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Behavior of Shannon's Sampling Series for Hardy Spaces

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Abstract In this paper the convergence behavior of the Shannon sampling series is analyzed for Hardy spaces. It is well known that the Shannon sampling series is locally uniformly convergent. However, for practical applications the global uniform convergence is important. It is shown that there are functions in the Hardy space such that the Shannon sampling series is not uniformly convergent on the whole real axis. In fact, there exists a function in this space such that the peak value of the Shannon sampling series diverges unboundedly. The proof uses Fefferman's theorem, which states that the dual space of the Hardy space is the space of functions of bounded mean oscillation.

Keywords Shannon sampling series · Hardy space · Uniform convergence · Stable reconstruction

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1 Introduction

Since Shannon introduced the sampling series in the context of communication in the landmark papers [14–16], the Shannon sampling series has become more and more

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important. By now, sampling theory has grown to an independent research area [17]. The developments comprise extensions to multidimensional sampling [9], which has applications in image processing and to bandpass or multi-band sampling [1], which is important in communications. A good overview article, containing numerous references itself, is [12]. However the analyses for the original Shannon sampling series and its extensions are often made only for the space of bandlimited and square-integrable functions. In contrast, we aim to extend the Shannon sampling theorem to hold for as large a space of functions as possible.

A stable reconstruction of signals from its samples, in the sense of a uniformly convergent Shannon sampling series, is important not only from the theoretical point of view but in particular for practical applications. If the series is uniformly convergent on the whole real axis, then it is possible to bound the maximum norm of the approximation error, which is made by using only finitely many samples. Thus it is desirable to have a stable reconstruction for as large a space of functions as possible. In this paper the convergence behavior of the Shannon sampling series is analyzed for the Hardy space. Since the space of functions is considerably enlarged, the Hilbert space techniques, which are eligible in the setting of square-integrable functions, cannot be used anymore. The Hardy space is interesting from the mathematical point of view, which is expressed by its importance in the Fourier and functional analysis [9, 13].

In order to continue the discussion, we need some preliminaries and notation. Let $\hat{f}$ denote the Fourier transform of a function f, where $\hat{f}$ is to be understood in the distributional sense. $L^p(\mathbb{R})$, $1 \le p < \infty$, denotes the space of all to the pth power Lebesgue integrable functions on $\mathbb{R}$ with the usual norm $\|\cdot\|_p$ and $L^\infty(\mathbb{R})$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. Furthermore, l^p, $1 \le p < \infty$, is the space of all sequences such that the p-norm $\|\cdot\|_{l^p}$ is finite, and l^∞ denotes the space of bounded sequences with the supremum norm $\|\cdot\|_{l^\infty}$. For $\sigma > 0$ and $1 \le p \le \infty$, we denote by $\mathcal{PW}_\sigma^p$ the Paley–Wiener space of functions f with representation $f(z) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} g(e^{i\omega})\, e^{iz\omega}\, \mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^p(\partial D)$. If $f \in \mathcal{PW}_\sigma^p$, then $g(e^{i\omega}) = \hat{f}(\omega)$. $D = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk and ∂D its boundary, i.e., the unit circle. The norm for $\mathcal{PW}_\sigma^1$ is given by $\|f\|_{\mathcal{PW}_\sigma^1} = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} |\hat{f}(\omega)|\, \mathrm{d}\omega$. Note that every function g defined on ∂D can be identified with a function $h(\omega) = g(e^{i\omega})$ defined on $\mathbb{R}$ and being 2π periodic. As a consequence of Hölder's inequality, we have $\mathcal{PW}_\sigma^p \supset \mathcal{PW}_\sigma^s$ for $1 \le p < s \le \infty$. Moreover, it holds $\|f\|_\infty \le \|f\|_{\mathcal{PW}_\pi^1}$.

2 Motivation

For functions $f \in \mathcal{PW}_\pi^p$, $p > 1$, the convergence of the Shannon sampling series

$$f(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \tag{2.1}$$

is uncritical: The Shannon sampling series is absolutely and uniformly convergent on $\mathbb{R}$ for all $f \in \mathcal{PW}_\pi^p$, $p > 1$ [6, p. 9].

