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A Simple Proof of the Matrix-Valued Fejér-Riesz Theorem

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Abstract A very short proof of the Fejér-Riesz lemma is presented in the matrix case.

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The following fundamental result in matrix spectral factorization theory belongs to Wiener [11–13] (see also [5, 6]):

Theorem *Let*

$$S(z) \sim \sum_{n=-\infty}^{\infty} \sigma_n z^n, \tag{1}$$

$|z| = 1$, σ_n *are* $r \times r$ *matrix coefficients, be a positive definite matrix-function with integrable entries*, $S(z) \in L_1(\mathbb{T})$. *If the logarithm of the determinant is integrable*, $\log\det S(z) \in L_1(\mathbb{T})$, *then there exists a factorization*

$$S(z) = \chi^+(z)(\chi^+(z))^*, \tag{2}$$

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where

$$\chi^+(z) = \sum_{n=0}^{\infty} \gamma_n z^n, \tag{3}$$

$|z| < 1$, *is an analytic matrix-function with entries from the Hardy space* H_2, $\chi^+(z) \in H_2$, *and the determinant of which is an outer function.*

The relation (2) is assumed to hold a.e. on the unit circle $\mathbb{T}$ and $(\chi^+)^* = (\overline{\chi^+})^T$ is the adjoint of χ^+.

It is well-known that if $S(z)$ in (1) is a Laurent polynomial of order m, then $\chi^+(z)$ in (3) is a polynomial of the same order m. This result is known as the Fejér-Riesz lemma in the scalar case and it was generalized by Rosenblatt [9] and Helson [5] to the matrix case using the linear prediction theory of multidimensional weakly stationary processes as in the proof of the existence theorem above. The result was further generalized to operator valued functions as well by Rosenblum [10]. Below, we present a simple, transparent, and natural proof of the Fejér-Riesz lemma in the matrix case.

Theorem 1 *If*

$$S(z) = \sum_{n=-m}^{m} \sigma_n z^n \tag{4}$$

is a matrix-valued Laurent polynomial which is positive definite (*a.e. on* $\mathbb{T}$), *then its spectral factor is a matrix-valued polynomial of the same order,*

$$\chi^+(z) = \sum_{n=0}^{m} \gamma_n z^n. \tag{5}$$

We use the generalization of Smirnov's theorem (see [8, p. 109]) which states that if $f(z) = g(z)/h(z)$, $|z| < 1$, where $g \in H_{p_1}$, $p_1 > 0$, and h is an outer function from H_{p_2}, $p_2 > 0$, and the boundary values $f(e^{i\theta}) \in L_p(\mathbb{T})$, $p > 0$, then $f \in H_p$.

Proof We know that

$$(\chi^+(z))^* \in L_2^-(\mathbb{T}) \tag{6}$$

in general, and it suffices to show that

$$z^m(\chi^+(z))^* \in L_2^+(\mathbb{T}) = H_2, \tag{7}$$

where $L_2^-(\mathbb{T})$ and $L_2^+(\mathbb{T})$ are the classes of square integrable functions with, respectively, positive and negative Fourier coefficients equal to 0, and the latter is naturally identified with H_2.

It follows from (2) that

$$(\chi^+(z))^{-1} z^m S(z) = z^m(\chi^+(z))^* \tag{8}$$

for a.a. $z \in \mathbb{T}$. The matrix-function

$$(\chi^+(z))^{-1} = \frac{1}{\det \chi^+(z)} A(z)$$

is analytic in the unit circle, where $A(z) \in H_{2/r}$. Consequently, since $z^m S(z) \in H_\infty$ by hypothesis, the entries of the left-hand side matrix in (8) can be represented as the ratios of some functions from $H_{2/r}$ and the outer function $\det \chi^+(z) \in H_{2/r}$, while their boundary values belong to $L_2(\mathbb{T})$ because of (8) and (6). Thus, by virtue of the above mentioned generalization of Smirnov's theorem, $(\chi^+(z))^{-1} z^m S(z) \in H_2 = L_2^+(\mathbb{T})$ and (7) follows again from (8). □

The same idea can be used to prove the uniqueness (up to a constant unitary matrix) of the spectral factorization (2). Indeed (cf. [1, p. 766]), assume $S(z) = \rho^+(z)(\rho^+(z))^*$ together with (2), where $\rho^+(z) \in H_2$ and $\det \rho^+(z)$ is outer. Then

$$(\chi^+(z))^{-1} \rho^+(z)((\chi^+(z))^{-1} \rho^+(z))^* = I, \tag{9}$$

so that the analytic matrix-function $U(z) := (\chi^+(z))^{-1} \rho^+(z)$, $|z| < 1$, is unitary for a.a. $z \in \mathbb{T}$. Thus, the boundary values of $U(z)$ belong to L_∞ and, as in the proof of Theorem 1, we have $U(z) \in H_\infty$. By changing the roles of χ^+ and ρ^+ in this discussion, we get $(\rho^+(z))^{-1} \chi^+(z) \in H_\infty$. But $(U(z))^* = (\rho^+(z))^{-1} \chi^+(z)$ for a.a. $z \in \mathbb{T}$, by virtue of (9). Thus the boundary values of $U(z)$ as well their conjugate belong to $L_\infty^+(\mathbb{T})$ which implies that $U(z)$ is constant.

