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Continuity Envelopes of Spaces of Generalized Smoothness in the Critical Case

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Abstract The continuity envelope for the Besov and Triebel-Lizorkin spaces of generalized smoothness $B_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ and $F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$, respectively, are computed in the critical case $s = n/p$, provided that Ψ satisfies an appropriate critical condition. Surprisingly, in this critical situation, the corresponding optimal index is ∞, when compared with all the known results. Moreover, in the particular case of the classical spaces, we solve an open problem posed by Haroske in Envelopes and Sharp Embeddings of Function Spaces, Research Notes in Mathematics, vol. 437, Chapman & Hall, Boca Raton, 2007. As an immediate application of our results we give an upper estimate for approximation numbers of related embeddings.

Keywords Function spaces of generalized smoothness · Continuity envelopes

Mathematics Subject Classification (2000) 46E35 · 47B06

1 Introduction

There is a variety of recent contributions to the theory of envelopes of spaces of generalized smoothness of Besov or Triebel-Lizorkin type, denoted by $B_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ and

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$F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$, respectively. In [19] the continuity envelopes for spaces of generalized smoothness have been studied in the case $n/p < s < 1 + n/p$. The so-called super-limiting case $s = 1 + n/p$ has been considered in [5]. For a more general setting we refer also to [20]. In the present paper we supplement these investigations by computing the continuity envelope functions for spaces of generalized smoothness in the remaining limiting case—the so-called critical case $s = n/p$ (cf. Proposition 3.2). Moreover, we prove that in such critical situation the corresponding optimal index is ∞ (cf. Theorems 3.4 & 3.5). This assertion is surprising in comparison with all the known results, where the optimal index was always q for the Besov spaces and p for the Triebel-Lizorkin spaces. In particular, we are able to complete the picture of the continuity envelopes for the classical spaces in the critical situation, improving the former result by Haroske in [18, Theorem 9.10], and answering the open question posed by her in [18, Remark 9.11].

The paper is organized as follows. Section 2 contains notation, definitions, preliminary assertions and auxiliary results. In Sect. 3 we state and prove our main results, give some examples and show an immediate application to approximation numbers.

2 Preliminaries

2.1 General Notation

For a real number a, let $a_+ := \max(a, 0)$ and let $[a]$ denote its integer part. For $p \in (0, \infty]$, the number p' is defined by $1/p' := (1 - 1/p)_+$ with the convention that $1/\infty = 0$. By c, c_1, c_2, etc. we denote positive constants independent of appropriate quantities. For two non-negative expressions (i.e. functions or functionals) $\mathcal{A}$, $\mathcal{B}$, the symbol $\mathcal{A} \lesssim \mathcal{B}$ (or $\mathcal{A} \gtrsim \mathcal{B}$) means that $\mathcal{A} \le c\,\mathcal{B}$ (or $c\,\mathcal{A} \ge \mathcal{B}$). If $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{A} \gtrsim \mathcal{B}$, we write $\mathcal{A} \sim \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are equivalent. Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding is bounded. If not otherwise indicated, log is always taken with respect to base 2.

2.2 Slowly Varying Functions

Definition 2.1 A positive and measurable function Ψ defined on the interval (0, 1] is said to be *slowly varying* if

$$\lim_{t\to 0} \frac{\Psi(st)}{\Psi(t)} = 1, \qquad s \in (0, 1]. \tag{1}$$

Remark 2.2 Any function of the form

$$\Psi(t) = \exp\left\{ -\int_t^1 \varrho(s) \frac{\mathrm{d}s}{s} \right\}, \qquad t \in (0, 1],$$

where ϱ is a measurable function on (0, 1] with $\lim_{s\to 0^+} \varrho(s) = 0$, is slowly varying. Actually this is a characterization: any slowly varying function is equivalent to a function Ψ of the above type for an appropriate function ϱ (that can even be taken to be a C^∞ function, cf. [1, Theorem 1.1.3]). Therefore, from now on we suppose that this will be always the case, when such properties are technically important in a proof.

The following functions are examples of slowly varying functions:

(i) $\Psi(x) = (1 + |\log x|)^a (1 + \log(1 + |\log x|))^b$, $x \in (0, 1]$, $a, b \in \mathbb{R}$,
(ii) $\Psi(x) = \exp(|\log x|^c)$, $x \in (0, 1]$, $c \in (0, 1)$.

We remark that the example in (i) is also an admissible function in the sense of [11, 12]. We recall that an admissible function Ψ is a positive monotone function defined on $(0, 1]$ such that $\Psi(2^{-2j}) \sim \Psi(2^{-j})$, $j \in \mathbb{N}$. It can be proved that an admissible function is, up to equivalence, a slowly varying function.

The proposition below gives some properties of slowly varying functions that will be useful in what follows. We refer to the monograph [1] for details and further properties, see also [27, Chap. V, p. 186], [13], and, more recently, [23, 24] and [2].

Proposition 2.3 *Let Ψ be a slowly varying function.*

(i) *For any $\delta > 0$ there exists $c = c(\delta) > 1$ such that*

$$\frac{1}{c} s^\delta \le \frac{\Psi(st)}{\Psi(t)} \le c\, s^{-\delta}, \quad t, s \in (0, 1].$$

(ii) *For each $\alpha > 0$ there is a decreasing function ϕ and an increasing function φ such that*

$$t^{-\alpha}\, \Psi(t) \sim \phi(t) \quad \text{and} \quad t^\alpha\, \Psi(t) \sim \varphi(t).$$

(iii) *If $\int_0^1 \Psi(s) \frac{ds}{s} < \infty$, then $\tilde{\Psi}$ defined by $\tilde{\Psi}(t) = \int_0^t \Psi(s) \frac{ds}{s}$, $t \in (0, 1]$, is a slowly varying function such that*

$$\lim_{t \to 0^+} \frac{\tilde{\Psi}(t)}{\Psi(t)} = \infty.$$

(iv) *Ψ^r, $r \in \mathbb{R}$, is a slowly varying function.*

Remark 2.4 It follows easily from the last proposition that

$$\Psi(t) \sim \Psi(2^{-j}) \sim \Psi(2^{-(j+1)}), \quad t \in [2^{-(j+1)}, 2^{-j}],\ j \in \mathbb{N}_0.$$

The next proposition provides a very useful discretization method.

Proposition 2.5 *Let $u \in (0, \infty]$ and let Ψ be a slowly varying function such that $(\Psi(2^{-j})^{-1})_{j \in \mathbb{N}} \in \ell_u$. Then*

$$\left(\int_0^t \Psi(s)^{-u} \frac{ds}{s} \right)^{1/u} \sim \left(\sum_{j=[|\log t|]}^{\infty} \Psi(2^{-j})^{-u} \right)^{1/u}, \quad t \in (0, 2^{-1}].$$

In the case $u = \infty$ the usual modification with supremum is required.

Proof Let $u \in (0, \infty)$. For $t \in (0, 1]$, we have that $2^{-([|\log t|]+1)} \le t \le 2^{-[|\log t|]}$. Taking advantage of the properties of slowly varying functions, as described in Proposition 2.3, we obtain, for $t \in (0, 2^{-1}]$

$$\begin{aligned}\sum_{j=[|\log t|]}^{\infty} \Psi(2^{-j})^{-u} &\lesssim \sum_{j=[|\log t|]+1}^{\infty} \Psi(2^{-j})^{-u} \lesssim \sum_{j=[|\log t|]+1}^{\infty} \int_{2^{-(j+1)}}^{2^{-j}} \Psi(s)^{-u} \frac{\mathrm{d}s}{s}\\ &\lesssim \int_0^{2^{-([|\log t|]+1)}} \Psi(s)^{-u} \frac{\mathrm{d}s}{s} \lesssim \int_0^t \Psi(s)^{-u} \frac{\mathrm{d}s}{s}\\ &\lesssim \int_0^{2^{-[|\log t|]}} \Psi(s)^{-u} \frac{\mathrm{d}s}{s} = \sum_{j=[|\log t|]}^{\infty} \int_{2^{-(j+1)}}^{2^{-j}} \Psi(s)^{-u} \frac{\mathrm{d}s}{s}\\ &\lesssim \sum_{j=[|\log t|]}^{\infty} \Psi(2^{-j})^{-u}.\end{aligned}$$

When $u = \infty$, the proof is analogous to the previous case. □

The next result is a discrete version of the estimate [23, (3.2)].