A well-known fact [5–7] about the convergence behavior of the Shannon sampling series for $f \in \mathcal{PW}_\pi^1$ is expressed by the following theorem.

Theorem 1 *For all $f \in \mathcal{PW}_\pi^1$ and $T > 0$ fixed, it holds*

$$\lim_{N\to\infty}\left(\max_{t\in[-T,T]}\left|f(t) - \sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|\right) = 0.$$

The result of Theorem 1 is very important, because it establishes the uniform convergence on compact subsets of $\mathbb{R}$ for a large class of functions, namely $\mathcal{PW}_\pi^1$, which is the largest space within the scale of Paley–Wiener spaces. Unfortunately, it is not possible to extend Theorem 1 in such a way that the uniform convergence holds on all of $\mathbb{R}$ for the space $\mathcal{PW}_\pi^1$.

Since the Hardy space has shown to be a good substitute for L^1 in many cases, it is reasonable to expect a better convergence behavior of the Shannon sampling series for the Hardy space. However, as we will show in Sect. 4, even for the subspaces $\mathcal{H}^1$ and $\operatorname{Re}\mathcal{H}^1$ of $\mathcal{PW}^1$, the Shannon sampling series does not converge uniformly on $\mathbb{R}$. It is interesting to discuss the proof technique: In order to obtain the results, we used Fefferman's Theorem [10, p. 245], which states that the dual space of H^1 is $\mathcal{BMO}$.

Note that there are modifications of the Shannon sampling series, for example, the Shannon sampling series with oversampling [8] or series that are centered around t [2], which are uniformly convergent on all of $\mathbb{R}$. However, those modifications have other drawbacks, most notably the increased bandwidth compared to the Shannon sampling series.

3 The Hardy Space

As already mentioned, the Hardy space H^1 has several nice properties. Therefore, it is an often used space, especially as a substitute for L^1. Since the Hardy space is closely related to the Hilbert transformation, we introduce the Hilbert transformation first.

Definition 1 The Hilbert transform Hf of a function f is defined by

$$(Hf)(t) := \frac{1}{2\pi}\,\text{V.P.}\int_{-\pi}^{\pi}\frac{f(\tau)}{\tan(\frac{t-\tau}{2})}\,\mathrm{d}\tau,$$

where V.P. denotes the Cauchy principal value.

H is known to map functions in $L^p[-\pi,\pi)$ into functions in $L^p[-\pi,\pi)$, $1 < p < \infty$. However, for $p = 1$, this property is not true [18].

Definition 2 The Hardy space H^p, $0 < p < \infty$, consists of functions f analytic in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ satisfying

$$\|f\|_{H^p} := \sup_{0\le r<1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f\left(re^{i\theta}\right)\right|^p\,\mathrm{d}\theta\right)^{\frac{1}{p}} < \infty,$$

where $\|\cdot\|_{H^p}$ denotes the Hardy space norm.

We will focus on H^1, which is the largest space out of all Hardy spaces that are Banach spaces, in the following. If $f \in H^1$, then

$$\|f\|_{H^1} = \sup_{0\le r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|f\left(re^{i\theta}\right)\right| \mathrm{d}\theta,$$

and, furthermore, the limit $g(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exists almost everywhere and $g \in L^1(\partial D)$.

Definition 3 The space of functions f with the representation $f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{i\omega})e^{iz\omega}\,\mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in H^1$ is denoted by $\mathcal{H}^1$, and the norm is given by $\|f\|_{\mathcal{H}^1} = \|g\|_{H^1}$.

Definition 4 The space $\operatorname{Re} H^1$ consists of the boundary values of the real parts of functions of H^1, that is, if $g \in H^1$, then $f(\theta) = \lim_{r\uparrow 1} \operatorname{Re} g(re^{i\theta})$ belongs to $\operatorname{Re} H^1$. The norm is given by $\|f\|_{\operatorname{Re} H^1} = \|f\|_{L^1[-\pi,\pi)} + \|Hf\|_{L^1[-\pi,\pi)}$.