Remark 1 The matrix-function $S(z)$ is defined in $\mathbb{C}\backslash\{0\}$ by (4), and the spectral factorization representation (2) on $\mathbb{T}$ can be extended to this set by the equation

$$S(z) = \chi^+(z)(\chi^+)^*(1/z), \quad z \in \mathbb{C}\backslash\{0\}, \tag{10}$$

where $(\chi^+)^*(z) = \sum_{n=0}^m \overline{\gamma_n}^T z^n$, $z \in \mathbb{C}$.

Remark 2 Theorem 1 can be used to handle a rank-deficient situation as well whenever (4) is positive semi-definite on $\mathbb{T}$ instead of positive definite. In this case the spectral factor is not unique anymore in general, but still there exists a polynomial matrix (5) such that the representation (10) holds (cf. [4]). To see this, we can factorize the positive definite matrix-valued Laurent polynomial $S(z) + \varepsilon I$, where $\varepsilon > 0$ and I is r-dimensional unit matrix, according to Theorem 1,

$$S(z) + \varepsilon I = \chi_\varepsilon^+(z)(\chi_\varepsilon^+)^*(1/z),$$

and, letting $\varepsilon \to 0$, extract a convergent subsequence from χ_ε^+. The equations

$$\sum_{j=1}^r \left|[\chi_\varepsilon^+]_{ij}(z)\right|^2 = [S]_{ii}(z) + \varepsilon, \quad z \in \mathbb{T},\ i, j = 1, 2, \ldots, r,$$

guarantee that the polynomials (of order m) $[\chi_\varepsilon^+]_{ij}(z)$, $i, j = 1, 2, \ldots, r$ are uniformly bounded on $\mathbb{T}$, so that convergent subsequences exist.

A further development of the circle of ideas presented in this paper leads to the constructive analytic proof of the existence theorem, formulated in the beginning, as well as to the efficient algorithm for approximate computation of the matrix coefficients γ_n in (3) for a given matrix-function (1) (see [2, 3, 7]). Such an algorithm is very important for practical applications and it has been searched since Wiener's period.

References

1. Barclay, S.: Continuity of the spectral factorization mapping. J. Lond. Math. Soc. (2) **70**, 763–779 (2004)
2. Ephremidze, L., Janashia, G., Lagvilava, E.: A new efficient matrix spectral factorization algorithm. In: Proceedings of the SICE Annual Conference, Kagawa University, Japan (2007)
3. Ephremidze, L., Janashia, G., Lagvilava, E.: An analytic proof of the matrix spectral factorization theorem. Georgian Math. J. **15**, 241–249 (2008)
4. Hardin, D.P., Hogan, T.A., Sun, Q.: The matrix-valued Riesz lemma and local orthonormal bases in shift-invariant spaces. Adv. Comput. Math. **20**, 367–384 (2004)
5. Helson, H.: Lectures on Invariant Subspaces. Academic, San Diego (1964)
6. Helson, H., Lowdenslager, D.: Prediction theory and Fourier series in several variables. Acta Math. **99**, 165–201 (1958)
7. Janashia, G., Lagvilava, E.: A method of approximate factorization of positive definite matrix functions. Stud. Math. **137**, 93–100 (1999)
8. Koosis, P.: Introduction to H_p Spaces. Cambridge University Press, Cambridge (1980)
9. Rosenblatt, M.: A multidimensional prediction problem. Ark. Mat. **3**, 407–424 (1958)
10. Rosenblum, M.: Vectorial Toeplitz operators and the Fejér-Riesz theorem. J. Math. Anal. Appl. **23**, 139–147 (1963)
11. Wiener, N.: On the factorization of matrices. Comment. Math. Helv. **29**, 97–111 (1955)
12. Wiener, N., Masani, P.: The prediction theory of multivariate stochastic processes, I. Acta Math. **98**, 111–150 (1957)
13. Wiener, N., Akutowicz, E.J.: A factorization of positive Hermitian matrices. J. Math. Mech. **8**, 111–120 (1959)