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Maximally Equiangular Frames and Gauss Sums

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Abstract In a finite-dimensional complex Euclidean space, a maximally equiangular frame is a tight frame which has a number of elements equal to the square of the dimension of the space, and in which the inner products of distinct elements are of constant magnitude. Though the general question of their existence remains open, many examples of maximally equiangular frames have been constructed as finite Gabor systems. These constructions involve number theory, specifically Schaar's identity, which provides a reciprocity formula for quadratic Gauss sums. To be precise, Zauner used Schaar's identity to compute the spectrum of a chirp-Fourier operator, the eigenvectors of which he conjectured to be well-suited for the construction of maximally equiangular Gabor frames. We provide two new characterizations of such frames, both of which further confirm the relevance of the theory of Gauss sums to this area of frame theory. We also show how the unique time-frequency properties of a particular cyclic chirp function may be exploited to provide a new, short and elementary proof of Schaar's identity.

Keywords Frames · Equiangular · SIC-POVM · Gauss sums

Mathematics Subject Classification (2000) 42C15 · 11L05

1 Introduction

For any $a \in \mathbb{Z}^+$, let $\mathbb{Z}_a = \mathbb{Z}/a\mathbb{Z}$ be the finite group of integers modulo a, and let $\ell(\mathbb{Z}_a)$ be the a-dimensional Hilbert space of a-periodic complex-valued functions

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over the integers:

$$\ell(\mathbb{Z}_a) = \{f : \mathbb{Z} \to \mathbb{C} \mid f(n+a) = f(n),\ \forall n \in \mathbb{Z}\}.$$

A collection of functions $\{f_k\}_{k=1}^{a^2} \subseteq \ell(\mathbb{Z}_a)$ is a maximally equiangular frame (MEF) for $\ell(\mathbb{Z}_a)$ if

$$|\langle f_j, f_k\rangle|^2 = \begin{cases} 1, & j = k, \\ \frac{1}{a+1}, & j \neq k. \end{cases}$$

Such frames are maximal in two respects [15]: if, for some $\alpha \geq 0$, a collection of unit vectors $\{f_k\}_{k=1}^{K} \subseteq \ell(\mathbb{Z}_a)$ satisfies the equiangularity condition $|\langle f_j, f_k\rangle|^2 = \alpha$ for all $j \neq k$, then one necessarily has that (i) $K \leq a^2$, and, when $K = a^2$, also necessarily has that (ii) $\alpha \geq 1/(a+1)$. As such, MEFs, should they exist, would pack the largest possible collection of equiangular points on a sphere in such a way that the angle between distinct points is also maximized. As noted in [15], any MEF is also necessarily a tight frame for $\ell(\mathbb{Z}_a)$, that is,

$$a\|f\|^2 = \sum_{m=1}^{a^2} |\langle f, f_m\rangle|^2$$

for all $f \in \ell(\mathbb{Z}_a)$. Conversely, one may also show that any equiangular tight frame of a^2 unit vectors is necessarily maximal, that is, in this case, the equiangularity constant α is necessarily $\alpha = 1/(a+1)$.

Despite their importance to information theory, both quantum [12] and traditional [15], it is an open question whether MEFs exist for all but the smallest $a \in \mathbb{Z}^+$ [1, 10]. In the physics literature [16], MEFs are usually referred to as symmetric informationally complete positive operator valued measures (SIC-POVMs), where the theory has some overlap with problems of mutually unbiased bases (MUBs). There is some numerical evidence that MEFs may exist for all a; to our knowledge, for no a has their existence been proven impossible.

Gabor theory provides a promising method for constructing MEFs. To be precise, for any $m \in \mathbb{Z}$, consider the corresponding translation and modulation operators: $\mathrm{T}^m, \mathrm{M}^m : \ell(\mathbb{Z}_a) \to \ell(\mathbb{Z}_a)$,

$$(\mathrm{T}^m f)(n) = f(n-m), \qquad (\mathrm{M}^m f)(n) = \mathrm{e}^{2\pi \mathrm{i} mn/N} f(n),$$

respectively. One attempts to find a single vector $f \in \ell(\mathbb{Z}_a)$ that generates an MEF under the action of the Weyl-Heisenberg group, that is, such that $\{\mathrm{M}^m \mathrm{T}^n f\}_{m,n\in\mathbb{Z}_a}$ is a maximally equiangular frame. In this case, we say f generates a Gabor maximally equiangular frame (GMEF). Here and throughout, the indexing "$n \in \mathbb{Z}_a$" denotes any collection of coset representatives of $\mathbb{Z}$ over $a\mathbb{Z}$. In the next section, we prove the following two characterizations of those $f \in \ell(\mathbb{Z}_a)$ which generate GMEFs, both of which involve the same "double autocorrelation" of f:

Theorem 1 *For any $a \in \mathbb{Z}^+$ and any $f \in \ell(\mathbb{Z}_a)$, the Gabor system $\{\mathrm{M}^m\mathrm{T}^n f\}_{m,n\in\mathbb{Z}_a}$ is a maximally equiangular frame for $\ell(\mathbb{Z}_a)$ if and only if*

$$\sum_{k\in\mathbb{Z}_a} f(k)\overline{f(k-m)f(k-n)}f(k-m-n) = \frac{1}{a+1}[\delta_0(m)+\delta_0(n)] \tag{1.1}$$

for all $m, n \in \mathbb{Z}_a$.

Theorem 2 *For any $a \in \mathbb{Z}^+$*

$$\sum_{m,n\in\mathbb{Z}_a}\left|\sum_{p\in\mathbb{Z}_a} f(p)\overline{f(p-n)f(p-m)}f(p-m-n)\right|^2 = \frac{1}{a}\sum_{m,n\in\mathbb{Z}_a}|\langle f, \mathrm{M}^m\mathrm{T}^n f\rangle|^4,$$

and moreover,

$$\sum_{m,n\in\mathbb{Z}_a}\left|\sum_{p\in\mathbb{Z}_a} f(p)\overline{f(p-n)f(p-m)}f(p-m-n)\right|^2 \geq \frac{2}{a+1},$$

with equality if and only if $\{\mathrm{M}^m\mathrm{T}^n f\}_{m,n\in\mathbb{Z}_a}$ is a maximally equiangular frame for $\ell(\mathbb{Z}_a)$.