Definition 5 The space of functions f with the representation $f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\omega)e^{iz\omega}\,\mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in \operatorname{Re} H^1$ is denoted by $\operatorname{Re}\mathcal{H}^1$, and the norm is $\|f\|_{\operatorname{Re}\mathcal{H}^1} = \|g\|_{\operatorname{Re} H^1}$.

The relationship between the real Hardy space $\operatorname{Re} H^1$ and the Hardy space H^1 is as follows: If $f \in \operatorname{Re} H^1$, then $f_+ = f + i(Hf)$ is in $L^1[-\pi,\pi)$, and f_+ is the boundary-value function of a function $F_+ \in H^1$, i.e., $f_+(\theta) = \lim_{r\uparrow 1} F_+(re^{i\theta})$ almost everywhere. Conversely, for every function $F_+ \in H^1$, it holds that $\operatorname{Im}(f_+) = H \operatorname{Re}(f_+)$, where $f_+(\theta) = \lim_{r\uparrow 1} F_+(re^{i\theta})$. f_+ is called analytical signal. Note that $L^p[-\pi,\pi) \subset \operatorname{Re} H^1 \subset L^1[-\pi,\pi)$ for all $1 < p \le \infty$.

If $f \in H^1$, then it has the representation $f(z) = \sum_{k=0}^{\infty} c_k z^k$, $z \in D$ and it holds [10, p. 93]

$$\sum_{k=0}^{\infty} \frac{|c_k|}{k+1} \le \pi \|f\|_{H^1}. \tag{3.1}$$

As already mentioned, the Hilbert transformation H does not map functions in $L^1[-\pi,\pi)$ into functions in $L^1[-\pi,\pi)$. However, it holds that H maps functions in $\operatorname{Re} H^1$ into functions in $\operatorname{Re} H^1$. Therefore, the real Hardy space $\operatorname{Re} H^1$ is a suitable substitute for L^1 concerning the Hilbert transformation. Thus it could be possible that for $\mathcal{H}^1$, which is a closed subspace of $\mathcal{PW}_\pi^1$, the Shannon sampling series has properties that are not true for $\mathcal{PW}_\pi^1$.

One interesting fact about the Hardy space $\mathcal{H}^1$ is the absolute convergence of the Shannon sampling series: For all $f \in \mathcal{H}^1$ and all $t \in \mathbb{R}$, it holds by the absolute convergence principle [11, p. 6] and (3.1) that

$$\sum_{k=-\infty}^{\infty} \left|f(k)\right| \left|\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| < \infty. \tag{3.2}$$

In contrast, for $f \in \mathcal{PW}_\pi^1$, the convergence can fail to be absolute [11, p. 53]. Equation (3.2) indicates that $\mathcal{H}^1$ and $\operatorname{Re}\mathcal{H}^1$ could be suitable substitutes for the space $\mathcal{PW}_\pi^1$. For example, the nonsymmetric Shannon sampling series, which was shown to be divergent for functions in $\mathcal{PW}_\pi^1$ [3], is convergent for these spaces, because of (3.1) and (3.2).

Although the series (3.2) is not uniformly convergent on $\mathbb{R}$ for all function in $\mathcal{PW}_\pi^1$, one could guess that this is true for all functions in $\mathcal{H}^1$. However, we will prove the opposite.

In order to define the $\mathcal{BMO}$ space, which is needed for the proof, we have to introduce the average of a function f on the bounded interval $I \subset \mathbb{R}$ given by

$$(f)_I := \frac{1}{\mu(I)} \int_I f(x)\,\mathrm{d}x \tag{3.3}$$

and the abbreviation $(f)_d = (f)_{[-d,d)}$ for $d > 0$. μ denotes the Lebesgue measure.

Definition 6 A 2π-periodic function f on $\mathbb{R}$ is said to be of bounded mean oscillation if

$$\|f\|_{\mathcal{BMO}} := \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(x)\right|\mathrm{d}x + \sup_{\mu(I)\le 2\pi} \frac{1}{\mu(I)}\int_I \left|f(x) - (f)_I\right|\mathrm{d}x < \infty,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$ with $\mu(I) \le 2\pi$. By $\mathcal{BMO}$ we denote the space of all functions of bounded mean oscillation.

