. \tag{2}$$

But

$$\begin{aligned}
&\int_{[R+i\tau, i\tau]} f_R(\zeta)d\zeta \\
&\quad = -\int_0^R (1-(i\tau+s)/R)^\lambda (i\tau+s)^\alpha (\mathrm{Log}(i\tau+s))^m \mathrm{e}^{ic(i\tau+s)} ds \\
&\quad = -R^{-\lambda}\mathrm{e}^{-c\tau} \int_0^R (R-s-i\tau)^\lambda (i\tau+s)^\alpha (\ln|i\tau+s| + i \operatorname{arccot}(s/\tau))^m \mathrm{e}^{ics} ds
\end{aligned}$$

and, due to the factor $\mathrm{e}^{-c\tau}$, we find

$$\lim_{\tau\to+\infty} \int_{[R+i\tau, i\tau]} f_R(\zeta)d\zeta = 0.$$

Similarly, the integrals of f_R along the quarters of circles around 0 and R which are parts of D converge to zero when ρ tends to zero. Hence, by (2), we have

$$I(R) = \int_{[0,R]} f_R(\zeta)d\zeta = \lim_{\tau\to+\infty} \left(-\int_{[R,R+i\tau]} f_R(\zeta)d\zeta - \int_{[i\tau,0]} f_R(\zeta)d\zeta \right).$$

Let us study these two last integrals. Firstly,

$$\begin{aligned}
&\int_{[R,R+i\tau]} f_R(\zeta)d\zeta \\
&\quad = \int_0^\tau (1-(R+it)/R)^\lambda (R+it)^\alpha (\log(R+it))^m \, \mathrm{e}^{ic(R+it)} i\, dt \\
&\quad = -(-i)^{\lambda+1} R^{\alpha-\lambda} \mathrm{e}^{icR} \int_0^\tau t^\lambda (1+it/R)^\alpha (\ln R + \ln|1+it/R| \\
&\qquad + i\, \mathrm{Arg}(1+it/R))^m \, \mathrm{e}^{-ct} dt.
\end{aligned}$$

Due to the factor e^{-ct}, this integral converges when τ tends to $+\infty$ and

$$\begin{aligned}
&\lim_{\tau\to+\infty} \int_{[R,R+i\tau]} f_R(\zeta)d\zeta \\
&\quad = -(-i)^{\lambda+1} R^{\alpha-\lambda} \mathrm{e}^{icR} \sum_{m_1+m_2+m_3=m} \frac{m!}{m_1!\,m_2!\,m_3!}
\end{aligned}$$

$$\times \int_0^{+\infty} t^\lambda (1+it/R)^\alpha (\ln R)^{m_1} (\ln|1+it/R|)^{m_2} (i\arctan(t/R))^{m_3}\, \mathrm{e}^{-ct}\, dt.$$

When R tends to $+\infty$, these integrals converge to 0 in the case $m_2 > 0$ or $m_3 > 0$ (since, for all $s \geq 0$, $\ln(1+s) \leq s$ and $\arctan(s) \leq s$). Therefore $\lim_{\tau\to+\infty} \int_{[R,R+i\tau]} f_R(\zeta)d\zeta$ behaves, when R tends to $+\infty$, as

$$-(-i)^{\lambda+1} R^{\alpha-\lambda} (\ln R)^m \mathrm{e}^{icR} \int_0^{+\infty} t^\lambda\, \mathrm{e}^{-ct}\, dt.$$

In particular, it converges to zero if $\lambda > \alpha$ and, if $\lambda = \alpha$, it gives the last term in ii). Secondly,

$$\begin{aligned}\int_{[i\tau,0]} f_R(\zeta)d\zeta &= -\int_0^\tau (1-it/R)^\lambda (it)^\alpha (\log(it))^m\, \mathrm{e}^{ic(it)} i\, dt\\ &= -i^{\alpha+1} \int_0^\tau (1-it/R)^\lambda t^\alpha (\ln t + \pi i/2)^m\, \mathrm{e}^{-ct}\, dt.\end{aligned}$$

Due to the factor e^{-ct}, this integral converges when τ tends to $+\infty$ and

$$\begin{aligned}&\lim_{\tau\to+\infty} \int_{[i\tau,0]} f_R(\zeta)d\zeta\\ &= -i^{\alpha+1} \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \left(\frac{\pi i}{2}\right)^{m_1} \int_0^{+\infty} (1-it/R)^\lambda t^\alpha (\ln t)^{m_2}\, \mathrm{e}^{-ct}\, dt;\end{aligned}$$

this converges, when R tends to $+\infty$, to

$$-i^{\alpha+1} \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \left(\frac{\pi i}{2}\right)^{m_1} \int_0^{+\infty} t^\alpha (\ln t)^{m_2}\, \mathrm{e}^{-ct}\, dt.$$

The conclusion follows. □

Remark 2 This proof is inspired by [8, p. 353].

Proposition 2 *Let* $\alpha \leq -1$, $m \in \mathbf{N}_0$, $c > 0$ *and* $R_0 > 0$. *The integral*

$$\int_0^{+\infty} \chi_{]R_0,+\infty[}(t)\, t^\alpha (\ln t)^m\, \mathrm{e}^{ict}\, dt$$

is (C,λ)*-summable for any* $\lambda \geq 0$.

Proof It suffices to prove that this integral is $(C,0)$-summable. An integration by parts gives, for $R > R_0$,

$$\int_{R_0}^R t^\alpha (\ln t)^m\, \mathrm{e}^{ict}\, dt$$

$$= \frac{1}{ic}\left[t^{\alpha}(\ln t)^{m}\,\mathrm{e}^{ict}\right]_{R_0}^{R} - \frac{1}{ic}\int_{R_0}^{R}\left(\alpha(\ln t)^{m} + m(\ln t)^{m-1}\right)t^{\alpha-1}\,\mathrm{e}^{ict}\,dt.$$

When R tends to $+\infty$, the right-hand side converges since $\alpha \le -1$. □

4 Oscillatory Integrals

Proposition 3 *Let $g \in C^{\infty}(\mathbf{R}^n, \mathbf{C})$ a function with compact support and $\varphi \in C^{\infty}(\mathbf{R}^n, \mathbf{R})$ a function which has no critical point on the support of g. Then, as $\tau \to +\infty$,*

$$\int_{\mathbf{R}^n} g(x)\mathrm{e}^{i\tau\varphi(x)}dx = O(\tau^{-p})$$

for any $p \in \mathbf{N}_0$.

