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Pointwise Summability of Gabor Expansions

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Abstract A general summability method, the so-called θ-summability method is considered for Gabor series. It is proved that if the Fourier transform of θ is in a Herz space then this summation method for the Gabor expansion of f converges to f almost everywhere when $f \in L_1$ or, more generally, when $f \in W(L_1, \ell_\infty)$ (Wiener amalgam space). Some weak type inequalities for the maximal operator corresponding to the θ-means of the Gabor expansion are obtained. Hardy-Littlewood type maximal functions are introduced and some inequalities are proved for these.

Keywords Wiener amalgam spaces · Herz spaces · θ-summability · Gabor expansions · Gabor frames · Time-frequency analysis · Hardy-Littlewood inequality

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1 Introduction

It is known that the partial sums of the Fourier series of an $f \in L_p(\mathbb{T})$ $(1 < p < \infty)$ converge to f in norm and almost everywhere (see Carleson [5]). This result is not true for $p = 1$ and $p = \infty$. However, for some well known summability methods, such as those named after Fejér, Riesz, Weierstrass, Abel, etc., the corresponding means $\sigma_n f$ of the Fourier series of f converge to f uniformly as $n \to \infty$ if f is

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continuous, and in L_p norm and almost everywhere if $f \in L_p(\mathbb{T})$ for some p, $1 \le p < \infty$ (see, e.g., Fejér [14], Zygmund [32], Butzer and Nessel [4], Stein and Weiss [28] or Trigub and Belinsky [29]).

A general method of summation, the so called θ-summation method, which is generated by a single function θ is studied intensively in the literature (see, e.g., Butzer and Nessel [4], Trigub and Belinsky [29] and Weisz [30, 31] and the references therein). If the Fourier transform of θ is integrable then the norm convergence results just mentioned hold for the θ-summation method, too (see Butzer and Nessel [4], Trigub and Belinsky [29] or Feichtinger and Weisz [10]). The issue of almost everywhere convergence for the θ-summation method is much more complicated. We proved in [11] that $\sigma_n f \to f$ a.e. (more precisely, at each p-Lebesgue point of f) for all $f \in L_p(\mathbb{T}^d)$, whenever $\hat{\theta}$ is in the homogeneous Herz space $\dot{E}_{p'}(\mathbb{R}^d)$ where $1 \le p < \infty$ and $1/p + 1/p' = 1$.

In this paper we will consider the analogous results for summation of Gabor expansions

$$\sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k} \gamma$$

of $f \in W(L_p, \ell_q)(\mathbb{R}^d)$, where $\alpha, \beta > 0$, $g, \gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$, and M denotes the modulation operator and T the translation operator. Here $W(L_p, \ell_q)(\mathbb{R}^d)$ denotes the Wiener amalgam spaces and note that $W(L_p, \ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$. We suppose that the window functions g and γ generate dual Gabor frames for $L_2(\mathbb{R}^d)$. Gröchenig, Heil and Okoudjou [19, 20] proved the norm convergence of the partial sums of the Gabor series of $f \in W(L_p, \ell_q)(\mathbb{R}^d)$, $1 < p < \infty$, $1 \le q < \infty$. The same result was also shown by Grafakos and Lennard [17] for $L_p(\mathbb{R}^d)$ $(1 < p < \infty)$ functions and for special window functions from the Schwartz class. If $\hat{\theta}$ is integrable and $1 \le p < \infty$ then Feichtinger and Weisz [12] verified the norm convergence of the θ-summation method (see also Grafakos and Lennard [17] for the Fejér means). The main result in [12] is that if $\hat{\theta}$ is in a suitable Herz space and $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$ with $1 < p < \infty$ then the θ-means of the Gabor series of f converge to f a.e. Note that the space $W(L_p, \ell_\infty)(\mathbb{R}^d)$ contains $W(L_p, \ell_q)(\mathbb{R}^d)$ for each $1 \le p < \infty$, $1 \le q \le \infty$.

Unfortunately, the proof of the a.e. convergence of the θ-means just mentioned does not work for $p = 1$. In the present paper this theorem will be proved for $p = 1$ which demands completely different ideas. This is a significant generalization of the result in [12] since $W(L_1, \ell_\infty)(\mathbb{R}^d) \supset W(L_p, \ell_\infty)(\mathbb{R}^d)$ for $1 < p < \infty$. First, we rewrite the θ-means $\sigma_{K,N}^\theta f$ of the Gabor series of f as an integral operator and prove some elementary results about the θ-kernels. From this it follows easily that the operator $\sigma_{K,N}^\theta$ is bounded on $L_p(\mathbb{R}^d)$ $(1 \le p < \infty)$ for all $K, N \in \mathbb{N}$, whenever $\hat{\theta} \in L_1(\mathbb{R}^d)$. We estimate the supremum of the θ-means (called maximal θ-operator) of the Gabor series by a new Hardy-Littlewood type operator pointwise (see Theorems 6 and 7). The most complex part of the paper, which needs essentially new ideas, is to estimate the weak (and strong) L_p-norms of this Hardy-Littlewood type operator by the corresponding norm of f (see Theorems 2 and 3). This generalizes the well known inequalities with respect to the classical Hardy-Littlewood operator. From this it follows that the maximal θ-operator is bounded on the Wiener amalgam spaces. As the main result of this paper we show that the θ-means $\sigma_{K,N}^\theta f$ of the Gabor series of f converge almost everywhere to f for all $f \in W(L_p, \ell_\infty)(\mathbb{R}^d) \supset L_p(\mathbb{R}^d)$, whenever $\hat{\theta}$ is in the Herz space $\dot{E}_{p'}(\mathbb{R}^d)$, where $1 \le p < \infty$, $1/p + 1/p' = 1$. Of course, this includes the convergence result for each $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$. As special cases of the θ-summation method we investigate all well known summability methods, such as the summation methods of Fejér, Riesz, Weierstrass, Abel, Picard, Bessel, Riemann, de La Vallée-Poussin and Rogosinski.

2 Wiener Amalgam Spaces

Let us fix $d \ge 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \ne \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \cdots \times \mathbb{Y}$ taken with itself d-times. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \dots, u_d) \in \mathbb{R}^d$ set

$$u \cdot x := \sum_{k=1}^{d} u_k x_k \quad \text{and} \quad |x| := \max_{k=1,\dots,d} |x_k|.$$

The norm (or quasi-norm) of the usual $L_p(\mathbb{R}^d)$ space is given by $\|f\|_p := (\int_{\mathbb{R}^d} |f|^p \, d\lambda)^{1/p}$ $(0 < p \le \infty)$, where λ is the Lebesgue measure. The space of continuous functions with the supremum norm is denoted by $C(\mathbb{R}^d)$.

The *weak* L_p space, $L_{p,\infty}(\mathbb{R}^d)$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}} := \sup_{\rho > 0} \rho \lambda(|f| > \rho)^{1/p} < \infty,$$

while we set $L_{\infty,\infty}(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$. Note that $L_{p,\infty}(\mathbb{R}^d)$ is a quasi-normed space (see Bergh and Löfström [3]). It is easy to see that for each $0 < p \le \infty$,

$$L_p(\mathbb{R}^d) \subset L_{p,\infty}(\mathbb{R}^d) \quad \text{and} \quad \|\cdot\|_{L_{p,\infty}} \le \|\cdot\|_p.$$

Translation and *modulation* of a function f are defined, respectively, by

$$T_x f(t) := f(t - x) \quad \text{and} \quad M_\omega f(t) := e^{2\pi \imath \omega \cdot t} f(t) \quad (x, \omega \in \mathbb{R}^d).$$

For a set H we use the notation $T_x H := H - x$. The *Fourier transform* of $f \in L_1(\mathbb{R}^d)$ is

$$\mathcal{F} f(x) := \hat{f}(x) := \int_{\mathbb{R}^d} f(t) e^{-2\pi \imath x \cdot t} \, dt \quad (x \in \mathbb{R}^d),$$

where $\imath = \sqrt{-1}$.

Let Q and Q_α denote the cubes

$$Q = [0, 1)^d, \qquad Q_\alpha = [0, \alpha)^d \quad (\alpha > 0).$$

A measurable function f belongs to the *Wiener amalgam space* $W(L_p, \ell_q)(\mathbb{R}^d)$ $(1 \le p, q \le \infty)$ if

$$\|f\|_{W(L_p,\ell_q)} := \left(\sum_{k \in \mathbb{Z}^d} \|f(\cdot + k)\|_{L_p(Q)}^q \right)^{1/q} = \left(\sum_{k \in \mathbb{Z}^d} \|f \cdot T_k \mathbf{1}_Q\|_p^q \right)^{1/q} < \infty,$$

with the obvious modification for $q=\infty$. It is easy to show that the norm

$$\|f\| := \left(\sum_{k\in\mathbb{Z}^d} \|f\cdot T_{\alpha k}\mathbf{1}_{Q_\alpha}\|_p^q \right)^{1/q} < \infty$$

is an equivalent norm on $W(L_p,\ell_q)(\mathbb{R}^d)$. If we replace the space $L_p(Q)$ by $L_{p,\infty}(Q)$ then we get the definition of $W(L_{p,\infty},\ell_q)(\mathbb{R}^d)$. The smallest closed subspace of $W(L_\infty,\ell_q)(\mathbb{R}^d)$ containing continuous functions is denoted by $W(C,\ell_q)(\mathbb{R}^d)$ $(1\le q\le\infty)$. The space $W(C,\ell_1)(\mathbb{R}^d)$ is called *Wiener algebra*.