Lemma 2.6 *Let* $0 < u < \infty$ *and* Ψ *be a slowly varying function. Then*

$$\left(\sum_{j=0}^{k} 2^{ju}\Psi(2^{-j})^{-u}\right)^{1/u} \sim 2^k\, \Psi(2^{-k})^{-1}, \quad k \in \mathbb{N} \tag{2}$$

and

$$\sup_{j\in\{0,\dots,k\}} 2^j \Psi(2^{-j})^{-1} \sim 2^k\, \Psi(2^{-k})^{-1}, \quad k \in \mathbb{N}. \tag{3}$$

Proof We only prove the estimates (2), since the proof of the estimates (3) is similar. Let $\varepsilon \in (0, u)$. Using the fact that $t^{-\varepsilon}\Psi(t)^{-u}$ is equivalent to a decreasing function, cf. Proposition 2.3(ii), we obtain, for $k \in \mathbb{N}$,

$$\begin{aligned}\sum_{j=0}^{k} 2^{ju}\, \Psi(2^{-j})^{-u} &= \sum_{j=0}^{k} 2^{j(u-\varepsilon)}\, (2^{-j})^{-\varepsilon}\, \Psi(2^{-j})^{-u}\\ &\lesssim (2^{-k})^{-\varepsilon}\, \Psi(2^{-k})^{-u} \sum_{j=0}^{k} 2^{j(u-\varepsilon)}\\ &\le 2^{uk}\, \Psi(2^{-k})^{-u}\, \frac{2^{u-\varepsilon}}{2^{u-\varepsilon}-1}\\ &\lesssim 2^{uk}\, \Psi(2^{-k})^{-u},\end{aligned}$$

this completes the proof since the reverse inequality is clear. □

2.3 Function Spaces of Generalized Smoothness

In the sequel, let $\mathcal{S}(\mathbb{R}^n)$ stand for the Schwartz space of all complex-valued rapidly decreasing C^∞ functions on $\mathbb{R}^n$ and we denote by $\mathcal{S}'(\mathbb{R}^n)$ its topological dual, the space of all tempered distributions. Furthermore, $L_p(\mathbb{R}^n)$, with $0 < p \le \infty$, is the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f \mid L_p(\mathbb{R}^n)\| := \left(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x\right)^{1/p},$$

with the usual modification if $p = \infty$. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ be a function such that

$$\varphi_0(x) = 1 \quad \text{for } |x| \le 1 \quad \text{and} \quad \operatorname{supp}\varphi_0 \subset \{x \in \mathbb{R}^n : |x| \le 2\}. \tag{4}$$

For each $j \in \mathbb{N}$ we define

$$\varphi_j(x) := \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x), \quad x \in \mathbb{R}^n. \tag{5}$$

Then since $\sum_{j=0}^\infty \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$, the sequence $(\varphi_j)_{j\in\mathbb{N}_0}$ is a dyadic resolution of unity. Given any $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $\widehat{f}$ and $f^\vee$ its Fourier transform and its inverse Fourier transform, respectively.

Definition 2.7 Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and let Ψ be a slowly varying function according to Definition 2.1.

(i) Then $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ is defined to be the set of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\| f \mid B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\right\| := \left(\sum_{j=0}^\infty 2^{jsq}\Psi(2^{-j})^q \left\|(\varphi_j \widehat{f})^\vee \mid L_p(\mathbb{R}^n)\right\|^q\right)^{1/q} \tag{6}$$

is finite. In the limiting case $q = \infty$ the usual modification is required.

(ii) Let $0 < p < \infty$. Then $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ is defined to be the set of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\left\| f \mid F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\right\| := \left\|\left(\sum_{j=0}^\infty 2^{jsq}\Psi(2^{-j})^q |(\varphi_j \widehat{f})^\vee(\cdot)|^q\right)^{1/q} \middle| L_p(\mathbb{R}^n)\right\| \tag{7}$$

is finite. In the limiting case $q = \infty$ the usual modification is required.

Remark 2.8 The above spaces were introduced by Edmunds and Triebel in [11, 12] and also considered by Moura in [21, 22] when Ψ is an admissible function. These spaces are independent of the resolution of unity $(\varphi_j)_{j\in\mathbb{N}_0}$, according to (4) and (5), in the sense of equivalent quasi-norms. Taking $\Psi \equiv 1$ brings us back to the classical Besov and Triebel-Lizorkin spaces denoted by $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$, respectively.

Denoting by A either B or F, we have for all $\varepsilon > 0$ the following elementary embeddings between classical spaces and spaces of generalized smoothness

$$A^{s+\varepsilon}_{pq}(\mathbb{R}^n) \hookrightarrow A^{(s,\Psi)}_{pq}(\mathbb{R}^n) \hookrightarrow A^{s-\varepsilon}_{pq}(\mathbb{R}^n).$$

The next assertion on embeddings between Besov and Triebel-Lizorkin spaces of generalized smoothness will enable us to handle continuity envelopes of Triebel-Lizorkin spaces of generalized smoothness in a very simple way by using the results for B-spaces. We refer to [7, Proposition 3.4, Example 3.5] for a proof in the case of Ψ being an admissible function and to [6, Lemma 1] for a more general situation.

Proposition 2.9 *Let Ψ be a slowly varying function. Let $0 < p_0 < p < p_1 \le \infty$, $0 < q \le \infty$ and let $s, s_0, s_1 \in \mathbb{R}$ be such that $s_0 - n/p_0 = s - n/p = s_1 - n/p_1$. Then*

$$B^{(s_0,\Psi)}_{p_0 u}(\mathbb{R}^n) \hookrightarrow F^{(s,\Psi)}_{pq}(\mathbb{R}^n) \hookrightarrow B^{(s_1,\Psi)}_{p_1 v}(\mathbb{R}^n) \quad \textit{if, and only if,} \quad 0 < u \le p \le v \le \infty.$$

The following result gives a characterization of the Besov spaces of generalized smoothness by means of Peetre's maximal functions. The proof runs in the same way as that of [21, Theorem 1.7(i)] for Ψ an admissible function.

Theorem 2.10 *Let $(\varphi_j)_{j\in\mathbb{N}_0}$ be a smooth dyadic resolution of unity as above. Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and let Ψ be a slowly varying function. Let $a > n/p$, then*

$$\left\| f \,|B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\right\|^* := \left(\sum_{j=0}^{\infty} 2^{jsq}\, \Psi(2^{-j})^q \left\| (\varphi_j^* f)_a \,|L_p(\mathbb{R}^n)\right\|^q \right)^{1/q}$$

(with usual modification for $q = \infty$) is an equivalent quasi-norm in $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$, where the Peetre's maximal function $(\varphi_j^* f)_a$ is defined by

$$(\varphi_j^* f)_a(x) := \sup_{z\in\mathbb{R}^n} \frac{|(\varphi_j \widehat{f})^\vee(x-z)|}{(1+2^j|z|)^a} \quad \textit{for } x \in \mathbb{R}^n.$$

An important tool in our later considerations is the characterization of the spaces of generalized smoothness by means of atomic decompositions. We state this here for the B-spaces only (cf. Theorem 2.13 below). We refer to [21, Theorem 1.18(ii)] in case of Ψ being an admissible function and to [2, 14] for a more general situation.

