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Sampling Theorem and Discrete Fourier Transform on the Riemann Sphere

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Abstract Using coherent-state techniques, we prove a sampling theorem for Majorana's (holomorphic) functions on the Riemann sphere and we provide an exact reconstruction formula as a convolution product of N samples and a given reconstruction kernel (a sinc-type function). We also discuss the effect of over- and under-sampling. Sample points are roots of unity, a fact which allows explicit inversion formulas for resolution and overlapping kernel operators through the theory of Circulant Matrices and Rectangular Fourier Matrices. The case of band-limited functions on the Riemann sphere, with spins up to J, is also considered. The connection with the standard Euler angle picture, in terms of spherical harmonics, is established through a discrete Bargmann transform.

Keywords Holomorphic functions · Coherent states · Discrete Fourier transform · Sampling · Frames

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1 Introduction

The Fourier transform on the sphere is applied in a wide variety of fields: geophysics, seismology, tomography, atmospheric science, computer vision, atomic physics, astrophysics, statistics, signal processing, crystallography, etc. It is therefore of great interest to develop efficient techniques for the computation of Fourier coefficients, spherical convolutions, etc.

Sometimes we have at our disposal just a set of samples of our signal and we ask ourselves whether the Fourier transform may be computed, or the whole signal be reconstructed (up to a certain degree of accuracy), from the discrete samples. In the case of band-limited functions on the line (or Abelian harmonic analysis in general), the classical (Shannon) sampling theorem provides the necessary and sufficient conditions for this problem. However, the establishment of sampling theorems for harmonic analysis on non-Abelian groups and their homogeneous spaces is still relatively scarce in the literature, apart from some important general results for compact groups [1, 2] and the (noncompact) motion group [3]. Moreover, we would want our algorithms to be *fast* and efficient. The Fast Fourier Transform (FFT), in the setting of Abelian harmonic analysis (i.e., the well-known Cooley–Tukey algorithm [4] for time series analysis), has been extensively studied in both the theoretical and applied literature but, again, there are few algorithms for the efficient computation of Fourier transforms associated with non-Abelian groups and their homogeneous spaces (see again Refs. [1, 2] for compact groups and [5] for the motion group and its engineering applications [3], namely in robotics [6]). For finite non-Abelian groups, like the symmetric group S_n, the reference [7] provides efficient algorithms to compute Fourier transforms.

For the two-dimensional sphere $\mathbb{S}^2$, the efficient computation of Fourier transforms of band-limited functions (those functions in $L^2(\mathbb{S}^2)$ which expansion requires only spherical harmonics of angular momentum at most J) has been achieved in, for instance, Refs. [8–12]. In references [8, 9], the authors develop a sampling theorem on the sphere, which reduces the computation of Fourier transforms and convolutions of band-limited functions to discrete (finite) calculations. Here, band-limited functions on $\mathbb{S}^2$, of bandwidth J, are expanded in terms of spherical harmonics and sampled at an equiangular grid of $4J^2$ points.

The point of view followed in these references is a group theoretic one. In this setting, the FFT on $\mathbb{S}^2$ is an algorithm for the efficient expansion of a function defined on the sphere $\mathbb{S}^2 = SO(3)/SO(2)$ in terms of a set of irreducible matrix coefficients for the special orthogonal group in three dimensions, $G = SO(3)$, which, in this case, are the standard family of spherical harmonics.

In this article we consider the group $G = SU(2)$ (double cover of $SO(3)$), which allows for (extra) half-integer angular momenta (spin). Moreover, we shall work in a different (holomorphic) picture and use, instead of spherical harmonics (based on an Euler angle characterization), another system of (less standard) orthogonal polynomials: "Majorana's (holomorphic) functions" [13, 14] on the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (one-point compactification of the complex plane). The advantage of using this "complex holomorphic picture", instead of the standard "Euler angle picture," is twofold: firstly, we can take advantage of the either diagonal or circulant structure of resolution and overlapping kernel operators, respectively, to provide explicit inversion formulas and, secondly, we can extend the sampling procedure to half-integer angular momenta s, which could be useful when studying, for example, discrete frames for coherent states of spinning particles in Atomic Physics (see, e.g., Refs. [15, 16] for a thorough exposition on coherent states and its applications in Physics). Moreover, for integer angular momenta $s = j$, we could always pass from one picture to another through the Bargmann transform (3.31).

Working with a fixed angular momentum (spin) s, we shall introduce a system of coherent states for $SU(2)$ (the spin coherent states), which is a set of states sharing similar properties with wavelets (in fact, they can be considered the same thing, see [18, 19]). We shall provide a generalized Bargmann Transform [17] relating both pictures (representations): the "holomorphic" one and the "standard" one, which is a particular case of coherent-state transform [15, 16]. Then we shall choose in $\bar{\mathbb{C}}$ the roots of unity as sampling points, so that the sampling of the coherent-state overlap (or Reproducing Kernel) has a "circulant" structure [20]. Using the properties of the Rectangular Fourier Matrices (RFM) and the theory of Circulant Matrices, we will be able to invert the (sampled) reproducing kernel $\mathcal{B}$ and provide a reconstruction formula for Majorana's (holomorphic) functions on the Riemann sphere. The inversion formula is accomplished through an eigen-decomposition $\mathcal{B} = \mathcal{F} D \mathcal{F}^{-1}$ of $\mathcal{B}$, where $\mathcal{F}$ turns out to be the standard discrete Fourier transform matrix. This fact allows for a straightforward fast extension of the reconstruction algorithm. The case of band-limited functions is also considered, but in this case the inversion should be done numerically, and no fast algorithm is available, for the moment.

In order to keep the article as self-contained as possible, we shall introduce in the next two sections general definitions and results about coherent states and frames based on a group G and the standard construction of spin coherent states for the case $G = SU(2)$. We refer the reader to Refs. [15, 16, 18, 21] for more information. In Sect. 4 we provide sampling theorems and reconstruction formulas for Majorana's functions on the Riemann sphere and discuss the effect of over- and under-sampling and the analogies with the so called "covariant interpolation." We also discuss the case of band-limited functions, where a negative result is proved in the case of sampling at roots of unity. A reconstruction theorem is provided for another set of sampling points, but the inversion should be done numerically. In Sect. 5 we provide explicit expressions (discrete Bargmann transforms) which connect our "complex holomorphic picture" and the standard "Euler angle picture," and we discuss some obstructions that arise. Appendices A and B are devoted to a brief review of rectangular Fourier matrices and circulant matrices, respectively.

2 A Brief on Coherent States and Frames

Let us consider a *unitary* representation U of a Lie group G on a Hilbert space $(\mathcal{H}, \langle\cdot|\cdot\rangle)$. Consider also the space $L^2(G, dg)$ of square-integrable complex functions Ψ on G, where $dg = d(g'g),\ \forall g' \in G$, stands for the left-invariant Haar measure, which defines the scalar product

$$(\Psi|\Phi) = \int_G \bar{\Psi}(g)\Phi(g)dg. \tag{2.1}$$

A nonzero function $\gamma \in \mathcal{H}$ is called *admissible* (or a *fiducial* vector) if $\Gamma(g) \equiv \langle U(g)\gamma|\gamma\rangle \in L^2(G, dg)$, that is, if

$$c_\gamma = \int_G \bar{\Gamma}(g)\Gamma(g)\,dg = \int_G \left|\langle U(g)\gamma|\gamma\rangle\right|^2 dg < \infty. \tag{2.2}$$

Let us assume that the representation U is *irreducible* and that there exists a function γ admissible. Then a system of coherent states (CS) of $\mathcal{H}$ associated to (or indexed by) G is defined as the set of functions in the orbit of γ under G

$$\gamma_g = U(g)\gamma, \quad g \in G. \tag{2.3}$$

We can also restrict ourselves to a suitable homogeneous space $Q = G/H$ for some closed subgroup H. Then, the nonzero function γ is said to be admissible mod(H,σ) (with $\sigma : Q \to G$ a Borel section), and the representation U square-integrable mod(H,σ) if the condition

$$\int_Q \left|\langle U(\sigma(q))\gamma|\psi\rangle\right|^2 dq < \infty, \quad \forall \psi \in \mathcal{H} \tag{2.4}$$

holds, where dq is a measure on Q "projected" from the left-invariant measure dg on the whole G. The coherent states indexed by Q are defined as $\gamma_{\sigma(q)} = U(\sigma(q))\gamma, q \in Q$, and they form an overcomplete set in $\mathcal{H}$.

Condition (2.4) could also be written as the "expectation value"

$$0 < \int_Q \left|\langle U(\sigma(q))\gamma|\psi\rangle\right|^2 dq = \langle\psi|A_\sigma|\psi\rangle < \infty, \quad \forall \psi \in \mathcal{H}, \tag{2.5}$$

where $A_\sigma = \int_Q |\gamma_{\sigma(q)}\rangle\langle\gamma_{\sigma(q)}|dq$ is a positive, bounded, and invertible operator.[1]

If the operator A_σ^{-1} is also bounded, then the set $S_\sigma = \{|\gamma_{\sigma(q)}\rangle, q \in Q\}$ is called a *frame*, and a *tight frame* if A_σ is a positive multiple of the identity, $A_\sigma = \lambda I, \lambda > 0$.

To avoid domain problems in the following, let us assume that γ generates a frame (i.e., that A_σ^{-1} is bounded). The *CS map* is defined as the linear map

$$\begin{aligned} T_\gamma :\quad & \mathcal{H} \longrightarrow L^2(Q, dq) \\ & \psi \longmapsto \Psi_\gamma(q) = [T_\gamma \psi](q) = \frac{\langle\gamma_{\sigma(q)}|\psi\rangle}{\sqrt{c_\gamma}}. \end{aligned} \tag{2.6}$$

Its range $L^2_\gamma(Q, dq) \equiv T_\gamma(\mathcal{H})$ is complete with respect to the scalar product $(\Phi|\Psi)_\gamma \equiv (\Phi|T_\gamma A_\sigma^{-1} T_\gamma^{-1}\Psi)_Q$, and T_γ is unitary from $\mathcal{H}$ onto $L^2_\gamma(Q, dq)$. Thus, the inverse map T_γ^{-1} yields the *reconstruction formula*

$$\psi = T_\gamma^{-1}\Psi_\gamma = \int_Q \Psi_\gamma(q) A_\sigma^{-1} \gamma_{\sigma(q)}\, dq, \quad \Psi_\gamma \in L^2_\gamma(Q, dq), \tag{2.7}$$

[1] In this paper we shall extensively use the Dirac notation in terms of "bra" and "kets" (see, e.g., [18, 22]). The Dirac notation is justified by the Riesz Representation Theorem and is valid in more general settings than Hilbert spaces of square-integrable functions.

which expands the signal ψ in terms of CS $A_\sigma^{-1}\gamma_{\sigma(q)}$ with wavelet coefficients $\Psi_\gamma(q) = [T_\gamma \psi](q)$. These formulas acquire a simpler form when A_σ is a multiple of the identity, as is for the case considered in this article.

When it comes to numerical calculations, the integral $A_\sigma = \int_Q |\gamma_{\sigma(q)}\rangle\langle\gamma_{\sigma(q)}|dq$ has to be discretized, which means to restrict ourself to a discrete subset $\mathcal{Q} \subset Q$. The question is whether this restriction will imply a loss of information, that is, whether the set $\mathcal{S} = \{|q_k\rangle \equiv |\gamma_{\sigma(q_k)}\rangle, q_k \in \mathcal{Q}\}$ constitutes a discrete frame itself, with the resolution operator

$$\mathcal{A} = \sum_{q_k \in \mathcal{Q}} |q_k\rangle\langle q_k|. \tag{2.8}$$

The operator $\mathcal{A}$ need not coincide with the original A_σ. In fact, a continuous tight frame might contain discrete nontight frames, as happens in our case (see later on Sect. 4).