We note that Theorem 1 has recently been independently discovered by other researchers [2, 11]. As explained in the next section, Theorems 1 and 2 imply any search for GMEF generators must necessarily consider the role of chirps, that is, unimodular functions of nonconstant frequency. Indeed, we shall see that of all functions of the form $\mathrm{e}^{\mathrm{i}\theta(n)}$, Theorems 1 and 2 assign a special significance to the case where θ is quadratic. For nonobvious reasons, the canonical chirp of this form is better defined as $\mathrm{e}^{\pi\mathrm{i}n^2/a}$ as opposed to $\mathrm{e}^{2\pi\mathrm{i}n^2/a}$ [7, 8]. However, we note the function $\mathrm{e}^{\pi\mathrm{i}n^2/a}$ is only a-periodic when a is even. As we wish our chirp to be well-defined in $\ell(\mathbb{Z}_a)$, this had led some to instead use $\mathrm{e}^{\pi\mathrm{i}n(n+1)/a}$ when a is odd. A more elegant solution is to consider a single definition that is equally valid for a both even and odd: for any $a \in \mathbb{Z}^+$, consider $\mathrm{c}_a \in \ell(\mathbb{Z}_a)$,

$$\mathrm{c}_a(n) = \mathrm{e}^{\pi\mathrm{i}(a+1)n^2/a} = (-1)^n\mathrm{e}^{\pi\mathrm{i}n^2/a}. \tag{1.2}$$

Some properties and advantages of definition (1.2) are discussed in [7], in which it is equivalently, but alternatively, defined as $\mathrm{e}^{\pi\mathrm{i}n(n-a)/a}$. Meanwhile in [8], definition (1.2) is derived from Weil's abstract theory of second degree characters. Other results and applications concerning finite chirps are given in [5, 17]. As shown in the next section, modulation by the cyclic chirp (1.2) indeed preserves the set of GMEF generators. That is, letting $\mathrm{C}: \ell(\mathbb{Z}_a) \to \ell(\mathbb{Z}_a)$, $(\mathrm{C}f)(n) = \mathrm{c}_a(n)f(n)$ denote chirp-modulation, we have that if $\{\mathrm{M}^m\mathrm{T}^n f\}_{m,n\in\mathbb{Z}_a}$ is an MEF, then $\{\mathrm{M}^m\mathrm{T}^n\mathrm{C}f\}_{m,n\in\mathbb{Z}_a}$ is also an MEF.

More striking is the fact that many of the known examples of MEF generators exhibit a chirp-Fourier symmetry [1, 10], that is, satisfy $\mathrm{F}f = \alpha\mathrm{C}f$ where $|\alpha| = 1$

and $\mathrm{F} : \ell(\mathbb{Z}_a) \to \ell(\mathbb{Z}_a)$ is the discrete Fourier transform:

$$(\mathrm{F}f)(n) = \frac{1}{\sqrt{a}} \sum_{m \in \mathbb{Z}_a} f(m) \mathrm{e}^{-2\pi \mathrm{i} mn/a}.$$

Such f are therefore eigenvectors of the chirp-Fourier operator $\mathrm{C}^{-1}\mathrm{F}$, whose cube is a scalar multiple of the identity. In particular, $(\mathrm{C}^{-1}\mathrm{F})^3 = \beta \mathrm{I}$ where β is the Gauss sum:

$$\beta = \frac{1}{\sqrt{a}} \sum_{n \in \mathbb{Z}_a} (-1)^n \mathrm{e}^{-\pi \mathrm{i} n^2/a}. \tag{1.3}$$

As explained below, a classical result of number theory gives the explicit value of (1.3) to be $\beta = \mathrm{e}^{\pi \mathrm{i}(a-1)/4}$. Absorbing β into the cube of the relation $(\mathrm{C}^{-1}\mathrm{F})^3 = \beta \mathrm{I}$ yields the chirp-Fourier operator $\mathrm{C}h\mathrm{F} : \ell(\mathbb{Z}_a) \to \ell(\mathbb{Z}_a)$,

$$(\mathrm{C}h\mathrm{F}f)(n) = \frac{\mathrm{e}^{\pi \mathrm{i}(1-a)/12}}{\sqrt{a}} \sum_{m \in \mathbb{Z}_a} f(m) \mathrm{e}^{-\pi \mathrm{i}[(a+1)n^2 + 2mn]/a},$$

which satisfies $\mathrm{C}h\mathrm{F}^3 = \mathrm{I}$, and whose eigenspaces appear well-suited for the search for GMEF generators. This led Zauner, in his investigation of MEFs [18], to compute the spectrum of $\mathrm{C}h\mathrm{F}$, namely the multiplicities of the eigenvalues $\{1, \mathrm{e}^{2\pi \mathrm{i}/3}, \mathrm{e}^{4\pi \mathrm{i}/3}\}$:

Theorem 3 (Zauner [18]) *For any $a \in \mathbb{Z}^+$, $\mathrm{C}h\mathrm{F}$ is a unitary operator and $\mathrm{C}h\mathrm{F}^3 = \mathrm{I}$. Moreover, the multiplicities of the eigenvalues of $\mathrm{C}h\mathrm{F}$ are*:

	1	$\mathrm{e}^{2\pi \mathrm{i}/3}$	$\mathrm{e}^{4\pi \mathrm{i}/3}$
$a = 3N$	$N+1$	$N-1$	N
$a = 3N+1$	$N+1$	N	N
$a = 3N+2$	$N+1$	N	$N+1$

In essence, we see that the operator $\mathrm{C}h\mathrm{F}$ divides complex Euclidean space into three eigenspaces of approximately equal dimension; Zauner conjectures that at least one of these eigenspaces contains a GMEF generator. In a similar manner, the discrete Fourier transform itself may be shown to have four eigenspaces of approximately equal size [4]. There has been some success in exploiting this and other related conjectures; for certain a, the structure of the Jacobi/Clifford group provides an advantageously small space in which to search for GMEF generators [1, 10].