Observe that $W(L_p,\ell_p)(\mathbb{R}^d)=L_p(\mathbb{R}^d)$,

$$W(L_{p_1},\ell_q)(\mathbb{R}^d) \hookleftarrow W(L_{p_2},\ell_q)(\mathbb{R}^d) \quad (p_1\le p_2)$$

and

$$W(L_p,\ell_{q_1})(\mathbb{R}^d) \hookrightarrow W(L_p,\ell_{q_2})(\mathbb{R}^d) \quad (q_1\le q_2)$$

$(1\le p_1,p_2,q_1,q_2\le\infty)$. Thus

$$W(L_\infty,\ell_1)(\mathbb{R}^d)\subset L_p(\mathbb{R}^d)\subset W(L_1,\ell_\infty)(\mathbb{R}^d) \quad (1\le p\le\infty).$$

In this paper the constants C and C_p may vary from line to line and the constants C_p are depending only on p, α and β.

3 Gabor Frames

Given a window $g\in L_2(\mathbb{R}^d)$ and $\alpha,\beta>0$ we say that the collection

$$\mathcal{G}(g,\alpha,\beta) := \{M_{\beta n}T_{\alpha k}g : k,n\in\mathbb{Z}^d\}$$

is a *Gabor frame* for $L_2(\mathbb{R}^d)$ if there exist constants $A,B>0$ such that

$$A\|f\|_2^2 \le \sum_{k,n\in\mathbb{Z}^d} |\langle f, M_{\beta n}T_{\alpha k}g\rangle|^2 \le B\|f\|_2^2$$

for all $f\in L_2(\mathbb{R}^d)$. In this case there exists a *dual window* $\gamma\in L_2(\mathbb{R}^d)$ such that $\mathcal{G}(\gamma,\alpha,\beta)$ is also a Gabor frame for $L_2(\mathbb{R}^d)$ and

$$f = \sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n}T_{\alpha k}g\rangle M_{\beta n}T_{\alpha k}\gamma = \sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n}T_{\alpha k}\gamma\rangle M_{\beta n}T_{\alpha k}g \tag{1}$$

for all $f\in L_2(\mathbb{R}^d)$. This series converges unconditionally in $L_2(\mathbb{R}^d)$ and the ℓ_2 norm of the Gabor coefficients $(\langle f, M_{\beta n}T_{\alpha k}\gamma\rangle)$ is an equivalent norm on $L_2(\mathbb{R}^d)$. For more details we refer to Daubechies [6, Chap. 3] or Gröchenig [18, Chap. 5].

Under some stronger condition on g and γ, (1) is valid for other function spaces, too. If g,γ is in the Feichtinger's algebra then (1) holds for modulation spaces (see Feichtinger and Zimmermann [13] and Gröchenig [18]) and if $g,\gamma\in W(L_\infty,\ell_1)$

then for L_p and amalgam spaces (Gröchenig, Heil and Okoudjou [19, 20] and Feichtinger and Weisz [12]). In the last case the convergence is conditional, first we sum over n and then over k.

Recall that the nth *Fourier coefficient* of a $1/\beta$ periodic function $h \in L_1(Q_{1/\beta})$ is given by

$$\hat{h}(n) := \beta^d \int_{Q_{1/\beta}} h(t) e^{-2\pi \imath \beta n \cdot t} \, dt \quad (n \in \mathbb{Z}^d).$$

For $g \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ and $f \in W(L_p, \ell_q)(\mathbb{R}^d)$ $(1 \le p, q \le \infty)$ define the $1/\beta$-periodic function $m_{g,k}$ by

$$m_{g,k}(x) := \beta^{-d} \sum_{n \in \mathbb{Z}^d} (f \cdot T_{\alpha k} \overline{g})(x - n/\beta) \quad (k \in \mathbb{Z}^d). \tag{2}$$

It is proved in Gröchenig, Heil and Okoudjou [20] and Feichtinger and Weisz [12] that $m_{g,k} \in L_p(Q_{1/\beta})$ and the sum converges unconditionally in $L_p(Q_{1/\beta})$. Moreover, $\hat{m}_{g,k}(n) = \langle f, M_{\beta n} T_{\alpha k} g \rangle$ for all $n \in \mathbb{Z}^d$ and so the Fourier series of $m_{g,k}$ is

$$m_{g,k}(x) \sim \sum_{n \in \mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k} g \rangle e^{2\pi \imath \beta n \cdot x}.$$

If $1 < p < \infty$ then the rectangular partial sums of this Fourier series converge to $m_{g,k}$ in $L_p(Q_{1/\beta})$ norm (cf. Zygmund [32] or Weisz [31]).

The following theorem is proved by Gröchenig, Heil and Okoudjou [19, 20] (see also Feichtinger and Weisz [12] and Balan and Daubechies [1]).

Theorem 1 ([19, 20]) *Assume that $g, \gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ and $f \in W(L_p, \ell_q)(\mathbb{R}^d)$ for some $1 \le p, q \le \infty$. Then the series defining the operator*

$$R_{g,\gamma} f := \sum_{k \in \mathbb{Z}^d} m_{g,k} T_{\alpha k} \gamma \tag{3}$$

converges unconditionally in $W(L_p, \ell_q)(\mathbb{R}^d)$ norm if $1 \le q < \infty$ and unconditionally in the weak topology of $W(L_p, \ell_\infty)(\mathbb{R}^d)$ if $q = \infty$. Moreover, $R_{g,\gamma}$ is bounded on $W(L_p, \ell_q)(\mathbb{R}^d)$, i.e.*

$$\|R_{g,\gamma} f\|_{W(L_p,\ell_q)} \le C \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_p,\ell_q)}. \tag{4}$$

If $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L_2(\mathbb{R}^d)$ with dual frame $\mathcal{G}(\gamma, \alpha, \beta)$ then $R_{g,\gamma} f = f$.

4 Hardy-Littlewood Type Operators

In this section we introduce new operators, which are analogous to the well known Hardy-Littlewood operator. The investigations about these operators will play an important role in the proof of the pointwise convergence of the θ-summability method.

First we introduce a slight modification of the classical Hardy-Littlewood operator. For $1 \le p \le \infty$ let us define

$$M_p f(x) := \sup_{x\in I}\left(\frac{1}{|I|}\int_I |f|^p\, d\lambda\right)^{1/p} \quad (x \in \mathbb{R}^d)$$

with the usual modification for $p = \infty$, where the supremum is taken over all cubes with sides parallel to the axes. Usually the operator with $p = 1$ is called Hardy-Littlewood operator. We have for $1 \le p \le \infty$ that

$$\|M_p f\|_{L_{p,\infty}} \le C_p \|f\|_p \quad (f \in L_p(\mathbb{R}^d)) \tag{5}$$

and

$$\|M_p f\|_r \le C_r \|f\|_r \quad (f \in L_r(\mathbb{R}^d),\ p < r \le \infty) \tag{6}$$

(see Stein [27] or Feichtinger and Weisz [11]).

Depending on the two window functions g and γ we introduce the *Hardy-Littlewood type maximal function* by

$$M_{g,\gamma,p} f(x) = \sup_{x\in I} |I|^{-1/p}\left(\int_I |f_{g,\gamma}(x,t)|^p\, dt\right)^{1/p}, \tag{7}$$

where

$$f_{g,\gamma}(x,t) := \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} |f(t+n/\beta)||T_{\alpha k} g(t+n/\beta)||T_{\alpha k}\gamma(x)|$$

and the supremum is taken over all cubes with sides parallel to the axes.

The classical inequalities (5) and (6) are extended in the next theorem.

Theorem 2 *Let $g, \gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ and $1 \le p \le \infty$. Then*

$$\|M_{g,\gamma,p} f\|_{L_{p,\infty}} \le C_p \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_p \tag{8}$$

for all $f \in L_p(\mathbb{R}^d)$. Moreover, for every $p < q \le \infty$ and $f \in L_q(\mathbb{R}^d)$,

$$\|M_{g,\gamma,p} f\|_q \le C_q \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_q. \tag{9}$$

By taking the supremum over all cubes I for which $|I| \le 1$, only, we define the *local Hardy-Littlewood type maximal function*, i.e.,

$$m_{g,\gamma,p} f(x) = \sup_{x\in I, |I|\le 1} |I|^{-1/p}\left(\int_I |f_{g,\gamma}(x,t)|^p\, dt\right)^{1/p}.$$

Recall that

$$\|f\|_{W(L_{p,\infty},\ell_\infty)} = \sup_{k\in\mathbb{Z}^d}\sup_{\rho>0} \rho\, \lambda(|f| > \rho, [k,k+1))^{1/p}.$$

The analogous results to Theorem 2 for the local maximal function are given in the next theorem.

Theorem 3 *Let* $g, \gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ *and* $1 \le p \le \infty$. *Then*

$$\|m_{g,\gamma,p} f\|_{W(L_{p,\infty},\ell_\infty)} \le C_p \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_p,\ell_\infty)} \tag{10}$$

for all $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$ *and*

$$\|m_{g,\gamma,p} f\|_{L_{p,\infty}} \le C_p \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_p \tag{11}$$

for all $f \in L_p(\mathbb{R}^d)$.

The next corollary follows easily from Theorem 2.

Corollary 1 *Let* $g, \gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$, $1 \le p < q \le \infty$ *and* $1 \le r \le \infty$. *Then*

$$\|m_{g,\gamma,p} f\|_{W(L_q,\ell_r)} \le C_q \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_q,\ell_r)}$$

for all $f \in W(L_q, \ell_r)(\mathbb{R}^d)$.