Let us assume that $\mathcal{S}$ generates a discrete frame, that is, there are two positive constants $0 < b < B < \infty$ (*frame bounds*) such that the admissibility condition

$$b\|\psi\|^2 \le \left|\sum_{q_k \in \mathcal{Q}} \langle q_k|\psi\rangle\right|^2 \le B\|\psi\|^2 \tag{2.9}$$

holds $\forall \psi \in \mathcal{H}$. To discuss the properties of a frame, it is convenient to define the frame (or sampling) operator $\mathcal{T} : \mathcal{H} \to \ell^2$ given by $\mathcal{T}(\psi) = \{\langle q_k|\psi\rangle, q_k \in \mathcal{Q}\}$. Then we can write $\mathcal{A} = \mathcal{T}^*\mathcal{T}$, and the admissibility condition (2.9) now adopts the form

$$bI \le \mathcal{T}^*\mathcal{T} \le BI, \tag{2.10}$$

where I denotes the identity operator in $\mathcal{H}$. This implies that $\mathcal{A}$ is invertible. If we define the *dual frame* $\{|\tilde{q}\rangle \equiv \mathcal{A}^{-1}|q\rangle\}$, one can easily prove that the expansion (*reconstruction formula*)

$$|\psi\rangle = \sum_{q_k \in \mathcal{Q}} \langle q_k|\psi\rangle|\tilde{q}_k\rangle \tag{2.11}$$

converges strongly in $\mathcal{H}$, that is, the expression

$$\mathcal{T}_l^{+}\mathcal{T} = \sum_{q_k \in \mathcal{Q}} |\tilde{q}_k\rangle\langle q_k| = \mathcal{T}^*\left(\mathcal{T}_l^{+}\right)^* = \sum_{q_k \in \mathcal{Q}} |q_k\rangle\langle\tilde{q}_k| = I \tag{2.12}$$

provides a resolution of the identity, where $\mathcal{T}_l^{+} \equiv (\mathcal{T}^*\mathcal{T})^{-1}\mathcal{T}^*$ is the (left) pseudoinverse (see, for instance, [23]) of $\mathcal{T}$ (see, e.g., [18, 21] for a proof, where they introduce the dual frame operator $\tilde{\mathcal{T}} = (\mathcal{T}_l^{+})^*$ instead).

It is interesting to note that the operator $P = \mathcal{T}\mathcal{T}_l^{+}$ acting on ℓ^2 is an orthogonal projector onto the range of $\mathcal{T}$.

We shall also be interested in cases where there are not enough points to completely reconstruct the signal, i.e., *undersampling*, but a partial reconstruction is still possible. In these cases, $\mathcal{S}$ does not generate a discrete frame, and the resolution operator $\mathcal{A}$ would not be invertible. But we can construct another operator from $\mathcal{T}$, $\mathcal{B} = \mathcal{T}\mathcal{T}^*$, acting on ℓ^2.

The matrix elements of $\mathcal{B}$ are $\mathcal{B}_{kl} = \langle q_k | q_l \rangle$, therefore $\mathcal{B}$ is the discrete reproducing kernel operator, see (3.30). If the set $\mathcal{S}$ is linearly independent, the operator $\mathcal{B}$ will be invertible and a (right) pseudoinverse can be constructed for $\mathcal{T}$, $\mathcal{T}_r^+ \equiv \mathcal{T}^*(\mathcal{T}\mathcal{T}^*)^{-1}$, in such a way that $\mathcal{T}\mathcal{T}_r^+ = I_{\ell^2}$. As in the previous case, there is another operator, $P_{\mathcal{S}} = \mathcal{T}_r^+\mathcal{T}$ acting on $\mathcal{H}$, which is an orthogonal projector onto the subspace spanned by $\mathcal{S}$. A pseudo-dual frame can be defined as

$$|\tilde{q}_k\rangle = \sum_{q_l \in \mathcal{Q}} \left(\mathcal{B}^{-1}\right)_{lk} |q_l\rangle \tag{2.13}$$

providing a resolution of the projector $P_{\mathcal{S}}$,

$$\mathcal{T}_r^+\mathcal{T} = \sum_{q_k \in \mathcal{Q}} |\tilde{q}_k\rangle\langle q_k| = \mathcal{T}^*\left(\mathcal{T}_r^+\right)^* = \sum_{q_k \in \mathcal{Q}} |q_k\rangle\langle \tilde{q}_k| = P_{\mathcal{S}}. \tag{2.14}$$

Using this, an "alias" $|\hat{\psi}\rangle$ of the signal $|\psi\rangle$ is obtained,

$$|\hat{\psi}\rangle = \sum_{q_k \in \mathcal{Q}} \langle q_k | \psi \rangle |\tilde{q}_k\rangle, \tag{2.15}$$

which is the orthogonal projection of $|\psi\rangle$ onto the subspace spanned by $\mathcal{S}$, $|\hat{\psi}\rangle = P_{\mathcal{S}}|\psi\rangle$. An example of this can be found in Sect. 4.1.2.

The two operators $\mathcal{A}$ and $\mathcal{B}$ are intertwined by the frame operator $\mathcal{T}$, $\mathcal{T}\mathcal{A} = \mathcal{B}\mathcal{T}$. If $\mathcal{T}$ is invertible, then both $\mathcal{A}$ and $\mathcal{B}$ are invertible and $\mathcal{T}_r^+ = \mathcal{T}_l^+ = \mathcal{T}^{-1}$. This case corresponds to critical sampling, where both operators $\mathcal{A}$ and $\mathcal{B}$ can be used to fully reconstruct the signal.

It should be noted that in the case in which there is a finite number N of sampling points q_k, the space ℓ^2 should be substituted by $\mathbb{C}^N$, and the operator $\mathcal{B}$ can be identified with its matrix once a basis has been chosen. If the Hilbert space $\mathcal{H}$ is finite dimensional, as it is the case for all irreducible and unitary representations of $SU(2)$, all operators appearing in this section can be identified with their matrices.

3 Representations of $SU(2)$: Spin Coherent States

The subject of Harmonic Analysis on the rotation group has been extensively treated in the literature. Here we shall try to summarize what is important for our purposes, in order to keep the article as self-contained as possible.

3.1 Coordinate Systems and Generators

The (two-dimensional) fundamental representation of the Lie group $SU(2)$ corresponds to the group of complex 2×2 unitary matrices with determinant one:

$$SU(2) = \{U(\zeta) = \begin{pmatrix} \zeta_1 & \zeta_2 \\ -\bar{\zeta}_2 & \bar{\zeta}_1 \end{pmatrix}, \quad \zeta_1, \zeta_2 \in \mathbb{C} : \det(U) = |\zeta_1|^2 + |\zeta_2|^2 = 1\}. \tag{3.1}$$

The coordinates ζ_1, ζ_2 are called "Cayley–Klein" parameters in the literature. Writing

$$\zeta_1 = \epsilon_0 + i\epsilon_3, \qquad \zeta_2 = \epsilon_2 + i\epsilon_1, \quad \epsilon_j \in \mathbb{R}, \tag{3.2}$$

we have that

$$\det(U) = |\zeta_1|^2 + |\zeta_2|^2 = \epsilon_0^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 = 1, \tag{3.3}$$

which tells us that $SU(2) \approx \mathbb{S}^3$ (the four-dimensional sphere) as a (three-dimensional) manifold. Denoting by

$$J_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad J_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.4}$$

a basis of 2×2 traceless Hermitian (halved Pauli) matrices, we can also write any matrix $U \in SU(2)$, in a compact way, as

$$U(\epsilon) = \epsilon_0 I + 2i \sum_{k=1}^{3} \epsilon_k J_k, \tag{3.5}$$

where I stands for the 2×2 identity matrix. The matrices (3.4) are also called the generators of infinitesimal (small) transformations $U = I + i\varepsilon A$, $\varepsilon \ll 1$, since $UU^* = I$ and $\det(U) = 1$ imply (up to quantities of order two) that A is a traceless Hermitian matrix, that is, it can be written as $A = \sum_{k=1}^{3} a_k J_k$. The Lie algebra of infinitesimal generators of $SU(2)$ is defined as the (real) vector space $su(2) = \text{Span}\{J_1, J_2, J_3\}$ of traceless Hermitian matrices satisfying the standard (angular momentum) commutation relations (easy to check):

$$[J_1, J_2] = iJ_3, \qquad [J_2, J_3] = iJ_1, \qquad [J_3, J_1] = iJ_2. \tag{3.6}$$

Any connected Lie group like $SU(2)$ can be built up by means of its infinitesimal generators via the exponential:

$$U(\alpha) = e^{i\sum_{k=1}^{3}\alpha_k J_k} = \cos\frac{\alpha}{2} I + 2i \sum_{k=1}^{3} n_k \sin\frac{\alpha}{2} J_k, \tag{3.7}$$

where $\alpha_k \in \mathbb{R}$, $k = 1, 2, 3$, are called canonical coordinates at the identity element $U = I$ and $\alpha = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$, $n_k = \frac{\alpha_k}{\alpha}$. Comparing (3.5) with (3.7) gives a relation between the Cayley–Klein parameters ϵ and the canonical coordinates α.

Let us introduce another complex parametrization of $SU(2)$, adapted to the *Hopf fibration* of $\mathbb{S}^3$, which will be of use in what follows. Let us define the following equivalence relation in $SU(2)$:

$$(\zeta_1', \zeta_2') \sim (\zeta_1, \zeta_2) \Leftrightarrow (\zeta_1', \zeta_2') = \eta(\zeta_1, \zeta_2); \qquad \eta \in \mathbb{C}, \ |\eta| = 1, \tag{3.8}$$

so that the quotient space (coset) $(SU(2)/\sim)$ coincides with the complex projective space $\mathbb{C}P^1$, which is isomorphic to $\mathbb{S}^2$. Indeed, let us denote by $[\zeta_1, \zeta_2]$ an element (equivalence class) of $\mathbb{C}P^1$. If $\zeta_2 \neq 0$, then $[\zeta_1, \zeta_2] = [\frac{\zeta_1}{\zeta_2}, 1] = [z, 1]$ represents a point $z \in \mathbb{C}$, which is related to the stereographic projection of the Riemann sphere on $\mathbb{C}$ (see later on this section). If $\zeta_2 = 0$, then $[\zeta_1, 0] = [1, 0]$ is just a point (the north/south pole). The other chart corresponds to $\zeta_1 \neq 0$, which contains the identity element $U = I$ of $SU(2)$. We shall work in this chart and define $z \equiv \frac{\zeta_2}{\zeta_1}$. The projection

$$\pi : SU(2) \to \mathbb{S}^2, \quad (z_1, z_2) \mapsto [z_1, z_2] \tag{3.9}$$

gives $SU(2)$ a principal fibre bundle structure with the structural group

$$\pi^{-1}([z_1, z_2]) = \{\eta \in \mathbb{C} : |\eta| = 1\} \simeq U(1). \tag{3.10}$$

In our chart, we can take $\eta = e^{i\varphi} = \frac{\zeta_1}{|\zeta_1|}$. The Cayley–Klein parameters can be written in these Hopf-fibration coordinates as

$$\zeta_1 = \mathcal{N}(z, \bar{z})\eta, \qquad \zeta_2 = \mathcal{N}(z, \bar{z})z\eta; \quad \mathcal{N}(z, \bar{z}) \equiv \sqrt{\frac{1}{1 + z\bar{z}}}, \tag{3.11}$$

where we have defined the suitable normalization factor $\mathcal{N}$ for convenience. Denoting by $J_\pm = J_1 \pm iJ_2$ the raising and lowering ladder operators, we can check that any group element $U \in SU(2)$ can also be written in complex coordinates z, η as

$$U(z, \bar{z}, \varphi) = \mathcal{N}(z, \bar{z}) e^{zJ_-} e^{-\bar{z}J_+} e^{-i\varphi J_3}. \tag{3.12}$$

We have discussed the (two-dimensional) fundamental representation of $SU(2)$. There is also a three-dimensional (adjoint) representation of $SU(2)$ on its Lie algebra

$$\begin{aligned} su(2) &= \left\{ X = \sum_{k=1}^{3} x_k J_k = \frac{1}{2}\begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}, x_k \in \mathbb{R} \right\} \\ &\simeq \mathbb{R}^3 = \{(x_1, x_2, x_3), x_k \in \mathbb{R}\} \end{aligned} \tag{3.13}$$

given by the action

$$U : su(2) \longrightarrow su(2), \quad X \mapsto UXU^*, \tag{3.14}$$

which reduces to the standard action of the rotation group $SO(3)$, of 3×3 orthogonal matrices, on $\mathbb{R}^3$. The fact that U and $-U$ give the same rotation in (3.14) is a consequence of the fact that $SO(3) = SU(2)/\mathbb{Z}_2$ or, in other words, $SU(2)$ is the double cover of $SO(3)$. It is usual to parametrize $SO(3)$ in terms of Euler angles, which correspond to the choice (in the arrangement $x_3(\varphi) \to x_2(\theta) \to x_3(\phi)$)

$$U(\theta, \phi, \varphi) = e^{-i\phi J_3} e^{-i\theta J_2} e^{-i\varphi J_3}. \tag{3.15}$$

After a little bit of algebra (power expansion of the exponentials), we can find a relation between Cayley–Klein parameters and Euler angles given by

$$\zeta_1 = e^{i\frac{\varphi+\phi}{2}} \cos\frac{\theta}{2}, \qquad \zeta_2 = e^{i\frac{\varphi-\phi}{2}} \sin\frac{\theta}{2}, \tag{3.16}$$

so that $z = \frac{\zeta_2}{\zeta_1} = e^{i\phi}\tan(\frac{\theta}{2})$ is the stereographic projection of the Riemann sphere on the complex plane, as anticipated before.

We have discussed the two-dimensional (spin $s = 1/2$) and three-dimensional (spin $s = 1$) representations of $SU(2)$ in order to introduce coordinate systems. Let us consider now higher-dimensional unitary irreducible representations of arbitrary spin s.