4 Convergence Behavior for $\operatorname{Re}\mathcal{H}^1$ and $\mathcal{H}^1$

Now we are in the position to analyze the behavior of the Shannon sampling series for functions in $\operatorname{Re}\mathcal{H}^1$ and $\mathcal{H}^1$, respectively. We start the analysis with the space $\operatorname{Re}\mathcal{H}^1$.

Theorem 2 *There exists a function $f_1 \in \operatorname{Re}\mathcal{H}^1$ such that*

$$\limsup_{N\to\infty}\left(\max_{t\in\mathbb{R}}\left|\sum_{k=-N}^{N} f_1(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|\right) = \infty.$$

The following corollary is a direct consequence of Theorem 2.

Corollary 1 *There exists a function $f_1 \in \operatorname{Re}\mathcal{H}^1$ such that*

$$\sup_{t\in\mathbb{R}} \sum_{k=-\infty}^{\infty} \left|f_1(k)\right|\left|\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| = \infty.$$

Proof of Theorem 2 Let $f \in \operatorname{Re}\mathcal{H}^1$. Then, by definition there exists a function $F \in \operatorname{Re} H^1$, which is 2π-periodic, such that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)e^{iz\omega}\,\mathrm{d}\omega, \quad z \in \mathbb{C}.$$

Consequently,

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega+\pi)e^{iz\omega}\,\mathrm{d}\omega, \quad z \in \mathbb{C},$$

also is in $\operatorname{Re}\mathcal{H}^1$, and we have $g(k) = f(k)(-1)^k$, $k \in \mathbb{Z}$. But since

$$\sum_{k=-N}^{N} g(k)\frac{\sin(\pi(t-k))}{\pi(t-k)} = \frac{\sin(\pi t)}{\pi}\sum_{k=-N}^{N}\frac{f(k)}{t-k}, \tag{4.1}$$

it is sufficient to analyze the sum on the right-hand side of (4.1), which we denote by

$$(A_N f)(t) := \sum_{k=-N}^{N}\frac{f(k)}{t-k}, \quad N \in \mathbb{N}.$$

For $t = N + 1/2$, we have

$$(A_N f)\left(N+\frac{1}{2}\right) = \sum_{k=-N}^{N}\frac{f(k)}{N+\frac{1}{2}-k} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)\sum_{k=-N}^{N}\frac{1}{N+\frac{1}{2}-k}e^{i\omega k}\,\mathrm{d}\omega.$$

Consider the functional $\Psi_N : \operatorname{Re} H^1 \to \mathbb{C}$ given by

$$\Psi_N\hat{f} := \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)\sum_{k=-N}^{N}\frac{1}{N+\frac{1}{2}-k}e^{i\omega k}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)\hat{h}_N(\omega)\,\mathrm{d}\omega.$$

Obviously, Ψ_N is a continuous linear functional on $\operatorname{Re} H^1$ for fixed N, and it holds [10, p. 245] $\sup_{\|\hat{f}\|_{\operatorname{Re} H^1}\le 1}|\Psi_N\hat{f}| = \|\hat{h}_N\|_{\mathcal{BMO}}$. Since the norm of the functional is determined by the $\mathcal{BMO}$-norm of $\hat{h}_N$, we analyze $\|\hat{h}_N\|_{\mathcal{BMO}}$ next. For all $d \in (0,\pi)$,

$$\|\hat{h}_N\|_{\mathcal{BMO}} \ge \frac{1}{2d}\int_{-d}^{d}\left|\hat{h}_N(\omega) - \left(\hat{h}_N\right)_d\right|\mathrm{d}\omega \ge \frac{1}{2d}\int_{-d}^{d}\left|b_N(\omega)-(b_N)_d\right|\mathrm{d}\omega$$

holds, where we used the abbreviation $b_N(\omega) = \operatorname{Im}\hat{h}_N(\omega)$ and $(\cdot)_d$ defined in (3.3) and below. Furthermore,