5 The Summability Kernel

The Nth square partial sum of a function $h \in L_p(Q_{1/\beta})$ is denoted by $S_N h$,

$$S_N h(x) := \sum_{|n| \le N} \hat{h}(n) e^{2\pi \imath \beta n \cdot x} \quad (N \in \mathbb{N}).$$

The $L_p(Q_{1/\beta})$ norm $(1 < p < \infty)$ convergence of $S_N h$ can be found in Zygmund [32] or Weisz [31]. Moreover, according to one of the deepest results in harmonic analysis, $S_N h$ converges a.e. to $h \in L_p(Q_{1/\beta})$ $(1 < p < \infty)$ (see Carleson [5], Hunt [23] and in the higher-dimensional case Fefferman [7] and also Grafakos [16]), i.e.

$$S_N h \to h \quad \text{in } L_p(Q_{1/\beta}) \text{ norm and a.e. as } N \to \infty. \tag{12}$$

If $p = 1$ then the results in (12) are not true. However, using a summability method, say the Fejér's method, we can extend (12). Summability methods are used quite often in Fourier analysis. For the theory of summation see e.g. Butzer and Nessel [4], Trigub and Belinsky [29] and Weisz [31]. The Nth Fejér mean of the Fourier series of $h \in L_1(Q_{1/\beta})$ is given by

$$\sigma_N h(x) := \sum_{|n| \le N} \left(\prod_{j=1}^d \left(1 - \frac{|n_j|}{N+1} \right) \right) \hat{h}(n) e^{2\pi \imath \beta n \cdot x} \quad (N \in \mathbb{N}).$$

Then

$$\sigma_N h \to h \quad \text{in } L_p(Q_{1/\beta}) \text{ norm and a.e. as } N \to \infty, \tag{13}$$

whenever $1 \le p < \infty$ (see Marcinkiewicz and Zygmund [26, 32] or Weisz [31]).

Instead of Fejér summation we may take a general summability method, the so called θ-summability defined by one single function θ. For $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ the Nth θ-mean of the Fourier series of $h \in L_1(Q_{1/\beta})$ is defined by

$$\sigma_N^\theta h(x) := \sum_{n\in\mathbb{Z}^d} \theta\left(\frac{-n}{N+1}\right)\hat{h}(n)e^{2\pi\imath\beta n\cdot x}.$$

If $\theta = \mathbf{1}_{(-1,1)^d}$ then we get the partial sums, if $\theta(x) = \prod_{j=1}^d \max(0, 1-|x_j|)$ then the Fejér means.

In Feichtinger and Weisz [10, 11] we verified the analogous statements to (13) for θ-summability. If $\hat{\theta} \in L_1(\mathbb{R}^d)$ then

$$\sigma_N^\theta h \to \theta(0)h \quad \text{in } L_p(Q_{1/\beta}) \text{ norm as } N \to \infty \tag{14}$$

for all $h \in L_p(Q_{1/\beta})$ $(1 \le p < \infty)$.

The a.e. convergence of $\sigma_N^\theta h$ is much more complicated. In Feichtinger and Weisz [11] we applied the *homogeneous Herz spaces* in summability theory. $\dot{E}_q(\mathbb{R}^d)$ contains all measurable functions f for which

$$\|f\|_{\dot{E}_q} := \sum_{k=-\infty}^{\infty} 2^{kd(1-1/q)}\|f\mathbf{1}_{P_k}\|_q < \infty,$$

where $P_k := \{x : 2^{k-1} \le |x| < 2^k\}$. These spaces are special cases of the Herz spaces [22] (see also Feichtinger [8], Garcia-Cuerva and Herrero [15]). It is easy to see that

$$L_1(\mathbb{R}^d) = \dot{E}_1(\mathbb{R}^d) \hookleftarrow \dot{E}_q(\mathbb{R}^d) \hookleftarrow \dot{E}_r(\mathbb{R}^d) \hookleftarrow \dot{E}_\infty(\mathbb{R}^d), \quad 1 < q < r < \infty.$$

In this way we obtained ([11]) the following result: if $\hat{\theta} \in \dot{E}_{p'}(\mathbb{R}^d)$ then

$$\sigma_N^\theta h \to \theta(0)h \quad \text{a.e. as } N \to \infty \tag{15}$$

for all $h \in L_p(Q_{1/\beta})$, where $1 \le p < \infty$ and $1/p + 1/p' = 1$. Actually, the convergence holds at every Lebesgue point. Some sufficient conditions for θ such that $\hat{\theta} \in \dot{E}_\infty(\mathbb{R}^d)$ and many examples can be found in Sect. 8.

These results are generalized for Gabor series as follows. The partial sums and the θ-means are defined for Gabor series by

$$S_{g,\gamma,K,N} f := S_{K,N} f := \sum_{|k|\le K}\sum_{|n|\le N} \langle f, M_{\beta n}T_{\alpha k}g\rangle M_{\beta n}T_{\alpha k}\gamma$$

and

$$\sigma_{g,\gamma,K,N}^\theta f := \sigma_{K,N}^\theta f := \sum_{|k|\le K}\sum_{n\in\mathbb{Z}^d} \theta\left(\frac{-n}{N+1}\right)\langle f, M_{\beta n}T_{\alpha k}g\rangle M_{\beta n}T_{\alpha k}\gamma,$$

respectively, where $g,\gamma \in W(L_\infty,\ell_1)(\mathbb{R}^d)$, $f \in W(L_1,\ell_\infty)(\mathbb{R}^d)$ and $\theta \in W(C,\ell_1)(\mathbb{R}^d)$. Note that the summation method just defined is not symmetric. It is taken in the frequency domain, only. As we can see later we obtain results for the partial sums in the time domain and for the summability means in the frequency domain, so it is not necessary to consider summability in the time domain, because we verify "better" results. However, we can define a symmetric summability method, see Theorem 9 later. Observe that the last series is absolutely convergent, because

$$|\langle f, M_{\beta n} T_{\alpha k} g\rangle| \le \|f\|_{W(L_1,\ell_\infty)} \|g\|_{W(L_\infty,\ell_1)}$$

and

$$\sum_{n\in\mathbb{Z}^d} \left|\theta\left(\frac{-n}{N+1}\right)\right| \le (N+1)^d \|\theta\|_{W(C,\ell_1)} < \infty. \tag{16}$$

The following convergence theorem is known for Gabor series (see Gröchenig, Heil and Okoudjou [19, 20] and Feichtinger and Weisz [12]). Note that Fejér summation of Gabor series for $L_1(\mathbb{R}^d)$ spaces and for special window functions from the Schwartz class was investigated in Grafakos and Lennard [17], too. Another type of summation method was considered for $L_2(\mathbb{R}^d)$ spaces in Lyubarskii and Seip [25].

Theorem 4 ([12, 19, 20]) *Assume that $g, \gamma \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L_2(\mathbb{R}^d)$ with dual frame $\mathcal{G}(\gamma, \alpha, \beta)$. Let $f \in W(L_p, \ell_q)(\mathbb{R}^d)$.*

(i) *If $1 < p < \infty$ and $1 \le q \le \infty$ then*

$$\lim_{K,N\to\infty} S_{K,N} f = f \quad a.e.$$

If $1 \le q < \infty$ then the convergence holds in $W(L_p, \ell_q)(\mathbb{R}^d)$ norm.

(ii) *If $\hat\theta \in \dot E_{r'}(\mathbb{R}^d)$, $1 \le r < p < \infty$, $1/r + 1/r' = 1$ and $1 \le q \le \infty$ then*

$$\lim_{K,N\to\infty} \sigma^\theta_{K,N} f = \theta(0) f \quad a.e.$$

If $\hat\theta \in L_1(\mathbb{R}^d)$, $1 \le p < \infty$ and $1 \le q < \infty$ then the convergence holds in $W(L_p, \ell_q)(\mathbb{R}^d)$ norm.

Unfortunately, the case $p = r$ in (ii) concerning the a.e. convergence is not included in Theorem 4 and it cannot even be proved with the method of [12]. This means, that the a.e. convergence is not proved for the most important case, i.e. for functions $f \in L_1(\mathbb{R}^d)$ or $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$. Note that $W(L_1, \ell_\infty)(\mathbb{R}^d)$ is the largest space between the spaces $W(L_p, \ell_q)(\mathbb{R}^d)$, $1 \le p, q \le \infty$.

In this paper we will recover this gap using new ideas. To this end we define and investigate first the summability kernel. Let

$$F^\theta_{K,N}(x,t) := \sum_{|k|\le K} \sum_{n\in\mathbb{Z}^d} \theta\left(\frac{-n}{N+1}\right) e^{2\pi\imath\beta n(x-t)} \overline{T_{\alpha k} g(t)} T_{\alpha k}\gamma(x)$$

be the θ-kernels.

Theorem 5 *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $g,\gamma \in W(L_\infty,\ell_1)(\mathbb{R}^d)$ and $f \in W(L_1,\ell_\infty)(\mathbb{R}^d)$ then*

$$\sigma_{K,N}^\theta f = \int_{\mathbb{R}^d} f(t) F_{K,N}^\theta(x,t)\,dt.$$

Moreover, for $K,N \in \mathbb{N}$,

$$\|\sigma_{K,N}^\theta f\|_{W(L_1,\ell_\infty)} \le C_\alpha (N+1)^d \|\theta\|_{W(C,\ell_1)} \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_1,\ell_\infty)}.$$

In the next result we can see that the θ-means are bounded on the $L_p(\mathbb{R}^d)$ spaces.