3.2 Higher-Spin Representations

Unitary irreducible representations of the Lie algebra $su(2)$ are $(2s+1)$-dimensional, where $s = 0, 1/2, 1, 3/2, \ldots$ is a half-integer parameter (spin or angular momentum) that labels each representation. Each carrier space $\mathcal{H}_s \simeq \mathbb{C}^{2s+1}$ is spanned by the common angular momentum orthonormal basis $B(\mathcal{H}_s) = \{|s, m\rangle, m = -s, \ldots, s\}$ (in bra-ket notation) of eigenvectors of J_3 and the Casimir (central) operator $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$, i.e.,

$$J_3|s, m\rangle = m|s, m\rangle, \qquad \vec{J}^2|s, m\rangle = s(s+1)|s, m\rangle. \tag{3.17}$$

From the commutation relations

$$[J_3, J_\pm] = \pm J_\pm \tag{3.18}$$

we see that $J_\pm$ play the role of raising and lowering ladder operators, respectively, whose action on the basis vectors proves to be

$$J_\pm|s, m\rangle = \sqrt{(s \mp m)(s \pm m + 1)}|s, m \pm 1\rangle. \tag{3.19}$$

Indeed, it can be easily checked that the action (3.19) preserves the commutation relations (3.6); for example:

$$[J_+, J_-]|s, m\rangle = \cdots = 2m|s, m\rangle = 2J_3|s, m\rangle, \tag{3.20}$$

and so on.

Note that the structure subgroup $U(1) \subset SU(2)$ in (3.10), generated by J_3, stabilizes any basis vector up to an overall multiplicative phase factor (a character of $U(1)$), i.e., $e^{-i\varphi J_3}|s, m\rangle = e^{-im\varphi}|s, m\rangle$. Thus, according to the general prescription explained in Sect. 2, letting $Q = SU(2)/U(1) = \mathbb{S}^2$, and taking the Borel section $\sigma : Q \to G$ with $\sigma(\phi, \theta) = (\theta, \phi, \varphi = 0)$, or $\sigma(z, \bar{z}) = (z, \bar{z}, 0)$, we shall define, from now on, families of covariant coherent states mod$(U(1), \sigma)$ (see [18]). In simple words, we shall set $\varphi = 0$ and drop it from the vectors: $U(\theta, \phi, \varphi)|s, m\rangle$ and $U(z, \bar{z}, \varphi)|s, m\rangle$.

Therefore, we have different characterizations of spin coherent states according to distinct choices of parameterizations. We shall concentrate on the (Hopf) complex (3.12) and Euler angle (3.15) parameterizations.

3.3 Euler Angle Characterization: Spherical Harmonics

For any choice of fiducial vector $|\gamma\rangle = |s, m\rangle$, the set of coherent states $|\theta, \phi; m\rangle \equiv U(\theta, \phi)|\gamma\rangle$ is overcomplete (for any m) in $\mathcal{H}_s$. They can be easily computed by exponentiating the relations (3.17, 3.19). This set of coherent states is also a tight frame with

$$A_\sigma = \frac{2s+1}{4\pi} \int_{\mathbb{S}^2} |\theta, \phi; m\rangle\langle\theta, \phi; m| d\Omega \tag{3.21}$$

a resolution of unity and $d\Omega = \sin\theta\, d\theta\, d\phi$ the standard invariant measure on the 2-sphere. Indeed, due to the invariance of the measure, it follows that $UA_\sigma = A_\sigma U$ for all $U \in SU(2)$. Since the representation is irreducible, we conclude from Schur's Lemma that $A_\sigma = \lambda I$ for some constant λ. Moreover, $\mathrm{Tr}(A_\sigma) = 2s + 1 = \mathrm{Tr}(I) \Rightarrow \lambda = 1 \Rightarrow A_\sigma = I$.

For the particular case of integer spin $s = j$ and fiducial vector $m = 0$, the standard *spherical harmonics* $Y_j^m(\theta, \phi)$ arise as the irreducible matrix coefficients (or Wigner $\mathcal{D}$-functions, see, e.g., Wigner's text book [24]):

$$\langle\theta, \phi; 0|j, m\rangle = \langle j, 0|U(\theta, \phi)^*|j, m\rangle = \sqrt{\frac{4\pi}{2j+1}} Y_j^m(\theta, \phi), \tag{3.22}$$

or, in other words, the components of spin coherent states $|\theta, \phi; 0\rangle$ over the orthonormal basis $\{|j, m\rangle\}$. Thus, for a general angular momentum j state $|\psi\rangle$ we have the standard spherical harmonic decomposition [the wavelet coefficients (2.6)]

$$\Psi(\theta, \phi) = \langle\theta, \phi; 0|\psi\rangle = \sqrt{\frac{4\pi}{2j+1}} \sum_{m=-j}^{j} \psi_m Y_j^m(\theta, \phi) \tag{3.23}$$

with Fourier coefficients $\psi_m = \langle j, m|\psi\rangle$.

Spherical harmonics are rather well-known special functions in the literature. In this article, we shall work with a less standard set of basis functions for the (complex) Riemann sphere: the Majorana functions.

3.4 Complex Holomorphic Characterization: Majorana Functions

In this case we shall use $|\gamma\rangle = |s, s\rangle$ as fiducial vector (i.e., the highest weight vector), so that $J_+|\gamma\rangle = 0$, and the coherent states

$$|z\rangle \equiv U(z, \bar{z})|\gamma\rangle = \mathcal{N}_s(z, \bar{z}) e^{zJ_-} e^{-\bar{z}J_+}|s, s\rangle = \mathcal{N}_s(z, \bar{z}) e^{zJ_-}|s, s\rangle \tag{3.24}$$

are holomorphic (only a function of z), apart from the normalization factor $\mathcal{N}_s$ which, for higher-spin representations $s > 1/2$, (slightly) differs from $\mathcal{N}$ in (3.12). In order to determine $\mathcal{N}_s$, we first recall the relation (3.19) which, by exponentiation, gives

$$\begin{aligned} e^{zJ_-}|s, s\rangle &= |s, s\rangle + z\sqrt{2s}|s, s-1\rangle + \frac{1}{2} z^2 \sqrt{2s}\sqrt{2(2s-1)}|s, s-2\rangle \\ &\quad + \cdots + z^{2s}|s, -s\rangle \equiv \mathcal{N}_s^{-1}|z\rangle. \end{aligned} \tag{3.25}$$

Then, imposing unitarity, i.e., $\langle z|z\rangle = 1$, we arrive at $\mathcal{N}_s = \mathcal{N}^{2s}$ with $\mathcal{N}$ given in (3.11).

As for the Euler angle case, the frame $\{|z\rangle, z \in \mathbb{C}\}$ also is tight in $\mathcal{H}_s$, with resolution of unity

$$I = \frac{2s+1}{\pi}\int_{\mathbb{C}} |z\rangle\langle z| \frac{d^2 z}{(1+z\bar{z})^2}, \tag{3.26}$$

where we denote $d^2z = d\,\mathrm{Re}(z)d\,\mathrm{Im}(z)$. Indeed, using (3.25), we have that

$$\begin{aligned}\frac{2s+1}{\pi}\int_{\mathbb{C}} |z\rangle\langle z| \frac{d^2 z}{(1+z\bar{z})^2} &= \frac{2s+1}{\pi}\int_{\mathbb{C}} \sum_{n,m=0}^{2s} \frac{z^n \bar{z}^m}{n!m!} J_-^n |s,s\rangle\langle s,s| J_+^m \\ &\quad \times \frac{d\,\mathrm{Re}(z) d\,\mathrm{Im}(z)}{(1+z\bar{z})^{2s+2}} \\ &= (2s+1)\sum_{n=0}^{2s} \binom{2s}{n} \int_0^\infty |s, s-n\rangle\langle s, s-n| \\ &\quad \times \frac{x^n dx}{(1+x)^{2s+2}} \\ &= \sum_{m=-s}^{s} |s,m\rangle\langle s,m| = I, \end{aligned} \tag{3.27}$$

where polar coordinates were used at intermediate stage. Also, the same argument as in Sect. 3.3, based on Schur's Lemma, is valid here.

Using (3.25), the decomposition of the coherent state $|z\rangle$ over the orthonormal basis $\{|s,m\rangle\}$ gives the irreducible matrix coefficients

$$\begin{aligned}\langle z|s,m\rangle = \langle s,s|U(z,\bar{z})^*|s,m\rangle &= \binom{2s}{s+m}^{1/2} (1+z\bar{z})^{-s}\bar{z}^{s+m} \\ &\equiv \mathcal{N}(z,\bar{z})^{2s}\Upsilon_s^m(\bar{z}),\end{aligned} \tag{3.28}$$

where now $\Upsilon_s^m(\bar{z})$ is just a monomial in $\bar{z}$ times a numeric (binomial) factor. A general spin s state $|\psi\rangle$ is represented in the present complex characterization by the so called *Majorana function* [13, 14]:

$$\Psi(z) \equiv \langle z|\psi\rangle = (1+z\bar{z})^{-s}\sum_{m=-s}^{s} \psi_m \Upsilon_s^m(\bar{z}) = \mathcal{N}(z,\bar{z})^{2s} f(\bar{z}), \tag{3.29}$$

which is an anti-holomorphic function of z (in this case, a polynomial).[2]

[2]Here we abuse notation when representing the nonanalytic function $\Psi(z,\bar{z})$ simply as $\Psi(z)$, which is indeed anti-holomorphic up to the normalizing, nonanalytic (real), pre-factor $\mathcal{N}^{2s} = (1+z\bar{z})^{-s}$. Usually, this pre-factor is absorbed in the integration measure in (3.26). If we choose the lowest weight fiducial vector $|\gamma\rangle = |s,-s\rangle$, we would obtain proper holomorphic functions $f(z)$.

Note that the set of CS $\{|z\rangle\}$ is not orthogonal. The CS overlap (or Reproducing Kernel) turns out to be

$$C\big(z, z'\big) = \big\langle z|z'\big\rangle = \frac{(1 + z'\bar{z})^{2s}}{(1 + z\bar{z})^s(1 + z'\bar{z}')^s}. \tag{3.30}$$

This quantity will be essential in our sampling procedure on the Riemann sphere.

For completeness, let us provide an expression which allows us to translate between both characterizations of coherent states for integer spin $s = j$. It is given by the Coherent State (or Bargmann-like) Transform (see, e.g., [15, 16]):

$$\begin{aligned} K(\theta, \phi; z) \equiv \langle \theta, \phi|z\rangle &= \sum_{m=-j}^{j} \langle \theta, \phi|j, m\rangle\langle j, m|z\rangle \\ &= (1 + z\bar{z})^{-j} \sqrt{\frac{4\pi}{2j + 1}} \sum_{m=-j}^{j} Y_j^m(\theta, \phi)\Upsilon_j^m(z) \\ &= (1 + z\bar{z})^{-j} \frac{\sqrt{(2j)!}}{2^j j!}\big(\sin\theta\, e^{-i\phi} + 2z\cos\theta - z^2 \sin\theta\, e^{i\phi}\big)^j, \end{aligned} \tag{3.31}$$

which can be seen as a generating function for the spherical functions $Y_j^m(\theta, \phi)$ when we drop the normalization factor $\mathcal{N}^{2j}$ from the last expression.

4 Sampling Theorem and DFT on $\mathbb{S}^2$

Sampling techniques consist in the evaluation of a continuous function (“signal”) on a discrete set of points and later (fully or partially) recovering the original signal without losing essential information in the process, and the criteria to that effect are given by various forms of Sampling Theorems. Basically, the density of sampling points must be high enough to ensure the reconstruction of the function in arbitrary points with reasonable accuracy. We shall concentrate on fixed spin holomorphic (Majorana’s) functions and sample them at the roots of unity.

4.1 Single Spin Case

Let us first restrict ourselves to functions in $\mathcal{H}_s$, i.e., with well-defined spin or angular momentum s. In this case there is a convenient way to select the sampling points in such a way that the resolution operator $\mathcal{A}$ and/or the reproducing kernel operator $\mathcal{B}$ are invertible and explicit formulas for their inverses are available. These are given by the N^{th} roots of unity in the complex plane, $N \in \mathbb{N}$, which would be associated, by inverse stereographic projection, to a uniformly distributed set of points in the equator of the Riemann sphere. The choice of roots of unity is made for convenience, since the N^{th} roots of any nonzero complex number would also be valid and would correspond to different parallels in the Riemann sphere, but then the formulas obtained are less symmetrical than the ones corresponding to roots of unity. The most important reason to select roots of unity is that they are associated with the discrete cyclic subgroup $\mathbb{Z}_N \subset U(1) \subset SU(2)$. The choice $N = 2s+1$ corresponds to *critical sampling*. We shall also discuss the consequences of *over-sampling*, with $N > 2s+1$, and *under-sampling*, with $N < 2s+1$, in the following subsections.