Proof This is [13, Proposition 4, p. 341]. □

Proposition 4 *Let $g \in C^{\infty}(\mathbf{R}^n, \mathbf{C})$ a function with compact support and $\varphi \in C^{\infty}(\mathbf{R}^n, \mathbf{R})$. We assume that φ has an isolated, non-degenerate critical point x_0. If the support of g is contained in a sufficiently small neighbourhood of x_0, we have the following asymptotic expansion, for $\tau \to +\infty$,*

$$\int_{\mathbf{R}^n} g(x)\mathrm{e}^{i\tau\varphi(x)}dx \sim \mathrm{e}^{i\tau\varphi(x_0)}\tau^{-n/2}\sum_{j=0}^{+\infty} a_j\tau^{-j}.$$

Proof This is [5, Theorem 1, p. 152]. □

Proposition 5 *Let $g \in C^{\infty}(\mathbf{R}^n, \mathbf{C})$ a function with compact support and $\varphi \in C^{\infty}(\mathbf{R}^n, \mathbf{R})$. We assume that φ has an isolated critical point x_0 and that φ is analytic in a neighbourhood of x_0. If the support of g is contained in a sufficiently small neighbourhood of x_0, we have the following asymptotic expansion, for $\tau \to +\infty$,*

$$\int_{\mathbf{R}^n} g(x)\mathrm{e}^{i\tau\varphi(x)}dx \sim \mathrm{e}^{i\tau\varphi(x_0)}\sum_{\alpha}\sum_{k=0}^{n-1} a_{\alpha,k}\tau^{\alpha}(\ln\tau)^k, \qquad (3)$$

where the parameter α runs through finitely many arithmetic progressions which depend only upon φ and are formed from negative rational numbers. The coefficients $a_{\alpha,k}$ depend upon g and φ.

Proof This is [2, Theorem 6.3, p. 181]. □

In connection with this proposition we introduce some definitions. The *set of indices* of a phase φ analytic around a critical point x_0 is the collection of the numbers α such that, for every neighbourhood V of x_0, there exists $g \in C^{\infty}(\mathbf{R}^n, \mathbf{C})$ with compact support included in V such that in the corresponding asymptotic expansion (3) there is a k in $\{0, 1, \dots, n-1\}$ with $a_{\alpha,k} \neq 0$. The *oscillation index* of a phase φ analytic around a critical point x_0 is the maximum of its set of indices; we write it β. The *singularity index* of a phase φ in n variables analytic around a critical point x_0 is its oscillation index plus $n/2$; we write it γ, i.e. $\gamma := \beta + n/2$.

5 Caustic and Curvature

Let us consider a general oscillatory integral

$$\int_{\mathbf{R}^n} \psi(x) e^{iR\varphi_t(x)} dx$$

where the phase φ is a function of $(t, x) \in \mathbf{R}^m \times \mathbf{R}^n$. We will consider x as a variable and t as a parameter (in particular, all partial derivatives of φ will be with respect to x_j, $j = 1, \dots, n$). The *caustic* K_φ of the phase is the set of parameters t such that the function $x \mapsto \varphi_t(x)$ has at least one degenerate critical point; in other words, writing $H\varphi_t$ the Hessian matrix of φ_t,

$$K_\varphi := \{t \mid \operatorname{grad} \varphi_t(x) = 0 \text{ and } \det H\varphi_t(x) = 0 \text{ for some } x\}.$$

Consider now a compact oriented $(N-1)$-dimensional analytic submanifold Σ of $\mathbf{R}^N$ (where $N \geq 3$), that is, a hypersurface. For simplicity, we first assume that Σ is given by only one analytic parametrisation $f : \Omega \to \mathbf{R}^N$, where $\Omega \subset \mathbf{R}^{N-1}$ is bounded and $f(\Omega) = \Sigma$. We will study the phase $\varphi : (\mathbf{R}^N \setminus \Sigma) \times \Omega \to \mathbf{R}$ defined by

$$(a, u) \mapsto \varphi_a(u) := \|f(u) - a\|,$$

so that $u \in \Omega \subset \mathbf{R}^{N-1}$ is the variable and $a \in \mathbf{R}^N \setminus \Sigma$ the parameter. For this, we recall that the first fundamental form of Σ is the matrix g of functions $g_{ij} : \Omega \to \mathbf{R}$ (where $1 \leq i, j \leq N-1$) defined by

$$g_{ij}(u) := (\partial_i f(u) | \partial_j f(u))$$

and that the second fundamental form of Σ is the matrix h of functions $h_{ij} : \Omega \to \mathbf{R}$ (where $1 \leq i, j \leq N-1$) defined by

$$h_{ij}(u) := (\partial^2_{ij} f(u) | n(u))$$

where $n(u)$ is the unitary normal vector to Σ at the point $f(u)$:

$$n(u) := \partial_1 f(u) \wedge \dots \wedge \partial_{N-1} f(u) / \sqrt{\det g(u)}$$

[3, p. 216]. The principal curvatures of Σ at $f(u)$, written $\kappa_i(u)$ (where $1 \leq i \leq N-1$) are the eigenvalues of the matrix $g^{-1}(u)h(u)$ [3, p. 242]. For every principal curvature κ_i, the set

$$\Gamma(\kappa_i) := \{x \in \mathbf{R}^N \mid \exists u \in \Omega \text{ with } \kappa_i(u) \neq 0 \text{ and } x = f(u) + n(u)/\kappa_i(u)\}$$

is the *focal surface* of Σ associated to the principal curvature κ_i. Finally we write Γ_Σ the union of all focal surfaces of Σ associated to the principal curvatures of Σ.

Proposition 6 *With the above assumptions and notations, the caustic of the phase* $\varphi_a(u) = \|f(u) - a\|$ *is equal to* Γ_Σ.