We need some preparation. As for $\mathbb{Z}^n$, it stands for the lattice of all points in $\mathbb{R}^n$ with integer components, $Q_{\nu m}$ denotes a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-\nu}m = (2^{-\nu}m_1, \ldots, 2^{-\nu}m_n)$, and with side length $2^{-\nu}$, where $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q.

Definition 2.11 Let $s \in \mathbb{R}$, $0 < p \le \infty$, $K \in \mathbb{N}_0$, $L+1 \in \mathbb{N}_0$ and $d > 1$. The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an $(s, p, \Psi)_{K,L}$*-atom* if for some $\nu \in \mathbb{N}_0$ the following assumptions are satisfied

(i) supp $a \subset dQ_{\nu m}$ for some $m \in \mathbb{Z}^n$,
(ii) $|\mathrm{D}^\alpha a(x)| \le 2^{-\nu(s-\frac{n}{p})+|\alpha|\nu}\Psi(2^{-\nu})^{-1}$ for $|\alpha| \le K,\ x \in \mathbb{R}^n$,
(iii) $\int_{\mathbb{R}^n} x^\beta a(x)\,\mathrm{d}x = 0$ for $|\beta| \le L$.

If conditions (i) and (ii) are satisfied for $\nu = 0$, then a is called an 1_K*-atom*.

Remark 2.12 In the sequel, we will write $a_{\nu m}$ instead of a, to indicate the localization and size of an $(s, p, \Psi)_{K,L}$-atom a, i.e. if supp $a \subset dQ_{\nu m}$. If $L = -1$, then (iii) simply means that no moment conditions are required.

Theorem 2.13 *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ *and* Ψ *be a slowly varying function. Put* $\sigma_p := n(1/p - 1)_+$. *Let* $K \in \mathbb{N}_0$ *and* $L + 1 \in \mathbb{N}_0$ *with*

$$K \ge (1 + [s])_+ \quad \textit{and} \quad L \ge \max(-1, [\sigma_p - s])$$

be fixed. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}, \quad \textit{convergence being in } \mathcal{S}'(\mathbb{R}^n), \tag{8}$$

where $a_{\nu m}$ *are* 1_K*-atoms* $(\nu = 0)$ *or* $(s, p, \Psi)_{K,L}$*-atoms* $(\nu \in \mathbb{N})$ *and* $\lambda \in b_{pq}$. *Furthermore*

$$\inf \|\lambda\ |b_{pq}\|, \tag{9}$$

where the infimum is taken over all admissible representations (8), *is an equivalent quasi-norm in* $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$.

The next result characterizes the embeddings of $B^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)$ and $F^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)$ into $C(\mathbb{R}^n)$, the space of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^n$, equipped with the sup-norm as usual. The case of Ψ being an admissible function is covered by [8, Proposition 3.11]. We refer to [3, Corollary 3.10 & Remark 3.11] and to [6, Proposition 4.4] for a more general situation.

Theorem 2.14 *Let* $0 < p, q \le \infty$ *and* Ψ *be a slowly varying function.*

(i) *Then*

$$B^{(n/p,\Psi)}_{pq}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \quad \textit{if, and only if,} \quad \left(\Psi(2^{-j})^{-1}\right)_{j \in \mathbb{N}} \in \ell_{q'}.$$

(ii) *Assume* $0 < p < \infty$. *Then*

$$F^{(n/p,\Psi)}_{pq}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \quad \textit{if, and only if,} \quad \left(\Psi(2^{-j})^{-1}\right)_{j \in \mathbb{N}} \in \ell_{p'}.$$

2.4 Continuity Envelopes

The concept of continuity envelopes has been introduced by Haroske in [17] and Triebel in [26]. Here we quote the basic definitions and results concerning continuity envelopes. For further details on this subject we refer to [17, 18, 26].

Recall that the classical Lipschitz space $\mathrm{Lip}^1(\mathbb{R}^n)$ is defined as the space of all functions $f \in C(\mathbb{R}^n)$ such that

$$\|f|\mathrm{Lip}^1(\mathbb{R}^n)\| := \|f|C(\mathbb{R}^n)\| + \sup_{t\in(0,1)} \frac{\omega(f,t)}{t} \tag{10}$$

is finite, where $\omega(f,t)$ stands for the modulus of continuity,

$$\omega(f,t) := \sup_{|h|\le t} \sup_{x\in\mathbb{R}^n} |\Delta_h f(x)| = \sup_{|h|\le t} \|\Delta_h f|L_\infty(\mathbb{R}^n)\|, \quad t>0,$$

with $\Delta_h f(x) := f(x+h) - f(x),\ x,h\in\mathbb{R}^n$.

Definition 2.15 Let X be some function space on $\mathbb{R}^n$ such that $X \hookrightarrow C(\mathbb{R}^n)$.

(i) The *continuity envelope function* $\mathcal{E}_{\mathrm{C}}^X : (0,\infty) \to [0,\infty)$ is defined by

$$\mathcal{E}_{\mathrm{C}}^X(t) := \sup_{\|f|X\|\le 1} \frac{\omega(f,t)}{t}, \quad t>0.$$

(ii) Assume $X \not\hookrightarrow \mathrm{Lip}^1(\mathbb{R}^n)$. Let $\varepsilon\in(0,1)$, $H(t) := -\log h(t)$, $t\in(0,\varepsilon]$, where h is a continuous monotonically decreasing function equivalent to $\mathcal{E}_{\mathrm{C}}^X$ in $(0,\varepsilon]$, and let μ_H be the associated Borel measure. The number u_X, $0<u_X\le\infty$, is defined as the infimum of all numbers v, $0<v\le\infty$, such that

$$\left(\int_0^\varepsilon \left(\frac{\omega(f,t)}{t\,\mathcal{E}_{\mathrm{C}}^X(t)}\right)^v \mu_H(\mathrm{d}t)\right)^{1/v} \le c\,\|f|X\| \tag{11}$$

(with the usual modification if $v=\infty$) holds for some $c>0$ and all $f\in X$. The couple

$$\mathfrak{E}_{\mathrm{C}}(X) = \left(\mathcal{E}_{\mathrm{C}}^X(\cdot), u_X\right)$$

is called *continuity envelope* for the function space X.

As it will be useful in the sequel, we recall some properties of the continuity envelopes. In view of (i) we obtain—strictly speaking—equivalence classes of continuity envelope functions when working with equivalent (quasi-) norms in X as we shall do in the sequel. However, for convenience we do not want to distinguish between representative and equivalence class in what follows and thus stick at the notation introduced in (i). Note that $\mathcal{E}_{\mathrm{C}}^X$ is equivalent to some monotonically decreasing function; for a proof and further properties we refer to [17, 18]. Concerning (ii) it is obvious that (11) holds for $v=\infty$ and any X. Moreover, one verifies that

$$\begin{aligned}\sup_{0<t\le\varepsilon} \frac{g(t)}{\mathcal{E}_{\mathrm{C}}^X(t)} &\le c_1\left(\int_0^\varepsilon \left(\frac{g(t)}{\mathcal{E}_{\mathrm{C}}^X(t)}\right)^{v_1} \mu_H(\mathrm{d}t)\right)^{1/v_1}\\ &\le c_2\left(\int_0^\varepsilon \left(\frac{g(t)}{\mathcal{E}_{\mathrm{C}}^X(t)}\right)^{v_0} \mu_H(\mathrm{d}t)\right)^{1/v_0}\end{aligned} \tag{12}$$

for $0 < v_0 < v_1 < \infty$ and all non-negative monotonically decreasing functions g on $(0, \varepsilon]$; cf. [26, Prop. 12.2, p. 183–184]. So—passing to a monotonically decreasing function equivalent to $\frac{\omega(f,t)}{t}$, see [9, Chap. 2, Lemma 6.1, p. 43]—we observe that the left-hand sides in (11) are monotonically ordered in v and it is natural to look for the smallest possible one.