4.1.1 Over-Sampling and Critical Sampling

In the case of over-sampling, the set $\mathcal{S}$ generates $\mathcal{H}_s$, and the resolution operator $\mathcal{A} = \mathcal{T}^*\mathcal{T}$ is invertible. The case of critical sampling is a particular case of this, and therefore the following discussion also applies to it.

The previous statements are formalized by the following lemma.

Lemma 4.1 *Let $\mathcal{Q} = \{z_k = e^{2\pi i k/N},\ N \geq 2s+1,\ k = 0, \ldots, N-1\}$ be the discrete subset of the homogeneous space $Q = SU(2)/U(1) = \mathbb{S}^2 = \bar{\mathbb{C}}$ made of the N^{th} roots of unity. The discrete set of CS $\mathcal{S} = \{|z_k\rangle, z_k \in \mathcal{Q}\}$ constitutes a discrete frame in $\mathcal{H}_s$, and the expression*

$$I_{2s+1} = \sum_{k=0}^{N-1} |z_k\rangle\langle\tilde{z}_k| = \sum_{k=0}^{N-1} |\tilde{z}_k\rangle\langle z_k| \tag{4.1}$$

provides a resolution of the identity in $\mathcal{H}_s$. Here $|\tilde{z}_k\rangle = \mathcal{A}^{-1}|z_k\rangle, k = 0, \ldots, N-1$, denotes the dual frame, and the resolution operator, $\mathcal{A}$, is diagonal in the angular momentum orthonormal basis $B(\mathcal{H}_s)$, $\mathcal{A} = \mathrm{diag}(\lambda_0, \ldots, \lambda_{2s})$, with $\lambda_n = \frac{N}{2^{2s}}\binom{2s}{n}$, $n = 0, \ldots, 2s$.

Proof First, from (3.28) the expression for the matrix elements of $\mathcal{T}$ can be obtained, $\mathcal{T}_{kn} = \langle z_k|s, n-s\rangle = 2^{-s}\sqrt{\binom{2s}{n}}e^{-i\frac{2\pi kn}{N}}$, $k = 0, 1, \ldots, N-1$, $n = 0, 1, \ldots, 2s$. Then, the resolution operator turns out to be

$$\mathcal{A}_{nm} = \sum_{k=0}^{N-1} (\mathcal{T}_{kn})^* \mathcal{T}_{km} = 2^{-2s}\sqrt{\binom{2s}{n}\binom{2s}{m}} \sum_{k=0}^{N-1} e^{2\pi i k(n-m)/N} = N 2^{-2s}\binom{2s}{n}\delta_{nm}, \tag{4.2}$$

where we have used the well-known orthogonality relation

$$\sum_{k=0}^{N-1} \left(e^{2\pi i(n-m)/N}\right)^k = \left\{\begin{matrix} N, & \text{if } n = m \bmod N \\ 0, & \text{if } n \neq m \bmod N \end{matrix}\right\} = N\delta_{nm}, \tag{4.3}$$

since $N \geq 2s+1$. Therefore $\mathcal{A}$ is diagonal with nonzero diagonal elements, thus it is invertible and a dual frame and a (left) pseudoinverse for $\mathcal{T}$ can be constructed, $\mathcal{T}_l^+ \equiv \mathcal{A}^{-1}\mathcal{T}^*$, providing, according to (2.12), a resolution of the identity. □

Remark 4.2 It is interesting to rewrite this proof in terms of Rectangular Fourier Matrices (see Appendix A). Let $D = \mathrm{diag}(\lambda_0, \ldots, \lambda_{2s})$ be a diagonal $(2s+1) \times (2s+1)$ matrix, then $\mathcal{T} = \mathcal{F}_{N,2s+1} D^{1/2} = \mathcal{F}_N \circ \iota_{N,2s+1} \circ D^{1/2}$. From this the expression $\mathcal{A} = \mathcal{T}^*\mathcal{T} = D$ is readily recovered, and also $\mathcal{B} = \mathcal{T}\mathcal{T}^*$ is seen to be $\mathcal{B} = \mathcal{F}_N\, D^{\uparrow}\, \mathcal{F}_N^*$, where $D^{\uparrow} = \left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right)_{N\times N}$ (see Appendix A). Note that $\mathcal{B}$ is a singular $N \times N$ matrix with only $2s+1$ nonzero eigenvalues λ_n, $n = 0, 1, \ldots, 2s$, and that they coincide with those of $\mathcal{A}$. In fact, $\mathcal{B} = \mathcal{F}_N\, \mathcal{A}^{\uparrow}\, \mathcal{F}_N^*$.

Lemma 4.3 *Under the conditions of the previous lemma, the operator* $P = \mathcal{T}\mathcal{T}_l^+ = \mathcal{F}_N\, P_{2s+1}\, \mathcal{F}_N^*$ *is an orthogonal projector onto a* $(2s+1)$*-dimensional subspace of* $\mathbb{C}^N$*, the range of* $\mathcal{T}$.

Proof By direct computation (and using Appendix A),

$$\begin{aligned} P = \mathcal{T}\mathcal{T}_l^+ = \mathcal{T}\mathcal{A}^{-1}\mathcal{T}^* &= \mathcal{F}_N\, \iota_{N,2s+1}\, D^{1/2}\, D^{-1}\, D^{1/2}\, p_{2s+1,N}\, \mathcal{F}_N^* \\ &= \mathcal{F}_N\, \iota_{N,2s+1}\, p_{2s+1,N}\, \mathcal{F}_N^* = \mathcal{F}_N\, P_{2s+1}\, \mathcal{F}_N^*, \end{aligned} \tag{4.4}$$

where $P_{2s+1} = (I_{2s+1})^{\uparrow} = \left(\begin{array}{c|c} I_{2s+1} & 0 \\ \hline 0 & 0 \end{array}\right)_{N\times N}$. This clearly shows that P is an orthogonal projector, unitarily equivalent to P_{2s+1} and that $P\mathcal{T} = \mathcal{T}$. □

Theorem 4.4 (Reconstruction formula) *Any function* $\psi \in \mathcal{H}_s$ *can be reconstructed from* $N \geq 2s+1$ *of its samples (the data)* $\Psi(z_k) \equiv \langle z_k|\psi\rangle$ *at the sampling points* $z_k = e^{2\pi i k/N}$, $k = 0, \ldots, N-1$, *by means of*

$$\Psi(z) = \langle z|\psi\rangle = \sum_{k=0}^{N-1} \Psi(z_k)\, \Xi\big(z z_k^{-1}\big), \tag{4.5}$$

where

$$\Xi(z) = \frac{2^s}{N}(1+z\bar{z})^{-s}\, \frac{1-\bar{z}^{2s+1}}{1-\bar{z}} \tag{4.6}$$

plays the role of a "sinc-type function."

Proof From the resolution of the identity (4.1), any $\psi \in \mathcal{H}_s$ can be written as $|\psi\rangle = \sum_{k=0}^{N-1} \Psi(z_k)|\tilde{z}_k\rangle$, and therefore $\Psi(z) = \langle z|\psi\rangle = \sum_{k=0}^{N-1} \Psi(z_k)\langle z|\tilde{z}_k\rangle$. Using that $|\tilde{z}_k\rangle = \mathcal{A}^{-1}|z_k\rangle$, we derive that

$$\begin{aligned} \langle z|\tilde{z}_k\rangle &= \frac{1}{\sqrt{N}} \sum_{n=0}^{2s} \lambda_n^{-1/2}\, e^{2\pi i k n/N} \langle z|s, n-s\rangle = \frac{2^s}{N} \mathcal{N}^{2s} \sum_{n=0}^{2s} \big(\bar{z}\bar{z}_k^{-1}\big)^n \\ &\equiv \Xi\big(z z_k^{-1}\big), \quad k = 0, 1, \ldots, N-1, \end{aligned} \tag{4.7}$$

where (3.28) has been used. □

Remark 4.5 It is interesting to note that (4.5) can be interpreted as a Lagrange-type interpolation formula, where the role of Lagrange polynomials is played by the functions $L_k(z) = \Xi(z z_k^{-1})$ satisfying the "orthogonality relations" $L_k(z_l) = \Xi(z_l z_k^{-1}) = P_{lk}$, where P is the projector of Lemma 4.3. In the case of critical sampling, $N = 2s+1$, the usual result $L_k(z_l) = \delta_{lk}$ is recovered, but for the strict oversampling case, $N > 2s+1$, a projector is obtained to account for the fact that an arbitrary set of overcomplete data $\Psi(z_k)$, $k = 0, \dots, N-1$, can be incompatible with $|\psi\rangle \in \mathcal{H}_s$.

A reconstruction in terms of the Fourier coefficients can be directly obtained by means of the (left) pseudoinverse of the frame operator $\mathcal{T}$:

Corollary 4.6 *The Fourier coefficients a_m of the expansion $|\psi\rangle = \sum_{m=-s}^{s} a_m |s, m\rangle$ of any $\psi \in \mathcal{H}_s$ in the angular momentum orthonormal basis $B(\mathcal{H}_s)$ can be determined in terms of the data $\Psi(z_k) = \langle z_k|\psi\rangle$ as*

$$a_{n-s} = \frac{2^s}{N}\binom{2s}{n}^{-1/2} \sum_{k=0}^{N-1} \Psi(z_k) e^{2\pi i k n/N}, \quad n = 0, \dots, 2s. \tag{4.8}$$

Proof Taking the scalar product with $\langle z_k|$ in the expression of $|\psi\rangle$, we arrive at the over-determined system of equations

$$\sum_{n=0}^{2s} \mathcal{T}_{kn} a_{n-s} = \Psi(z_k), \quad \mathcal{T}_{kn} = \langle z_k|s, n-s\rangle, \tag{4.9}$$

which can be solved by left multiplying it by the (left) pseudoinverse of $\mathcal{T}$, $\mathcal{T}_l^{+} = (\mathcal{T}^*\mathcal{T})^{-1}\mathcal{T}^* = \mathcal{A}^{-1}\mathcal{T}^*$. Using the expressions of $\mathcal{A}^{-1} = \mathrm{diag}(\lambda_0^{-1}, \lambda_1^{-1}, \dots, \lambda_{2s}^{-1})$ given in Lemma 4.1 and the matrix elements $\mathcal{T}_{kn}$ given by formula (3.28), we arrive at the desired result. □

Remark 4.7 Actually, using the vector notation, we have $\mathcal{T}\vec{a} = \vec{\Psi}$, where $\vec{a} = (a_{-s}, \dots, a_s)$, and $\vec{\Psi}$ denotes the vector of samples $\Psi(z_k)$, $k = 0, \dots, N-1$. Using the (left) pseudoinverse of $\mathcal{T}$, we can solve it obtaining $\vec{a} = D^{-1/2}\, p_{2s+1,N}\, \mathcal{F}_N^* \vec{\Psi}$, which coincides with (4.8). Note also that the last expression is a map from $\mathbb{C}^N$ to $\mathbb{C}^{2s+1} \approx \mathcal{H}_s$ due to the presence of the projector $p_{2s+1,N}$ (see Appendix A), and this prevents the appearance of infinities in the reciprocal of the binomial coefficient $\binom{2s}{n}^{-1/2}$ with $n > 2s$. This is clearer if we apply $\mathcal{T}$ to the expression of $\vec{a}$ to obtain $\mathcal{T}\vec{a} = \mathcal{F}_N\, \iota_{N,2s+1}\, D^{1/2}\, D^{-1/2}\, p_{2s+1,N}\, \mathcal{F}_N^* \vec{\Psi} = P\vec{\Psi}$, that is, the data $\vec{\Psi}$ should be first projected in order to obtain a compatible set of data.

Next we provide an interesting expression.

