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Beurling Dimension of Gabor Pseudoframes for Affine Subspaces

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Abstract Pseudoframes for subspaces have been recently introduced by Li and Ogawa as a tool to analyze lower dimensional data with arbitrary flexibility of both the analyzing and the dual sequence.

In this paper we study Gabor pseudoframes for affine subspaces by focusing on geometrical properties of their associated sets of parameters. We first introduce a new notion of Beurling dimension for discrete subsets of $\mathbb{R}^d$ by employing a certain generalized Beurling density. We present several properties of Beurling dimension including a comparison with other notions of dimension showing, for instance, that our notion includes the mass dimension as a special case. Then we prove that Gabor pseudoframes for affine subspaces satisfy a certain Homogeneous Approximation Property, which implies invariance under time–frequency shifts of an approximation by elements from the pseudoframe.

The main result of this paper is a classification of Gabor pseudoframes for affine subspaces by means of the Beurling dimension of their sets of parameters. This provides us, in particular, with a Nyquist dimension which separates sets of parameters

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of pseudoframes from those of non-pseudoframes and which links a fixed value to sets of parameters of pseudo-Riesz sequences. These results are even new for the special case of Gabor frames for an affine subspace.

Keywords Beurling density · Beurling dimension · Frame · Gabor system · Discrete Hausdorff dimension · Homogeneous Approximation Property · Mass dimensions · Nyquist density · Pseudoframe · Pseudoframe for subspaces

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1 Introduction

Frames are a well-established tool both in applied and pure mathematics which provides robust and stable—but usually nonunique—representations of vectors. For instance, in wireless communications frames ensure robustness of transmission against erasures [22], in image processing they serve as building blocks for novel directional representation systems [3], and just recently it has been discovered that the theory of frames may provide a means to attack the Kadison–Singer Conjecture in operator theory from 1959 [5].

However, for some applications, frames lack enough flexibility, for instance, in the design of the dual frame needed for reconstruction. Several different approaches have been recently proposed to circumvent this problem, e.g., fusion frames [4], g-frames [23], and oblique frames [7]. In this paper we will focus on *pseudoframes* introduced by Li and Ogawa [16] as a tool to analyze lower dimensional data with arbitrary flexibility of both the analyzing and the dual sequence. In the situation of a pseudoframe, the "dual sequence" is only required to provide a reconstruction formula. In some instances, data that we wish to analyze is contained in a subspace, which naturally leads to the idea of *pseudoframes for subspaces* [17, 18].

In this paper we study a special class of pseudoframes for subspaces which are of particular interest in time–frequency analysis. A *Gabor system* consists of a collection of time–frequency shifts of a single function or a finite family of functions in $L^2(\mathbb{R}^d)$ with respect to a discrete set of parameters in $\mathbb{R}^d \times \mathbb{R}^d$. Due to this structure, Gabor systems are especially suitable for applications involving time-dependent frequency content, for example, for the analysis of acoustic signals such as music [9].

Classical Gabor systems, which employ a lattice as set of parameters, have been studied extensively over the past 20 years. Recently, the more general *irregular Gabor systems* with arbitrary sets of parameters in $\mathbb{R}^d \times \mathbb{R}^d$ have attracted increasing attention, in particular, due to applications from sampling and perturbation theory. Questions concerning (frame) properties of irregular Gabor systems lead naturally to the study of the associated sets of parameters. A very elegant way to classify sets of parameters is the consideration of their Beurling densities. For a recent survey article with extensive list of references, we refer to [10].

However, we intend to focus on the study of *Gabor pseudoframes for (affine) subspaces*. Beurling density alone does not serve our needs here, and instead the "dimensionality" of the set of parameters will play an essential role. Beurling density only serves as a classifying tool for sets of parameters of the same "dimension." Comparable problems occur in the study of sets of parameters of *wave packet systems*, which are systems consisting of dilates, translates, and modulates of a single function in $L^2(\mathbb{R})$ with their sets of parameters being contained in the affine Weyl–Heisenberg group. Progress on the problem of classifying sets of parameters of wave packet systems was recently made by the authors through the introduction of *upper and lower Beurling dimensions* based on Beurling density and inspired by the notion of Hausdorff dimension [8].