Zauner's proof of Theorem 3 relies upon explicit computations of both the Gauss sum (1.3) and the trace of $\mathrm{C}h\mathrm{F}$, both of which he finds using Schaar's identity. This identity, also known as the Landsberg-Schaar relation, is a classical result of number theory [6] which provides a reciprocity formula for quadratic Gauss sums:

Schaar's Identity *For any $a, b \in \mathbb{Z}^+$ such that ab is even*:

$$\frac{1}{\sqrt{b}} \sum_{n \in \mathbb{Z}_b} \mathrm{e}^{\pi \mathrm{i} a n^2/b} = \frac{\mathrm{e}^{\pi \mathrm{i}/4}}{\sqrt{a}} \sum_{n \in \mathbb{Z}_a} \mathrm{e}^{-\pi \mathrm{i} b n^2/a}.$$

The preceding pages are thus summarized: open problems of Gabor maximally equiangular frames lead to cyclic chirps (1.2) and the chirp-Fourier transform $\mathrm{C}^{-1}\mathrm{F}$, which, in turn, lead to Gauss sums and Schaar's identity. We note however, that Schaar's identity, as used in [18], is traditionally proven using theta functions and other techniques of analytic number theory [6], though elementary proofs do exist [3]. Beautiful as they may be, these classical proofs fall somewhat outside the frame theorist's usual toolbox. As such, after proving Theorems 1 and 2, the remainder of our work is devoted to providing a new, short and elementary proof of Schaar's identity, one which uses only standard techniques of finite time-frequency analysis, albeit applied to the rather remarkable cyclic chirp (1.2). In particular, we show how a straightforward application of the finite Poisson summation formula to (1.2) yields:

Theorem 4 *For any $a, b \in \mathbb{Z}^+$ and any $m \in \mathbb{Z}$,*

$$\frac{1}{\sqrt{b}} \sum_{n \in \mathbb{Z}_b} \mathrm{e}^{\pi \mathrm{i}[a(b+1)n^2 - 2mn]/b} = \mathrm{e}^{\pi \mathrm{i}(1-ab)/4} \mathrm{e}^{-\pi \mathrm{i}(ab+1)m^2/(ab)} \frac{1}{\sqrt{a}} \sum_{n \in \mathbb{Z}_a} \mathrm{e}^{-\pi \mathrm{i}[b(a+1)n^2 - 2mn]/a}.$$

We then mitigate the ugliness of the above result by showing its equivalence to the clean, classical extension of Schaar's identity given in [6, 14]:

Theorem 5 *For any $a, b \in \mathbb{Z}^+$ and any $c \in \mathbb{Z}$ such that $ab + c$ is even*:

$$\frac{1}{\sqrt{b}} \sum_{n \in \mathbb{Z}_b} \mathrm{e}^{\pi \mathrm{i}(an^2 + cn)/b} = \mathrm{e}^{\pi \mathrm{i}(ab - c^2)/(4ab)} \frac{1}{\sqrt{a}} \sum_{n \in \mathbb{Z}_a} \mathrm{e}^{-\pi \mathrm{i}(bn^2 + cn)/a}.$$

As the proof of the equivalence of Theorems 4 and 5 is rather straightforward, we point out that Theorem 4 is essentially a classical result, in an area of mathematics in which the literature is vast. Indeed, the Gauss sums in Schaar's identity can be explicitly computed in terms of the Jacobi symbol [9], and as such, Schaar's identity is closely related to quadratic reciprocity, whose immense literature contains many elementary proofs. As such the novelty of this work lies in Theorems 1, 2 and in the new cyclic chirp-based proof of Theorem 4, rather than in the statement of Theorem 4 itself.

Throughout this work, we make use of the basic time-frequency relations $\mathrm{T}^m\mathrm{F} = \mathrm{FM}^m$ and $\mathrm{FT}^m = \mathrm{M}^{-m}\mathrm{F}$, as well as the fact that for any $f, g \in \ell(\mathbb{Z}_a)$ we have $\mathrm{F}(f * g) = (\mathrm{F}f)(\mathrm{F}g)$ and $\mathrm{F}(fg) = (\mathrm{F}f) * (\mathrm{F}g)$ where juxtaposition of functions denotes a pointwise product, and "$*$" denotes normalized convolution:

$$(f * g)(n) = \frac{1}{\sqrt{a}} \sum_{m \in \mathbb{Z}_a} f(n-m)g(m).$$

In the following section, we prove Theorems 1 and 2, further explain the importance of quadratic phase chirps to GMEFs, and study the chirp-Fourier operator. In the

final section, we then apply standard techniques of finite time-frequency analysis to the cyclic chirp, and obtain a weak version of Theorem 4. This result, when combined with a classical computation of the quadratic Gauss sum, gives Theorem 4 which, in turn, gives Theorem 5.

2 Maximally Equiangular Gabor Frames

There are two main reasons why Gabor theory has become a valuable tool in the construction of maximally equiangular frames. The first reason is that the symmetry of Gabor systems significantly reduces the complexity of the problem. To be precise, a general MEF for an a-dimensional space consists of a^2 unit vectors whose $a^2(a^2-1)/2$ distinct inner products must be of equal magnitude. Meanwhile, a GMEF for an a-dimensional space consists of all translations and modulations of a single unit vector, and only $a(a-1)/2$ inner products must be checked for equality; as $\langle \mathrm{M}^{m_1}\mathrm{T}^{n_1} f, \mathrm{M}^{m_2}\mathrm{T}^{n_2} f\rangle = \mathrm{e}^{2\pi \mathrm{i}(m_1-m_2)n_1/a}\langle f, \mathrm{M}^{m_2-m_1}\mathrm{T}^{n_2-n_1} f\rangle$, the collection $\{\mathrm{M}^m\mathrm{T}^n f\}_{m,n\in\mathbb{Z}_a}$ is an MEF if and only if

$$|(\mathrm{A}f)(m,n)|^2 = \begin{cases} 1, & m,n=0 \mod a, \\ (a+1)^{-1}, & \text{else}, \end{cases} \tag{2.1}$$

where Af is the discrete ambiguity function of f, namely $\mathrm{A}f \in \ell(\mathbb{Z}_a^2)$,

$$(\mathrm{A}f)(m,n) = \langle f, \mathrm{M}^m\mathrm{T}^n f\rangle = \sum_{p\in\mathbb{Z}_a} f(p)\overline{f(p-n)}\mathrm{e}^{-2\pi \mathrm{i} mp/a}.$$

The second reason why MEFs are often constructed as GMEFs is that any complete Gabor system automatically satisfies a particular necessary requirement of MEFs. In particular, any MEF is necessarily a tight frame [15], while the complete Gabor system generated by any $f \in \ell(\mathbb{Z}_a)$ of unit norm is necessarily tight; for any $f, g \in \ell(\mathbb{Z}_a)$ with $\|f\| = 1$,

$$\begin{aligned} \sum_{m,n\in\mathbb{Z}_a} |\langle g, \mathrm{M}^m\mathrm{T}^n f\rangle|^2 &= a \sum_{m,n\in\mathbb{Z}_a} |(\mathrm{F}g\overline{\mathrm{T}^n f})(m)|^2 \\ &= a \sum_{m,n\in\mathbb{Z}_a} |(g\overline{\mathrm{T}^n f})(m)|^2 \\ &= a \sum_{m,n\in\mathbb{Z}_a} |g(m)|^2 |f(m-n)|^2 \\ &= a\|g\|^2. \end{aligned} \tag{2.2}$$