In the important case when H happens to be continuously differentiable in $(0, \varepsilon]$, we have $\mu_H(\mathrm{d}t) = H'(t)\,\mathrm{d}t$, and for the functions we want to integrate we can calculate the integrals as improper Riemann integrals.

The proposition below gives some properties of continuity envelope functions which will be useful in the sequel. A proof can be seen in [18, Propositions 5.3 & 6.4].

Proposition 2.16

(i) *Let $X_i \hookrightarrow C(\mathbb{R}^n)$, $i = 1, 2$, be some function spaces on $\mathbb{R}^n$. Then $X_1 \hookrightarrow X_2$ implies that there is some positive constant c such that for all $t > 0$,*

$$\mathcal{E}_{\mathrm{C}}^{X_1}(t) \le c\mathcal{E}_{\mathrm{C}}^{X_2}(t).$$

(ii) *Let $X_i \hookrightarrow C(\mathbb{R}^n)$, $i = 1, 2$, be some function spaces on $\mathbb{R}^n$ with $X_1 \hookrightarrow X_2$. Assume for their continuity envelope functions*

$$\mathcal{E}_{\mathrm{C}}^{X_1}(t) \sim \mathcal{E}_{\mathrm{C}}^{X_2}(t), \quad t \in (0, \varepsilon),$$

for some $\varepsilon > 0$. Then, for the corresponding indices u_{X_i}, $i = 1, 2$, we have

$$u_{X_1} \le u_{X_2}.$$

3 Main Results

In this section we present our main results. We first derive the envelope functions for the Besov and Triebel-Lizorkin space of generalized smoothness $B_{pq}^{(s,\Psi)}(\mathbb{R}^n)$ and $F_{pq}^{(s,\Psi)}(\mathbb{R}^n)$, respectively, in the critical case $s = n/p$, provided that Ψ satisfies an appropriate critical condition. We start with the next result about extremal functions, which plays a key role in obtaining the lower estimates for the envelope functions. For related assertions, but different, see [26, pp. 220–221] and [5, Proposition 2.4].

Proposition 3.1 *Let h be the compactly supported C^∞ function on $\mathbb{R}$ defined by $h(y) = e^{-\frac{1}{1-y^2}}$ for $|y| < 1$ and $h(y) = 0$ otherwise. Let $0 < p \le \infty$, $0 < q \le \infty$ and let Ψ be a slowly varying function with $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \in \ell_{q'}$. For each $b = (b_j)_{j\in\mathbb{N}} \in \ell_q$, let f^b be defined by*

$$f^b(x) := \sum_{j=1}^{\infty} b_j\, \Psi(2^{-j})^{-1} \prod_{k=1}^{n} h(2^j x_k), \quad x = (x_k)_{k=1}^n \in \mathbb{R}^n. \tag{13}$$

(i) *Then* $f^b \in B^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)$ *and*

$$\|f^b \,|B^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)\| \le c_1 \,\|b\,|\ell_q\| \tag{14}$$

for some $c_1 > 0$ *independent of* b.

(ii) *If* $b_j \ge 0$, $j \in \mathbb{N}$, *then*

$$\frac{\omega(f^b, 2^{-k})}{2^{-k}} \ge c_2 2^k \sum_{j=k}^{\infty} b_j \Psi(2^{-j})^{-1}, \quad k \in \mathbb{N}, \tag{15}$$

for some $c_2 > 0$ *depending only on the function* h.

Proof Since the functions

$$a_j(x) := \Psi(2^{-j})^{-1} \prod_{k=1}^{n} h(2^j x_k), \quad x = (x_k)_{k=1}^n \in \mathbb{R}^n, \; j \in \mathbb{N},$$

are (up to constants, independently of j) $(n/p, p, \Psi)_{K,-1}$-atoms, for some fixed $K \in \mathbb{N}$ with $K > n/p$, and $b \in \ell_q$, then (14) is an immediate consequence of the atomic decomposition theorem, cf. Theorem 2.13.

Let us now prove (ii). Let $k \in \mathbb{N}$ and let $\eta \in (0,1)$. Then, putting temporarily $c = \prod_{k=2}^n h(0)$, we obtain

$$\begin{aligned}
\frac{\omega(f^b, 2^{-k})}{2^{-k}} &\ge 2^k \big(f^b(0) - f^b(-\eta 2^{-k}, 0, \ldots, 0)\big) \\
&= 2^k \sum_{j=1}^{\infty} b_j \Psi(2^{-j})^{-1} \big(h(0) - h(-\eta 2^{j-k})\big) \cdot c \\
&\ge 2^k \sum_{j=k}^{\infty} b_j \Psi(2^{-j})^{-1} \big(h(0) - h(-\eta 2^{j-k})\big) \cdot c \\
&\ge c_1 2^k \sum_{j=k}^{\infty} b_j \Psi(2^{-j})^{-1}.
\end{aligned}$$

The last but one estimate above holds true, since $h(0) - h(-\eta 2^{j-k}) > 0$ for $j < k$. The last inequality above follows from the fact that $h(0) - h(-\eta 2^{j-k}) \ge h(0) - h(-\eta) > 0$ for all $j \ge k$. This shows the estimate (15). □

In the next proposition we present the continuity envelope functions for the Besov an Triebel-Lizorkin spaces of generalized smoothness in the critical case.

Proposition 3.2 *Let* $0 < p, q \le \infty$ (*with* $p < \infty$ *in the* F-*case*) *and* Ψ *be a slowly varying function. Let* ϕ_u *be defined in* $(0,1]$ *by*

$$\phi_u(t) := \frac{1}{t} \left(\int_0^t \Psi(s)^{-u} \frac{\mathrm{d}s}{s} \right)^{1/u} \quad \textit{if } 0 < u < \infty$$

and

$$\phi_u(t) := \frac{1}{t} \sup_{s\in(0,t)} \Psi(s)^{-1} \quad \textit{if } u = \infty,$$

provided that $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \in \ell_u$.

(i) *Assume* $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \in \ell_{q'}$. *Then*

$$\mathcal{E}_{\mathrm{C}}^{B_{pq}^{(n/p,\Psi)}}(t) \sim \phi_{q'}(t), \quad 0 < t \le \frac{1}{2}.$$

(ii) *Assume* $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \in \ell_{p'}$. *Then, if* $p < \infty$,

$$\mathcal{E}_{\mathrm{C}}^{F_{pq}^{(n/p,\Psi)}}(t) \sim \phi_{p'}(t), \quad 0 < t \le \frac{1}{2}.$$

Proof Taking into account the embeddings

$$B_{p_0p}^{(n/p_0,\Psi)}(\mathbb{R}^n) \hookrightarrow F_{pq}^{(n/p,\Psi)}(\mathbb{R}^n) \hookrightarrow B_{p_1p}^{(n/p_1,\Psi)}(\mathbb{R}^n) \tag{16}$$

for $0 < p_0 < p < p_1 \le \infty$—cf. Proposition 2.9—together with the properties of the continuity envelope function described in Proposition 2.16(i), we shall be concerned only with the Besov space case.

Step 1. We start by proving the upper estimate for the envelope function, that is,

$$\mathcal{E}_{\mathrm{C}}^{B_{pq}^{(n/p,\Psi)}}(t) \lesssim \frac{1}{t}\left(\int_0^t \Psi(s)^{-q'} \frac{\mathrm{d}s}{s}\right)^{1/q'} \quad \text{for } 0 < t < 1 \tag{17}$$

(usual modification if $q' = \infty$, that is, if $0 < q \le 1$).

The first ingredient is the following estimate. Let $k \in \mathbb{N}_0$, $a > 0$ and $|h| \le 2^{-k}$, then

$$\|\Delta_h f|L_\infty(\mathbb{R}^n)\| \le \sum_{j=0}^{k} 2^{j-k} \|(\varphi_j^* f)_a|L_\infty(\mathbb{R}^n)\| + \sum_{j=k+1}^{\infty} \|(\varphi_j^* f)_a|L_\infty(\mathbb{R}^n)\| \tag{18}$$

(cf. [25, formulas (8), (9) on p. 111]).