Proof Let a be a point in the caustic of φ; this means there exists $\bar{u} \in \Omega$ such that $\operatorname{grad}\varphi_a(\bar{u}) = 0$ and $\det H\varphi_a(\bar{u}) = 0$. But

$$\partial_i \varphi_a(u) = \frac{1}{\varphi_a(u)}(\partial_i f(u) | f(u) - a).$$

Hence $\operatorname{grad}\varphi_a(\bar{u}) = 0$ implies that the vector $f(\bar{u}) - a$ is normal to the tangent space to Σ at $f(\bar{u})$ or, equivalently, that there exists $\beta_{\alpha,\bar{u}} \in \mathbf{R} \setminus \{0\}$ with $f(\bar{u}) - a = \beta_{\alpha,\bar{u}} n(\bar{u})$. Moreover

$$\begin{aligned}\partial^2_{ij}\varphi_a(u) &= -\frac{\partial_j \varphi_a(u)}{(\varphi_a(u))^2}(\partial_i f(u) | f(u) - a) \\ &\quad + \frac{1}{\varphi_a(u)}\Big((\partial_i f(u) | \partial_j f(u)) + (\partial^2_{ij} f(u) | f(u) - a)\Big) \\ &= \frac{1}{\varphi_a(u)}\Big(-\partial_j\varphi_a(u)\partial_i\varphi_a(u) + g_{ij}(u) + (\partial^2_{ij} f(u) | f(u) - a)\Big).\end{aligned}$$

Since $\varphi_a(\bar{u}) = \|f(\bar{u}) - a\| = \|\beta_{\alpha,\bar{u}} n(\bar{u})\| = |\beta_{\alpha,\bar{u}}|$, we get

$$\partial^2_{ij}\varphi_a(\bar{u}) = \frac{1}{|\beta_{\alpha,\bar{u}}|}(g_{ij}(\bar{u}) + \beta_{\alpha,\bar{u}} h_{ij}(\bar{u})).$$

By assumption, $\det H\varphi_a(\bar{u}) = 0$ which implies $\det(g(\bar{u}) + \beta_{\alpha,\bar{u}} h(\bar{u})) = 0$ or, since $g(\bar{u})$ is an invertible matrix, $\det(I + \beta_{\alpha,\bar{u}} g^{-1}(\bar{u})h(\bar{u})) = 0$. In other words, $-\beta^{-1}_{\alpha,\bar{u}}$ is an eigenvalue of the matrix $g^{-1}(\bar{u})h(\bar{u})$, that is, there exists $1 \le l \le N - 1$ such that $\beta_{\alpha,\bar{u}} = -1/\kappa_l(\bar{u})$ and therefore

$$f(\bar{u}) - a = -n(\bar{u})/\kappa_l(\bar{u}).$$

We have thus showed that $a \in \Gamma(\kappa_l) \subset \Gamma_\Sigma$.

The other inclusion is easily proved by going backwards in the above calculations. □

In case the hypersurface Σ is given by a finite family of parametrisations $\{(f_i, \Omega_i) \,|\, i = 1, \ldots, i_0\}$, we put $\Gamma_\Sigma := \bigcup_{i=1}^{i_0} \Gamma_{\Sigma_i}$, where $\Sigma_i := f(\Omega_i)$. To each parametrisation (f_i, Ω_i) corresponds the phase $\varphi^i_a : (\mathbf{R}^N \setminus \Sigma_i) \times \Omega_i \to \mathbf{R}$ defined by $(a, u) \mapsto \varphi^i_a(u) := \|f_i(u) - a\|$. We write $K(\Sigma) := \bigcup_{i=1}^{i_0} K_{\varphi_i}$ and call it the *caustic* of the hypersurface Σ. These definitions do not depend upon the choice of the parametrisations (f_i, Ω_i) and we have $\Gamma_\Sigma = K(\Sigma)$.

6 Fourier Inversion of Distributions

Let Σ be a compact oriented $(N - 1)$-dimensional analytic submanifold of $\mathbf{R}^N$ ($N \ge 3$). There exist a finite family $\{(\Omega_i, \phi_i) \,|\, i = 1, \ldots, i_0\}$ of local analytic parametrisations of Σ and a finite partition of unity $\{\theta_i \mid i=1,\dots,i_0\}$ on Σ such that Ω_i is bounded for every $i=1,\dots,i_0$, $\Sigma \subset \bigcup_{i=1}^{i_0} \phi_i(\Omega_i)$ and $\operatorname{supp}\theta_i \subset \phi_i(\Omega_i)$ for every $i=1,\dots,i_0$. We write $W_i := \phi_i(\Omega_i)$ for every $i=1,\dots,i_0$.

Fix l in $\{1,\dots,i_0\}$. Let $g^l(u) = (g^l_{ij}(u))_{1\le i,j\le N-1}$ be the matrix of the first fundamental form of ϕ_l: $g^l_{ij}(u) := (\partial_i\phi_l(u)|\partial_j\phi_l(u))$, and $J\phi_l(u)\,du := \sqrt{\det g^l(u)}\,du$ the measure on Ω_l induced by the parametrisation ϕ_l. Given a function $f:\Sigma\to\mathbf{C}$ with support in W_l, if $(f\circ\phi_l)\cdot J\phi_l$, is integrable on Ω_l we define

$$\int_{W_l} f(y)d\sigma(y) := \int_{\Omega_l} f(\phi_l(u))J\phi_l(u)du.$$

With no assumption on the support of $f:\Sigma\to\mathbf{C}$, suppose that $((f\cdot\theta_i)\circ\phi_i)\cdot J\phi_i$ is integrable on Ω_i for every $i=1,\dots,i_0$; we then say that f is integrable on Σ and define

$$\mu_\Sigma(f) = \int_\Sigma f(y)d\sigma(y) := \sum_{i=1}^{i_0}\int_{W_i} f(y)\theta_i(y)d\sigma(y).$$

In this way we define the *natural measure* μ_Σ on Σ (it is independent of the choice of the parametrisations and the partition of unity); it can be seen as a distribution on $\mathbf{R}^N$ of order 0 and compact support included in Σ.

The Fourier transform of a distribution T on $\mathbf{R}^N$ with compact support is the analytic function $\mathcal{F}T$ defined on $\mathbf{R}^N$ by $\mathcal{F}T(x) := T(y\mapsto \mathrm{e}^{-2\pi i(x|y)})$.

Let $P(D) := \sum_{|\alpha|\le m} c_\alpha D^\alpha$ be a partial differential operator in N variables with constant coefficients of order m. Let also $\psi\in C^\infty(\mathbf{R}^N,\mathbf{R})$. We will study, for a fixed point a in $\mathbf{R}^N\setminus\Sigma$, the behaviour of

$$\int_{\mathbf{R}^N}(1-\|\xi\|/R)_+^\lambda \mathcal{F}[P(D)\psi\mu_\Sigma](\xi)\mathrm{e}^{2\pi i(a|\xi)}d\xi.$$

More precisely, we want to find $\lambda_0\in\mathbf{R}$ such that, for every $\lambda>\lambda_0$, this integral converges when $R\to+\infty$.