In this paper, we present a notion of Beurling dimension for discrete subsets of $\mathbb{R}^d$ by employing a certain generalized Beurling density. First, we study several of its properties, including behavior under perturbations, monotonicity, stability, and geometric invariance. We further compare the new notion with the most well-known notions of dimension for discrete subsets of $\mathbb{R}^d$, such as the mass dimension, which we show to be a special case of our notion. Secondly, we apply the new notion of Beurling dimension to Gabor pseudoframes for (affine) subspaces. In particular, inspired by techniques from [11, 21], we determine the Beurling dimensions of the sets of parameters of Gabor pseudoframes for affine subspaces. This leads to a classification of Gabor pseudoframes for affine subspaces and Gabor pseudo-Riesz sequences by means of Beurling dimensions of their associated sets of parameters (Theorem 3.3).

These main results and the techniques of their proofs have several interesting implications. One implication concerns an improvement of the applicability of Gabor pseudoframes. In fact, we prove that Gabor pseudoframes for affine subspaces always satisfy a certain *Homogeneous Approximation Property*. This property which can be interpreted as invariance of the quality of an approximation by elements from a Gabor pseudoframe for an affine subspace associated with a boxed set of parameters under time–frequency shifts of this box. This, in turn, provides us with more flexibility to approximate signals by means of Gabor pseudoframes for an affine subspace. The other implication we mention is of theoretical nature. It is well known that Gabor frames exhibit a Nyquist density phenomenon, i.e., the Beurling density separates sets of parameters of frames from those of non-frames [6], whereas for wavelet frames this was recently shown to be false [12, 14] by employing a notion of density adapted to affine systems. Due to the fact that a suitable notion of density does not provide useful information as already discussed earlier, Gabor pseudoframes for affine subspaces cannot be expected to possess a Nyquist density. However, our results show that, instead, there exists a *Nyquist dimension* which separates sets of parameters of pseudoframes from those of non-pseudoframes and yields a fixed value for sets of parameters of pseudo-Riesz sequences. This result is new even in the special case of Gabor frames for an affine subspace and provides us with a deeper understanding of the nature of Nyquist phenomena.

This paper is organized as follows. The definition and general theory of Beurling dimension for discrete subsets of $\mathbb{R}^d$ is given in Sect. 2, including a comparison of Beurling dimension with the mass dimension and the discrete Hausdorff dimension. In Sect. 3 we determine the Beurling dimensions of the sets of parameters of Gabor pseudoframes for affine subspaces and prove the classification result.

2 Beurling Dimension

2.1 Definition of Beurling Dimension

We will define a notion of Beurling dimension for sequences Λ in $\mathbb{R}^d$ suited to Euclidean geometry by using as a backbone a generalization of Beurling density. Notice that throughout this paper, although Λ will always denote a sequence of points in $\mathbb{R}^d$ and not merely a subset, for simplicity we will write $\Lambda \subset \mathbb{R}^d$.

First we require some notation. Let Q denote the box $[-1,1]^d$ and, for $h > 0$, let Q_h be the dilation of Q by the factor of h:

$$Q_h = hQ = [-h, h]^d.$$

For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we let $Q_h(x)$ be the set Q_h translated in such a way that it is centered at x, i.e.,

$$Q_h(x) = \prod_{i=1}^{d} [x_i - h, x_i + h].$$

Employing these notions, we define a generalization of Beurling density.

Definition 2.1 We say that $S \subset \mathbb{R}^d$ is an *affine subspace of* $\mathbb{R}^d$ if it is a coset of some linear subspace of $\mathbb{R}^d$.

Let $\Lambda \subset \mathbb{R}^d$, let S be an affine subspace of $\mathbb{R}^d$, and let $r > 0$. The *lower Beurling density of* Λ *with respect to* S *and* r is defined by

$$\mathcal{D}^-_{S,r}(\Lambda) = \liminf_{h\to\infty} \inf_{x\in S} \frac{\#(\Lambda \cap Q_h(x))}{h^r},$$

and the *upper Beurling density of* Λ *with respect to* S *and* r is defined by

$$\mathcal{D}^+_{S,r}(\Lambda) = \limsup_{h\to\infty} \sup_{x\in S} \frac{\#(\Lambda \cap Q_h(x))}{h^r}.$$

If $\mathcal{D}^-_{S,r}(\Lambda) = \mathcal{D}^+_{S,r}(\Lambda)$, then we say that Λ has *uniform Beurling density with respect to* S *and* r and we denote this density by $\mathcal{D}_{S,r}(\Lambda)$.