Suppose that $q \in [1,\infty]$. By (18), Hölder's inequality, Theorem 2.10 and Lemma 2.6, we have

$$\begin{aligned}\frac{\omega(f,2^{-k})}{2^{-k}} &\le \sum_{j=0}^{k} 2^j \Psi(2^{-j})\Psi(2^{-j})^{-1} \|(\varphi_j^* f)_a|L_\infty(\mathbb{R}^n)\| \\ &\quad + 2^k \sum_{j=k+1}^{\infty} \Psi(2^{-j})\Psi(2^{-j})^{-1} \|(\varphi_j^* f)_a|L_\infty(\mathbb{R}^n)\| \\ &\le \left(\sum_{j=0}^{k} 2^{jq'} \Psi(2^{-j})^{-q'}\right)^{1/q'} \left(\sum_{j=0}^{\infty} \Psi(2^{-j})^q \|(\varphi_j^* f)_a|L_\infty(\mathbb{R}^n)\|^q\right)^{1/q}\end{aligned}$$

$$
\begin{aligned}
&+2^k\left(\sum_{j=k+1}^{\infty}\Psi(2^{-j})^{-q'}\right)^{1/q'}\left(\sum_{j=0}^{\infty}\Psi(2^{-j})^{q}\|(\varphi_j^*f)_a|L_\infty(\mathbb{R}^n)\|^q\right)^{1/q}\\
&\sim\left(\sum_{j=0}^{k}2^{jq'}\Psi(2^{-j})^{-q'}\right)^{1/q'}\|f|B^{(0,\Psi)}_{\infty q}(\mathbb{R}^n)\|\\
&\quad+2^k\left(\sum_{j=k+1}^{\infty}\Psi(2^{-j})^{-q'}\right)^{1/q'}\|f|B^{(0,\Psi)}_{\infty q}(\mathbb{R}^n)\|\\
&\sim 2^k\Psi(2^{-k})^{-1}\|f|B^{(0,\Psi)}_{\infty q}(\mathbb{R}^n)\|\\
&\quad+2^k\left(\sum_{j=k+1}^{\infty}\Psi(2^{-j})^{-q'}\right)^{1/q'}\|f|B^{(0,\Psi)}_{\infty q}(\mathbb{R}^n)\|\\
&\lesssim 2^k\left(\sum_{j=k}^{\infty}\Psi(2^{-j})^{-q'}\right)^{1/q'}\|f|B^{(0,\Psi)}_{\infty q}(\mathbb{R}^n)\| \qquad (19)
\end{aligned}
$$

(with the usual modification if $q=\infty$ or $q'=\infty$). Now, (19) together with the elementary embedding $B^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)\hookrightarrow B^{(0,\Psi)}_{\infty q}(\mathbb{R}^n)$ yields

$$
\frac{\omega(f,2^{-k})}{2^{-k}}\lesssim 2^k\left(\sum_{j=k}^{\infty}\Psi(2^{-j})^{-q'}\right)^{1/q'}\|f|B^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)\|. \qquad (20)
$$

Let us now prove a corresponding upper estimate when $0<q\le 1$. Making use of inequality (19) with $q=1$, we have

$$
\frac{\omega(f,2^{-k})}{2^{-k}}\lesssim 2^k\Big(\sup_{j\ge k}\Psi(2^{-j})^{-1}\Big)\|f|B^{(0,\Psi)}_{\infty 1}(\mathbb{R}^n)\|.
$$

This together with the elementary embeddings $B^{(0,\Psi)}_{\infty q}(\mathbb{R}^n)\hookrightarrow B^{(0,\Psi)}_{\infty 1}(\mathbb{R}^n)$, since $0<q\le 1$, and $B^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)\hookrightarrow B^{(0,\Psi)}_{\infty q}(\mathbb{R}^n)$ gives

$$
\begin{aligned}
\frac{\omega(f,2^{-k})}{2^{-k}}&\lesssim 2^k\Big(\sup_{j\ge k}\Psi(2^{-j})^{-1}\Big)\|f|B^{(0,\Psi)}_{\infty q}(\mathbb{R}^n)\|\\
&\lesssim 2^k\Big(\sup_{j\ge k}\Psi(2^{-j})^{-1}\Big)\|f\ |B^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)\|. \qquad (21)
\end{aligned}
$$

Now, for $t\in(0,1]$ let $k\in\mathbb{N}_0$ be such that $2^{-(k+1)}<t\le 2^{-k}$. Then, by (20), (21) and Proposition 2.5 we arrive at

$$
\frac{\omega(f,t)}{t}\lesssim\frac{\omega(f,2^{-(k+1)})}{2^{-(k+1)}}\lesssim 2^{k+1}\left(\sum_{j=k+1}^{\infty}\Psi(2^{-j})^{-q'}\right)^{1/q'}\|f\ |B^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)\|
$$

$$\sim 2^k \left(\int_0^{2^{-(k+1)}} \Psi(s)^{-q'} \frac{ds}{s} \right)^{1/q'} \| f \mid B_{pq}^{(n/p,\Psi)}(\mathbb{R}^n) \|$$

$$\lesssim \frac{1}{t} \left(\int_0^t \Psi(s)^{-q'} \frac{ds}{s} \right)^{1/q'} \| f \mid B_{pq}^{(n/p,\Psi)}(\mathbb{R}^n) \|$$

(with the usual modifications if $q' = \infty$, that is $0 < q \le 1$) which finishes the proof of (17).

Step 2. Let us now prove the lower estimate. We make use of Proposition 3.1.

Suppose that $1 < q \le \infty$. For each $J \in \mathbb{N}$ we denote by f_J^b the function f^b in (13) with $b = (b_j)_{j\in\mathbb{N}}$ given by

$$b_j := \begin{cases} \Psi(2^{-j})^{1-q'} (\sum_{k=J}^{\infty} \Psi(2^{-k})^{-q'})^{-1/q} & \text{for } j \ge J, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $b_j \ge 0,\ j \in \mathbb{N}$. Moreover, we check at once that $\|b|\ell_q\| = 1$. By (14) and (15) we obtain

$$\mathcal{E}_{\mathrm{C}}^{B_{pq}^{(n/p,\Psi)}}(2^{-J}) \ge \frac{\omega(c\, f_J^b, 2^{-J})}{2^{-J}} \gtrsim 2^J \sum_{j=J}^{\infty} b_j \Psi(2^{-j})^{-1}$$

$$= 2^J \left(\sum_{j=J}^{\infty} \Psi(2^{-j})^{-q'} \right)^{1/q'}. \tag{22}$$

Suppose now that $0 < q \le 1$. Let $J \in \mathbb{N}$. For each $k \ge J$, let us consider the sequence $b^k = (b_j^k)_{j\in\mathbb{N}}$ defined by

$$b_j^k := \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{otherwise} \end{cases}$$

For each $k \ge J$, we denote by f^{b^k} the function f^b in (13) with $b = b^k$. For each $k \ge J$ it is clear that $b_j^k \ge 0,\ j \in \mathbb{N}$. Moreover, we check at once that $\|b^k|\ell_q\| = 1$. From (14) and (15) we obtain

$$\mathcal{E}_{\mathrm{C}}^{B_{pq}^{(n/p,\Psi)}}(2^{-J}) \ge \sup_{k\ge J} \frac{\omega(c\, f^{b^k}, 2^{-J})}{2^{-J}} \gtrsim 2^J \sup_{k\ge J} \Psi(2^{-k})^{-1}. \tag{23}$$

Finally, for $0 < t \le 1/2$ choose $J \in \mathbb{N}$ such that $2^{-J-1} < t \le 2^{-J}$. Now, Proposition 2.5, the properties of $\mathcal{E}_{\mathrm{C}}^{B_{pq}^{(n/p,\Psi)}}$ as described in Sect. 2.4, (22) and (23) yield

$$\mathcal{E}_{\mathrm{C}}^{B_{pq}^{(n/p,\Psi)}}(t) \gtrsim \mathcal{E}_{\mathrm{C}}^{B_{pq}^{(n/p,\Psi)}}(2^{-J}) \gtrsim \frac{1}{t} \left(\int_0^t \Psi(s)^{-q'} \frac{ds}{s} \right)^{1/q'}$$

(with the usual modification if $q' = \infty$, that is $0 < q \le 1$) which finishes the proof. □

We refer to [15, Theorem 3.2] and [16, Remark 5 (iv)] for a similar envelope function in the context of Bessel potential type spaces modelled upon Lorentz-Karamata spaces.