Corollary 2 *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\hat\theta \in L_1(\mathbb{R}^d)$ and $g,\gamma \in W(L_\infty,\ell_1)(\mathbb{R}^d)$ then*

$$\|\sigma_{K,N}^\theta f\|_p \le C_p \|\hat\theta\|_1 \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_p$$

for all $K,N \in \mathbb{N}$ and $1 \le p \le \infty$.

6 Pointwise Summability

We consider the *maximal θ-operator* of Gabor series defined by

$$\sigma_{g,\gamma,*}^\theta := \sigma_*^\theta f := \sup_{K,N\in\mathbb{N}} |\sigma_{K,N}^\theta f|.$$

It is known for Fourier series that

$$\Big\|\sup_{N\in\mathbb{N}} |S_N h|\Big\|_p \le C_p \|h\|_p \quad (1<p<\infty),$$

$$\Big\|\sup_{N\in\mathbb{N}} |\sigma_N^\theta h|\Big\|_q \le C_q \|\hat\theta\|_{\dot E_{p'}} \|h\|_q \quad (1\le p<q<\infty)$$

and

$$\Big\|\sup_{N\in\mathbb{N}} |\sigma_N^\theta h|\Big\|_{L_{p,\infty}} \le C_p \|\hat\theta\|_{\dot E_{p'}} \|f\|_p \quad (1\le p<\infty),$$

whenever $\hat\theta \in \dot E_{p'}$ (see Carleson [5], Hunt [23], Fefferman [7], Grafakos [16] and Feichtinger and Weisz [11]). Now we prove similar inequalities for Gabor series.

Theorem 6 *Let $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $g,\gamma \in W(L_\infty,\ell_1)(\mathbb{R}^d)$, $1 \le p \le \infty$ and $1/p + 1/p' = 1$. If $\hat\theta \in \dot E_{p'}(\mathbb{R}^d)$ then*

$$\|\sigma_*^\theta f\|_{L_{p,\infty}} \le C_p \|\hat\theta\|_{\dot E_{p'}} \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_p$$

for all $f \in L_p(\mathbb{R}^d)$. Moreover, for every $p<q\le\infty$ and $f \in L_q(\mathbb{R}^d)$,

$$\|\sigma_*^\theta f\|_q \le C_q \|\hat\theta\|_{\dot E_{p'}} \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_q.$$

A similar result to Theorem 6 for Wiener amalgam spaces reads as follows.

Theorem 7 *Let* $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $g,\gamma \in W(L_\infty,\ell_1)(\mathbb{R}^d)$, $1 \le p \le \infty$ *and* $1/p + 1/p' = 1$. *If* $\hat\theta \in \dot E_{p'}(\mathbb{R}^d)$ *then*

$$\|\sigma_*^\theta f\|_{W(L_{p,\infty},\ell_\infty)} \le C_p \|\hat\theta\|_{\dot E_{p'}} \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_p,\ell_\infty)}$$

for all $f \in W(L_p,\ell_\infty)(\mathbb{R}^d)$. *Moreover, for every* $p < q \le \infty$ *and* $f \in W(L_q,\ell_\infty)(\mathbb{R}^d)$,

$$\|\sigma_*^\theta f\|_{W(L_q,\ell_\infty)} \le C_q \|\hat\theta\|_{\dot E_{p'}} \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_q,\ell_\infty)}.$$

Using Theorem 7 we will prove some pointwise convergence results for Gabor series. Note that $W(L_p,\ell_\infty)(\mathbb{R}^d) \supset W(L_p,\ell_q)(\mathbb{R}^d)$ for $1 \le q \le \infty$.

Theorem 8 *Assume that* $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ *and* $g,\gamma \in W(L_\infty,\ell_1)(\mathbb{R}^d)$. *If* $1 \le p < \infty$ *and* $\hat\theta \in \dot E_{p'}(\mathbb{R}^d)$ *then*

$$\lim_{K,N\to\infty} \sigma_{K,N}^\theta f = \theta(0) R_{g,\gamma} f \quad a.e. \tag{17}$$

for all $f \in W(L_p,\ell_\infty)(\mathbb{R}^d)$. *If* $\mathcal{G}(g,\alpha,\beta)$ *is a Gabor frame for* $L_2(\mathbb{R}^d)$ *with dual frame* $\mathcal{G}(\gamma,\alpha,\beta)$ *then* $\sigma_{K,N}^\theta f$ *converges to* $\theta(0)f$ *a.e.*

Finally we remark that we can define a symmetric summation method in the time and frequency domain by

$$\sigma_{K,N}^{\eta,\theta} f := \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d} \eta\left(\frac{-k}{K+1}\right) \theta\left(\frac{-n}{N+1}\right) \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k}\gamma,$$

where $g,\gamma \in W(L_\infty,\ell_1)(\mathbb{R}^d)$, $f \in W(L_1,\ell_\infty)(\mathbb{R}^d)$ and $\eta,\theta \in W(C,\ell_1)(\mathbb{R}^d)$ with $\eta(0) = 1$. (Instead of $\eta \in W(C,\ell_1)$ it is enough to suppose that $\eta \in W(L_\infty,\ell_1)(\mathbb{R}^d)$ is continuous at 0.) If $\eta(x) = \mathbf{1}_{[0,1]}(|x|)$ then $\sigma_{K,N}^{\eta,\theta} f = \sigma_{K,N}^\theta f$. Let

$$\sigma_*^{\eta,\theta} f := \sup_{K,N\in\mathbb{N}} |\sigma_{K,N}^{\eta,\theta} f|.$$

Under these conditions on η all the above results can be shown for $\sigma_{K,N}^{\eta,\theta} f$ and $\sigma_*^{\eta,\theta} f$ without difficulties in the same way.

Theorem 9 *If in addition* $\eta \in W(C,\ell_1)(\mathbb{R}^d)$ *with* $\eta(0) = 1$ *then in Theorems* 6, 7 *and* 8 $\sigma_*^\theta f$ (*resp.* $\sigma_{K,N}^\theta f$) *can be replaced by* $\sigma_*^{\eta,\theta} f$ (*resp.* $\sigma_{K,N}^{\eta,\theta} f$).

7 Proofs

In this section we present the proofs of the results mentioned above. First we prove Theorem 3 and then, by estimating $M_{g,\gamma,p} f$ by $m_{g,\gamma,p} f$ we give the proof of Theorem 2.

Proof of Theorem 3 If $p=\infty$ then

$$\begin{aligned} m_{g,\gamma,\infty} f(x) &\le \sup_{I:x\in I} \sup_{t\in I} \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d} |f(t+n/\beta)||T_{\alpha k} g(t+n/\beta)||T_{\alpha k}\gamma(x)| \\ &\le \|f\|_\infty \sup_{I:x\in I} \sup_{t\in I} \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d} |T_{\alpha k} g(t+n/\beta)||T_{\alpha k}\gamma(x)| \\ &\le C\|g\|_{W(L_\infty,\ell_1)}\|\gamma\|_{W(L_\infty,\ell_1)}\|f\|_\infty, \end{aligned} \tag{18}$$

which implies

$$\|m_{g,\gamma,\infty} f\|_\infty \le C\|g\|_{W(L_\infty,\ell_1)}\|\gamma\|_{W(L_\infty,\ell_1)}\|f\|_\infty.$$

Assume that $1\le p<\infty$. Let $E_i := \{x\in T_i Q : m_{g,\gamma,p} f(x) > \rho\}$ and $K_i \subset E_i$ be a compact subset ($i\in\mathbb{Z}^d$). For each $x\in K_i$ there exists a cube I_x such that $x\in I_x$, $|I_x|\le 1$, $I_x\cap T_i Q\ne\emptyset$ and

$$|I_x|^{-1/p}\left(\int_{I_x} |f_{g,\gamma}(x,t)|^p\,dt\right)^{1/p} > \rho. \tag{19}$$

Since $x\in I_x$ and K_i is compact, we can select a finite collection of these cubes covering K_i, say $\bigcup_j I_{i,j}\supset K_i$. By Vitali covering lemma (see e.g. Stein [27] or Weisz [31]) we can choose a finite disjoint subcollection $I_{i,j}$, $j=1,\dots,m_i$ of this covering with $|K_i|\le 3^d\sum_{j=1}^{m_i}|I_{i,j}|$. Since each cube $I_{i,j}$ satisfies (19), we have

$$\int_{I_{i,j}} |f_{g,\gamma}(x_{i,j},t)|^p\,dt > \rho^p |I_{j,i}|,$$

where $x_{i,j}\in K_i\subset T_i Q$. Thus

$$\sum_{j=1}^{m_i}\int_{I_{i,j}} |f_{g,\gamma}(x_{i,j},t)|^p\,dt > \rho^p\sum_{j=1}^{m_i}|I_{j,i}| > 3^{-d}\rho^p|K_i|.$$

We estimate the left hand side by

$$\begin{aligned} &\left(\sum_{j=1}^{m_i}\int_{I_{i,j}} |f_{g,\gamma}(x_{i,j},t)|^p\,dt\right)^{1/p} \\ &= \left(\sum_{j=1}^{m_i}\int_{I_{i,j}} \left(\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} |f(t+n/\beta)||T_{\alpha k} g(t+n/\beta)||T_{\alpha k}\gamma(x_{i,j})|\right)^p dt\right)^{1/p} \\ &\le \left(\sum_{j=1}^{m_i}\int_{I_{i,j}} \left(\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} |f(t+n/\beta)||T_{\alpha k} g(t+n/\beta)|\sup_{T_i Q}|T_{\alpha k}\gamma|\right)^p dt\right)^{1/p} \\ &\le \left(\int_{[-i-1,-i+2]} \left(\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} |f(t+n/\beta)||T_{\alpha k} g(t+n/\beta)|\sup_{T_i Q}|T_{\alpha k}\gamma|\right)^p dt\right)^{1/p}, \end{aligned}$$

where $-i-1=(-i_1-1,\dots,-i_d-1)$. Hence

$$\rho|K_i|^{1/p} \le 3^{d/p}\biggl(\int_{[-i-1,-i+2]}\biggl(\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}|f(t+n/\beta)||T_{\alpha k}g(t+n/\beta)| \times \sup_{T_i Q}|T_{\alpha k}\gamma|\biggr)^p dt\biggr)^{1/p}.$$