We say that *the distance to Σ of the point $a\in\mathbf{R}^N\setminus\Sigma$ has a finite number of critical points* if, for every $i=1,\dots,i_0$, the function $u\mapsto\|\phi_i(u)-a\|$ has only finitely many critical points in Ω_i. From now on, we will suppose that a satisfies this condition.

First step. Let us calculate

$$\int_{\mathbf{R}^N}(1-\|\xi\|/R)_+^\lambda \mathcal{F}[D^\alpha\psi\mu_\Sigma](\xi)\mathrm{e}^{2\pi i(a|\xi)}d\xi$$

$$\int_{\mathbf{R}^N}(1-\|\xi\|/R)_+^\lambda (2\pi i\xi)^\alpha\mathcal{F}[\psi\mu_\Sigma](\xi)\mathrm{e}^{2\pi i(a|\xi)}d\xi$$

$$\int_{\mathbf{R}^N}(1-\|\xi\|/R)_+^\lambda (2\pi i\xi)^\alpha\left(\int_\Sigma \mathrm{e}^{-2\pi i(y|\xi)}\psi(y)d\sigma(y)\right)\mathrm{e}^{2\pi i(a|\xi)}d\xi$$

$$\int_{\mathbf{R}^N}(1-\|\xi\|/R)_+^\lambda (2\pi i\xi)^\alpha\left(\int_\Sigma \mathrm{e}^{-2\pi i(y-a|\xi)}\psi(y)d\sigma(y)\right)d\xi$$

$$\int_0^{+\infty}\left[\int_{S^{N-1}}(1-r/R)_+^{\lambda}(2\pi i r\eta)^{\alpha}\left(\int_{\Sigma}\mathrm{e}^{-2\pi i(y-a|r\eta)}\psi(y)d\sigma(y)\right)d\sigma(\eta)\right]r^{N-1}dr$$
$$\int_0^{+\infty}(1-r/R)_+^{\lambda}r^{N-1}\int_{\Sigma}\left(\int_{S^{N-1}}(2\pi i r\eta)^{\alpha}\mathrm{e}^{-2\pi i(y-a|r\eta)}d\sigma(\eta)\right)\psi(y)d\sigma(y)dr.$$

Define for every multiindex $\alpha\in\mathbf{N}_0^N$ and every $q\in\mathbf{Z}$ a polynomial $P_q^{\alpha}(x)$ in $x\in\mathbf{R}^N$ by $P_q^{\alpha}(x):=0$ for $q<0$ or $q>|\alpha|$, $P_0^0(x):=1$, $P_1^{\epsilon_k}(x):=x^{\epsilon_k}=x_k$, $P_0^{\epsilon_k}(x):=0$ and the recurrence relation $P_q^{\alpha+\epsilon_k}(x)=x_kP_{q-1}^{\alpha}(x)+\partial_kP_q^{\alpha}(x)$, where $1\le k\le N$. Then $\deg P_q^{\alpha}\le q$, $P_{|\alpha|}^{\alpha}(x)=x^{\alpha}$ and

$$\begin{aligned}&\int_{S^{N-1}}(2\pi i r\eta)^{\alpha}\mathrm{e}^{2\pi i(y-a|r\eta)}d\sigma(\eta)\\&\quad=\sum_{q=0}^{|\alpha|}(-1)^q(2\pi)^{q+1}r^{-N/2+q+1}\|x\|^{-N/2+1-q}J_{N/2-1+q}(2\pi r\|x\|)P_q^{\alpha}(x)\end{aligned}$$

[6, proof of Lemma 6, p. 295]. Hence

$$\begin{aligned}&\int_{\mathbf{R}^N}(1-\|\xi\|/R)_+^{\lambda}\mathcal{F}[P(D)\psi\mu_{\Sigma}](\xi)\mathrm{e}^{2\pi i(a|\xi)}d\xi\\&\quad=\sum_{|\alpha|\le m}c_{\alpha}\int_{\mathbf{R}^N}(1-\|\xi\|/R)_+^{\lambda}\mathcal{F}[D^{\alpha}\psi\mu_{\Sigma}](\xi)\mathrm{e}^{2\pi i(a|\xi)}d\xi\\&\quad=\sum_{|\alpha|\le m}c_{\alpha}\int_0^{+\infty}(1-r/R)_+^{\lambda}r^{N-1}\left[\int_{\Sigma}\sum_{q=0}^{|\alpha|}(-1)^q(2\pi)^{q+1}r^{-N/2+q+1}\right.\\&\qquad\left.\times\|y-a\|^{-N/2+1-q}J_{N/2-1+q}(2\pi r\|y-a\|)P_q^{\alpha}(a-y)\psi(y)d\sigma(y)\right]dr\\&\quad=\sum_{q=0}^{m}\int_0^{+\infty}(1-r/R)_+^{\lambda}r^{N/2+q}\left[\int_{\Sigma}\|y-a\|^{-N/2+1-q}J_{N/2-1+q}(2\pi r\|y-a\|)\right.\\&\qquad\left.\times Q_q^{\alpha}(y-a)\psi(y)d\sigma(y)\right]dr,\end{aligned}$$

where $Q_q(x):=(-1)^q(2\pi)^{q+1}\sum_{q\le|\alpha|\le m}c_{\alpha}P_q^{\alpha}(-x)$. We use now the partition of unity $\{\theta_i\mid i=1,\dots,i_0\}$ on Σ to define, for all $i=1,\dots,i_0$,

$$\begin{aligned}I_i^q(r):=r^{N/2+q}\int_{W_i}&\|y-a\|^{-N/2+1-q}J_{N/2-1+q}(2\pi r\|y-a\|)\\&\times Q_q^{\alpha}(y-a)\psi(y)\theta_i(y)d\sigma(y).\end{aligned}\tag{4}$$

Therefore

$$\int_{\mathbf{R}^N}(1-\|\xi\|/R)_+^\lambda \mathcal{F}[P(D)\psi\mu_\Sigma](\xi)\mathrm{e}^{2\pi i(a|\xi)}d\xi=\sum_{q=0}^{m}\sum_{i=0}^{i_0}\int_0^{+\infty}(1-r/R)_+^\lambda I_i^q(r)dr$$

and it will suffice to study the summability of $\int_0^{+\infty} I_i^q(r)dr$ for all $0\le q\le m$ and $0\le i\le i_0$.