Now, we give some examples which follow from Proposition 3.2.

Examples 3.3 Let $0 < p, q \leq \infty$ (with $p < \infty$ in the F-case).

1. Let $a, b \in \mathbb{R}$ and let Ψ be the slowly varying function defined in $(0, 1]$ by

$$\Psi(x) = (1 + |\log x|)^a (1 + \log(1 + |\log x|))^b, \quad x \in (0, 1].$$

(i) If $a > \frac{1}{q'}$, $b \in \mathbb{R}$, then

$$\mathcal{E}_{\mathrm{C}}^{B_{pq}^{(n/p,\Psi)}}(t) = \frac{1}{t}\,(1 + |\log t|)^{-a+\frac{1}{q'}}(1 + \log(1 + |\log t|))^{-b}, \quad 0 < t < \frac{1}{2}.$$

(ii) If $a = \frac{1}{q'}$ and $b > \frac{1}{q'}$, then

$$\mathcal{E}_{\mathrm{C}}^{B_{pq}^{(n/p,\Psi)}}(t) = \frac{1}{t}\,(1 + \log(1 + |\log t|))^{-b+\frac{1}{q'}}, \quad 0 < t < \frac{1}{2}.$$

(We can take $a = 0$ and $b \geq 0$ when $0 < q \leq 1$.)

(iii) If $a > \frac{1}{p'}$, $b \in \mathbb{R}$, then

$$\mathcal{E}_{\mathrm{C}}^{F_{pq}^{(n/p,\Psi)}}(t) = \frac{1}{t}\,(1 + |\log t|)^{-a+\frac{1}{p'}}(1 + \log(1 + |\log t|))^{-b}, \quad 0 < t < \frac{1}{2}.$$

(iv) If $a = \frac{1}{p'}$ and $b > \frac{1}{p'}$, then

$$\mathcal{E}_{\mathrm{C}}^{F_{pq}^{(n/p,\Psi)}}(t) = \frac{1}{t}\,(1 + \log(1 + |\log t|))^{-b+\frac{1}{p'}}, \quad 0 < t < \frac{1}{2}.$$

(We can take $a = 0$ and $b \geq 0$ when $0 < p \leq 1$.)

2. Let $a \in (0, 1)$ and let $b \in \mathbb{R}$.

(i) Let Ψ be the slowly varying function defined in $(0, 1]$ by

$$\Psi(x) = (1 + |\log x|)^{-(a-1)/q'} \exp(b(1 + |\log x|)^a), \quad x \in (0, 1].$$

If $b > 0$, then

$$\mathcal{E}_{\mathrm{C}}^{B_{pq}^{(n/p,\Psi)}}(t) = \frac{1}{t}\exp(-b(1 + |\log t|)^a), \quad 0 < t < \frac{1}{2}.$$

(We can take $b \geq 0$ when $0 < q \leq 1$.)

(ii) Let Ψ be the slowly varying function defined in $(0, 1]$ by

$$\Psi(x) = (1 + |\log x|)^{-(a-1)/p'} \exp(b(1 + |\log x|)^a), \quad x \in (0, 1].$$

If $b > 0$, then

$$\mathcal{E}_{\mathrm{C}}^{F_{pq}^{(n/p,\Psi)}}(t) = \frac{1}{t}\,\exp(-b(1 + |\log t|)^a), \quad 0 < t < \frac{1}{2}.$$

(We can take $b \geq 0$ when $0 < p \leq 1$.)

More examples of slowly varying functions (with appropriate modifications) that satisfy the conditions of Proposition 3.2, can be seen in [24, Remarks 5.7 & 5.8].

In the next theorem we calculate the continuity envelope for the spaces $B_{pq}^{(n/p,\Psi)}(\mathbb{R}^n)$ with $q > 1$ and for the spaces $F_{pq}^{(n/p,\Psi)}(\mathbb{R}^n)$ with $p > 1$.

Theorem 3.4 *Let* $0 < p, q \leq \infty$ (*with* $p < \infty$ *in the* F*-case*) *and* Ψ *be a slowly varying function.*

(i) *Assume* $1 < q \leq \infty$ *and* $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \in \ell_{q'}$. *Then*

$$\mathfrak{E}_{\mathrm{C}}(B_{pq}^{(n/p,\Psi)}(\mathbb{R}^n)) = (\phi_{q'}(t), \infty)$$

(ii) *Assume* $1 < p < \infty$ *and* $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \in \ell_{p'}$. *Then*

$$\mathfrak{E}_{\mathrm{C}}(F_{pq}^{(n/p,\Psi)}(\mathbb{R}^n)) = (\phi_{p'}(t), \infty)$$

Proof By virtue of Proposition 3.2, it only remains to prove that $u_{B_{pq}^{(n/p,\Psi)}(\mathbb{R}^n)} = \infty$ and $u_{F_{pq}^{(n/p,\Psi)}(\mathbb{R}^n)} = \infty$. Furthermore, taking into account the embedding (16), Proposition 3.2 and Proposition 2.16, it is sufficient to prove that assertion for the Besov space case.

Let $1 < q \leq \infty$. Assume that for some $v \in (0, \infty)$ there is a positive constant $c(v)$ such that

$$\left(\int_0^{\varepsilon}\left(\frac{\omega(f,t)}{t\,\phi_{q'}(t)}\right)^{v}\mu_{q'}(\mathrm{d}t)\right)^{\frac{1}{v}} \leq c(v)\|f \mid B_{pq}^{(n/p,\Psi)}(\mathbb{R}^n)\| \tag{24}$$

holds for all $f \in B_{pq}^{(n/p,\Psi)}(\mathbb{R}^n)$, where $\mu_{q'}$ is the Borel measure associated with $-\log\phi_{q'}$ in $(0, \varepsilon]$ for some small $\varepsilon > 0$.

Since $\phi_{q'}$ is continuously differentiable in $(0, \varepsilon]$ (cf. Remark 2.2), the integral on the left-hand side of (24) can be calculated as the improper Riemann integral

$$\int_0^{\varepsilon}\left(\frac{\omega(f,t)}{t\phi_{q'}(t)}\right)^{v}\frac{-\phi_{q'}'(t)}{\phi_{q'}(t)}\,\mathrm{d}t. \tag{25}$$

By the definition of $\phi_{q'}$, we have

$$-\frac{\phi_{q'}'(t)}{\phi_{q'}(t)} = t^{-1}\left(1 - \frac{1}{q'}\frac{\Psi(t)^{-q'}}{\int_0^t \Psi(s)^{-q'}\frac{\mathrm{d}s}{s}}\right), \quad \text{for } t \in (0, \varepsilon),$$

and since

$$\lim_{t\to 0^+}\frac{\Psi(t)^{-q'}}{\int_0^t \Psi(s)^{-q'}\frac{\mathrm{d}s}{s}} = 0$$

(cf. Proposition 2.3 (iii)&(iv)), there is $\delta > 0$ such that

$$-\frac{\phi'_{q'}(t)}{\phi_{q'}(t)} \sim \frac{1}{t}, \quad 0 < t < \delta. \tag{26}$$

We can assume, without loss of generality, that $\varepsilon \le \delta$ and let $k_0 \in \mathbb{N}$ be such that $2^{-k_0} < \varepsilon \le \delta$.