Taking the supremum over all K_i we obtain on the left hand side $\rho|E_i|^{1/p}$. Then

$$\begin{aligned}\|m_{g,\gamma,p}f\|_{W(L_{p,\infty},\ell_q)} &= \biggl(\sum_{i\in\mathbb{Z}^d}\biggl(\sup_{\rho>0}\rho|E_i|^{1/p}\biggr)^q\biggr)^{1/q}\\ &\le 3^{d/p}\biggl(\sum_{i\in\mathbb{Z}^d}\biggl(\int_{[-i-1,-i+2]}\biggl(\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}|f(t+n/\beta)|\\ &\qquad\times|T_{\alpha k}g(t+n/\beta)|\sup_{T_i Q}|T_{\alpha k}\gamma|\biggr)^p dt\biggr)^{q/p}\biggr)^{1/q}. \end{aligned} \tag{20}$$

If $q=\infty$,

$$\begin{aligned}\|m_{g,\gamma,p}f\|_{W(L_{p,\infty},\ell_\infty)} &\le C_p \sup_{i\in\mathbb{Z}^d}\biggl(\int_{[-i-1,-i+2]}\biggl(\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}|f(t+n/\beta)|\\ &\qquad\times|T_{\alpha k}g(t+n/\beta)|\sup_{T_i Q}|T_{\alpha k}\gamma|\biggr)^p dt\biggr)^{1/p}\\ &\le C_p \sup_{i\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\sup_{t\in[-i-1,-i+2]}|T_{\alpha k}g(t+n/\beta)|\\ &\qquad\times\sup_{T_i Q}|T_{\alpha k}\gamma|\|f\|_{W(L_p,\ell_\infty)}\\ &\le C_p\|g\|_{W(L_\infty,\ell_1)}\|\gamma\|_{W(L_\infty,\ell_1)}\|f\|_{W(L_p,\ell_\infty)},\end{aligned}$$

which proves (10).

We have for $q=p$ that

$$\|m_{g,\gamma,p}f\|_{W(L_{p,\infty},\ell_q)} = \|m_{g,\gamma,p}f\|_{L_{p,\infty}} \quad (1\le p<\infty).$$

Using (20) and a duality theorem we obtain

$$\begin{aligned}\|m_{g,\gamma,p}f\|_{L_{p,\infty}} &\le 3^{d/p}\biggl(\sum_{i\in\mathbb{Z}^d}\int_{[-i-1,-i+2]}\biggl(\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}|f(t+n/\beta)|\\ &\qquad\times|T_{\alpha k}g(t+n/\beta)|\sup_{T_i Q}|T_{\alpha k}\gamma|\biggr)^p dt\biggr)^{1/p}\\ &= 3^{d/p}\sup_{\|h\|\le 1}\sum_{i\in\mathbb{Z}^d}\int_{[-i-1,-i+2]}\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}|f(t+n/\beta)|\\ &\qquad\times|T_{\alpha k}g(t+n/\beta)|\sup_{T_i Q}|T_{\alpha k}\gamma||h_i(t)|\,dt,\end{aligned} \tag{21}$$

where $h=(h_i, i\in\mathbb{Z}^d)$ and

$$\|h\| := \left(\sum_{i\in\mathbb{Z}^d}\int_{[-i-1,-i+2]}|h_i(t)|^{p'}\,dt\right)^{1/p'}.$$

If $\|h\|\le 1$ then Hölder's inequality imply

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}^d}\int_{[-i-1,-i+2]}\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}|f(t+n/\beta)||T_{\alpha k}g(t+n/\beta)|\sup_{T_iQ}|T_{\alpha k}\gamma||h_i(t)|\,dt\\
&\quad\le \sum_{i\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\int_{[-i-1,-i+2]}|f(t+n/\beta)||T_{\alpha k}g(t+n/\beta)|^{1/p}\sup_{T_iQ}|T_{\alpha k}\gamma|^{1/p}\\
&\qquad\times|T_{\alpha k}g(t+n/\beta)|^{1/p'}\sup_{T_iQ}|T_{\alpha k}\gamma|^{1/p'}|h_i(t)|\,dt\\
&\quad\le \sum_{i\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\left(\int_{[-i-1,-i+2]}|f(t+n/\beta)|^p|T_{\alpha k}g(t+n/\beta)|\sup_{T_iQ}|T_{\alpha k}\gamma|\,dt\right)^{1/p}\\
&\qquad\times\left(\int_{[-i-1,-i+2]}|T_{\alpha k}g(t+n/\beta)|\sup_{T_iQ}|T_{\alpha k}\gamma||h_i(t)|^{p'}\,dt\right)^{1/p'}\\
&\quad\le \left(\sum_{i\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\int_{[-i-1,-i+2]}|f(t+n/\beta)|^p|T_{\alpha k}g(t+n/\beta)|\sup_{T_iQ}|T_{\alpha k}\gamma|\,dt\right)^{1/p}\\
&\qquad\times\left(\sum_{i\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\int_{[-i-1,-i+2]}|T_{\alpha k}g(t+n/\beta)|\sup_{T_iQ}|T_{\alpha k}\gamma||h_i(t)|^{p'}\,dt\right)^{1/p'}.
\end{aligned}$$

By the definition of $W(L_\infty,\ell_1)(\mathbb{R}^d)$ the right hand side can be estimated by

$$\begin{aligned}
&C_p\left(\sum_{i\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}|f(t)|^p|T_{\alpha k}g(t)|\sup_{T_iQ}|T_{\alpha k}\gamma|\,dt\right)^{1/p}\\
&\qquad\times\left(\sum_{i\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\int_{[-i-1,-i+2]}\sup_{T_iQ}|T_{\alpha k}\gamma||h_i(t)|^{p'}\,dt\right)^{1/p'}\|g\|^{1/p'}_{W(L_\infty,\ell_1)}\\
&\quad\le C_p\left(\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}|f(t)|^p|T_{\alpha k}g(t)|\,dt\right)^{1/p}\|\gamma\|^{1/p}_{W(L_\infty,\ell_1)}\\
&\qquad\times\left(\sum_{i\in\mathbb{Z}^d}\int_{[-i-1,-i+2]}|h_i(t)|^{p'}\,dt\right)^{1/p'}\|g\|^{1/p'}_{W(L_\infty,\ell_1)}\|\gamma\|^{1/p'}_{W(L_\infty,\ell_1)}\\
&\quad\le C_p\|f\|_p\|g\|_{W(L_\infty,\ell_1)}\|\gamma\|_{W(L_\infty,\ell_1)}\|h\|
\end{aligned}$$

and this, together with (21) prove (11). □

Proof of Theorem 2 The first inequality in Theorem 2 for $p=\infty$ can be proved as in (18). Assume that $1\le p<\infty$. Of course, if $|I|\le 1$ and $x\in I$ then

$$|I|^{-1/p}\left(\int_I |f_{g,\gamma}(x,t)|^p\,dt\right)^{1/p}\le m_{g,\gamma,p}f(x). \tag{22}$$

If $l^d\le |I|<(l+1)^d$ for some $l\ge 1$ and $x\in I$ then

$$\begin{aligned}|I|^{-1}\int_I |f_{g,\gamma}(x,t)|^p\,dt&\le |I|^{-1}\sum_{j:T_jQ\cap I\neq\emptyset}\int_{T_jQ}|f_{g,\gamma}(x,t)|^p\,dt\\&\le (l+2)^d|I|^{-1}\sup_{j:T_jQ\cap I\neq\emptyset}\int_{T_jQ}|f_{g,\gamma}(x,t)|^p\,dt.\end{aligned} \tag{23}$$

Hence

$$M_{g,\gamma,p}f(x)\le m_{g,\gamma,p}f(x)+C\sup_{j\in\mathbb{Z}^d}\left(\int_{T_jQ}|f_{g,\gamma}(x,t)|^p\,dt\right)^{1/p}. \tag{24}$$

Obviously,

$$\begin{aligned}&\rho\left|\left\{x:\sup_{j\in\mathbb{Z}^d}\left(\int_{T_jQ}|f_{g,\gamma}(x,t)|^p\,dt\right)^{1/p}>\rho\right\}\right|\\&\quad\le\left(\int_{\mathbb{R}^d}\sup_{j\in\mathbb{Z}^d}\int_{T_jQ}|f_{g,\gamma}(x,t)|^p\,dt\,dx\right)^{1/p}.\end{aligned}$$

Again by duality,

$$\begin{aligned}&\left(\int_{\mathbb{R}^d}\sup_{j\in\mathbb{Z}^d}\int_{T_jQ}|f_{g,\gamma}(x,t)|^p\,dt\,dx\right)^{1/p}\\&\quad=\sup_{\|h\|\le 1}\left|\int_{\mathbb{R}^d}\sum_{j\in\mathbb{Z}^d}\int_{T_jQ}f_{g,\gamma}(x,t)h_j(x,t)\,dt\,dx\right|,\end{aligned}$$

where $h=(h_j,\,j\in\mathbb{Z}^d)$ and

$$\|h\|:=\left(\int_{\mathbb{R}^d}\left(\sum_{j\in\mathbb{Z}^d}\left(\int_{T_jQ}|h_j(x,t)|^{p'}\,dt\right)^{1/p'}\right)^{p'}dx\right)^{1/p'}.$$