Proposition 4.8 *If we define the "dual data" as $\Gamma(k) \equiv \langle \tilde{z}_k|\psi\rangle$, then they are related to the data $\Psi(k) \equiv \Psi(z_k) = \langle z_k|\psi\rangle$ through the convolution product*

$$\Gamma(k) = [\Delta * \Psi](k) = \sum_{l=0}^{N-1} \Delta(k-l)\Psi(l), \tag{4.10}$$

where $\Delta(k)$ (*the filter*) *turns out to be the Rectangular Fourier Transform of* $\vec{\delta} \equiv (\lambda_0^{-1}, \ldots, \lambda_{2s}^{-1})$, *i.e.*,

$$\Delta(k) = [\mathcal{F}_{N,2s+1}\delta](k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{2s} \lambda_n^{-1} e^{-i2\pi nk/N} = \frac{2^{2s}}{N^{3/2}} \sum_{n=0}^{2s} \binom{2s}{n}^{-1} e^{-i2\pi nk/N}. \tag{4.11}$$

Proof Applying (4.1) to ψ, we obtain:

$$|\psi\rangle = \sum_{k=0}^{N-1} \Gamma(k)|z_k\rangle.$$

Taking the scalar product with $\langle z_l|$ in the last equation, we arrive at $\vec{\Psi} = \mathcal{B}\vec{\Gamma}$, where $\mathcal{B} = \mathcal{T}\mathcal{T}^*$ shows a circulant matrix structure (see Appendix B). Using the diagonalization $\mathcal{B} = \mathcal{F}_N D^{\uparrow} \mathcal{F}_N^*$ of $\mathcal{B}$, a Moore–Penrose pseudoinverse can be computed as $\mathcal{B}^+ = \mathcal{F}_N (D^{-1})^{\uparrow} \mathcal{F}_N^*$, and this allows us to obtain $\vec{\Gamma} = \mathcal{B}^+\vec{\Psi} = \mathcal{F}_N (D^{-1})^{\uparrow} \mathcal{F}_N^* \vec{\Psi}$. This last expression, by duality, can be interpreted as the convolution $\vec{\Gamma} = \vec{\Delta} * \vec{\Psi}$ between the data and the filter (4.11). □

Remark 4.9 The relation between $\Psi(k)$ and $\Gamma(k)$ is simply a "change of basis," but with nonorthogonal sets of generators $\{|z_k\rangle\}$ and $\{|\tilde{z}_k\rangle\}$. Due to the particular choice of sampling points, the change of basis involves Fourier transforms, and this can be interpreted as a convolution.

Remark 4.10 For high spin values $s \gg 1$ (and $N \geq 2s+1$), it is easy to realize that the filter (4.11) acquires the simple form

$$\Delta(k) = \frac{2^{2s}}{N^{3/2}} \left(1 + e^{i2\pi rk/N} + O\left(\frac{1}{2s}\right)\right), \tag{4.12}$$

where $r = N - 2s$. There is also a more manoeuvrable closed expression for the exact value of the filter zero mode

$$\Delta(0) = \frac{2^{2s}}{N^{3/2}} \sum_{n=0}^{2s} \binom{2s}{n}^{-1} = \frac{2s+1}{N^{3/2}} \sum_{n=0}^{2s} \frac{2^n}{n+1}, \tag{4.13}$$

where we have used the result of Ref. [25] concerning sums of the reciprocals of binomial coefficients. For large values of s, we can also prove that

$$\lim_{s\to\infty} \sum_{n=0}^{2s} \binom{2s}{n}^{-1} = 2. \tag{4.14}$$

In the case of critical sampling all formulae are still valid, we only have to substitute $N = 2s+1$, the difference being that $\mathcal{T}$ is directly invertible and $\mathcal{T}^{-1} = \mathcal{T}_l^+$.

The projector P is the identity, and $\mathcal{A}$ and $\mathcal{B}$ are both invertible. The reason for considering the case of oversampling is twofold: first, by its intrinsic interest leading to overcomplete frames, and second, in order to apply fast extensions (as FFT, see [4]) of the reconstruction algorithms, it would be useful to consider N the smallest power of 2 greater or equal to $2s+1$.

4.1.2 Under-Sampling and Critical Sampling

Let us suppose now that the number of sampling points is $N \leq 2s+1$. We shall see that, for $N < 2s+1$, we cannot reconstruct exactly an arbitrary function $\psi \in \mathcal{H}_s$ but its orthogonal projection $\hat{\psi} \equiv P_N \psi$ onto the subspace $\hat{\mathcal{H}}_s$ of $\mathcal{H}_s$ spanned by the discrete set $\mathcal{S} = \{|z_k\rangle, k = 0, \dots, N\}$ of CS. In other words, the restriction to this discrete subset implies a loss of information.

This loss of information translates to the fact that the resolution operator $\mathcal{A}$ is not invertible, and therefore we do not have a frame nor a resolution of the identity like in the previous subsection, see the discussion at the end of Sect. 2. But, since the set $\mathcal{S}$ is linearly independent, we can construct another operator, the overlapping kernel $\mathcal{B} = \mathcal{T}\mathcal{T}^*$, which is invertible and provides a partial reconstruction formula. In addition, the overlapping kernel operator has a circulant structure, and this provides a deep insight in the reconstruction process.

Let us formalize again the previous assertions.

Lemma 4.11 *Let $\mathcal{Q} = \{z_k = e^{2\pi i k/N},\ k = 0, \dots, N-1\}$ be the discrete subset of the homogeneous space $Q = SU(2)/U(1) = \mathbb{S}^2 = \bar{\mathbb{C}}$ made of the N^{th} $(N \leq 2s+1)$ roots of unity. The pseudo-frame operator $\mathcal{T} : \mathcal{H}_s \to \mathbb{C}^N$ given by $\mathcal{T}(\psi) = \{\langle z_k|\psi\rangle, z_k \in \mathcal{Q}\}$ [remember the construction after (2.9)] is such that the overlapping kernel operator $\mathcal{B} = \mathcal{T}\mathcal{T}^*$ is an $N \times N$ Hermitian positive definite invertible matrix, admitting the eigen-decomposition $\mathcal{B} = \mathcal{F}_N \hat{D} \mathcal{F}_N^*$, where $\hat{D} = \mathrm{diag}(\hat{\lambda}_0, \dots, \hat{\lambda}_{N-1})$ is the diagonal matrix with $\hat{\lambda}_k = \frac{N}{2^{2s}} \sum_{j=0}^{\bar{q}-1} \binom{2s}{k+jN}$, $\bar{q}$ being the ceiling of $(2s+1)/N$.*

Proof Let us see that the eigenvalues $\hat{\lambda}_k$ of $\mathcal{B} = \mathcal{T}\mathcal{T}^*$ are indeed all strictly positive, and hence $\mathcal{B}$ is invertible. This can be done by using RFM or taking advantage of the circulant structure of $\mathcal{B}$ (see Appendix B). With RFM we start with the expression of $\mathcal{T} = \mathcal{F}_{N,2s+1} D^{1/2}$ to obtain $\mathcal{B} = \mathcal{T}\mathcal{T}^* = \mathcal{F}_{N,2s+1} D \mathcal{F}_{N,2s+1}^*$, which should be further worked on in order to fully diagonalize it.

This can be done by using the "trick" mentioned in Appendix A consisting in enlarging the RFM $\mathcal{F}_{N,2s+1}$ to $\mathcal{F}_{N\bar{M}}$, where $\bar{M}$ is the smaller multiple of N greater or equal to $2s+1$, and $\bar{q} = \bar{M}/N$ is the ceiling of $(2s+1)/N$ (see Appendix A). In this way, $\mathcal{F}_{N\bar{M}}$ always contains an integer number of ordinary Fourier matrices $\mathcal{F}_N$.

Using this, we obtain that $\mathcal{B} = \mathcal{F}_{N\bar{M}} D^{\uparrow} \mathcal{F}_{N\bar{M}}^*$, where $D^{\uparrow}$ is the extension of D to a $\bar{M} \times \bar{M}$ matrix, and with a little of algebra the expression $\mathcal{B} = \mathcal{F}_N \hat{D} \mathcal{F}_N^*$ is obtained, where $\hat{D} = \mathrm{diag}(\hat{\lambda}_0, \dots, \hat{\lambda}_{N-1})$ and

$$\hat{\lambda}_k = \sum_{l=0}^{\bar{q}-1} \lambda_{k+lN} = \frac{N}{2^{2s}} \sum_{l=0}^{\bar{q}-1} \binom{2s}{k+lN}. \tag{4.15}$$

All the eigenvalues are strictly positive, and therefore $\mathcal{B}$ is invertible. □

Following Sect. 2, we introduce the following result:

Lemma 4.12 *Under the conditions of the previous Lemma, the set* $\{|\tilde{z}_k\rangle = \sum_{l=0}^{N-1}(\mathcal{B}^{-1})_{lk}|z_l\rangle,\ k = 0,\ldots,N-1\}$ *constitutes a dual pseudo-frame for* $\mathcal{S}$, *the operator* $P_{\mathcal{S}} = \mathcal{T}_r^+\mathcal{T}$ *is an orthogonal projector onto the subspace of* $\mathcal{H}_s$ *spanned by* $\mathcal{S}$, *where* $\mathcal{T}_r^+ = \mathcal{T}^*\mathcal{B}^{-1}$ *is a (right) pseudoinverse for* $\mathcal{T}$, *and*

$$\sum_{k=0}^{N-1}|\tilde{z}_k\rangle\langle z_k| = \sum_{k=0}^{N-1}|z_k\rangle\langle\tilde{z}_k| = P_{\mathcal{S}} \tag{4.16}$$

provides a resolution of the projector $P_{\mathcal{S}}$.

Proof If we define $\mathcal{T}_r^+ = \mathcal{T}^*\mathcal{B}^{-1}$, it is easy to check that $\mathcal{T}\mathcal{T}_r^+ = I_N$ is the identity in $\mathbb{C}^N$. In the same way, $P_{\mathcal{S}} = \mathcal{T}_r^+\mathcal{T}$ is a projector, since $P_{\mathcal{S}}^2 = \mathcal{T}_r^+\mathcal{T}\mathcal{T}_r^+\mathcal{T} = \mathcal{T}_r^+\mathcal{T} = P_{\mathcal{S}}$ and it is orthogonal, $P_{\mathcal{S}}^* = (\mathcal{T}^*\mathcal{B}^{-1}\mathcal{T})^* = \mathcal{T}^*\mathcal{B}^{-1}\mathcal{T} = P_{\mathcal{S}}$, since $\mathcal{B}$ is self-adjoint. The resolution of the projector is provided by (2.14). □

Although the full reconstruction of the original signal is not possible in the case of undersampling, a partial reconstruction is still possible in the following sense.

Theorem 4.13 (Partial reconstruction formula) *Any function* $\psi \in \mathcal{H}_s$ *can be partially reconstructed from* $N \le 2s+1$ *of its samples (the data)* $\Psi(z_k) \equiv \langle z_k|\psi\rangle$ *at the sampling points* $z_k = e^{2\pi ik/N}$, $k = 0,\ldots,N-1$, *by the alias* $|\hat{\psi}\rangle = P_{\mathcal{S}}|\psi\rangle$, *by means of*

$$\hat{\Psi}(z) = \langle z|\hat{\psi}\rangle = \sum_{k=0}^{N-1}\Psi(z_k)\hat{\Xi}(zz_k^{-1}), \tag{4.17}$$

where

$$\hat{\Xi}(z) = \frac{2^s}{N}(1+z\bar{z})^{-s}\sum_{p=0}^{N-1}\hat{\lambda}_p^{-1}\sum_{l=0}^{\bar{q}-1}\lambda_{p+lN}\bar{z}^{p+lN} \tag{4.18}$$

plays the role of a "sinc-type function.".

Proof The proof follows the same lines as in Theorem 4.4. From the resolution of the projector (4.16), any $\psi \in \mathcal{H}_s$ has a unique alias $\hat{\psi} = P_{\mathcal{S}}\psi$ which can be written as $|\hat{\psi}\rangle = \sum_{k=0}^{N-1}\Psi(z_k)|\tilde{z}_k\rangle$, and therefore $\hat{\Psi}(z) = \langle z|\hat{\psi}\rangle = \sum_{k=0}^{N-1}\Psi(z_k)\langle z|\tilde{z}_k\rangle$. Using that $|\tilde{z}_k\rangle = \sum_{l=0}^{N-1}(\mathcal{B}^{-1})_{lk}|z_l\rangle$, we derive that

$$\langle z|\tilde{z}_k\rangle = \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1}(\mathcal{B}^{-1})_{lk}\sum_{n=0}^{2s}\lambda_n^{1/2}e^{\frac{2\pi ikn}{N}}\langle z|s,n-s\rangle$$

$$= \frac{2^s}{N}\mathcal{N}^{2s}\sum_{p=0}^{N-1}\hat{\lambda}_p^{-1}\sum_{l=0}^{\bar{q}-1}\lambda_{p+lN}\big(\bar{z}\bar{z}_k^{-1}\big)^{p+lN} = \hat{\Xi}\big(zz_k^{-1}\big), \tag{4.19}$$

where (3.28) and the orthogonality relations (4.3) have been used (but with $N \leq 2s+1$, as in Appendix B). □

Remark 4.14 As in the case of oversampling, (4.17) can be interpreted as a Lagrange-type interpolation formula, where the role of Lagrange polynomials is played by the functions $\hat{L}_k(z) = \hat{\Xi}(zz_k^{-1})$, this time satisfying the proper orthogonality relations $\hat{L}_k(z_l) = \hat{\Xi}(z_l z_k^{-1}) = \delta_{lk}$. The reason for this is that, in the case of under-sampling, there is no an overcomplete set of data, and therefore the "Lagrange functions" are orthogonal, although not complete.