Take $R_0>1$ and write

$$\int_0^{+\infty}(1-r/R)_+^\lambda I_i^q(r)dr$$
$$=\int_0^{+\infty}(1-r/R)_+^\lambda \chi_{[0,R_0]}(r)I_i^q(r)dr+\int_0^{+\infty}(1-r/R)_+^\lambda \chi_{[R_0,+\infty]}(r)I_i^q(r)dr.$$

For $\lambda=0$, the first integral of the right-hand side is equal to

$$\int_0^{+\infty}\chi_{[0,R_0]}(r)I_i^q(r)dr=\int_0^{R_0}I_i^q(r)dr$$

and so is (C,λ)-summable for every $\lambda\ge 0$.

Second step. Let d_a be the distance from a to Σ, so that

$$r\|y-a\|\ge R_0 d_a>0$$

for all $r\ge R_0$ and $y\in\Sigma$. Let $l_0:=\lfloor (N+3)/2\rfloor+m$, so that, for all $0\le q\le m$, $(N-1)/2+q-l_0\le -3/2$. Using (4) and the asymptotic expansion of the Bessel functions (1), we see that $I_i^q(r)$ is equal to

$$\Re\mathrm{e}\Bigg(\sum_{l=0}^{l_0-1}c_l^q r^{(N-1)/2+q-l}\int_{W_i}\|y-a\|^{-(N-1)/2-q-l}$$
$$\times Q_q(y-a)\psi(y)\theta_i(y)\mathrm{e}^{2\pi i r\|y-a\|}d\sigma(y)$$
$$+r^{N/2+q}\int_{W_i}\|y-a\|^{-N/2+1-q}\Upsilon_{l_0}^q(r\|y-a\|)Q_q(y-a)\psi(y)\theta_i(y)d\sigma(y)\Bigg);$$

moreover this last term can be bounded above as follows:

$$\left|r^{N/2+q}\int_{W_i}\|y-a\|^{-N/2+1-q}\Upsilon_{l_0}^q(r\|y-a\|)Q_q(y-a)\psi(y)\theta_i(y)d\sigma(y)\right|$$
$$\le C_{l_0}^m r^{N/2+q}\int_{W_i}\|y-a\|^{-N/2+1-q}(r\|y-a\|)^{-l_0-1/2}$$
$$\times|Q_q(y-a)\psi(y)|\theta_i(y)d\sigma(y)$$
$$\le C_{l_0}^m r^{-3/2}\int_{W_i}\|y-a\|^{-(N-1)/2-q-l_0}|Q_q(y-a)\psi(y)|\theta_i(y)d\sigma(y),$$

and is therefore integrable on $[R_0, +\infty[$. Hence it will suffice to study the summability, for all $0 \le q \le m$, $0 \le i \le i_0$ and $0 \le l \le l_0 - 1$, of

$$I_i^{ql}(r) := r^{(N-1)/2+q-l} \int_{W_i} \|y-a\|^{-(N-1)/2-q-l} \times Q_q(y-a)\psi(y)\theta_i(y)\mathrm{e}^{2\pi i r\|y-a\|} d\sigma(y).$$

Third step. Let K_i be the support of $\theta_i \circ \phi_i$; this is a compact subset of $\mathbf{R}^{N-1}$ contained in Ω_i. Hence there exists a bounded open set V_i such that

$$K_i \subset V_i \subset \overline{V_i} \subset \Omega_i.$$

We can then find $\omega_i \in C^\infty(\mathbf{R}^{N-1}, \mathbf{R})$ with $0 \le \omega_i \le 1$, $\omega_i = 1$ on V_i and $\omega_i = 0$ on $\mathbf{R}^{N-1} \setminus \Omega_i$. We now define $\varphi \in C^\infty(\mathbf{R}^{N-1}, \mathbf{R})$ by $\varphi_i(u) := 0$ if $u \in \mathbf{R}^{N-1} \setminus \Omega_i$ and $\varphi_i(u) := 2\pi\|\phi_i(u) - a\|\omega_i(u)$ if $u \in \Omega_i$. Clearly $\varphi_i(u) = 2\pi\|\phi_i(u) - a\|$ on V_i and so φ_i is analytic on V_i, since the parametrisation ϕ_i is analytic by hypothesis. Finally we define, for every $l = 0, \dots, l_0 - 1$, $g_i^{ql} \in C^\infty(\mathbf{R}^{N-1}, \mathbf{C})$ by

$$g_i^{ql}(u) := c_l^q \|\phi_i(u) - a\|^{-(N-1)/2-q-l} Q_q(\phi_i(u) - a)\psi(\phi_i(u))\theta_i(\phi_i(u)) J\phi_i(u)$$

if $u \in \Omega_i$ and $g_i^{ql}(u) := 0$ if $u \in \mathbf{R}^{N-1} \setminus \Omega_i$; clearly $\operatorname{supp} g_i^{ql} \subset K_i$. We are thus able to write I_i^{ql} as an oscillatory integral:

$$I_i^{ql}(r) = r^{(N-1)/2+q-l} \int_{\mathbf{R}^{N-1}} g_i^{ql}(u)\mathrm{e}^{ir\varphi_i(u)} du.$$

Fourth step. Let u_{ij} with $j = 1, \dots, j_i$ be the critical points of ϕ_i which are in K_i (we let $j_i := 0$ if φ_i has no critical point in K_i). We want to 'isolate' these critical points. For that we write

$$d_1 := \min_{1 \le i \le i_0} \min\{d(u_{ij}, u_{ik}), d(u_{ij}, \partial V_j) \mid 1 \le j \ne k \le j_i\}.$$

For all $1 \le i \le i_0$, $1 \le j \le j_i$, $0 \le q \le m$ and $0 \le l \le l_0 - 1$, let $\eta_{ij}^{ql} > 0$ be such that $\overline{B(u_{ij}, \eta_{ij}^{ql})}$ is contained in a neighbourhood of u_{ij} which satisfies the assumptions of the Proposition 5 with the phase φ_i. Put

$$d_2 := \min\{\eta_{ij}^{ql} \mid 1 \le i \le i_0, 1 \le j \le j_i, 0 \le q \le m, 0 \le l \le l_0 - 1\}.$$

We choose $d_0 \in \mathbf{R}$ with $0 < d_0 \le \min\{d_1, d_2\}$. For all $1 \le i \le i_0$ and $1 \le j \le j_i$ there exist functions $\omega_{ij} \in C^\infty(\mathbf{R}^{N-1}, \mathbf{R})$ with $0 \le \omega_{ij} \le 1$, $\omega_{ij} = 1$ on $\overline{B(u_{ij}, d_0/2)}$ and $\omega_{ij} = 0$ on $\mathbf{R}^{N-1} \setminus B(u_{ij}, d_0)$; in particular $\operatorname{supp} \omega_{ij} \subset V_i$. We also define $\omega_{i0} := 1 - \sum_{j=1}^{j_i} \omega_{ij}$.