Though the MEF problem is somewhat simplified by restricting ourselves to GMEFs, proving or disproving the existence of $f \in \ell(\mathbb{Z}_a)$ which satisfies (2.1) is by no means a simple problem, and remains open. One approach to this problem is to ignore constructions for the most part, and instead focus on those operations which

preserve the set of all GMEF generators, should it prove to be nonempty. In particular, one may show that if a given vector generates a GMEF, then its translations, modulations, and its discrete Fourier transform do likewise. From (2.1) one may also see that chirp-modulation preserves the maximal equiangularity of a Gabor system, that is, if $\{\mathrm{M}^m\mathrm{T}^n f\}_{m,n\in\mathbb{Z}_a}$ is an MEF, then $\{\mathrm{M}^m\mathrm{T}^n\mathrm{C}f\}_{m,n\in\mathbb{Z}_a}$ is also an MEF, where $(\mathrm{C}f)(n) = (-1)^n \mathrm{e}^{\pi \mathrm{i} n^2/a} f(n)$. Indeed, as $\mathrm{MC} = \mathrm{CM}$ while $\mathrm{T}^n\mathrm{C} = (-1)^n \mathrm{e}^{\pi \mathrm{i} n^2/a}\mathrm{C}\mathrm{M}^{-n}\mathrm{T}^n$, see [7, 8], we see that chirp-modulation skews the ambiguity function:

$$(\mathrm{AC}f)(m,n) = \mathrm{e}^{-\pi \mathrm{i} m(m-a)/a}(\mathrm{A}f)(m-n,n),$$

and therefore preserves property (2.1). Though elegant and useful, the above argument fails to highlight the importance of chirps of linear frequency, that is, the special properties of $\mathrm{e}^{\pi \mathrm{i} n^2/a}$ as opposed to $\mathrm{e}^{\pi \mathrm{i} n^3/a}$ or, more generally, $\mathrm{e}^{\mathrm{i}\theta(n)}$. Instead, the unique significance of the quadratic exponent is more easily seen from Theorems 1 and 2. Indeed, if f generates a GMEF, then f satisfies (1.1). As such, $\mathrm{C}f$ also satisfies (1.1):

$$\begin{aligned}
&\sum_{p\in\mathbb{Z}_a} (\mathrm{C}f)(p)\overline{(\mathrm{C}f)(p-m)(\mathrm{C}f)(p-n)}(\mathrm{C}f)(p-m-n)\\
&\quad= \sum_{p\in\mathbb{Z}_a} (-1)^{p-(p-m)-(p-n)+(p-m-n)} \mathrm{e}^{\pi \mathrm{i}[p^2-(p-m)^2-(p-n)^2-(p-m-n)^2]}\\
&\qquad\times f(p)\overline{f(p-n)f(p-m)}f(p-m-n)\\
&\quad= \mathrm{e}^{2\pi \mathrm{i} mn/a} \sum_{p\in\mathbb{Z}_a} f(p)\overline{f(p-n)f(p-m)}f(p-m-n)\\
&\quad= \mathrm{e}^{2\pi \mathrm{i} mn/a}\frac{1}{a+1}[\delta_0(m)+\delta_0(n)]\\
&\quad= \frac{1}{a+1}[\delta_0(m)+\delta_0(n)], \qquad (2.3)
\end{aligned}$$

and therefore also generates a GMEF. The last equality in (2.3) follows from the fact that if either m or n is divisible by a then $\mathrm{e}^{2\pi \mathrm{i} mn/a} = 1$, while if neither is divisible by a then both sides of the equation are zero. We further note that if we apply the same line of reasoning (2.3) to $\mathrm{e}^{\mathrm{i}\theta(n)} f(n)$, we see that the ability to generate a GMEF, that is, property (1.1), will only be preserved provided $\theta(p) - \theta(p-m) - \theta(p-n) + \theta(p-m-n)$ is independent of p, modulo 2π. Such a second-order difference condition suggests that θ should either be constant, linear, or quadratic, and not of higher-order. Having demonstrated the usefulness of Theorems 1 and 2, we now give their proofs:

Proof of Theorem 1 Consider $g_n \in \ell(\mathbb{Z}_a)$, $g_n(p) = f(p)\overline{f(p-n)}$, and its involution $\widetilde{g}_n(m) = \overline{g_n(-m)}$. We have:

$$|(\mathrm{A}f)(m,n)|^2 = \left|\sum_{p\in\mathbb{Z}_a} g_n(p)\mathrm{e}^{-2\pi \mathrm{i} mp/a}\right|^2$$

$$
\begin{aligned}
&= a|(\mathrm{F}g_n)(m)|^2 \\
&= a(\mathrm{F}g_n)(m)\overline{(\mathrm{F}g_n)(m)} \\
&= a(\mathrm{F}g_n)(m)(\mathrm{F}\widetilde{g}_n)(m) \\
&= a[\mathrm{F}(g_n * \widetilde{g}_n)](m). \qquad (2.4)
\end{aligned}
$$

Next, let $\chi \in \ell(\mathbb{Z}_a)$ be the constant function $\chi(n) = 1$ for all $n \in \mathbb{Z}$, and note $\mathrm{F}\chi = \sqrt{a}\delta_0$. Considering (2.1), we have that $\{\mathrm{M}^m\mathrm{T}^n f\}_{m,n\in\mathbb{Z}_a}$ is an MEF if and only if

$$
|A(m,n)|^2 = \frac{1}{a+1}\chi(m) = \frac{\sqrt{a}}{a+1}(\mathrm{F}\delta_0)(m), \qquad (2.5)
$$

for all $m \in \mathbb{Z}_a$ when $n \neq 0 \bmod a$, while for $n = 0 \mod a$:

$$
|A(m,n)|^2 = \frac{1}{a+1}(a\delta_0 + \chi)(m) = \frac{\sqrt{a}}{a+1}[\mathrm{F}(\chi + \delta_0)](m), \qquad (2.6)
$$