A partial reconstruction can also be obtained, in a natural way, from the "dual data":

Proposition 4.15 *If we define the "dual data" as $\Gamma(k) \equiv \langle \tilde{z}_k|\psi\rangle$, then they are related to the data $\Psi(k) = \langle z_k|\psi\rangle$ through the convolution product*

$$\Gamma(k) = \big[\hat{\Delta} * \Psi\big](k) = \sum_{l=0}^{N-1}\hat{\Delta}(k-l)\Psi(l), \tag{4.20}$$

where $\hat{\Delta}(k)$ (the filter) turns out to be the discrete Fourier transform of $\hat{\delta} \equiv (\hat{\lambda}_0^{-1}, \ldots, \hat{\lambda}_{N-1}^{-1})$, where $\hat{\lambda}_k$ are the eigenvalues (4.15) of the overlapping kernel operator $\mathcal{B}$:

$$\hat{\Delta}(k) = \big[\mathcal{F}_N\hat{\delta}\big](k) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\hat{\lambda}_k^{-1}\,e^{-i2\pi nk/N}. \tag{4.21}$$

Proof The proof follows the same lines as in Proposition 4.8, with the difference that now $\mathcal{B}$ is not singular, and there is no need for a pseudoinverse. From $\vec{\Psi} = \mathcal{B}\vec{\Gamma}$ and using the diagonalization $\mathcal{B} = \mathcal{F}_N\hat{D}\mathcal{F}_N^*$ of $\mathcal{B}$, the inverse is directly $\mathcal{B}^{-1} = \mathcal{F}_N\hat{D}^{-1}\mathcal{F}_N^*$, and this allows one to obtain $\vec{\Gamma} = \mathcal{B}^{-1}\vec{\Psi} = \mathcal{F}_N\hat{D}^{-1}\mathcal{F}_N^*\vec{\Psi}$. This last expression, by duality, can be interpreted as the convolution $\vec{\Gamma} = \vec{\hat{\Delta}} * \vec{\Psi}$ between the data and the filter (4.21). □

The comments made in Remark 4.9 also apply here.

Again, a reconstruction in terms of the Fourier coefficients can be directly obtained by means of the (right) pseudoinverse of the frame operator $\mathcal{T}$:

Corollary 4.16 *The Fourier coefficients $\hat{a}_m$ of the expansion $|\hat{\psi}\rangle = \sum_{m=-s}^{s}\hat{a}_m|s,m\rangle$ of the alias of any $\psi \in \mathcal{H}_s$ in the angular momentum orthonormal basis $B(\mathcal{H}_s)$ can* *be determined in terms of the data* $\Psi(k) = \langle z_k|\psi\rangle$ *as*

$$\hat{a}_{n-s} = \frac{N}{2^s}\binom{2s}{n}^{1/2}\sum_{k=0}^{N-1} e^{2\pi i k n/N}\sum_{l=0}^{N-1}\left(\mathcal{B}^{-1}\right)_{kl}\Psi(l), \quad n = 0, \ldots, 2s. \tag{4.22}$$

Proof Taking the scalar product with $\langle z_k|$ in the expression of $|\hat{\psi}\rangle$, we arrive at the system of equations

$$\sum_{n=0}^{2s}\mathcal{T}_{kn}\hat{a}_{n-s} = \Psi(k), \quad \mathcal{T}_{kn} = \langle z_k|s, n-s\rangle, \tag{4.23}$$

which can be solved by left multiplying it by the (right) pseudoinverse of $\mathcal{T}$, $\mathcal{T}_r^+ = \mathcal{T}^* = \mathcal{T}^*\mathcal{B}^{-1}$. Using the expressions of $\mathcal{B}$ given in Lemma 4.11 and the matrix elements $\mathcal{T}_{kn}$ given by the formula (3.28), we arrive at the desired result by noting that $\mathcal{T}_r^+\mathcal{T} = P_{\mathcal{S}}$, and this acts as the identity on $\hat{a}_{n-s}$. □

Remark 4.17 Using the vector notation, this can be written as $\vec{\hat{a}} = \mathcal{T}_r^+\mathcal{T}\vec{a} = \mathcal{T}_r^+\vec{\Psi} = \mathcal{T}^*\mathcal{B}^{-1}\vec{\Psi}$, and this is even simpler in terms of the dual data, $\vec{\hat{a}} = \mathcal{T}^*\vec{\Gamma} = D^{1/2}\,\mathcal{F}^*_{N,2s+1}\vec{\Gamma}$.

It is interesting to establish the connection between our results and others in the literature [26].

Corollary 4.18 (Covariant interpolation) *For* $0 \le k \le N-1$, *define on* Q *the functions* $\Phi_k(z) \equiv \langle z|z_k\rangle$, $z \in \mathbb{C}$. *Let* $\zeta_0, \ldots, \zeta_{N-1}$ *be* N *complex numbers and* $\mathcal{B}_{kl}$ *the overlapping kernel operator. Define on* Q *the function*

$$\begin{aligned}\Phi(z) &= \Phi(z_0, \ldots, z_{N-1}; \zeta_0, \ldots, \zeta_{N-1}; z) \\ &\equiv -\frac{1}{\det(\mathcal{B})}\det\begin{pmatrix} 0 & \Phi_1(z) & \ldots & \Phi_{N-1}(z) \\ \zeta_0 & \mathcal{B}_{0,0} & \ldots & \mathcal{B}_{0,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{N-1} & \mathcal{B}_{N-1,0} & \ldots & \mathcal{B}_{N-1,N-1}\end{pmatrix}.\end{aligned} \tag{4.24}$$

Then we have that

1. $\Phi(z) = \langle z|\phi\rangle$ *for some* $\phi \in \mathcal{H}_s$.
2. Φ *is a solution of the interpolation problem, i.e.,* $\Phi(z_k) = \zeta_k$, $z = 0, \ldots, N-1$.
3. Φ *is of minimal norm, in the sense that if* $\tilde{\Phi}$ *is any other function on* Q *with* $\tilde{\Phi}(z) = \langle z|\tilde{\phi}\rangle$ *for some* $\tilde{\phi} \in \mathcal{H}_s$ *and* $\tilde{\Phi}(z_k) = \zeta_k$, *then* $\|\tilde{\Phi}\| \ge \|\Phi\|$.
4. *The interpolation procedure is invariant under left multiplication in G in the sense that* $U(g)\mathcal{B}U(g)^* = \mathcal{B}$ *and*

$$\Phi(gz_0, \ldots, gz_{N-1}; \zeta_0, \ldots, \zeta_{N-1}; gz) = \Phi(z_0, \ldots, z_{N-1}; \zeta_0, \ldots, \zeta_{N-1}; z)$$

   (gz *denotes the natural action of the group G on its homogeneous space* $Q = G/H$), *so that the left-displaced interpolation problem* $\check{\Phi}(gz_k) = \zeta_k$ *is solved by the function* $\check{\Phi}(z) = \Phi(g^{-1}z)$.

Proof It is a direct consequence of Theorem 4.13 if we identify the data $\zeta_k = \Psi(z_k)$, $\Phi(z) = \hat{\Psi}(z)$, and $\sum_{k=0}^{N-1}(\mathcal{B}^{-1})_{kl}\Phi_l(z) = \hat{\Xi}(z z_k^{-1})$. The fact that Φ is of minimal norm is a direct consequence of the orthogonality of the projector $P_{\mathcal{S}}$. The invariance under left multiplication is a consequence of the invariance of the overlapping kernel $\mathcal{B}$ under left multiplication. □

4.2 Several Spin Case

The case of several spins, i.e., band limited functions, is more involved than the single spin case, and it is not so easy to select the sampling points in such a way that an explicit expression for the inverse of the resolution or overlapping kernel operators be available.

Denoting by[3] $\mathcal{H}^{(J)} = \bigoplus_{s=0}^{J} \mathcal{H}_s$ the Hilbert space of band-limited functions, up to spin J, the set of coherent states can be defined in an analogous way to the single spin case.

First, let us denote by $U^{(J)}(z,\bar{z}) = \bigoplus_{s=0}^{J} U_s(z,\bar{z})$ the unitary and reducible representation of $SU(2)$ acting on $\mathcal{H}^{(J)}$, where $U_s(z,\bar{z})$ stands for the unitary and irreducible representation of spin s. The Hilbert space $\mathcal{H}^{(J)}$ has an orthogonal basis given by $\{|s,m\rangle\}$, in such a way that $I_{\mathcal{H}^{(J)}} = \frac{1}{J+1}\sum_{s=0}^{J}\sum_{m=-s}^{s}|s,m\rangle\langle s,m|$ is a resolution of the identity. Selecting the fiducial vector $|\gamma\rangle^J = \frac{1}{\sqrt{J+1}}\bigoplus_{s=0}^{J}|s,s>$, the set of coherent states is defined as $|z\rangle^J = U^{(J)}(z,\bar{z})|\gamma\rangle^J$.

The CS overlap, for the several spins case, is now

$$C^{(J)}(z,z') = \langle z|z'\rangle^J = \frac{1}{J+1}\sum_{s=0}^{J}\frac{(1+z'\bar{z})^{2s}}{(1+z\bar{z})^s(1+z'\bar{z}')^s}. \tag{4.25}$$

The first, naive choice, of sampling points would be the N^{th} roots of unity, $z_k = e^{2\pi ik/N}$, where now $N = \dim\mathcal{H}^{(J)} = (J+1)^2$ in order to have critical sampling. In this way, the operators $\mathcal{A}^{(J)}$ and $\mathcal{B}^{(J)}$ would have nice structure, and their inverse matrices would be easily computed.

However, the following negative result prevents us from proceeding in this way:

Proposition 4.19 *For $N \geq 2J+1$, the overlapping kernel operator $\mathcal{B}^{(J)}$ has rank* $2J+1$.

Proof Let λ^s, $\mathcal{T}^s$, $\mathcal{B}^s$, and D_{2s+1} be the eigenvalues, frame, overlapping kernel operators, and diagonal matrix appearing in the previous sections corresponding to angular momentum s. Then the frame operator $\mathcal{T}^{(J)}: \mathcal{H}^{(J)} \to \mathbb{C}^N$ can be written as a $N \times (J+1)^2$ matrix given by $\mathcal{T}^{(J)}_{k,(s,n)} = \mathcal{T}^s_{kn} = \mathcal{F}_{N,2s+1}\, D^{1/2}_{2s+1}$.

[3] For the time being, we shall restrict ourselves to integer values of spin, in order to compare with standard Fourier analysis on the sphere.

Then $\mathcal{B}^{(J)} = \mathcal{T}^{(J)}\mathcal{T}^{(J)*} = \mathcal{B}^0 + \mathcal{B}^1 + \cdots + \mathcal{B}^J = \mathcal{F}_N (D_1^{\uparrow} + D_3^{\uparrow} + \cdots + D_{2J+1}^{\uparrow}) \mathcal{F}_N^* = \mathcal{F}_N \tilde{D}_{2J+1}^{\uparrow} \mathcal{F}_N^*$, where $\tilde{D}_{2J+1}$ is a diagonal matrix with eigenvalues

$$\tilde{\lambda}_n = \frac{1}{J+1} \sum_{s=\overline{(n-1)/2}}^{J} \lambda_n^s, \quad n = 0, 1, \ldots, 2J, \tag{4.26}$$

where $\overline{(n-1)/2}$ stands for the ceiling of $\frac{n-1}{2}$. □

The proof could also have been done by using the circulant structure of $\mathcal{B}^{(J)}$, which can be written as $\mathcal{B}_{kl}^{(J)} = \mathcal{C}_{l-k}$, where now

$$\mathcal{C}_k \equiv \frac{1}{J+1} \sum_{s=0}^{J} \frac{1}{2^{2s}} \big(1 + e^{2\pi i k/N}\big)^{2s},$$

and computing its eigenvalues as in Appendix B.

Therefore, putting all sampling points in the equator of the Riemann sphere is not a good choice, and other alternatives should be looked for. The problem is that other choices of sampling points lead to resolution operators with less structure and therefore without the possibility of having an explicit inverse.

Another possibility is to use an equiangular grid in (θ, ϕ), as the one used in [8, 9]. If $(\theta_j, \phi_k) = (\frac{\pi}{N} j, \frac{2\pi}{N} k)$, $j, k = 0, 1, \ldots, N-1$, is a grid of N^2 points in the sphere, where $N \geq J+1$, the corresponding points in the complex plane by stereographic projection are given by $z_j^k = e^{i\phi_k} \tan\frac{\theta_j}{2} = e^{i\frac{2\pi}{N}k} \tan(\frac{\pi}{2N} j) = r_j e^{i\frac{2\pi}{N}k}$. However, it can be checked that in this case the resolution operator also is singular.