We remark that the Beurling density with respect to $\mathbb{R}^d$ and d coincides with the classical Beurling density. In particular, for $S = \mathbb{R}^d$ and $r = d$, the definition of Beurling density is independent of the particular choice of the set Q (see [15]). By using a similar argument for each $r > 0$, we obtain:

Proposition 2.2 *Let* $\Lambda \subset \mathbb{R}^d$, *and let* $U \subset \mathbb{R}^d$ *be a compact set of measure* 2^d *whose boundary has measure zero. Then, for any* $r > 0$ *and any affine subspaces* S *of* $\mathbb{R}^d$,

$$\mathcal{D}^-_{S,r}(\Lambda) = \liminf_{h\to\infty} \inf_{x\in S} \frac{\#(\Lambda \cap (x + hU))}{h^r}$$

and

$$\mathcal{D}^+_{S,r}(\Lambda) = \limsup_{h\to\infty} \sup_{x\in S} \frac{\#(\Lambda \cap (x + hU))}{h^r}.$$

For $y \in \mathbb{R}^d$, the Beurling density with respect to $S = \{y\}$ and $r = d$ is also a commonly used density, and it is known that this density does not depend on y. More generally, we have the following:

Proposition 2.3 *Let $\Lambda \subset \mathbb{R}^d$, T a linear subspace of $\mathbb{R}^d$, and $y \in \mathbb{R}^d$ be given. Then, for all $r > 0$,*

$$\mathcal{D}^+_{T+y,r}(\Lambda) = \mathcal{D}^+_{T,r}(\Lambda)$$

and

$$\mathcal{D}^-_{T+y,r}(\Lambda) = \mathcal{D}^-_{T,r}(\Lambda).$$

Proof Note that there exists $R > 0$ such that, for all $x \in T$ and $h > 0$, $Q_{h+R}(x) \supset Q_h(x + y)$ and $Q_{h+R}(x + y) \supset Q_h(x)$. The result follows. □

We now define the Beurling dimension of a set, which will be used to characterize the Nyquist dimension of Gabor pseudoframes for an affine subspace.

Definition 2.4 Let $\Lambda \subset \mathbb{R}^d$, and let S be an affine subspace of $\mathbb{R}^d$. The *lower Beurling dimension of $\Lambda \subset \mathbb{R}^d$ with respect to S* is defined by

$$\dim^-_S(\Lambda) = \inf\bigl\{r > 0 : \mathcal{D}^-_{S,r}(\Lambda) < \infty\bigr\},$$

and the *upper Beurling dimension of $\Lambda \subset \mathbb{R}^d$ with respect to S* is

$$\dim^+_S(\Lambda) = \sup\bigl\{r > 0 : \mathcal{D}^+_{S,r}(\Lambda) > 0\bigr\}.$$

When these two quantities are equal, we refer to them as the *Beurling dimension of Λ with respect to S* and we denote them by $\dim_S(\Lambda)$.

It follows immediately from the definition that we always have $\dim^-_S(\Lambda) \le \dim^+_S(\Lambda)$.

The following result presents possible equivalent definitions of Beurling dimensions. Since the proof is rather technical and uses the same arguments as the proof of a similar result in [8], we omit it here.

Proposition 2.5 *Let $\Lambda \subset \mathbb{R}^d$, and let S be an affine subspace of $\mathbb{R}^d$. Then,*

(i) $\dim^-_S(\Lambda) = \sup\{r > 0 : \mathcal{D}^-_{S,r}(\Lambda) > 0\}$,
(ii) $\dim^+_S(\Lambda) = \inf\{r > 0 : \mathcal{D}^+_{S,r}(\Lambda) < \infty\}$.

We note here that, while Beurling dimension is defined above for arbitrary subsets of $\mathbb{R}^d$, the upper Beurling dimension will be infinite unless Λ is discrete. Indeed, if x is an accumulation point of Λ, then, for all $h > 0$, $\#(\Lambda \cap Q_h(x)) = \infty$. Thus, we will restrict our attention to discrete subsets of Λ.

2.2 Properties of Beurling Dimension

In this section we present several properties of the Beurling dimension with respect to $S \subset \mathbb{R}^d$.

2.2.1 Perturbation

We first show that Beurling density is robust against perturbations of the elements in the set under consideration. Given $\epsilon > 0$, we say that $\Delta \subset \mathbb{R}^d$ is an *ϵ-perturbation* of $\Lambda \subset \mathbb{R}^d$ if $\Delta = \{\lambda + \delta_\lambda : \lambda \in \Lambda,\ \delta_\lambda \in [-\epsilon, \epsilon]\}$. As before, Δ should be considered as a sequence rather than a subset of $\mathbb{R}^d$.