By Hölder's inequality,

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\sum_{j\in\mathbb{Z}^d}\int_{T_jQ}f_{g,\gamma}(x,t)h_j(x,t)\,dt\,dx\right|\\&\quad\le\sum_{j\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}|T_{\alpha k}\gamma(x)|\int_{T_jQ}|f(t+n/\beta)||T_{\alpha k}g(t+n/\beta)||h_j(x,t)|\,dt\,dx\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}|T_{\alpha k}\gamma(x)|^{1/p}\left(\int_{T_jQ}|f(t+n/\beta)|^p|T_{\alpha k}g(t+n/\beta)|\,dt\right)^{1/p}\\
&\quad\times|T_{\alpha k}\gamma(x)|^{1/p'}\left(\int_{T_jQ}|T_{\alpha k}g(t+n/\beta)||h_j(x,t)|^{p'}\,dt\right)^{1/p'}dx\\
&\le \sum_{j\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\left(\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}|T_{\alpha k}\gamma(x)|\int_{T_jQ}|f(t+n/\beta)|^p|T_{\alpha k}g(t+n/\beta)|\,dt\right)^{1/p}\\
&\quad\times\left(\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}|T_{\alpha k}\gamma(x)|\int_{T_jQ}|T_{\alpha k}g(t+n/\beta)||h_j(x,t)|^{p'}\,dt\right)^{1/p'}dx.
\end{aligned}
$$

Using the definition of the amalgam spaces, the inequality $\|\gamma\|_1\le\|\gamma\|_{W(L_\infty,\ell_1)}$ and Hölder's inequality again we conclude that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^d}\sum_{j\in\mathbb{Z}^d}\int_{T_jQ}f_{g,\gamma}(x,t)h_j(x,t)\,dt\,dx\right|\\
&\quad\le C_p\int_{\mathbb{R}^d}\left(\sum_{k\in\mathbb{Z}^d}|T_{\alpha k}\gamma(x)|\int_{\mathbb{R}^d}|f(t)|^p|T_{\alpha k}g(t)|\,dt\right)^{1/p}\\
&\qquad\times\|g\|^{1/p'}_{W(L_\infty,\ell_1)}\|\gamma\|^{1/p'}_{W(L_\infty,\ell_1)}\sum_{j\in\mathbb{Z}^d}\left(\int_{T_jQ}|h_j(x,t)|^{p'}\,dt\right)^{1/p'}dx\\
&\quad\le C_p\left(\int_{\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}|T_{\alpha k}\gamma(x)|\int_{\mathbb{R}^d}|f(t)|^p|T_{\alpha k}g(t)|\,dt\,dx\right)^{1/p}\\
&\qquad\times\|g\|^{1/p'}_{W(L_\infty,\ell_1)}\|\gamma\|^{1/p'}_{W(L_\infty,\ell_1)}\left(\int_{\mathbb{R}^d}\left(\sum_{j\in\mathbb{Z}^d}\left(\int_{T_jQ}|h_j(x,t)|^{p'}\,dt\right)^{1/p'}\right)^{p'}dx\right)^{1/p'}\\
&\quad\le C_p\|f\|_p\|g\|_{W(L_\infty,\ell_1)}\|\gamma\|_{W(L_\infty,\ell_1)}\|h\|.
\end{aligned}
$$

Inequality (8) follows from (24) and (11).

If $q=\infty$ and $f\in L_\infty(\mathbb{R}^d)$ then

$$
\begin{aligned}
M_{g,\gamma,p}f(x)&\le\|f\|_\infty\sup_{x\in I}|I|^{-1/p}\left(\int_I\left(\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}|T_{\alpha k}g(t+n/\beta)||T_{\alpha k}\gamma(x)|\right)^p dt\right)^{1/p}\\
&\le C\|f\|_\infty\|g\|_{W(L_\infty,\ell_1)}\|\gamma\|_{W(L_\infty,\ell_1)}.
\end{aligned}
$$

Inequality (9) for $p<q<\infty$ follows by interpolation. □

Proof of Theorem 5 By the definition,

$$\sigma_{K,N}^{\theta} f(x) = \sum_{|k|\le K} \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \theta\left(\frac{-n}{N+1}\right) f(t) e^{2\pi\imath\beta n(x-t)} \overline{T_{\alpha k} g(t)} T_{\alpha k}\gamma(x)\, dt.$$

Using (16) and the definition of the Wiener amalgam spaces we get for all $l \in \mathbb{Z}^d$ that

$$\begin{aligned}
&\int_{T_{\alpha l} Q_\alpha} |\sigma_{K,N}^{\theta} f(x)|\, dx \\
&\le \sum_{|k|\le K} \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} \int_{T_{\alpha m} Q_\alpha} \left|\theta\left(\frac{-n}{N+1}\right)\right| |f(t)||T_{\alpha k} g(t)|\, dt \int_{T_{\alpha l} Q_\alpha} |T_{\alpha k}\gamma(x)|\, dx \\
&\le C_\alpha (N+1)^d \|\theta\|_{W(C,\ell_1)} \sum_{|k|\le K} \sum_{m\in\mathbb{Z}^d} \sup_{T_{\alpha m} Q_\alpha} |T_{\alpha k} g| \sup_{T_{\alpha l} Q_\alpha} |T_{\alpha k}\gamma| \int_{T_{\alpha m} Q_\alpha} |f(t)|\, dt \\
&\le C_\alpha (N+1)^d \|\theta\|_{W(C,\ell_1)} \sum_{|k|\le K} \sup_{T_{\alpha l} Q_\alpha} |T_{\alpha k}\gamma| \|g\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_1,\ell_\infty)} \\
&\le C_\alpha (N+1)^d \|\theta\|_{W(C,\ell_1)} \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_1,\ell_\infty)},
\end{aligned}$$

which proves the theorem. □

Lemma 1 *If $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\hat\theta \in L_1(\mathbb{R}^d)$ and $g,\gamma \in W(L_\infty,\ell_1)(\mathbb{R}^d)$ then*

$$F_{K,N}^{\theta}(x,t) = (N+1)^d \sum_{|k|\le K} \sum_{n\in\mathbb{Z}^d} \hat\theta\big((N+1)(\beta(x-t)+n)\big) \overline{T_{\alpha k} g(t)} T_{\alpha k}\gamma(x)$$

for all $K, N \in \mathbb{N}$.

Proof In Feichtinger and Weisz [10] or [31] we proved that

$$\int_Q f(x-t) \sum_{n\in\mathbb{Z}^d} \theta\left(\frac{-n}{N+1}\right) e^{2\pi\imath n t}\, dt = (N+1)^d \int_{\mathbb{R}^d} f(x-t)\hat\theta((N+1)t)\, dt$$

for all periodic functions $f \in L_1(Q)$. Therefore

$$\begin{aligned}
&\int_Q f(x-t) \sum_{n\in\mathbb{Z}^d} \theta\left(\frac{-n}{N+1}\right) e^{2\pi\imath n t}\, dt \\
&= (N+1)^d \sum_{n\in\mathbb{Z}^d} \int_{T_n Q} f(x-t)\hat\theta((N+1)t)\, dt \\
&= (N+1)^d \sum_{n\in\mathbb{Z}^d} \int_Q f(x-t-n)\hat\theta((N+1)(t+n))\, dt \\
&= (N+1)^d \int_Q f(x-t) \sum_{n\in\mathbb{Z}^d} \hat\theta((N+1)(t+n))\, dt,
\end{aligned}$$

which yields that

$$\sum_{n\in\mathbb{Z}^d}\theta\left(\frac{-n}{N+1}\right)e^{2\pi\imath nt}=(N+1)^d\sum_{n\in\mathbb{Z}^d}\hat{\theta}((N+1)(t+n)).$$

Now Lemma 1 follows from Theorem 5. □

Note that the last equation is a version of the Poisson summation formula, however, for the sake of completeness we presented the proof of Lemma 1.

Lemma 2 *If* $\theta\in W(C,\ell_1)(\mathbb{R}^d)$, $\hat{\theta}\in L_1(\mathbb{R}^d)$ *and* $g,\gamma\in W(L_\infty,\ell_1)(\mathbb{R}^d)$ *then*

$$\int_{\mathbb{R}^d}|F_{K,N}^{\theta}(x,t)|\,dx+\int_{\mathbb{R}^d}|F_{K,N}^{\theta}(x,t)|\,dt\le C_\beta\|\hat{\theta}\|_1\|g\|_{W(L_\infty,\ell_1)}\|\gamma\|_{W(L_\infty,\ell_1)}$$

for all $K,N\in\mathbb{N}$, $x,t\in\mathbb{R}^d$.