for all $m \in \mathbb{Z}_a$. By combining (2.4), (2.5) and (2.6) and then applying inverse Fourier transforms, we see that $\{\mathrm{M}^m\mathrm{T}^n f\}_{m,n\in\mathbb{Z}_a}$ is an MEF if and only if

$$
a(g_n * \widetilde{g}_n)(m) = \frac{\sqrt{a}}{a+1}\begin{cases} \chi(m) + \delta_0(m), & n = 0, \\ \delta_0(m), & n \neq 0, \end{cases}
$$

for all $m, n \in \mathbb{Z}_a$. That is, $\{\mathrm{M}^m\mathrm{T}^n f\}_{m,n\in\mathbb{Z}_a}$ is an MEF if and only if

$$
\begin{aligned}
&\sum_{p\in\mathbb{Z}_a} f(p)\overline{f(p-n)f(p-m)}f(p-m-n) \\
&\quad = \sum_{p\in\mathbb{Z}_a} g_n(p)\overline{g_n(p-m)} \\
&\quad = \sqrt{a}(g_n * \widetilde{g}_n)(m) \\
&\quad = \frac{1}{a+1}\begin{cases} \chi(m) + \delta_0(m), & n = 0, \\ \delta_0(m), & n \neq 0 \end{cases} \\
&\quad = \frac{1}{a+1}[\delta_0(m) + \delta_0(n)],
\end{aligned}
$$

for all $m, n \in \mathbb{Z}_a$, as claimed. □

Proof of Theorem 2 Again letting $g_n(p) = f(p)\overline{f(p-n)}$, we have:

$$
\begin{aligned}
&\sum_{m,n\in\mathbb{Z}_a}\left|\sum_{p\in\mathbb{Z}_a} f(p)\overline{f(p-n)f(p-m)}f(p-m-n)\right|^2 \\
&\quad = \sum_{m,n\in\mathbb{Z}_a}\left|\sum_{p\in\mathbb{Z}_a} g_n(p)\overline{g_n(p-m)}\right|
\end{aligned}
$$

$$= \sum_{m,n\in\mathbb{Z}_a} |\sqrt{a}(g_n * \widetilde{g}_n)(m)|$$

$$= a \sum_{n\in\mathbb{Z}_a} \|g_n * \widetilde{g}_n\|^2. \tag{2.7}$$

As the Fourier transform is unitary and $\mathrm{F}\widetilde{g}_n = \overline{\mathrm{F}g_n}$, (2.7) becomes:

$$a \sum_{n\in\mathbb{Z}_a} \|g_n * \widetilde{g}_n\|^2 = a \sum_{n\in\mathbb{Z}_a} \||\mathrm{F}g_n|^2\|^2 = a \sum_{m,n\in\mathbb{Z}_a} |(\mathrm{F}g_n)(m)|^4. \tag{2.8}$$

As the terms in the right-hand sum (2.8) are, in fact:

$$(\mathrm{F}g_n)(m) = \frac{1}{\sqrt{a}} \sum_{p\in\mathbb{Z}_a} f(p)\overline{f(p-n)}\mathrm{e}^{-2\pi \mathrm{i} mp/N} = \frac{1}{\sqrt{a}}\langle f, \mathrm{M}^m\mathrm{T}^n f\rangle, \tag{2.9}$$

we may put (2.7), (2.8) and (2.9) together to obtain our first conclusion:

$$\sum_{m,n\in\mathbb{Z}_a} \left| \sum_{p\in\mathbb{Z}_a} f(p)\overline{f(p-n)f(p-m)}f(p-m-n)\right|^2 = \frac{1}{a} \sum_{m,n\in\mathbb{Z}_a} |\langle f, \mathrm{M}^m\mathrm{T}^n f\rangle|^4.$$

To obtain our second conclusion, note that letting $g = f$ in (2.2) gives:

$$a = \sum_{m,n\in\mathbb{Z}_a} |\langle f, \mathrm{M}^m\mathrm{T}^n f\rangle|^2.$$

As $\langle f, \mathrm{M}^0\mathrm{T}^0\rangle = \|f\|^2 = 1$, the constrained minimization problem:

$$\min \frac{1}{a} \sum_{m,n\in\mathbb{Z}_a} |\langle f, \mathrm{M}^m\mathrm{T}^n f\rangle|^4 \quad \text{s.t.} \quad \|f\| = 1, \quad \sum_{m,n\in\mathbb{Z}_a} |\langle f, \mathrm{M}^m\mathrm{T}^n f\rangle|^2 = a,$$

has a lower bound that is achieved precisely when the remaining $a^2 - 1$ values $\{|\langle f, \mathrm{M}^m\mathrm{T}^n\rangle|^2\}_{(m,n)\neq(0,0)}$ are of equal value, that is, if and only if $\{\mathrm{M}^m\mathrm{T}^n f\}_{m,n\in\mathbb{Z}_a}$ is an MEF for $\ell(\mathbb{Z}_a)$. Moreover, as the constraint implies this constant value of $|\langle f, \mathrm{M}^m\mathrm{T}^n\rangle|^2$ is $(a-1)/(a^2-1) = a+1$, the bound on the objective function is:

$$\frac{1}{a} \sum_{m,n\in\mathbb{Z}_a} |\langle f, \mathrm{M}^m\mathrm{T}^n f\rangle|^4 \geq \frac{1}{a}\left[1 + \frac{a^2-1}{(a+1)^2}\right] = \frac{2}{a+1}$$

as claimed. □

Having proved two of our main results, we now note a seeming paradox: for a given dimension a, the set of GMEF generators (modulo unit scalar multiples) for $\ell(\mathbb{Z}_a)$ seems to be exceedingly small, and yet this set is nevertheless preserved by translations, modulations, chirp-modulations and Fourier transforms. It is therefore not surprising that many GMEF generators seem to exhibit a symmetry under these operations. That is, given a GMEF generator f, it is not uncommon that f

is some nontrivial translation/modulation/chirp-modulation/Fourier transform of itself. Indeed, as noted in the introduction, many GMEF generators are eigenvectors of $\mathrm{C}^{-1}\mathrm{F}$. We therefore conclude this section by showing how the cyclic chirp (1.2) may be exploited to provide a short proof of the fact that the cube of $\mathrm{C}^{-1}\mathrm{F}$ is a scalar multiple of the identity. This fact, when combined with Schaar's identity, then quickly implies Theorem 3 [18].

Theorem 6 $(\mathrm{C}^{-1}\mathrm{F})^3 = \overline{(\mathrm{F}\mathrm{c}_a)(0)}\mathrm{I}$.