We shall follow a mixture of both approaches consisting in using, as sampling points, the $(2s+1)^{\text{th}}$ roots of $(r_s)^{2s+1}$ for $s = 0, 1, \ldots, J$. Here r_s is a positive number depending on s, in such a way that if $s \neq s'$, then $r_s \neq r_{s'}$. Thus, we shall continue to use $N = (J+1)^2$ sampling points but distributed in circles of different radius. In the Riemann sphere, these would be distributed in different parallels, one for each value of spin. These points are given by

$$z_m^{(s)} = r_s \, e^{\frac{2\pi i m}{2s+1}}, \quad s = 0, \ldots, J, \ m = 0, \ldots, 2s,$$

where s denotes spin index, and m the index for the roots.

The frame operator $\mathcal{T}$ is a $(J+1)^2 \times (J+1)^2$ square matrix with a block structure given by $(2s'+1) \times (2s+1)$ blocks

$$\mathcal{T}^{s',s} = \mathcal{F}_{2s'+1,2s+1} \big(D_{2s+1}^{s',s}\big)^{1/2} \quad \text{with } D_{2s+1}^{s',s} = \operatorname{diag}\big(\lambda_0^{s',s}, \ldots, \lambda_{2s}^{s',s}\big), \tag{4.27}$$

where $\lambda_n^{s',s} = \frac{1}{J+1} \frac{2s'+1}{(1+r_{s'}^2)^{2s}} \binom{2s}{n} r_{s'}^{2n}$. The diagonal blocks are the frame operators for the case of critical sampling with fixed spin s for each value of $s = 0, 1, \ldots, J$, and they coincide with the previous expressions fixing $r_s = 1$ (up to the factor $\frac{1}{J+1}$).

The resolution operator $\mathcal{A}$ and the overlapping kernel operator $\mathcal{B}$ share this block structure with blocks given by

$$\begin{aligned}
\mathcal{A}^{s',s} &= \sum_{s''=0}^{J} \big(\mathcal{T}^{s'',s'}\big)^* \mathcal{T}^{s'',s} = \sum_{s''=0}^{J} \big(D^{s'',s'}_{2s'+1}\big)^{1/2} \mathcal{F}^*_{2s''+1,2s'+1}\mathcal{F}_{2s''+1,2s+1}\big(D^{s'',s}_{2s+1}\big)^{1/2},\\
\mathcal{B}^{s',s} &= \sum_{s''=0}^{J} \mathcal{T}^{s',s''}\big(\mathcal{T}^{s,s''}\big)^* = \sum_{s''=0} \mathcal{F}_{2s'+1,2s''+1}\big(D^{s',s''}_{2s''+1}\big)^{1/2}\big(D^{s,s''}_{2s''+1}\big)^{1/2}\mathcal{F}^*_{2s+1,2s''+1}.
\end{aligned} \tag{4.28}$$

The overlapping kernel operator $\mathcal{B}$ can be computed directly from the CS overlap (4.25) evaluated a the sampling points, turning out to be

$$\begin{aligned}
\mathcal{B}^{a,b}_{m,n} \equiv \langle z^{(a)}_m | z^{(b)}_n \rangle^J &= \frac{1}{J+1}\sum_{s=0}^{J}\left(\frac{(1+r_a r_b\, e^{2\pi i \frac{n(2a+1)-m(2b+1)}{(2a+1)(2b+1)}})^2}{(1+r_a^2)(1+r_b^2)}\right)^s \\
&= \begin{cases} 1 & \text{if } z^{(a)}_m = z^{(b)}_n, \\ \frac{1}{J+1}\frac{1-(\kappa^{a,b}_{m,n})^{J+1}}{1-\kappa^{a,b}_{m,n}} & \text{otherwise,} \end{cases}
\end{aligned} \tag{4.29}$$

where

$$\kappa^{a,b}_{m,n} \equiv \frac{(1+r_a r_b\, e^{2\pi i \frac{n(2a+1)-m(2b+1)}{(2a+1)(2b+1)}})^2}{(1+r_a^2)(1+r_b^2)} \tag{4.30}$$

is the multiplier of a geometric sum.

The overlapping kernel operator $\mathcal{B}^{a,b}_{m,n}$ is an Hermitian matrix having the following structure:

$$\left(\begin{array}{c|c|c|c|c|c|c}
\mathrm{circ}(1) & B_{01} & B_{02} & \cdots & B_{0k} & B_{0\,k+1} & \cdots \\ \hline
B^*_{01} & \mathrm{circ}(\mathcal{C}^{(1)}_0, \mathcal{C}^{(1)}_1, \mathcal{C}^{(1)}_2) & B_{12} & \cdots & B_{1k} & B_{1\,k+1} & \cdots \\ \hline
B^*_{02} & B^*_{12} & \mathrm{circ}(\mathcal{C}^{(2)}_0, \ldots, \mathcal{C}^{(2)}_4) & \cdots & B_{2k} & B_{2\,k+1} & \cdots \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline
B^*_{0k} & B^*_{1k} & B^*_{2k} & \cdots & \mathrm{circ}(\mathcal{C}^{(k)}_0, \ldots, \mathcal{C}^{(k)}_{2k}) & B_{k\,k+1} & \cdots \\ \hline
B^*_{0\,k+1} & B^*_{1\,k+1} & B^*_{2\,k+1} & \cdots & B^*_{k\,k+1} & \ddots & \cdots \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

The diagonal blocks are circulant matrices of dimension $2s+1$ with $\mathcal{C}^{(s)}_n = \frac{(1+r_s^2 e^{2\pi i n/(2s+1)})^2}{(1+r_s^2)^2}$, and the nondiagonal blocks B_{pq} are matrices of dimension $(2p+1)\times(2q+1)$.

The overlapping kernel operator $\mathcal{B}^{a,b}_{m,n}$ is not a circulant matrix,[4] not even a block circulant matrix, therefore the computation of its inverse needed for the reconstruction formula must be done numerically. Even the checking that it is nonsingular must be done numerically.

For different choices of r_s, we have checked that the overlapping kernel operator is invertible, and therefore the reconstruction formula can be used, although nice expressions like (4.5) or (4.17) are not available.

Once the overlapping kernel operator has been inverted, the Fourier coefficients of the signal can be obtained in the same fashion as in Corollary 4.6 or 4.16, using the vector notation of Remarks 4.7 and 4.17.

Corollary 4.20 *The Fourier coefficients* a^s_m *of the expansion* $|\psi\rangle = \sum_{s=0}^{J}\sum_{m=-s}^{s} a^s_m |s,m\rangle$ *of any* $\psi \in \mathcal{H}^{(J)}$ *in the angular momentum orthonormal basis* $B(\mathcal{H}^{(J)})$ *can be determined in terms of the data* $\Psi(z^{(s)}_k) = \langle z^{(s)}_k|\psi\rangle$ *as*

$$\vec{a} = \mathcal{T}^* \mathcal{B}^{-1} \vec{\Psi}. \tag{4.31}$$

In this expression, $\vec{a}$ and $\vec{\Psi}$ stand for vectors formed by gathering the vectors $\vec{a}$ and $\vec{\Psi}$ of Remarks 4.7 and 4.17 for each spin s.

Proof As in Corollaries 4.6 and 4.16, using the vector notation, we have $\mathcal{T}\vec{a} = \vec{\Psi}$. Applying the right pseudoinverse for $\mathcal{T}$, the desired result is obtained. □

Remark 4.21 For this choice of sampling points, both $\mathcal{A}$ and $\mathcal{T}$ turn out to be invertible, therefore we could have used the left pseudoinverse of $\mathcal{T}$ for the reconstruction, which requires the inverse of $\mathcal{A}$, or we could have directly inverted $\mathcal{T}$.

From the computational point of view, the most expensive step is the inversion of $\mathcal{B}$ (or $\mathcal{A}$ or $\mathcal{T}$), which is of order $O(N^3)$ with $N = (J+1)^2$. But this is done only once and can be stored for future uses. The determination of the Fourier coefficients from the data requires $O(N^2)$ operations. More efficient algorithms, to compete with the $O(N \log(N)^2)$ of [8, 9] would require taking advantage of the block structure of the matrices $\mathcal{B}$ or $\mathcal{A}$, or maybe choosing a different set of sampling points that leads to more structured matrices in such a way that the inverse is easily computed.

5 Connection with the Euler Angle Picture

We have provided reconstruction formulas for Majorana functions $\Psi(z)$ from N of its samples $\Psi_k = \langle z_k|\psi\rangle$ at the sampling points $z_k = e^{2\pi i k/N}$ in the Riemann sphere. As stated in the Introduction, the advantage of using this "complex holomorphic picture,"

[4] This is traced back to the fact that the sampling points do not form an Abelian group. Only the sets of the form $z^{(s)}_m$, $m = 0, 1, \ldots, 2s$, with fixed s form cyclic subgroups, and they are responsible for the appearance of circulant blocks at the diagonal.

instead of the standard "Euler angle picture," is twofold: firstly, we can take advantage of the either diagonal or circulant structure of resolution and overlapping kernel operators, respectively, to provide explicit inversion formulas and, secondly, we can extend the sampling procedure to half-integer angular momenta s, which could be useful when studying, for example, discrete frames for coherent states of spinning particles in Quantum Mechanics.

Moreover, for integer angular momenta $s = j$, we could always pass from one picture to another through the Bargmann transform (3.31). Indeed, let us work for simplicity in the critical case $N = 2j + 1$ and let us denote by $\Phi_k = \langle\theta_0, \phi_k|\psi\rangle$ the samples of the function (3.23), in the Euler angle characterization, at the sampling points $\theta_0 \neq 0, \pi$ and $\phi_k = -\frac{2\pi}{N}k$, $k = 0, \dots, N-1$ (i.e., a uniformly distributed set of N points in a parallel of the sphere $\mathbb{S}^2$ but counted clockwise). Denoting by

$$\begin{aligned}\mathcal{K}_{kl} &\equiv K(\theta_0, \phi_k; z_l) = \langle\theta_0, \phi_k|z_l\rangle \\ &= \frac{\sqrt{(2j)!}}{2^{2j}j!} e^{i\frac{2\pi}{N}jk} \sin^j(\theta_0)\big(1 + 2\cot(\theta_0)\, e^{i\frac{2\pi}{N}(l-k)} - e^{i\frac{4\pi}{N}(l-k)}\big)^j \end{aligned} \tag{5.1}$$

the discrete $N \times N$ matrix version of the Bargmann transform (3.31) and inserting the resolution of identity (4.1) in $\langle\theta_0, \phi_k|\psi\rangle$, we easily arrive at the following expression:

$$\Phi_k = \sum_{l,m=0}^{N-1} \mathcal{K}_{kl}\mathcal{B}^{-1}_{lm}\Psi_m, \tag{5.2}$$

which relates data between both characterizations or pictures through the CS transform and CS overlap matrices $\mathcal{K}$ and $\mathcal{B}$ in (5.1) and (B.1), respectively.

Except for some values of θ_0 (see later in this section), the transformation (5.2) is invertible, and explicit formulas of $\mathcal{K}^{-1}$ are available. Actually, $\mathcal{K}$ can be written as the product $\mathcal{K} = \Lambda\mathcal{Q}$,

$$\begin{aligned}\Lambda_{kp} &= \frac{\sqrt{(2j)!}}{2^{2j}j!} e^{i\frac{2\pi}{N}jk} \sin^j(\theta_0)\delta_{kp}, \\ \mathcal{Q}_{pl} &= \big(1 + 2\cot(\theta_0)\, e^{i\frac{2\pi}{N}(l-p)} - e^{i\frac{4\pi}{N}(l-p)}\big)^j \equiv q_{l-p}, \end{aligned} \tag{5.3}$$

of a diagonal matrix Λ times a circulant matrix $\mathcal{Q}$, which can be easily inverted (following the procedure of Appendix B) as $\mathcal{Q}^{-1} = \mathcal{F}_N\Omega^{-1}\mathcal{F}^*_N$, where $\Omega = \mathrm{diag}(\omega_0, \dots, \omega_{N-1})$ with eigenvalues

$$\omega_k = \sum_{n=0}^{N-1} q_n\, e^{-i\frac{2\pi}{N}kn} = N\sum_{p=0}^{j}\sum_{r=0}^{p}{}'(-1)^r\binom{p}{r}\big(2\cot(\theta_0)\big)^{p-r}, \tag{5.4}$$

where the prime over $\sum$ implies the restriction $p + r = k$. Therefore, we can also obtain data in the holomorphic characterization, $\vec{\Psi}$, from data in the Euler angle characterization, $\vec{\Phi}$, through the formula

$$\vec{\Psi} = \mathcal{B}\mathcal{Q}^{-1}\Lambda^{-1}\vec{\Phi} = \mathcal{F}_N D\Omega^{-1}\mathcal{F}^*_N\Lambda^{-1}\vec{\Phi}, \tag{5.5}$$

which can be seen as a convolution $\vec{\Psi} = \vec{\Theta} * \vec{\Phi}'$ of the re-scaled data $\vec{\Phi}' = \Lambda^{-1}\vec{\Phi}$ times the filter $\vec{\Theta} = \mathcal{F}_N\vec{\theta}$ with $\theta_k = \lambda_k/\omega_k$ the quotient of eigenvalues of $\mathcal{B}$ and $\mathcal{Q}$.