$$b_N(\omega) = \underbrace{\left(\sum_{k=0}^{2N}\frac{1}{k+\frac{1}{2}}\cos(k\omega)\right)}_{=c_N(\omega)}\sin(N\omega) - \underbrace{\left(\sum_{k=0}^{2N}\frac{1}{k+\frac{1}{2}}\sin(k\omega)\right)}_{=d_N(\omega)}\cos(N\omega),$$

where $|d_N(\omega)|$ can be upper bounded by a constant C_1 for all $N \in \mathbb{N}$ and $\omega \in [-\pi,\pi)$ by using the fact that there exists a constant C_1 such that

$$\left|\sum_{k=0}^{2N}\frac{1}{k+\frac{1}{2}}\sin(k\omega)\right| \le C_1$$

for all $N \in \mathbb{N}$ and $\omega \in [-\pi, \pi)$ [18, p. 5]. Thus we obtain

$$\frac{1}{2d}\int_{-d}^{d}\left|d_N(\omega)-(d_N)_d\right|\,\mathrm{d}\omega \le \frac{1}{2d}\int_{-d}^{d}\left|d_N(\omega)\right|\,\mathrm{d}\omega + \left|(d_N)_d\right| \le C_1,$$

because d_N is the product of an odd and an even function and hence itself an odd function, which leads to $|(d_N)_d| = |\frac{1}{2d}\int_{-d}^{d} d_N(\omega)\,\mathrm{d}\omega| = 0$. Consequently, for all $d \in (0,\pi)$, we have $\|\hat{h}_N\|_{\mathcal{BMO}} \ge \frac{1}{2d}\int_{-d}^{d}|c_N(\omega)-(c_N)_d|\,\mathrm{d}\omega - C_1$. Since c_N is an odd function, it follows $(c_N)_d = 0$, and therefore, for all $d \in (0,\pi)$,

$$\|\hat{h}_N\|_{\mathcal{BMO}} \ge \frac{1}{2d}\int_{-d}^{d}\left|c_N(\omega)\right|\,\mathrm{d}\omega - C_1 = \frac{1}{d}\int_{0}^{d}\left|c_N(\omega)\right|\,\mathrm{d}\omega - C_1. \tag{4.2}$$

Using $\cos(\omega) \ge 1 - w^2/2$, we obtain, for all $\omega \in [0, \frac{\pi}{N}]$,

$$\begin{aligned}\sum_{k=0}^{2N}\frac{1}{k+\frac{1}{2}}\cos(k\omega) &\ge \sum_{k=0}^{2N}\frac{1}{k+\frac{1}{2}} - \frac{\pi^2}{2N^2}\sum_{k=1}^{2N}\frac{k^2}{k+\frac{1}{2}}\\ &\ge \log\bigl(2(2N+1)\bigr) - 2\pi^2,\end{aligned} \tag{4.3}$$

where $\sum_{k=0}^{2N} 1/(k+1/2) \ge \sum_{k=0}^{2N}\int_k^{k+1} 1/(\tau+1/2)\,\mathrm{d}\tau = \int_0^{2N+1} 1/(\tau+1/2)\,\mathrm{d}\tau \ge \log(2(2N+1))$ and $\sum_{k=1}^{2N} k^2/(k+1/2) < \sum_{k=1}^{2N} k = N(2N+1)$ have been used. For N large enough, the right-hand side of (4.3) is positive, and it follows that

$$\left|c_N(\omega)\right| \ge \bigl(\log\bigl(2(2N+1)\bigr) - 2\pi^2\bigr)\left|\sin(N\omega)\right|. \tag{4.4}$$