Proof For any $f \in \ell(\mathbb{Z}_a)$ and any $q \in \mathbb{Z}$,

$$\begin{aligned}
&[(\mathrm{C}^{-1}\mathrm{F})^3 f](q)\\
&\quad = a^{-3/2} \sum_{m,n,p\in\mathbb{Z}_a} f(m)(-1)^{n+p+q}\mathrm{e}^{-\pi\mathrm{i}(2mn+n^2+2np+p^2+2pq+q^2)/a}\\
&\quad = a^{-3/2} \sum_{m,n,p\in\mathbb{Z}_a} f(m)(-1)^{n+p+q}\mathrm{e}^{-\pi\mathrm{i}[2mn+(n+p+q)^2-2nq]/a}\\
&\quad = \frac{1}{\sqrt{a}}\sum_{r\in\mathbb{Z}_a}(-1)^r\mathrm{e}^{-\pi\mathrm{i}r^2/a}\frac{1}{\sqrt{a}}\sum_{n\in\mathbb{Z}_a}\frac{1}{\sqrt{a}}\sum_{m\in\mathbb{Z}_a} f(m)\mathrm{e}^{-2\pi\mathrm{i}mn/a}\mathrm{e}^{2\pi\mathrm{i}nq/a}\\
&\quad = \frac{1}{\sqrt{a}}\sum_{r\in\mathbb{Z}_a}\overline{\mathrm{c}_a(r)}\frac{1}{\sqrt{a}}\sum_{n\in\mathbb{Z}_a}(\mathrm{F}f)(n)\mathrm{e}^{2\pi\mathrm{i}nq/a}\\
&\quad = \overline{(\mathrm{F}\mathrm{c}_a)(0)}(\mathrm{F}^{-1}\mathrm{F}f)(q)\\
&\quad = \overline{(\mathrm{F}\mathrm{c}_a)(0)}f(q),
\end{aligned}$$

as claimed. □

3 An Elementary Proof of Schaar's Identity

In this section, we show how the cyclic chirp (1.2) may be used to quickly prove the generalizations of Schaar's identity given in Theorems 4 and 5. As Theorem 5 is a classical result of number theory, the contribution of this work is not the result itself, but rather its proof. Indeed, as Schaar's identity has recently been shown to be relevant to the GMEF problem [18], there is some value in showing how Theorem 5 may be easily proven by applying basic algebraic techniques of finite time-frequency analysis to cyclic chirps.

For any $a \in \mathbb{Z}^+$ and any $b \in \mathbb{Z}$, consider the bth order chirp $\mathrm{c}_a^b \in \ell(\mathbb{Z}_a)$,

$$\mathrm{c}_a^b(n) = \mathrm{e}^{\pi\mathrm{i}b(a+1)n^2/a} = (-1)^{bn}\mathrm{e}^{\pi\mathrm{i}bn^2/a},$$

whose corresponding chirp-modulation operator is $\mathrm{C}^b : \ell(\mathbb{Z}_a) \to \ell(\mathbb{Z}_a)$, $(\mathrm{C}^b f)(n) = \mathrm{c}_a^b(n)f(n)$. When $b = 1$, we shall usually omit the superscript, that is, $\mathrm{c}_a = \mathrm{c}_a^1$. We shall make use of the finite Poisson summation formula, which has an elementary

proof using geometric sums. Here and throughout, the amended indexing "$a|n$" indicates that we only consider those coset representatives n which are divisible by a.

Poisson Summation Formula *For any $a, b \in \mathbb{Z}^+$ and any $f \in \ell(\mathbb{Z}_{ab})$,*

$$\sqrt{a} \sum_{\substack{m \in \mathbb{Z}_{ab} \\ a|m}} f(m) = \sqrt{b} \sum_{\substack{n \in \mathbb{Z}_{ab} \\ b|n}} (\mathrm{F}f)(n).$$

We shall also use the fact that the Fourier transform of the first-order chirp c_a is a scalar multiple of the complex conjugate of itself:

Lemma 1 $(\mathrm{Fc}_a)(m) = (\mathrm{Fc}_a)(0)\overline{\mathrm{c}_a(m)}$ *for any $a \in \mathbb{Z}^+, m \in \mathbb{Z}$.*

Proof For any $a \in \mathbb{Z}^+, m \in \mathbb{Z}$,

$$\begin{aligned}
(\mathrm{Fc}_a)(m) &= \frac{1}{\sqrt{a}} \sum_{n \in \mathbb{Z}_a} \mathrm{c}_a(n) \mathrm{e}^{-2\pi \mathrm{i} mn/a} \\
&= \frac{1}{\sqrt{a}} \sum_{n \in \mathbb{Z}_a} (-1)^n \mathrm{e}^{\pi \mathrm{i}(n^2 - 2mn)/a} \\
&= \frac{1}{\sqrt{a}} \sum_{n \in \mathbb{Z}_a} (-1)^n \mathrm{e}^{\pi \mathrm{i}[(n-m)^2 - m^2]/a} \\
&= (-1)^m \mathrm{e}^{-\pi \mathrm{i} m^2/a} \frac{1}{\sqrt{a}} \sum_{n \in \mathbb{Z}_a} (-1)^{n-m} \mathrm{e}^{\pi \mathrm{i}(n-m)^2/a} \\
&= \overline{\mathrm{c}_a(m)} (\mathrm{Fc}_a)(0),
\end{aligned}$$

as claimed. □

We now prove a slightly ambiguous version of Theorem 4:

Theorem 7 $(\mathrm{Fc}_b^a)(m) = (\mathrm{Fc}_{ab})(m)\overline{(\mathrm{Fc}_a^b)(m)}$ *for any $a, b \in \mathbb{Z}^+, m \in \mathbb{Z}$.*

Proof We apply the Poisson Summation Formula to an arbitrary modulation of c_{ab}. That is, for any $k \in \mathbb{Z}$,

$$\frac{\sqrt{a}}{\sqrt{b}} \sum_{\substack{m \in \mathbb{Z}_{ab} \\ a|m}} (\mathrm{M}^k \mathrm{c}_{ab})(m) = \sum_{\substack{n \in \mathbb{Z}_{ab} \\ b|n}} (\mathrm{FM}^k \mathrm{c}_{ab})(n). \tag{3.1}$$