Note that there are values of θ_0 for which $\mathcal{K}$ is not invertible. Such is the case of $\theta_0 = \pi/2$ (the equator), for which $\omega_k = N(-1)^{k/2}\binom{j}{k/2}$ if k even and zero otherwise. Let us show that this situation is linked to the fact that general functions (3.23) in the Euler angle picture cannot be reconstructed from its samples Φ_k on a uniformly distributed set of N points in the equator of the sphere. Indeed, let us insert this time the resolution of unity $I_{2j+1} = \sum_{m=-j}^{j} |j,m\rangle\langle j,m|$ in $\langle\theta_0,\phi_k|\psi\rangle$ with $|\psi\rangle = \sum_{m=-j}^{j} a_m|j,m\rangle$, which results in

$$\Phi_k = \sum_{m=-j}^{j} \sqrt{\frac{4\pi}{2j+1}} Y_j^m(\theta_0,\phi_k)\, a_m, \tag{5.6}$$

where we have used the definition (3.22). Denoting $\mathcal{Y}_{kn}(\theta_0) \equiv \sqrt{\frac{4\pi}{N}} Y_j^{n-j}(\theta_0,\phi_k)$ and knowing from Remark 4.7 that the Fourier coefficients a_{n-j} are given in terms of data Ψ_k trough $\vec{a} = D^{-1/2}\mathcal{F}_N^*\vec{\Psi}$, we arrive at a variant of the formula (5.2):

$$\vec{\Phi} = \mathcal{Y}(\theta_0) D^{-1/2}\mathcal{F}_N^*\vec{\Psi}, \tag{5.7}$$

which again connects data between both pictures. Knowing that spherical harmonics can be expressed in terms of associated Legendre functions P_j^m by

$$Y_j^m(\theta,\phi) = e^{im\phi} P_j^m(\cos\theta), \tag{5.8}$$

whose value at the equator $\theta_0 = \pi/2$ is given in terms of Gamma functions as

$$P_j^m(0) = \frac{2^m}{\sqrt{\pi}} \cos\left(\frac{1}{2}\pi(j+m)\right) \frac{\Gamma(\frac{1}{2}j + \frac{1}{2}m + \frac{1}{2})}{\Gamma(\frac{1}{2}j - \frac{1}{2}m + 1)}, \tag{5.9}$$

we immediately realize that $\mathcal{Y}(\pi/2)_{kn} = 0$ for n odd. In other words, for $\theta_0 = \pi/2$, the reconstruction process in the Euler angle picture fails unless we restrict to the subspace of functions ψ with null odd Fourier coefficients (i.e., $a_{n-j} = 0$ for n odd).

Appendix A: Rectangular Fourier Matrices

Let $N, M \in \mathbb{N}$, and let $\mathcal{F}_{NM}$ be the $N \times M$ matrix

$$(\mathcal{F}_{NM})_{nm} = \frac{1}{\sqrt{N}} e^{-i2\pi nm/N}, \quad n = 0,1,\ldots,N-1,\ m = 0,1,\ldots,M-1. \tag{A.1}$$

We shall denote these matrices Rectangular Fourier Matrices (RFM). For $N = M$, we recover the standard Fourier matrix $\mathcal{F}_N$. Let us study the properties of these matrices in the other two cases, $N > M$ and $N < M$.

A.1 Case $N > M$

The case $N > M$ is the one corresponding to oversampling, and it is the easiest one, since it is very similar to the $N = M$ case. Let us first introduce some notation.

Let $\iota_{NM} : \mathbb{C}^M \to \mathbb{C}^N$ be the inclusion into the first M rows of $\mathbb{C}^N$ (i.e., padding a M-vector with zeros). And let $p_{MN} : \mathbb{C}^N \to \mathbb{C}^M$ be the projection onto the first M rows of $\mathbb{C}^N$ (i.e., truncating an N-vector). The matrix expression for this applications are

$$\iota_{NM} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \qquad p_{MN} = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 \end{pmatrix}. \tag{A.2}$$

It can be easily checked that

$$p_{MN} = (\iota_{NM})^*, \qquad p_{MN}\,\iota_{NM} = I_M, \qquad \iota_{NM}\,p_{MN} = P_M \equiv \left(\begin{array}{c|c} I_M & 0 \\ \hline 0 & 0 \end{array}\right)_{N\times N}, \tag{A.3}$$

where I_M stands for the identity matrix in $\mathbb{C}^M$.

Given the square matrices A and B acting on $\mathbb{C}^N$ and $\mathbb{C}^M$, respectively, we define the square matrices $A^\downarrow$ and $B^\uparrow$ through the commutative diagrams

$$\begin{array}{ccc} \mathbb{C}^M & \xrightarrow{A^\downarrow} & \mathbb{C}^M \\ {\scriptstyle \iota_{NM}}\downarrow & & \uparrow{\scriptstyle p_{MN}} \\ \mathbb{C}^N & \xrightarrow{A} & \mathbb{C}^N \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{C}^N & \xrightarrow{B^\uparrow} & \mathbb{C}^N \\ {\scriptstyle p_{MN}}\downarrow & & \uparrow{\scriptstyle \iota_{NM}} \\ \mathbb{C}^M & \xrightarrow{B} & \mathbb{C}^M \end{array} \tag{A.4}$$

The matrix $A^\downarrow = p_{MN}\,A\,\iota_{NM}$ is the truncation of A to an $M \times M$ matrix, and the matrix $B^\uparrow = \iota_{NM}\,B\,p_{MN} \equiv \left(\begin{array}{c|c} B & 0 \\ \hline 0 & 0 \end{array}\right)_{N\times N}$ is the padded version of B. Also note that $P_M = (I_M)^\uparrow$.

From these definitions the following properties of the Rectangular Fourier Matrices are derived:

$$\begin{aligned} \mathcal{F}_{NM} &= \mathcal{F}_N\,\iota_{NM}, & \mathcal{F}^*_{NM} &= p_{MN}\,\mathcal{F}^*_N, \\ \mathcal{F}^*_{NM}\,\mathcal{F}_{NM} &= I_M, & \mathcal{F}_{NM}\,\mathcal{F}^*_{NM} &= \mathcal{F}_N\,P_M\,\mathcal{F}^*_N. \end{aligned} \tag{A.5}$$

A.2 Case $N < M$

The case $N < M$ is the one corresponding to undersamplig, and it is not as easy as the $N > M$ case. The same definitions as in the previous case also apply here, but interchanging the roles of N and M. Thus, in this case we have

$$p_{NM}\,\iota_{MN} = I_N, \qquad \iota_{MN}\,p_{NM} = P_N \equiv \left(\begin{array}{c|c} I_N & 0 \\ \hline 0 & 0 \end{array}\right)_{M\times M} = (I_N)^\uparrow. \tag{A.6}$$

The RFM in this case now read

$$\mathcal{F}_{NM} = \left(\mathcal{F}_N \middle| \mathcal{F}_N \middle| \overset{q \text{ times}}{\cdots} \middle| \mathcal{F}_N \middle| \mathcal{F}_{Np}\right),$$

$$\mathcal{F}^*_{NM} = \begin{pmatrix} \mathcal{F}^*_N \\ \hline \mathcal{F}^*_N \\ \hline \vdots \text{ q times} \\ \hline \mathcal{F}^*_N \\ \hline \mathcal{F}^*_{Np} \end{pmatrix}, \tag{A.7}$$

where $p = M \bmod N$ and $q = M \operatorname{div} N$.

Instead of working with these matrices, it is more convenient to "complete" them so as to have an integer multiple of Fourier matrices. Let $\bar{M}$ be the smaller multiple of N greater or equal to M, and $\bar{q} = \bar{M}/N$ the ceiling of M/N. Note that

$$\bar{q} = \begin{cases} q & p = 0, \\ q + 1, & p \neq 0. \end{cases} \tag{A.8}$$

Then

$$\mathcal{F}_{N\bar{M}} = \left(\mathcal{F}_N \middle| \mathcal{F}_N \middle| \overset{\bar{q} \text{ times}}{\cdots} \middle| \mathcal{F}_N\right), \tag{A.9}$$

a similar expression for $\mathcal{F}^*_{N\bar{M}}$ is obtained, and

$$\begin{aligned} &\mathcal{F}_{NM} = \mathcal{F}_{N\bar{M}}\, \iota_{\bar{M}M}, \qquad \mathcal{F}^*_{NM} = p_{M\bar{M}}\, \mathcal{F}^*_{N\bar{M}}, \\ &\mathcal{F}_{NM}\, \mathcal{F}^*_{NM} = q I_N + \mathcal{F}_N\, P_p\, \mathcal{F}^*_N, \qquad \mathcal{F}^*_{NM}\, \mathcal{F}_{NM} = \left(\hat{I}_{\bar{M}}\right)^{\downarrow} \equiv \hat{I}_M, \end{aligned} \tag{A.10}$$

where

$$\hat{I}_{\bar{M}} = \left(\begin{array}{c|c|c|c} I_N & I_N & \overset{\bar{q} \text{ times}}{\cdots} & I_N \\ \hline I_N & I_N & \cdots & I_N \\ \hline \vdots\ \bar{q} \text{ times} & \vdots & \ddots & \vdots \\ \hline I_N & I_N & \cdots & I_N \end{array}\right)_{\bar{M} \times \bar{M}}. \tag{A.11}$$

Appendix B: Circulant Matrices

The overlapping kernel operator $\mathcal{B}$ has a circulant matrix structure which gives a deep insight into the process taking place, and we may take advantage of this fact to diagonalize it in the case of undersampling, where RFM are more difficult to handle.

Indeed, note that

$$\mathcal{B}_{kl} = \langle z_k | z_l \rangle = \frac{1}{2^{2s}} \left(1 + e^{2\pi i (l-k)/N}\right)^{2s} = \mathcal{C}_{l-k}, \quad k, l = 0, \ldots, N-1, \tag{B.1}$$

where $\mathcal{C}_n = \frac{1}{2^{2s}}(1+e^{2\pi i n/N})^{2s}$, which shows a circulant matrix structure

$$\mathcal{B} = \mathrm{circ}(\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_{N-1}) = \begin{pmatrix} \mathcal{C}_0 & \mathcal{C}_1 & \dots & \mathcal{C}_{N-1} \\ \mathcal{C}_{N-1} & \mathcal{C}_0 & \dots & \mathcal{C}_{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_1 & \mathcal{C}_2 & \dots & \mathcal{C}_0 \end{pmatrix} = \sum_{j=0}^{N-1} \mathcal{C}_j \Pi^j \equiv P_c(\Pi), \tag{B.2}$$

where

$$\Pi = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}, \quad \left(\Pi^N = I_N,\ \Pi^t = \Pi^* = \Pi^{-1} = \Pi^{N-1}\right),$$

is the *generating matrix* of the circulant matrices, and $P_c(t)$ is the *representative polynomial* of the circulant (we put $\Pi^0 \equiv I_N$). According to the general theory (see, e.g., [20]), every circulant matrix is diagonalizable, whose eigenvectors are the columns of the Vandermonde matrix $V_N = V(z_0, \dots, z_{N-1}) = \sqrt{N}\mathcal{F}^*$ and whose eigenvalues $\hat{\lambda}_k$ can be computed through its representative polynomial as[5]

$$\begin{aligned} \hat{\lambda}_k = P_c(\bar{z}_k) &= \sum_{l=0}^{N-1} \mathcal{C}_l z_k^{-l} = 2^{-(2s)} \sum_{l=0}^{N-1} \sum_{n=0}^{2s} \binom{2s}{n} e^{2\pi i l(2s-n)/N} e^{-2\pi i k l/N} \\ &= \frac{N}{2^{2s}} \sum_{l=0}^{\bar{q}-1} \binom{2s}{k+lN}, \quad k = 0, \dots, N-1, \end{aligned} \tag{B.3}$$

where $\bar{q}$ is the ceiling of $(2s+1)/N$, and we have used the orthogonality relation (4.3), although in this case, since $N \le 2s+1$, there can be more terms in the sum. All of them are strictly positive, and it is easy to prove that $\mathcal{B} = \mathcal{F}_N \hat{D} \mathcal{F}_N^*$, where $\hat{D} = \mathrm{diag}(\hat{\lambda}_0, \dots, \hat{\lambda}_{N-1})$.

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[5]We are using the complex conjugate $\bar{z}_k$ instead of z_k in the representative polynomial to obtain the more convenient factorization $\mathcal{B} = \mathcal{F}_N D \mathcal{F}_N^*$ instead of the standard one $\mathcal{B} = \mathcal{F}_N^* D' \mathcal{F}_N$, where the eigenvalues in D' have reversed order with respect to those of D.

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