For all $1 \le i \le i_0$, $0 \le j \le j_i$, $0 \le q \le m$ and $0 \le l \le l_0 - 1$, let

$$g_{ij}^{ql} := g_i^{ql} \cdot \omega_{ij};$$

we have $g_{ij}^{ql} \in C^\infty(\mathbf{R}^{N-1}, \mathbf{C})$ and $\operatorname{supp} g_{ij}^{ql} \subset V_i$. (Note that in case $j_i = 0$ there are only the functions $\omega_{i0} = 1$ and $g_{i0}^{ql} = g_i^{ql}$.) Finally we let

$$I_{ij}^{ql}(r) := r^{(N-1)/2+q-l} \int_{\mathbf{R}^{N-1}} g_{ij}^{ql}(u) \mathrm{e}^{ir\varphi_i(u)} du,$$

so that $I_i^{ql}(r) = \sum_{j=0}^{j_i} I_{ij}^{ql}(r)$.

Fifth step. We first consider the oscillatory integral $I_{i0}^{ql}(r)$. Here no critical point of the phase φ is included in the support of the amplitude g_{i0}^{ql}. Hence, by the proposition 3, $I_{i0}^{ql}(r) = O(r^{-3/2})$ as $r \to +\infty$, which implies that $\int_0^{+\infty} \chi_{[R_0,+\infty]}(r) I_{i0}^{ql}(r) dr$ is $(C,0)$-summable.

Next, we consider the oscillatory integral $I_{ij}^{ql}(r)$ where $1 \le j \le j_i$. In this case the phase φ has exactly one critical point, u_{ij}, in the support of the amplitude g_{ij}^{ql} and, by construction, we can apply the Proposition 5:

$$I_{ij}^{ql}(r) \sim \mathrm{e}^{ir\varphi_i(u_{ij})} \sum_{p \in \mathcal{A}_{ij}} \sum_{k=0}^{N-2} a_{p,k} r^{(N-1)/2+q-l+p} (\ln r)^k, \tag{5}$$

where $\mathcal{A}_{ij}$ is the union of arithmetic progressions formed by strictly negative rational numbers only depending on φ_i (and u_{ij}); we write

$$\beta_{ij} := \max \mathcal{A}_{ij} \tag{6}$$

the oscillation index of φ_i at u_{ij}. Choosing

$$p_0 = p_0(m) := -(N+3)/2 + m,$$

we can write

$$I_{ij}^{ql}(r) = \mathrm{e}^{ir\varphi_i(u_{ij})} \sum_{p \in \mathcal{A}_{ij},\, p \ge p_0} \sum_{k=0}^{N-2} a_{p,k} r^{(N-1)/2+q-l+p} (\ln r)^k + \Upsilon_{ij}^{qlp_0}(r),$$

where

$$\Upsilon_{ij}^{qlp_0}(r) = o(r^{(N-1)/2+q-l+p_0} (\ln r)^{N-2}) = o(r^{-3/2})$$

as $r \to +\infty$; in particular $\int_0^{+\infty} \chi_{[R_0,+\infty]}(r) \Upsilon_{ij}^{qlp_0}(r) dr$ is $(C,0)$-summable. Finally we define

$$\sigma(a, \Sigma) := \max_{1 \le i \le i_0} \max_{1 \le j \le j_i} \beta_{ij} \tag{7}$$

the maximum of the oscillation indices for Σ and the chosen point a (this is a strictly negative rational number) and

$$\lambda_0 = \lambda_0(a, \Sigma, m) := (N-1)/2 + m + \sigma(a, \Sigma).$$

For all $1 \le i \le i_0$, $1 \le j \le j_i$, $0 \le q \le m$, $0 \le l \le l_0 - 1$, $0 \le k \le N-2$ and $p \in \mathcal{A}_{ij}$ with $p \ge p_0$ we have

$$(N-1)/2 + q - l + p \le (N-1)/2 + m - 0 + \beta_{ij} \le (N-1)/2 + m + \sigma(a, \Sigma) \le \lambda_0.$$

Since moreover $\varphi_i(u_{ij}) = 2\pi \|\phi_i(u_{ij}) - a\| \ge 2\pi d_a > 0$, the Propositions 1 and 2 imply that

$$\int_0^{+\infty} \chi_{[R_0,+\infty]}(r) r^{(N-1)/2+q-l+p} (\ln r)^k e^{ir\varphi_i(u_{ij})} dr$$

is (C, λ)-summable for every $\lambda \ge 0$ with $\lambda > \lambda_0$.

We can now state our result.

Theorem 1 *Let Σ be a compact oriented $(N-1)$-dimensional analytic submanifold of $\mathbf{R}^N$ $(N \ge 3)$, $a \in \mathbf{R}^N \setminus \Sigma$ such that its distance to Σ has only a finite number of critical points, $P(D)$ a partial differential operator with constant coefficients of order m and $\psi \in C^\infty(\mathbf{R}^N, \mathbf{R})$. The Fourier integral of $P(D)\psi\mu_\Sigma$ at a is summable in Cesàro means of order λ to zero, that is,*

$$\lim_{R \to +\infty} \int_{\mathbf{R}^N} (1 - \|\xi\|/R)_+^\lambda \mathcal{F}[P(D)\psi\mu_\Sigma](\xi) e^{2\pi i (a|\xi)} d\xi = 0, \quad (8)$$

for all $\lambda \ge 0$ with $\lambda > \lambda_0$, where $\lambda_0 = (N-1)/2 + m + \sigma(a, \Sigma)$ and $\sigma(a, \Sigma)$ is the maximum of the oscillation indices for a and Σ given by (6) and (7).

Proof The previous calculations show that the above limit exists for all $\lambda \ge 0$ with $\lambda > \lambda_0$. The fact that this limit is equal to zero follows from [6, Proposition 1, p. 293]. □

Remark 3 Since $\sigma(a, \Sigma) < 0$, this result improves the one just cited for the distributions considered here.