Making the change of variables $m = -aj$, the left-hand side of (3.1) is:

$$\frac{\sqrt{a}}{\sqrt{b}} \sum_{\substack{m \in \mathbb{Z}_{ab} \\ a|m}} (\mathrm{M}^k \mathrm{c}_{ab})(m) = \frac{\sqrt{a}}{\sqrt{b}} \sum_{\substack{m \in \mathbb{Z}_{ab} \\ a|m}} \mathrm{e}^{2\pi \mathrm{i} km/(ab)} (-1)^m \mathrm{e}^{\pi \mathrm{i} m^2/(ab)}$$

$$= \frac{\sqrt{a}}{\sqrt{b}} \sum_{j\in\mathbb{Z}_b} \mathrm{e}^{-2\pi \mathrm{i} kaj/(ab)} (-1)^{-aj} \mathrm{e}^{\pi \mathrm{i}(-aj)^2/(ab)}$$

$$= \frac{\sqrt{a}}{\sqrt{b}} \sum_{j\in\mathbb{Z}_b} (-1)^{aj} \mathrm{e}^{\pi \mathrm{i} aj^2/b} \mathrm{e}^{-2\pi \mathrm{i} kj/b}$$

$$= \sqrt{a}(\mathrm{F}_b \mathrm{c}_b^a)(k).$$

Meanwhile, letting $n = bp$, the right-hand side of (3.1) becomes:

$$\sum_{\substack{n\in\mathbb{Z}_{ab}\\ b|n}} (\mathrm{FM}^k \mathrm{c}_{ab})(n) = \sum_{p\in\mathbb{Z}_a} (\mathrm{FM}^k \mathrm{c}_{ab})(bp)$$

$$= \sum_{p\in\mathbb{Z}_a} (\mathrm{T}^k \mathrm{Fc}_{ab})(bp)$$

$$= \sum_{p\in\mathbb{Z}_a} (\mathrm{Fc}_{ab})(bp-k).$$

To continue simplifying the right-hand side of (3.1), we apply Lemma 1:

$$\sum_{\substack{n\in\mathbb{Z}_{ab}\\ b|n}} (\mathrm{FM}^k \mathrm{c}_{ab})(n) = \sum_{p\in\mathbb{Z}_a} (\mathrm{Fc}_{ab})(0)\overline{\mathrm{c}_{ab}(bp-k)}$$

$$= (\mathrm{Fc}_{ab})(0) \sum_{p\in\mathbb{Z}_a} (-1)^{bp-k} \mathrm{e}^{-\pi \mathrm{i}(bp-k)^2/(ab)}$$

$$= (\mathrm{Fc}_{ab})(0)(-1)^k \mathrm{e}^{-\pi \mathrm{i} k^2/(ab)} \sum_{p\in\mathbb{Z}_a} (-1)^{bp} \mathrm{e}^{-\pi \mathrm{i} bp^2/a} \mathrm{e}^{2\pi \mathrm{i} pk/a}$$

$$= \sqrt{a}(\mathrm{Fc}_{ab})(0)\overline{\mathrm{c}_{ab}(k)} \frac{1}{\sqrt{a}} \overline{\sum_{p\in\mathbb{Z}_a} \mathrm{c}_a^b(p) \mathrm{e}^{-2\pi \mathrm{i} pk/a}}$$

$$= \sqrt{a}(\mathrm{Fc}_{ab})(k)\overline{(\mathrm{Fc}_a^b)(k)}.$$

Equating the two sides and dividing by $\sqrt{a}$ gives the result. □

We note that Theorem 7 may be more symmetrically stated as saying $(\mathrm{Fc}_a^b)(m)(\mathrm{Fc}_b^a)(m) = \eta(a,b)(\mathrm{Fc}_{ab})(m)$, where $\eta(a,b) = |(\mathrm{Fc}_a^b)(m)|^2$ is explicitly given in [7]. By finding the explicit expressions for $(\mathrm{Fc}_a^b)(m)$ and $(\mathrm{Fc}_b^a)(m)$, one may immediately see the connection between Theorems 4 and 7. Indeed, as Lemma 1 gives

$$(\mathrm{Fc}_{ab})(m) = (\mathrm{Fc}_{ab})(0)\overline{\mathrm{c}_{ab}(m)} = (\mathrm{Fc}_{ab})(0)\mathrm{e}^{-\pi \mathrm{i}(ab+1)m^2/(ab)},$$

we note that Theorem 4 follows directly from Theorem 7, provided one also has the Gauss sum formula $(\mathrm{Fc}_{ab})(0) = \mathrm{e}^{\pi \mathrm{i}(1-ab)/4}$, or, more generally:

Theorem 8 $(\mathrm{Fc}_a)(0) = \mathrm{e}^{\pi \mathrm{i}(1-a)/4}$ *for all* $a \in \mathbb{Z}^+$.

Proof We shall use the traditional Gauss sum formula

$$(\mathrm{Fc}_{4b}^2)(0) = \frac{1}{\sqrt{4b}} \sum_{n \in \mathbb{Z}_{4b}} \mathrm{e}^{2\pi \mathrm{i} n^2/4b} = 1 + \mathrm{i}, \tag{3.2}$$

for any $b \in \mathbb{Z}$, a fact for which many elementary proofs are known [6]. For any $a \in \mathbb{Z}^+$, Theorem 7 gives:

$$(\mathrm{Fc}_{2a}^2)(a) = (\mathrm{Fc}_{4a})(a)\overline{(\mathrm{Fc}_2^{2a})(a)}. \tag{3.3}$$

The left-hand side of (3.3) may be simplified as:

$$\begin{aligned}(\mathrm{Fc}_{2a}^2)(a) &= \frac{1}{\sqrt{2a}} \sum_{n \in \mathbb{Z}_{2a}} (-1)^{2n} \mathrm{e}^{\pi \mathrm{i} 2n^2/(2a)} \mathrm{e}^{-2\pi \mathrm{i} an/(2a)} \\ &= \frac{1}{\sqrt{2a}} \sum_{n \in \mathbb{Z}_{2a}} \mathrm{e}^{\pi \mathrm{i} n^2/a} (-1)^n \\ &= \frac{\sqrt{2}}{\sqrt{a}} \sum_{n \in \mathbb{Z}_a} \mathrm{c}_a(n) \\ &= \sqrt{2}(\mathrm{Fc}_a)(0),\end{aligned} \tag{3.4}$$

where the second to last equality follows from the fact that c_a is a-periodic. Meanwhile, we may also simplify the first term on the right-hand side of (3.3) using Lemma 1:

$$(\mathrm{Fc}_{4a})(a) = (\mathrm{Fc}_{4a})(0)(-1)^a \mathrm{e}^{-\pi \mathrm{i} a^2/(4a)} = \mathrm{e}^{3\pi \mathrm{i} a/4} (\mathrm{Fc}_{4a})(0), \tag{3.5}$$

and evaluate the second term of the right-hand side of (3.3) directly:

$$\begin{aligned}\overline{(\mathrm{Fc}_2^{2a})(a)} &= \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}_2} (-1)^{2an} \mathrm{e}^{-\pi \mathrm{i} 2an^2/2} \mathrm{e}^{2\pi \mathrm{i} an/2} \\ &= \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}_b} (-1)^{an^2} (-1)^{an} \\ &= \sqrt{2}.\end{aligned} \tag{3.6}$$