Proof It is enough to show that the first summand can be estimated by the right hand side. By Lemma 1,

$$\begin{aligned}\int_{\mathbb{R}^d}|F_{K,N}^{\theta}(x,t)|\,dx&\le(N+1)^d\sum_{|k|\le K}\sum_{n\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}\\&\quad\times\int_{T_{l/\beta}Q_{1/\beta}}\left|\hat{\theta}\big((N+1)(\beta(x-t)+n)\big)\right|\left|\overline{T_{\alpha k}g(t)}\right||T_{\alpha k}\gamma(x)|\,dx\\&\le(N+1)^d\sum_{|k|\le K}\sum_{n\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}|T_{\alpha k}g(t)|\sup_{T_{l/\beta}Q_{1/\beta}}|T_{\alpha k}\gamma|\\&\quad\times\int_{T_{l/\beta}Q_{1/\beta}}\left|\hat{\theta}\big((N+1)(\beta(x-t+n/\beta)\big)\right|dx\\&\le C_\beta\|\hat{\theta}\|_1\sum_{|k|\le K}\sum_{l\in\mathbb{Z}^d}|T_{\alpha k}g(t)|\sup_{T_{l/\beta}Q_{1/\beta}}|T_{\alpha k}\gamma|\\&\le C_\beta\|\hat{\theta}\|_1\sum_{|k|\le K}|T_{\alpha k}g(t)|\|\gamma\|_{W(L_\infty,\ell_1)}\\&\le C_\beta\|\hat{\theta}\|_1\|g\|_{W(L_\infty,\ell_1)}\|\gamma\|_{W(L_\infty,\ell_1)},\end{aligned}$$

which proves the lemma. □

Proof of Corollary 2 The result follows from Schur's test (see e.g. Gröchenig [18, p. 106]) and from Theorem 5 and Lemma 2. □

Proof of Theorem 6 By Theorem 5 and Lemma 1,

$$|\sigma_{K,N}^{\theta}f(x)|=\left|\int_{\mathbb{R}^d}f(x-t)F_{K,N}^{\theta}(x,x-t)\,dt\right|$$

$$
\begin{aligned}
&\le (N+1)^d \sum_{|k|\le K}\sum_{n\in\mathbb{Z}^d}\int_{\mathbb{R}^d} |f(x-t)| \\
&\quad \times |\hat{\theta}((N+1)(\beta(t+n/\beta))||T_{\alpha k}g(x-t)||T_{\alpha k}\gamma(x)|\,dt \\
&= (N+1)^d \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\int_{\mathbb{R}^d} |f(x-t+n/\beta)| \\
&\quad \times |\hat{\theta}((N+1)\beta t)||T_{\alpha k}g(x-t+n/\beta)||T_{\alpha k}\gamma(x)|\,dt \\
&= (N+1)^d \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\sum_{l=-\infty}^{\infty}\int_{P_l(1/\beta(N+1))} |f(x-t+n/\beta)| \\
&\quad \times |\hat{\theta}((N+1)\beta t)||T_{\alpha k}g(x-t+n/\beta)||T_{\alpha k}\gamma(x)|\,dt,
\end{aligned}
$$

where

$$
P_l(1/\beta(N+1)) := \left\{x : |x| < \frac{2^l}{\beta(N+1)}\right\}\Big\backslash\left\{x : |x| \ge \frac{2^{l-1}}{\beta(N+1)}\right\} \quad (l\in\mathbb{Z}).
$$

By Hölder's inequality,

$$
\begin{aligned}
&|\sigma^{\theta}_{K,N} f(x)| \\
&\quad \le \sum_{l=-\infty}^{\infty}\left(\int_{P_l(1/\beta(N+1))}\left(\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} |f(x-t+n/\beta)||T_{\alpha k}g(x-t+n/\beta)|\right.\right. \\
&\quad\quad \left.\left.\times |T_{\alpha k}\gamma(x)|\right)^p dt\right)^{1/p} \\
&\quad\quad \times \left(\int_{P_l(1/\beta(N+1))} |(N+1)^d\hat{\theta}((N+1)\beta t)|^{p'}\,dt\right)^{1/p'}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\left(\int_{P_l(1/\beta(N+1))} |(N+1)^d\hat{\theta}((N+1)\beta t)|^{p'}\,dt\right)^{1/p'} \\
&\quad = \beta^{-d}((N+1)\beta)^{d(1-1/p')}\left(\int_{P_l} |\hat{\theta}(t)|^{p'}\,dt\right)^{1/p'}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\int_{P_l(1/\beta(N+1))}\left(\sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} |f(x-t+n/\beta)||T_{\alpha k}g(x-t+n/\beta)||T_{\alpha k}\gamma(x)|\right)^p dt\right)^{1/p} \\
&\quad \le 2^{ld/p}((N+1)\beta)^{-d/p} M_{g,\gamma,p}f(x),
\end{aligned}
$$

where the *Hardy-Littlewood type maximal function* $M_{g,\gamma,p}f$ was defined in (7). Hence

$$\sigma_*^\theta f(x) \le \beta^{-d} M_{g,\gamma,p} f(x) \sum_{l=-\infty}^{\infty} 2^{ld/p} \left(\int_{P_l} |\hat{\theta}(t)|^{p'}\, dt \right)^{1/p'}$$

$$= \beta^{-d} \|\hat{\theta}\|_{\dot{E}_{p'}} M_{g,\gamma,p} f(x). \tag{25}$$

Now Theorem 2 finishes the proof. □

Proof of Theorem 7 We start the proof with the estimation of the right hand side of (23):

$$\int_{T_j Q} |f_{g,\gamma}(x,t)|^p\, dt$$

$$\le \left(\int_{T_j Q} \left(\sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d} |f(t+n/\beta)||T_{\alpha k} g(t+n/\beta)||T_{\alpha k}\gamma(x)| \right)^p dt \right)^{1/p}$$

$$\le \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{Z}^d} |T_{\alpha k}\gamma(x)| \sup_{t\in T_j Q} |T_{\alpha k} g(t+n/\beta)| \left(\int_{T_j Q} |f(t+n/\beta)|^p\, dt \right)^{1/p}$$

$$\le C_p \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_p,\ell_\infty)}.$$

Inequalities (22) and (23) imply

$$M_{g,\gamma,p} f(x) \le m_{g,\gamma,p} f(x) + C_p \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_p,\ell_\infty)}$$

and, by (25),

$$\sigma_*^\theta f(x) \le C_p \beta^{-d} \|\hat{\theta}\|_{\dot{E}_{p'}} (m_{g,\gamma,p} f(x) + C_p \|g\|_{W(L_\infty,\ell_1)} \|\gamma\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_p,\ell_\infty)}).$$

Theorem 7 follows from Theorem 3 and Corollary 1. □

Proof of Theorem 8 First suppose that γ has compact support. In this case the sum in (3) is a finite sum for each fixed $x \in \mathbb{R}^d$. It is easy to see that

$$\sigma_{K,N}^\theta f = \sum_{|k|\le K} (\sigma_N^\theta m_{g,k}) T_{\alpha k}\gamma.$$

If K is large enough then

$$|\theta(0) R_{g,\gamma} f(x) - \sigma_{K,N}^\theta f(x)| = \left| \sum_{|k|\le K} (\theta(0) m_{g,k}(x) - \sigma_N^\theta m_{g,k}(x)) T_{\alpha k}\gamma(x) \right|$$

$$\le \|\gamma\|_\infty \sum_{|k|\le K} |\theta(0) m_{g,k}(x) - \sigma_N^\theta m_{g,k}(x)|$$

and (15) proves (17).

If the support of γ is not compact then choose $\gamma_m \in W(L_\infty, \ell_1)(\mathbb{R}^d)$ with compact support such that $\|\gamma - \gamma_m\|_{W(L_\infty,\ell_1)} \to 0$ as $m \to \infty$ ($m \in \mathbb{N}$). Fix $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$ and set

$$\xi := \limsup_{K,N\to\infty} |\sigma^\theta_{g,\gamma,K,N} f - \theta(0) R_{g,\gamma} f|.$$

It is sufficient to show that $\xi = 0$ a.e. Since (17) holds for γ_m, we have

$$\begin{aligned}\xi &\le \limsup_{K,N\to\infty} |\sigma^\theta_{g,\gamma,K,N} f - \sigma^\theta_{g,\gamma_m,K,N} f| \\ &\quad + \limsup_{K,N\to\infty} |\sigma^\theta_{g,\gamma_m,K,N} f - \theta(0) R_{g,\gamma_m} f| + |\theta(0) R_{g,\gamma_m} f - \theta(0) R_{g,\gamma} f| \\ &\le \sigma^\theta_{g,\gamma-\gamma_m,*} f + |\theta(0) R_{g,\gamma-\gamma_m} f|\end{aligned}$$

for all $m \in \mathbb{N}$. Taking into account Theorems 1 and 7 we conclude

$$\begin{aligned}\|\xi\|_{W(L_{p,\infty},\ell_\infty)} &\le \|\sigma^\theta_{g,\gamma-\gamma_m,*} f\|_{W(L_{p,\infty},\ell_\infty)} + \|\theta(0) R_{g,\gamma-\gamma_m} f\|_{W(L_p,\ell_\infty)} \\ &\le C_p \big(\|\hat\theta\|_{\dot E_{p'}} + \theta(0)\big) \|g\|_{W(L_\infty,\ell_1)} \|\gamma - \gamma_m\|_{W(L_\infty,\ell_1)} \|f\|_{W(L_p,\ell_\infty)}\end{aligned}$$

for all $m \in \mathbb{N}$. Since $\gamma_m \to \gamma$ in $W(L_\infty, \ell_1)(\mathbb{R}^d)$ norm as $m \to \infty$, $\|\xi\|_{W(L_{p,\infty},\ell_\infty)} = 0$ and so $\xi = 0$ a.e. □