Lemma 2.6 *Let $\Lambda \subset \mathbb{R}^d$, S an affine subspace of $\mathbb{R}^d$, and $\epsilon > 0$ be given. For any ϵ-perturbation Δ of Λ and any $r > 0$, we have*

$$\mathcal{D}^-_{S,r}(\Lambda) = \mathcal{D}^-_{S,r}(\Delta) \quad \textit{and} \quad \mathcal{D}^+_{S,r}(\Lambda) = \mathcal{D}^+_{S,r}(\Delta).$$

Proof For $h > 0$ and $x \in S$, we obtain the following estimates for $\#(\Lambda \cap Q_h(x))$:

$$\#\big(\Delta \cap Q_{h-\epsilon}(x)\big) \le \#\big(\Lambda \cap Q_h(x)\big) \le \#\big(\Delta \cap Q_{h+\epsilon}(x)\big).$$

Dividing the terms by h^r for $r > 0$ and observing that

$$\limsup_{h\to\infty} \frac{\sup_{x\in S} \#(\Delta \cap Q_{h-\epsilon}(x))}{h^r} = \mathcal{D}^+_r(\Delta) = \limsup_{h\to\infty} \frac{\sup_{x\in S} \#(\Delta \cap Q_{h+\epsilon}(x))}{h^r}$$

implies $\mathcal{D}^+_{S,r}(\Lambda) = \mathcal{D}^+_{S,r}(\Delta)$.

The claim concerning the lower Beurling density with respect to S and r can be treated similarly. □

Combining Lemma 2.6 and the definition of upper and lower Beurling dimension yields the following perturbation result.

Theorem 2.7 *Let $\Lambda \subset \mathbb{R}^d$, S an affine subspace of $\mathbb{R}^d$, and $\epsilon > 0$ be given. For any ϵ-perturbation Δ of Λ, we have*

$$\dim^-_S(\Lambda) = \dim^-_S(\Delta) \quad \textit{and} \quad \dim^+_S(\Lambda) = \dim^+_S(\Delta).$$

2.2.2 Geometric Properties

Our next result shows that the upper, lower, and uniform Beurling dimensions satisfy properties which are typically associated with dimensions.

Proposition 2.8 *Let $\Lambda_1, \Lambda_2, \Lambda \subset \mathbb{R}^d$, and S an affine subspace of $\mathbb{R}^d$ be given. Then the following conditions hold*:

(i) Monotonicity: *If $\Lambda_1 \subseteq \Lambda_2$, then*

$$\dim^-_S(\Lambda_1) \le \dim^-_S(\Lambda_2) \quad \textit{and} \quad \dim^+_S(\Lambda_1) \le \dim^+_S(\Lambda_2).$$

(ii) Stability: *We have*

$$\dim_S^-(\Lambda_1 \cap \Lambda_2) \le \min\bigl(\dim_S^-(\Lambda_1), \dim_S^-(\Lambda_2)\bigr)$$

and

$$\dim_S^+(\Lambda_1 \cup \Lambda_2) = \max\bigl(\dim_S^+(\Lambda_1), \dim_S^+(\Lambda_2)\bigr).$$

(iii) Geometric invariance: *Let* $f : \mathbb{R}^d \to \mathbb{R}^d$ *be a uniform homeomorphism such that* $f(S) = S + y$ *for some* $y \in \mathbb{R}^d$. *Then*

$$\dim_S^-\bigl(f(\Lambda)\bigr) = \dim_S^-(\Lambda) \quad \text{and} \quad \dim_S^+\bigl(f(\Lambda)\bigr) = \dim_S^+(\Lambda).$$

Proof We will only prove the claims for the upper Beurling dimension. The lower Beurling dimension can be treated similarly.

(i) If $\Lambda_1 \subseteq \Lambda_2$, it follows that $\mathcal{D}_{S,r}^+(\Lambda_1) \le \mathcal{D}_{S,r}^+(\Lambda_2)$ for all $r > 0$. Using the definition of Beurling dimension proves the claim.