Remark 4 The distance to Σ of any point $a \in \mathbf{R}^N \setminus \Sigma$ has at least two critical points; it suffices to consider the points on Σ which are farthest away from a or nearest to a.

Remark 5 If $a \in \mathbf{R}^N \setminus \Sigma$ is not on the caustic of Σ, then $\lambda_0 = \lambda_0(a, \Sigma, m) = m$. Indeed, in this case all critical points are non degenerate (see Sect. 5) and therefore, by Proposition 4, $\beta_{ij} = -(N-1)/2$ for all $1 \le i \le i_0$ and $1 \le j \le j_i$.

Remark 6 Due to the rather unprecise nature of Proposition 5, it is in general not possible to know whether our estimate $\lambda > \lambda_0$ is sharp, that is, to know whether, given Σ and $a \in \mathbf{R}^N \setminus \Sigma$, there exists a distribution $P(D)\psi\mu_\Sigma$ such that (8) does not hold if $0 \le \lambda \le \lambda_0$. However, in the case $N = 3$ a refinement of Proposition 5 makes it possible to get an example where our estimate is indeed sharp (see next section).

Remark 7 We have used the Cesàro summation for integrals because its consistency (Remark 1.2) is well documented, whereas the consistency of the more usual

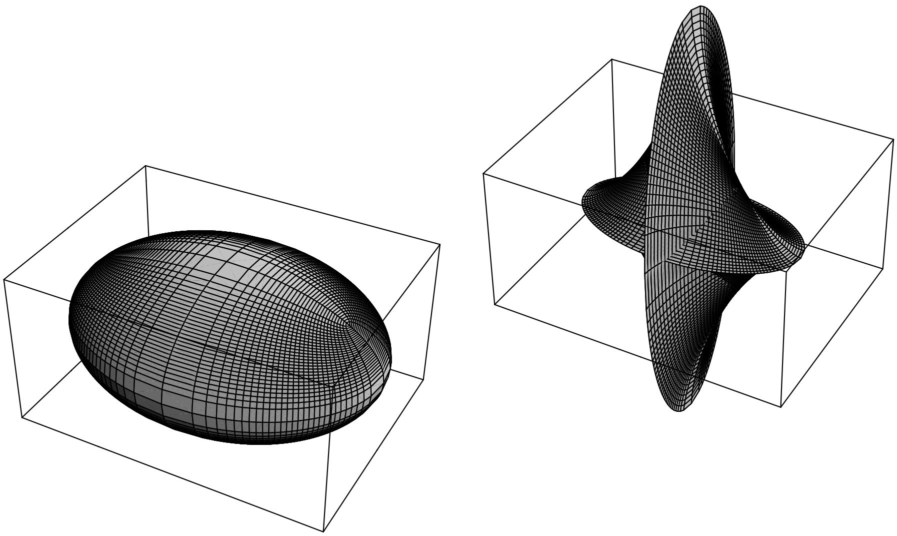

Fig. 1 The ellipsoid and its caustic

(in multidimensional Fourier analysis) Bochner-Riesz summation for integrals (with $(1-\|\xi\|^2/R^2)_+^\lambda$ instead of $(1-\|\xi\|/R)_+^\lambda$) does not seem to have been much studied. From our experience with both summation methods, we think that the theorem could be proved for the Bochner-Riesz summation as well; but this would mean establishing the corresponding results in Sect. 3: we leave it to the interested reader (for the proof of the consistency follow [4, pp. 206–208]).

7 An example

Let $t_1 > t_2 > t_3 > 0$. Consider the ellipsoid Σ in $\mathbf{R}^3$ given by the equation

$$\frac{x^2}{t_1} + \frac{y^2}{t_2} + \frac{z^2}{t_3} = 1.$$

The caustic $K(\Sigma)$ of Σ is formed by two focal surfaces. Each focal surface is itself formed by two symmetric sheets joined at their border and this border is a closed planar curve which is an edge of the focal surface. Each sheet has in its middle an edge which is a planar arc; these four arcs together form another closed curve; and the four points at which they connect are the focal points of the umbilics of the ellipsoid, that is, of the points on Σ where the principal curvatures coincide (see figures). In particular, the distance to Σ of any $a \in \mathbf{R}^3 \setminus \Sigma$ has only a finite number of critical points.

The singularity index associated to $a \in \mathbf{R}^3 \setminus \Sigma$ has been calculated in [2, p. 185], [1, pp. 217–218] and we find:

1. If $a \notin K(\Sigma)$, $\lambda_0 = m$ as seen in Remark 5.

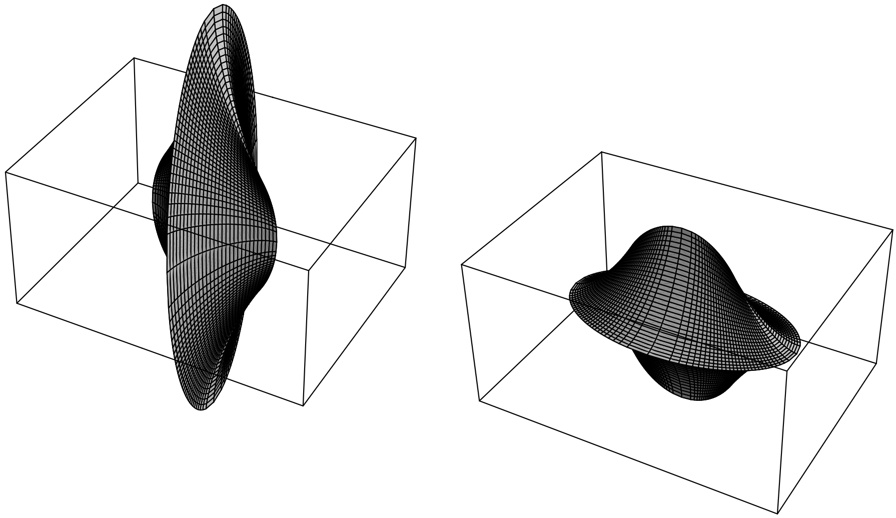

Fig. 2 The two focal surfaces

2. If $a \in K(\Sigma)$ is the focal point of an umbilic, the singularity is of type D_4^+ (index $\gamma = 1/3$) and $\lambda_0 = 1/3 + m$.
3. If $a \in K(\Sigma)$ is on one of the three edges of the caustic but not the focal point of an umbilic, the singularity is of type A_3 (index $\gamma = 1/4$) and $\lambda_0 = 1/4 + m$.
4. If $a \in K(\Sigma)$ is out of the three edges of the caustic, the singularity is of type A_2 (index $\gamma = 1/6$) and $\lambda_0 = 1/6 + m$.