Proof of Theorem 9 The result can be proved similarly to Theorems 6, 7 and 8. □

8 Some Summability Methods

In this section we consider some summability methods as special cases of the θ-summation. We have shown in Feichtinger and Weisz [11, Theorem 4.1] that if

$$\eta(x) := \sup_{\|t\|_r \ge \|x\|_r} |\hat\theta(t)| \in L_1(\mathbb{R}^d) \quad \text{for some } 1 \le r \le \infty \text{ then } \hat\theta \in \dot E_\infty(\mathbb{R}^d).$$

This is the case, if, e.g., $\hat\theta \in L_1(\mathbb{R})$ is even and $\hat\theta$ is non-increasing on $(0,\infty)$. Moreover, if θ is in the *weighted modulation space* $M_1^{v_d}(\mathbb{R}^d)$ then $\hat\theta \in \dot E_\infty(\mathbb{R}^d)$ (see Theorem 5.5 and Lemma 5.3 in [11]). The weighted modulation space or Feichtinger's algebra is defined via the *short-time Fourier transform* (STFT). Recall that for $f \in L_2(\mathbb{R}^d)$ the STFT with respect to a window function $g \in L_2(\mathbb{R}^d)$ is defined by

$$S_g f(x,\omega) := \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi\imath\omega t}\, dt \quad (x,\omega \in \mathbb{R}^d).$$

Using the STFT with respect to the Gauss function $g_0(x) := e^{-\pi|x|^2}$ we define $M_1^{v_d}(\mathbb{R}^d)$ by

$$M_1^{v_d}(\mathbb{R}^d) := \big\{ f \in L^2(\mathbb{R}^d) : \|f\|_{M_1^{v_d}} := \|S_{g_0} f \cdot v_d\|_{L_1(\mathbb{R}^{2d})} < \infty \big\},$$

where $v_d(x,\omega) := v_d(\omega) = (1+|\omega|)^d$ $(x,\omega \in \mathbb{R}^d)$.

The *Sobolev-type space* $V_1^k(\mathbb{R}^d)$ $(k \geq 2, k \in \mathbb{N})$ is a useful space to give some sufficient conditions for a function to be in $M_1^{v_d}(\mathbb{R}^d)$. A function θ is in $V_1^k(\mathbb{R})$ $(k \geq 2, k \in \mathbb{N})$, if there are numbers $-\infty = a_0 < a_1 < \cdots < a_n < a_{n+1} = \infty$ such that $n = n(\theta)$ is depending on θ and

$$\theta \in C^{k-2}(\mathbb{R}), \quad \theta \in C^k(a_i, a_{i+1}), \quad \theta^{(j)} \in L_1(\mathbb{R})$$

for all $i = 0, \dots, n$ and $j = 0, \dots, k$. Here C^k denotes the set of k-times continuously differentiable functions. The norm of this space is introduced by

$$\|\theta\|_{V_1^k} := \sum_{j=0}^{k} \|\theta^{(j)}\|_1 + \sum_{i=1}^{n} |\theta^{(k-1)}(a_i + 0) - \theta^{(k-1)}(a_i - 0)|,$$

where $\theta^{(k-1)}(a_i \pm 0)$ denote the right and left limits of $\theta^{(k-1)}$. It is easy to see that these limits do exist.

For $d > 1$ and $k \geq 2$ let $\theta \in V_1^k(\mathbb{R}^d)$ if θ is even in each variable and

$$\theta \in C^{k-2}(\mathbb{R}^d), \quad \theta \in C^k([0,\infty)^d \setminus \{(0,\dots,0)\}), \quad \partial_1^{i_1} \cdots \partial_d^{i_d} \theta(t) \in L_1([0,\infty)^l)$$

for each $i_j = 0, \dots, k$ $(j = 1, \dots, d)$ and fixed $0 < t_{m_1}, \dots, t_{m_{d-l}} < \infty$ $(1 \leq m_1 < m_2 < \cdots < m_{d-l} \leq d)$ and $1 \leq l \leq d$. We have proved in [11] that if $\theta \in V_1^k(\mathbb{R}^d)$ for $k > d+1$ then $\theta \in M_1^{v_d}(\mathbb{R}^d)$ and hence $\hat{\theta} \in \dot{E}_\infty(\mathbb{R})$. $V_1^2(\mathbb{R})$ is not contained in $M_1^{v_1}(\mathbb{R})$, however, if $\theta \in V_1^2(\mathbb{R})$ then $\hat{\theta} \in \dot{E}_\infty(\mathbb{R})$.

Applying these results, after some computation we can prove that $\hat{\theta} \in \dot{E}_\infty(\mathbb{R}^d)$ all in the following examples. For the details see [11].

Example 1 (Weierstrass summation) $\theta(x) = e^{-|x|^\gamma}$ $(x \in \mathbb{R}, 1 \leq \gamma < \infty)$. If $\gamma = 1$ then it is called Abel summation.

Example 2 $\theta(x) = e^{-(1+|x|^q)^\gamma}$ $(x \in \mathbb{R}, 1 \leq q < \infty, 0 < \gamma < \infty)$.

Example 3 $\theta(x) = e^{-\|x\|_q^q}$ $(x \in \mathbb{R}^d, 1 \leq q < \infty)$.

Example 4 $\theta(x) = e^{-(1+\|x\|_q^q)^\gamma}$ $(x \in \mathbb{R}^d, 1 \leq q < \infty, 0 < \gamma < \infty)$.

Example 5 If $\theta(x) = e^{-2\pi\|x\|_2}$ $(x \in \mathbb{R}^d)$ then $\hat{\theta}(x) = c_d/(1 + \|x\|_2^2)^{(d+1)/2}$ (see Stein and Weiss [28, p. 6.]).

Example 6 For $\theta(x) = 1/(1 + \|x\|_2^2)^{(d+1)/2}$ $(x \in \mathbb{R}^d)$ we have $\hat{\theta}(x) = c_d e^{-2\pi\|x\|_2}$.

Example 7 (Picard and Bessel summations) $\theta(x) = (1 + |x|^\gamma)^{-\alpha}$ $(x \in \mathbb{R}, 0 < \alpha < \infty, 1 \leq \gamma < \infty, \alpha\gamma > 1)$.

Example 8 $\theta(x) = (1 + \|x\|_\gamma^\gamma)^{-\alpha}$ $(x \in \mathbb{R}^d, 0 < \alpha < \infty, 1 \leq \gamma < \infty, \alpha\gamma > d)$.

Example 9

$$\theta(x) := \begin{cases} 1 & \text{if } \|x\|_q \le 1 \\ \|x\|_q^{-\alpha} & \text{if } \|x\|_q > 1 \end{cases} \quad (x \in \mathbb{R}^d, 1 \le q < \infty, d < \alpha < \infty).$$

Example 10 Let

$$\theta(x) := \begin{cases} 1 & \text{if } x = 0 \\ \frac{1-e^{-|x|^\alpha}}{|x|^\alpha} & \text{if } |x| > 0 \end{cases} \quad (x \in \mathbb{R}, 1 < \alpha < \infty).$$

Example 11 (de La Vallée-Poussin summation) Let

$$\theta(x) = \begin{cases} 1 & \text{if } |x| \le 1/2 \\ -2|x| + 2 & \text{if } 1/2 < |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases} \quad (x \in \mathbb{R}).$$

Example 12 (Jackson-de La Vallée-Poussin summation) Let

$$\theta(x) = \begin{cases} 1 - 3x^2/2 + 3|x|^3/4 & \text{if } |x| \le 1 \\ (2 - |x|)^3/4 & \text{if } 1 < |x| \le 2 \\ 0 & \text{if } |x| > 2 \end{cases} \quad (x \in \mathbb{R}).$$

Example 13 Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m$ and $\beta_0, \ldots, \beta_m$ $(m \in \mathbb{N})$ be real numbers, $\beta_0 = 1$, $\beta_m = 0$. Suppose that θ is even, $\theta(\alpha_j) = \beta_j$ $(j = 0, 1, \ldots, m)$, $\theta(x) = 0$ for $x \ge \alpha_m$, θ_9 is a polynomial on the interval $[\alpha_{j-1}, \alpha_j]$ $(j = 1, \ldots, m)$.

Example 14 (Rogosinski summation) Let

$$\theta(x) = \begin{cases} \cos \pi x/2 & \text{if } |x| \le 1 + 2j \\ 0 & \text{if } |x| > 1 + 2j \end{cases} \quad (j \in \mathbb{N}).$$

Example 15 (Riesz summation I)

$$\theta(x) := \begin{cases} (1 - |x|^\gamma)^\alpha & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases} \quad (x \in \mathbb{R}, 0 < \alpha < \infty, 1 \le \gamma < \infty).$$

It is called Fejér summation if $\alpha = \gamma = 1$. We have proved in [31] that

$$|\hat{\theta}(x)| \le \frac{C}{|x|^{(\alpha \wedge 1)+1}}.$$

Example 16 (Riesz summation II)

$$\theta(x) := \begin{cases} (1 - \|x\|_2^k)^\alpha & \text{if } \|x\|_2 \le 1 \\ 0 & \text{if } \|x\|_2 > 1 \end{cases} \quad (x \in \mathbb{R}^d, (d-1)/2 < \alpha < \infty, k \in \mathbb{N}, k > 0).$$

One can find in Stein and Weiss [28, p. 171] and Lu [24, p. 132] that

$$|\hat{\theta}(x)| \le C\|x\|_2^{-d/2-\alpha-1/2} \quad (x \neq 0)$$

and this is integrable if $\alpha > (d-1)/2$.

Example 17 (Riemann summation) If $\theta(x) = (\frac{\sin x/2}{x/2})^2$ then $\hat{\theta} = \max(0, 1-|x|)$.

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