Inserting (4.4) into (4.2) gives, for $d = \pi/N$,

$$\|\hat{h}_N\|_{\mathcal{BMO}} \ge \frac{\log(2(2N+1)) - 2\pi^2}{\pi} N \int_0^{\pi/N}\left|\sin(N\omega)\right|\,\mathrm{d}\omega - C_1,$$

which can be further simplified to $\|\hat{h}_N\|_{\mathcal{BMO}} \ge (2/\pi)\log(2(2N+1)) - 4\pi - C_1$. Hence, for every N large enough, there exists an $\hat{f}_N \in \operatorname{Re} H^1$, $\|\hat{f}_N\|_{\operatorname{Re} H^1} = 1$, such that

$$\Psi_N \hat{f}_N \ge \frac{2}{\pi}\log\bigl(2(2N+1)\bigr) - 4\pi - C_1$$

and consequently, for g_N defined by $\hat{g}_N(\omega) = \hat{f}_N(\omega+\pi)$,

$$\begin{aligned}\left|\sum_{k=-N}^{N} g_N(k)\frac{\sin(\pi(N+\frac{1}{2}-k))}{\pi(N+\frac{1}{2}-k)}\right| &= \frac{1}{\pi}\sum_{k=-N}^{N}\frac{f_N(k)}{N+\frac{1}{2}-k}\\ &= \Psi_N \hat{f}_N \ge \frac{2}{\pi}\log\bigl(2(2N+1)\bigr) - 4\pi - C_1\end{aligned}$$

with $\|g_N\|_{H^1} = 1$. Applying the Banach–Steinhaus theorem completes the proof. □

The space $\operatorname{Re} H^1$ is a good substitute for L^1 regarding the Hilbert transformation and Cesàro operators [4]. One reason for the nice properties of $\operatorname{Re} H^1$ is the fact that its dual space $\mathcal{BMO}$ contains unbounded functions. Nevertheless, the proof has shown that these properties cannot impede the divergence of the peak value of the Shannon sampling series.

Theorem 2 was formulated for the space $\operatorname{Re}\mathcal{H}^1$ but it can be easily extended to the space $\mathcal{H}^1$.

Theorem 3 *Theorem 2 is also true if* $\operatorname{Re}\mathcal{H}^1$ *is replaced by* $\mathcal{H}^1$.

Proof Before we start with the proof, let us introduce the abbreviation

$$(S_N f)(t) := \sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}$$

for the finite Shannon sampling series. By the same arguments as in the proof of Theorem 2, we see that there exists a function $f \in \operatorname{Re}\mathcal{H}^1$ such that

$$\limsup_{N\to\infty}\left|(S_N f)\left(-N-\frac{1}{2}\right)\right| = \infty. \tag{4.5}$$

The analytic signal $\hat{f}_+ = \frac{1}{2}(\hat{f} + iH\hat{f})$, $\hat{f} \in \operatorname{Re} H^1$, is the boundary-value function of some function in H^1. Therefore, $f_+ \in \mathcal{H}^1$, and we have

$$f_+(k) = \begin{cases} 0, & k \ge 1,\\ \frac{f(0)}{2}, & k = 0,\\ f(k), & k \le -1, \end{cases}$$

from which we obtain

$$(S_N f_+)(t) = (S_N f)(t) - \frac{f(0)}{2}\frac{\sin(\pi t)}{\pi t} - \sum_{k=1}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}. \tag{4.6}$$

The second term on the right-hand side of (4.6) has the estimate

$$\sup_{t\in\mathbb{R}}\left|\frac{f(0)}{2}\frac{\sin(\pi t)}{\pi t}\right| = \frac{|f(0)|}{2} < \infty. \tag{4.7}$$

As for the last term of the right-hand side of (4.6), for $t = -N - \frac{1}{2}$, we have

$$\begin{aligned}&\limsup_{N\in\mathbb{N}}\left|\sum_{k=1}^{N} f(k)\frac{\sin(\pi(-N-\frac{1}{2}-k))}{\pi(-N-\frac{1}{2}-k)}\right|\\ &\quad\le \limsup_{N\in\mathbb{N}}\sum_{k=1}^{N}\frac{|f(k)|}{\pi(N+\frac{1}{2}+k)} \le \sum_{k=1}^{\infty}\frac{|f(k)|}{\pi(\frac{3}{2}+k)} < \infty,\end{aligned} \tag{4.8}$$

by virtue of Hardy's inequality (3.1). Combining (4.5), (4.6), (4.7), and (4.8), we see that

$$\limsup_{N\to\infty}\left|(S_N f_+)\left(-N-\frac{1}{2}\right)\right| = \infty,$$

which completes the proof. □

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