Let $b = (b_j)_{j\in\mathbb{N}} \in \ell_q$ be a sequence of non-negative numbers and let f^b be the corresponding function according to (13). Then Proposition 3.1 yields

$$\| f^b | B^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)\| \le c_1 \|b|\ell_q\| \tag{27}$$

and

$$\frac{\omega(f^b, 2^{-k})}{2^{-k}} \ge c_2 2^k \sum_{j=k}^{\infty} b_j \Psi(2^{-j})^{-1}, \quad k \in \mathbb{N}, \tag{28}$$

with constants that do not depend on b and k, respectively. From (24), (25), (26), using (27), (28), the monotonicity of $\phi_{q'}$ and $\frac{\omega(f^b,t)}{t}$ (up to equivalence), we obtain

$$\begin{aligned}
\|b \mid \ell_q\| &\gtrsim \left(\int_0^{2^{-k_0}} \left(\frac{\omega(f^b,t)}{t\phi_{q'}(t)}\right)^v \mu_{q'}(\mathrm{d}t)\right)^{1/v} \\
&\sim \left(\int_0^{2^{-k_0}} \left(\frac{\omega(f^b,t)}{t\phi_{q'}(t)}\right)^v \frac{\mathrm{d}t}{t}\right)^{1/v} \\
&\sim \left(\sum_{k=k_0}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} \left(\frac{\omega(f^b,t)}{t\phi_{q'}(t)}\right)^v \frac{\mathrm{d}t}{t}\right)^{1/v} \\
&\gtrsim \left(\sum_{k=k_0}^{\infty} \left(\frac{\omega(f^b,2^{-k})}{2^{-k}}\right)^v 2^{-kv} \left(\sum_{j=k}^{\infty} \Psi(2^{-j})^{-q'}\right)^{-v/q'}\right)^{1/v} \\
&\gtrsim \left(\sum_{k=k_0}^{\infty} \left(\sum_{j=k}^{\infty} b_j \Psi(2^{-j})^{-1}\right)^v \left(\sum_{j=k}^{\infty} \Psi(2^{-j})^{-q'}\right)^{-v/q'}\right)^{1/v}.
\end{aligned} \tag{29}$$

Let $\ell \in \mathbb{N}$ with $\ell \ge k_0$. Then the right-hand side of (29) can be further estimated from below by

$$\begin{aligned}
&\left(\sum_{k=k_0}^{\ell} \left(\sum_{j=k}^{\infty} b_j \Psi(2^{-j})^{-1}\right)^v \left(\sum_{j=k}^{\infty} \Psi(2^{-j})^{-q'}\right)^{-v/q'}\right)^{1/v} \\
&\quad \gtrsim \left(\sum_{j=\ell}^{\infty} b_j \Psi(2^{-j})^{-1}\right) \left(\sum_{k=k_0}^{\ell} \left(\sum_{j=k}^{\infty} \Psi(2^{-j})^{-q'}\right)^{-v/q'}\right)^{1/v}.
\end{aligned}$$

Consequently, we arrive at

$$\|b \mid \ell_q\| \gtrsim \left(\sum_{j=\ell}^{\infty} b_j\, \Psi(2^{-j})^{-1}\right)\left(\sum_{k=k_0}^{\ell}\left(\sum_{j=k}^{\infty}\Psi(2^{-j})^{-q'}\right)^{-v/q'}\right)^{1/v}. \tag{30}$$

Let us take

$$b_j := \begin{cases} \Psi(2^{-j})^{1-q'}, & \text{for } j \geq \ell \\ 0, & \text{otherwise.}\end{cases}$$

Then clearly $b = (b_j)_{j\in\mathbb{N}} \in \ell_q$, due to the hypothesis on Ψ. After plugging in this sequence into (30), we conclude that

$$\left(\sum_{j=\ell}^{\infty}\Psi(2^{-j})^{-q'}\right)\left(\sum_{k=k_0}^{\ell}\left(\sum_{j=k}^{\infty}\Psi(2^{-j})^{-q'}\right)^{-v/q'}\right)^{1/v} \lesssim \left(\sum_{j=\ell}^{\infty}\Psi(2^{-j})^{-q'}\right)^{1/q}.$$

This is equivalent to saying that

$$\left(\sum_{k=k_0}^{\ell}\left(\sum_{j=k}^{\infty}\Psi(2^{-j})^{-q'}\right)^{-v/q'}\right)^{1/v}\left(\sum_{j=\ell}^{\infty}\Psi(2^{-j})^{-q'}\right)^{1/q'} \leq c,$$

for some positive constant that doesn't depend on ℓ. Therefore

$$\sup_{\ell\geq k_0}\left(\sum_{k=k_0}^{\ell}\left(\sum_{j=k}^{\infty}\Psi(2^{-j})^{-q'}\right)^{-v/q'}\right)^{1/v}\left(\sum_{j=\ell}^{\infty}\Psi(2^{-j})^{-q'}\right)^{1/q'} < \infty$$

which is equivalent to

$$\sup_{\ell\geq k_0}\frac{\sum_{k=k_0}^{\ell}(\sum_{j=k}^{\infty}\Psi(2^{-j})^{-q'})^{-v/q'}}{(\sum_{j=\ell}^{\infty}\Psi(2^{-j})^{-q'})^{-v/q'}} < \infty. \tag{31}$$

Now remark that, using l'Hôpital rule (since the numerator and the denominator tend to ∞),

$$\lim_{x\to 0+}\frac{\int_x^{2^{-k_0}}(\int_0^t \Psi(s)^{-q'}\frac{ds}{s})^{-v/q'}\frac{dt}{t}}{(\int_0^x \Psi(s)^{-q'}\frac{ds}{s})^{-v/q'}} \sim \lim_{x\to 0+}\frac{\int_0^x \Psi(s)^{-q'}\frac{ds}{s}}{\Psi(x)^{-q'}} = \infty,$$

(cf. Proposition 2.3 (iii)&(iv)). By Proposition 2.5 and a similar discretization procedure for the first integral in the numerator on the left-hand side of the previous estimate, this contradicts (31). Therefore, there is no $v \in (0,\infty)$ such that (24) holds and hence $u_{B_{pq}^{(n/p,\Psi)}(\mathbb{R}^n)} = \infty$. □

Theorem 3.5 *Let* $0 < p, q \leq \infty$ *and* Ψ *be a slowly varying function with* $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \in \ell_\infty$.

(i) *Let* $0 < q \le 1$ *and assume that either* $\Psi \sim 1$ *or* Ψ *is monotonically decreasing with* $\lim_{t\to 0^+} \Psi(t) = \infty$. *Then*

$$\mathfrak{E}_{\mathrm{C}}\big(B^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)\big) = \big(\phi_\infty(t), \infty\big)$$

(ii) *Let* $0 < p \le 1$ *and assume that either* $\Psi \sim 1$ *or* Ψ *is monotonically decreasing with* $\lim_{t\to 0^+} \Psi(t) = \infty$. *Then,*

$$\mathfrak{E}_{\mathrm{C}}\big(F^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)\big) = \big(\phi_\infty(t), \infty\big).$$

Proof Analogously to Theorem 3.4, we only need to prove that $u_{B^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)} = \infty$. Let $0 < q \le 1$. Recall that

$$\phi_\infty(t) = \frac{1}{t} \sup_{s\in(0,t)} \Psi(s)^{-1}, \quad t \in (0,1],$$

and, due to Remark 2.2, we can assume that

$$\Psi(t) = \exp\left\{-\int_t^1 \varrho(s)\frac{\mathrm{d}s}{s}\right\}, \quad t \in (0,1], \tag{32}$$

with ϱ being a C^∞ function on $(0,1]$ with $\lim_{t\to 0^+} \varrho(t) = 0$. Under our conditions, we can take either $\varrho(t) = 0$, $t \in (0,1]$, or $\lim_{t\to 0^+} \varrho(t) = 0^-$.