Of course, (8) may be true for $0 \le \lambda \le \lambda_0$. For example, take $m \in \mathbf{N}_0$ arbitrary and choose $P(D) = \partial^m/\partial x_3^m$ and $\psi = 1$ if m is odd, or $\psi(x) = x_3$ if m is even. A direct calculation using the symmetries of the ellipsoid gives

$$\int_{\mathbf{R}^N} \mathcal{F}[P(D)\psi\mu_\Sigma](\xi) \mathrm{e}^{2\pi i(a|\xi)} d\xi = 0$$

for all $a \in \mathbf{R}^3 \setminus \Sigma$ with $a_3 = 0$: at such points (8) is true for every $\lambda \ge 0$.

By contrast, there are cases when (8) is true if and only if $\lambda > \lambda_0$ and $\lambda \ge 0$. For that take $\psi = 1$ and $P(D)$ homogeneous elliptic and even (i.e. $P(D) = \sum_{|\alpha|=m} c_\alpha D^\alpha$ is such that the associated polynomial $\sum_{|\alpha|=m} c_\alpha x^\alpha$ has real coefficients, is zero only at $x = 0$ and is even with respect to x_1, x_2 and x_3). Now we use the following refinement of Proposition 5 which is valid in the case $n = 2$ [15, Theorem 0.6, p. 177]: there exists in the asymptotic expansion (3) a non zero coefficient $a_{\beta,k}$ where β is the oscillation index of φ around x_0 (and $k = 0$ or 1). To conclude that $\lambda > \lambda_0$ is a necessary condition, it suffices then, by the Proposition 1(ii) to show that, given $a \in \mathbf{R}^3 \setminus \Sigma$, the oscillatory integrals around the different critical points u_{ij} $(1 \le i \le i_0,\ 1 \le j \le j_i)$ with the same and highest oscillation index β_{ij} cannot cancel each other out (see (5)). Such cancellation can only occur between oscillatory integrals which have the same $\varphi_i(u_{ij})$ (same distance from a to the point on Σ corresponding to u_{ij}). Hence we must consider four cases.

1. Suppose $a \notin K(\Sigma)$. All critical points correspond to the feet on Σ of the normals to Σ through a. Cancellation may occur if there is no foot of such a normal whose distance from a is different to the distances from a of the feet of the other normals. By the symmetries of Σ, this is possible only if $a = 0$. But then the contributions of each pair of critical points add up, due to the symmetries of Σ and $P(D)$.
2. Suppose $a \in K(\Sigma)$ is the focal point of an umbilic. This gives the unique critical point corresponding to an oscillatory integral with oscillation index equal to $-2/3$ (i.e. singularity index $\gamma = 1/3$). Hence no cancellation is here possible.
3. Suppose $a \in K(\Sigma)$ is on one of the edges of the caustic. A reasoning analogous to the preceding applies.
4. Suppose $a \in K(\Sigma)$ is out of the edges of the caustic. Then there are at most two critical points corresponding to an oscillatory integral with oscillation index equal to $-5/6$ (i.e. singularity index $\gamma = 1/6$): this is the case when a is at the intersection of the two focal surfaces (see figures). Assume then that v and w are the feet of the normals to Σ through a that give these two critical points. Consider the plane containing a, v and w; its intersection with Σ is an ellipse and the line through a and v (respectively through a and w) is a normal to this ellipse. But it is not difficult (albeit tedious) to verify that, in this configuration, $d(a, v) = d(a, w)$ if and only if a is on one of the axes of the ellipse and v, w are symmetric with respect to this axis, which is not possible here.

We remark that in the three last cases the assumption that $P(D)$ be even with respect to x_1, x_2 and x_3 is not needed.

Acknowledgements The second author was partially supported by the Swiss National Science Foundation. We would like to thank the referee for the suggested improvements.

References

1. Arnol'd, V.I.: Catastrophe theory. In: Dynamical Systems. Encyclopaedia of Mathematical Sciences, vol. V, pp. 207–271. Springer, Berlin (1994)
2. Arnol'd, V.I., Guseĭn-Zade, S.M., Varchenko, A.N.: Singularities of Differentiable Maps. Vol. II (Monodromy and Asymptotics of Integrals). Birkhäuser, Boston (1988)
3. Brauner, H.: Differentialgeometrie. Vieweg, Braunschweig (1981)
4. Chapman, S., Hardy, G.H.: A general view of the theory of summable series. Q. J. Math. **42**, 181–215 (1911)
5. Fedoryuk, M.V.: The stationary phase method for multidimensional integrals. U.S.S.R. Comput. Math. Math. Phys. **2**, 152–157 (1962)
6. González Vieli, F.J.: Inversion de Fourier ponctuelle des distributions à support compact. Arch. Math. **75**, 290–298 (2000)
7. González Vieli, F.J.: Fourier inversion of distributions with support on a plane curve. Integral Transform. Spec. Funct. **13**, 93–100 (2002)
8. Hardy, G.H.: Divergent Series. Clarendon, Oxford (1949)
9. Hewitt, E.: Remarks on the inversion of Fourier-Stieltjes transforms. Ann. Math. (2) **57**, 458–474 (1953)
10. Pinsky, M.A.: Fourier inversion for piecewise smooth functions in several variables. Proc. Am. Math. Soc. **118**, 903–910 (1993)
11. Pinsky, M.A., Taylor, M.E.: Pointwise Fourier inversion: a wave equation approach. J. Fourier Anal. Appl. **3**, 647–703 (1997)
12. Popov, D.A.: Spherical convergence of the Fourier integral of the indicator function of an N-dimensional domain. Sb. Math. **189**, 1101–1113 (1998)

13. Stein, E.M.: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton (1993)
14. Titchmarsh, E.C.: Introduction to the Theory of Fourier Integrals. Clarendon, Oxford (1948)
15. Varchenko, A.N.: Newton polyhedra and estimation of oscillating integrals. Funct. Anal. Appl. **10**, 175–196 (1976)
16. Watson, G.N.: A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge (1944)