Substituting (3.4), (3.5) and (3.6) into (3.3) gives

$$(\mathrm{Fc}_a)(0) = \mathrm{e}^{3\pi \mathrm{i} a/4} (\mathrm{Fc}_{4a})(0). \tag{3.7}$$

Now, replacing a with $4a$ in (3.7) gives $(\mathrm{Fc}_{4a})(0) = \mathrm{e}^{3\pi \mathrm{i} 4a/4} (\mathrm{Fc}_{16a})(0)$, which, taken together with (3.7), implies:

$$(\mathrm{Fc}_a)(0) = \mathrm{e}^{3\pi \mathrm{i} a/4} \mathrm{e}^{3\pi \mathrm{i} 4a/4} (\mathrm{Fc}_{16a})(0) = \mathrm{e}^{-\pi \mathrm{i} a/4} (\mathrm{Fc}_{16a})(0). \tag{3.8}$$

Theorem 7 then allows us to continue simplifying (3.8) in terms of a sum of form (3.5) and the traditional Gauss sum (3.2) where $b = 2a$:

$$(\mathrm{Fc}_a)(0) = \mathrm{e}^{-\pi \mathrm{i} a/4}\left[\overline{(\mathrm{Fc}_2^{8a})(0)}\right]^{-1}(\mathrm{Fc}_{8a}^2)(0) = \mathrm{e}^{-\pi \mathrm{i} a/4}\frac{1}{\sqrt{2}}(1+\mathrm{i}) = \mathrm{e}^{\pi \mathrm{i}(1-a)/4},$$

as claimed. □

Though the proof of Theorem 8 relies upon the computation of the classical Gauss sum $\mathrm{Fc}_a^2(0)$ in the case where $4|a$, we note this assumption can be somewhat weakened. Indeed, by noting both $\mathrm{Fc}_a^2(0) = \mathrm{tr}(\mathrm{F})$ (see [4]) and the Vandermonde determinant $\det \mathrm{F} = \mathrm{e}^{\pi \mathrm{i}(a-1)(a+2)/4}$ (see [13]) one may, with a little work, recover Gauss's original result that $\mathrm{Fc}_a^2(0) = 1$ when $a = 1 \mod 4$ and $\mathrm{Fc}_a^2(0) = \mathrm{i}$ when $a = 3 \mod 4$. By using Theorems 6 and 7 as well as a calculation of $\det \mathrm{C}^{-1}\mathrm{F}$, one may then build on the computation of $\mathrm{Fc}_a(0)$ when $a = 1, 3 \mod 4$ to prove Theorem 8 from basic principles. We conclude our work by noting that our generalization of Schaar's identity given in Theorem 4 is equivalent to the classical generalization given in Theorem 5.

Proof of the equivalence of Theorems 4 and 5 To show Theorem 4 implies Theorem 5, take any $a, b \in \mathbb{Z}^+$ and any $c \in \mathbb{Z}$ such that $ab + c$ is even. Letting $m = (ab - c)/2$, the left-hand side of Theorem 4 becomes the left-hand side of Theorem 5:

$$\begin{aligned}
\frac{1}{\sqrt{b}}\sum_{n\in\mathbb{Z}_b}\mathrm{e}^{\pi \mathrm{i}[a(b+1)n^2-2mn]/b} &= \frac{1}{\sqrt{b}}\sum_{n\in\mathbb{Z}_b}\mathrm{e}^{\pi \mathrm{i}[a(b+1)n^2-(ab-c)n]/b} \\
&= \frac{1}{\sqrt{b}}\sum_{n\in\mathbb{Z}_b}\mathrm{e}^{\pi \mathrm{i}(abn^2-abn)/b}\mathrm{e}^{\pi \mathrm{i}(an^2+cn)/b} \\
&= \frac{1}{\sqrt{b}}\sum_{n\in\mathbb{Z}_b}\mathrm{e}^{\pi \mathrm{i} an(n-1)}\mathrm{e}^{\pi \mathrm{i}(an^2+cn)/b} \\
&= \frac{1}{\sqrt{b}}\sum_{n\in\mathbb{Z}_b}\mathrm{e}^{\pi \mathrm{i}(an^2+cn)/b}.
\end{aligned}$$

The summations of the right-hand sides of Theorems 4 and 5 are shown to be equal in a similar fashion, and the equality of the coefficients is given by:

$$\begin{aligned}
&\mathrm{e}^{\pi \mathrm{i}(1-ab)/4}\mathrm{e}^{-\pi \mathrm{i}(ab+1)m^2/(ab)} \\
&\quad= \mathrm{e}^{\pi \mathrm{i} ab(1-ab)/(4ab)}\mathrm{e}^{-\pi \mathrm{i}(ab+1)(ab-c)^2/(4ab)} \\
&\quad= \mathrm{e}^{\pi \mathrm{i}[-(ab)^2-(ab)(ab-c)^2-(ab)^2+2abc]/(4ab)}\mathrm{e}^{\pi \mathrm{i}(ab-c^2)/(4ab)} \\
&\quad= \mathrm{e}^{-\pi \mathrm{i}(ab-c)(2+ab-c)/4}\mathrm{e}^{\pi \mathrm{i}(ab-c^2)/(4ab)} \\
&\quad= \mathrm{e}^{-\pi \mathrm{i} m(1+m)}\mathrm{e}^{\pi \mathrm{i}(ab-c^2)/(4ab)} \\
&\quad= \mathrm{e}^{\pi \mathrm{i}(ab-c^2)/(4ab)}.
\end{aligned}$$

Having shown that Theorem 4 implies Theorem 5, we note that the converse follows from a similar argument: take any $a, b \in \mathbb{Z}^+$ and any $m \in \mathbb{Z}$, and let $c = ab - 2m$ in Theorem 5. □

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