In our case, we have

$$\phi_\infty(t) = \frac{1}{t}\Psi(t)^{-1}, \quad t \in (0,1],$$

and, using also (32), we obtain

$$-\frac{\phi'_\infty(t)}{\phi_\infty(t)} = \big(1 + \varrho(t)\big)\frac{1}{t} \sim \frac{1}{t}, \quad 0 < t < \delta,$$

for some $\delta > 0$. Following the ideas of the proof of Theorem 3.4, let us assume that for some $v \in (0,\infty)$ there is a positive constant $c(v)$ such that

$$\left(\int_0^\varepsilon \left(\frac{\omega(f,t)}{t\phi_\infty(t)}\right)^v \mu_\infty(\mathrm{d}t)\right)^{\frac{1}{v}} \le c(v)\|f \mid B^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)\|$$

holds for all $f \in B^{(n/p,\Psi)}_{pq}(\mathbb{R}^n)$, where μ_∞ is the Borel measure associated with $-\log\phi_\infty$ in $(0,\varepsilon]$ for some small $\varepsilon > 0$ with $\varepsilon < \delta$. Let $k_0 \in \mathbb{N}$ be such that $2^{-k_0} < \varepsilon$, let $b = (b_j)_{j\in\mathbb{N}} \in \ell_q$ be a sequence of non-negative numbers and let f^b be the corresponding function according to (13). We have again (27) and (28), and as the counterpart of (29) we have

$$\|b \mid \ell_q\| \gtrsim \left(\int_0^{2^{-k_0}} \left(\frac{\omega(f^b,t)}{t\phi_\infty(t)}\right)^v \mu_\infty(\mathrm{d}t)\right)^{1/v}$$

$$\sim \left(\int_0^{2^{-k_0}} \left(\frac{\omega(f^b,t)}{t\phi_\infty(t)}\right)^v \frac{\mathrm{d}t}{t}\right)^{1/v}$$

$$\sim \left(\sum_{k=k_0}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} \left(\frac{\omega(f^b,t)}{t\phi_\infty(t)}\right)^v \frac{\mathrm{d}t}{t}\right)^{1/v}$$

$$\gtrsim \left(\sum_{k=k_0}^{\infty} \left(\frac{\omega(f^b,2^{-k})}{2^{-k}}\right)^v 2^{-kv} \Big(\sup_{0<s<2^{-k}} \Psi(s)^{-1}\Big)^{-v}\right)^{1/v}$$

$$\gtrsim \left(\sum_{k=k_0}^{\infty} \left(\sum_{j=k}^{\infty} b_j \Psi(2^{-j})^{-1}\right)^v \left(\Psi(2^{-k})^{-1}\right)^{-v}\right)^{1/v}.$$

Then, for $\ell \in \mathbb{N}$ with $\ell \geq k_0$, instead of (30), we obtain

$$\|b \mid \ell_q\| \gtrsim \left(\sum_{j=\ell}^{\infty} b_j \Psi(2^{-j})^{-1}\right) \left(\sum_{k=k_0}^{\ell} \Psi(2^{-k})^v\right)^{1/v}. \tag{33}$$

Taking

$$b_j := \begin{cases} 1, & \text{for } j = \ell \\ 0, & \text{otherwise,} \end{cases}$$

leads us to

$$1 \gtrsim \Psi(2^{-\ell})^{-1} \left(\sum_{k=k_0}^{\ell} \Psi(2^{-k})^v\right)^{1/v} \qquad \text{for all } \ell \geq k_0,$$

which is clearly impossible if $\Psi \sim 1$. If $\lim_{t\to 0^+} \Psi(t) = \infty$, it cannot hold as well because, by l'Hôpital rule (since the numerator and the denominator tend to ∞),

$$\lim_{t\to 0^+} \frac{\int_t^{2^{-k_0}} \Psi(s)^v \frac{\mathrm{d}s}{s}}{\Psi(t)^v} = \lim_{t\to 0^+} \frac{-1}{v\varrho(t)} = \infty.$$

Therefore, there is no $v \in (0,\infty)$ such that (32) holds and hence $u_{B_{pq}^{(n/p,\Psi)}(\mathbb{R}^n)} = \infty$. □

Remark 3.6 Note that under the condition $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \in \ell_\infty$, we have that $\liminf_{t\to 0^+} \Psi(t) \neq 0$. Additionally, if $\limsup_{t\to 0^+} \Psi(t) \neq \infty$ then $\Psi \sim 1$ in $(0,1]$.

As an immediate consequence of our results, we give an upper estimate for approximation numbers.

The following result can be found in [4].

Proposition 3.7 *Let X be some Banach space of functions defined on the unit ball U in $\mathbb{R}^n$ with $X(U) \hookrightarrow C(U)$, where $C(U)$ stands for the space of all complex-valued* *bounded uniformly continuous functions on* $\overline{U}$ *equipped with the sup-norm. Then there is some* $c > 0$ *such that for all* $k \in \mathbb{N}$

$$a_k\big(\mathrm{id} : X(U) \longrightarrow C(U)\big) \le c k^{-1/n} \mathcal{E}_{\mathrm{C}}^X(k^{-1/n}),$$

where the k*-th approximation number* a_k *of* $\mathrm{id} : X(U) \longrightarrow C(U)$ *is defined by*

$$a_k(\mathrm{id} : X(U) \longrightarrow C(U)) := \inf\big\{\|\mathrm{id} - L\| : L \in L(X(U), C(U)), \mathrm{rank}\, L < k\big\},$$

with $\mathrm{rank}\, L$ *the dimension of the range of* L.

Combining Proposition 3.2 with the above stated proposition, we obtain the following upper estimate for approximation numbers.

Proposition 3.8 *Let* $0 < p, q \le \infty$ (*with* $p < \infty$ *in the* F*-case*) *and* Ψ *be a slowly varying function.*

(i) *Assume* $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \in \ell_{q'}$. *Then*

$$a_k\big(\mathrm{id} : B_{pq}^{(n/p,\Psi)}(U) \longrightarrow C(U)\big) \le c\left(\sum_{j=[\frac{\log k}{n}]}^{\infty} \Psi(2^{-j})^{-q'}\right)^{1/q'}$$

(*usual modification if* $q' = \infty$).

(ii) *Assume* $(\Psi(2^{-j})^{-1})_{j\in\mathbb{N}} \in \ell_{p'}$. *Then*

$$a_k\big(\mathrm{id} : F_{pq}^{(n/p,\Psi)}(U) \longrightarrow C(U)\big) \le c\left(\sum_{j=[\frac{\log k}{n}]}^{\infty} \Psi(2^{-j})^{-p'}\right)^{1/p'}.$$

(*usual modification if* $p' = \infty$).

Remark 3.9 Assume the conditions from Proposition 3.8.

If $q > 1$, it follows from Proposition 3.8(i) that

$$a_k\big(\mathrm{id} : B_{pq}^{(n/p,\Psi)}(U) \longrightarrow C(U)\big) \to 0 \quad \text{as } k \to \infty.$$

Consequently, the embedding $\mathrm{id} : B_{pq}^{(n/p,\Psi)}(U) \longrightarrow C(U)$ is compact if $q > 1$.

Similarly, by Proposition 3.8(ii), the embedding $\mathrm{id} : F_{pq}^{(n/p,\Psi)}(U) \longrightarrow C(U)$ is compact if $1 < p < \infty$.

When $0 < q \le 1$ for the Besov spaces or $0 < p \le 1$ for the Triebel-Lizorkin spaces, the embeddings mentioned above are not compact (cf. [10, Theorem 2.7.3] for the particular case of $\Psi \sim 1$, and [18, Proposition 11.13] for the more general case).

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