(ii) For all $r > 0$, we have $\mathcal{D}_{S,r}^+(\Lambda_1 \cup \Lambda_2) \ge \max(\mathcal{D}_{S,r}^+(\Lambda_1), \mathcal{D}_{S,r}^+(\Lambda_2))$. Thus

$$\dim_S^+(\Lambda_1 \cup \Lambda_2) \ge \max(\dim_S^+(\Lambda_1), \dim_S^+(\Lambda_2)). \tag{1}$$

Now fix $r > 0$. Since

$$\sup_{x\in S} \#\bigl((\Lambda_1 \cup \Lambda_2) \cap Q_h(x)\bigr) \le \sup_{x\in S} \#\bigl(\Lambda_1 \cap Q_h(x)\bigr) + \sup_{x\in S} \#\bigl(\Lambda_2 \cap Q_h(x)\bigr),$$

it follows that $\mathcal{D}_{S,r}^+(\Lambda_1 \cup \Lambda_2) \le \mathcal{D}_{S,r}^+(\Lambda_1) + \mathcal{D}_{S,r}^+(\Lambda_2)$. This implies $\mathcal{D}_{S,r}^+(\Lambda_1 \cup \Lambda_2) \le 2\max(\mathcal{D}_{S,r}^+(\Lambda_1), \mathcal{D}_{S,r}^+(\Lambda_2))$. Due to the definition of $\dim_S^+$, this yields

$$\dim_S^+(\Lambda_1 \cup \Lambda_2) \le \max\bigl(\dim_S^+(\Lambda_1), \dim_S^+(\Lambda_2)\bigr). \tag{2}$$

Equations (1) and (2) settle the claim.

(iii) Note that, if f is a uniformly continuous surjection of one normed linear space onto another, then it is Lipschitz for large distances, see, for example, [2, Lemma 5.1]. That is, for all $h_0 > 0$, there exists $C > 0$ such that $f(Q_h(x)) \subset Q_{Ch}(f(x))$ for all $x \in \mathbb{R}^d$ and $h \ge h_0$. In particular, we conclude that, for each $r > 0$,

$$\begin{aligned}\limsup_{h\to\infty} \sup_{x\in S} \frac{\#(f^{-1}(\Lambda) \cap Q_h(x))}{h^r} &\le \limsup_{h\to\infty} \sup_{x\in S} \frac{\#(\Lambda \cap Q_{Ch}(f(x)))}{h^r} \\ &\le C^r \limsup_{h\to\infty} \sup_{x\in S+y} \frac{\#(\Lambda \cap Q_h(x))}{h^r}.\end{aligned}$$

It follows from Proposition 2.3 that $\dim_S^+(f^{-1}(\Lambda)) \le \dim_S^+(\Lambda)$. A similar argument shows that $\dim_S^+(f(f^{-1}(\Lambda))) \le \dim_S^+(f^{-1}(\Lambda))$, which yields the result. □

2.2.3 Range of Values

In this section, we will restrict attention to the case $S = \mathbb{R}^d$. We present the possible range of values of Beurling dimension, making use of Proposition 2.9 below. This result provides a variety of different interpretations of finite upper and positive lower Beurling density with respect to $r = d$. Recall that $\Lambda \subset \mathbb{R}^d$ is *uniformly separated* if $\inf_{\lambda_1,\lambda_2\in\Lambda,\lambda_1\neq\lambda_2} |\lambda_1 - \lambda_2| > 0$. It is *relatively uniformly separated* if it is a finite union of uniformly separated sets. Further, Λ is *h-dense* if $\bigcup_{x\in\Lambda} Q_h(x) = \mathbb{R}^d$.

Proposition 2.9 *Let S, V be affine subspaces of $\mathbb{R}^d$, and suppose that $\Lambda \subset V$ and that there exists z_0 such that $V + z_0 \subset S$. Let n be the affine dimension of V.*

(i) *The following conditions are equivalent*:
 (a) $\mathcal{D}^+_{S,n}(\Lambda) < \infty$.
 (b) $\mathcal{D}^+_{S,\dim(S)}(\Lambda) < \infty$.
 (c) *There exists some $h > 0$ such that* $\sup_{x\in S} \#(\Lambda \cap Q_h(x)) < \infty$.
 (d) *For all $h > 0$,* $\sup_{x\in S} \#(\Lambda \cap Q_h(x)) < \infty$.
 (e) *Λ is relatively uniformly separated.*
 (f) *For all $h > 0$,* $\sup_{x\in S} \#\{\lambda \in \Lambda : x \in Q_h(\lambda)\} < \infty$.

(ii) *Also the following conditions are equivalent*:
 (a) $\mathcal{D}^-_{S,n}(\Lambda) > 0$.
 (b) $\mathcal{D}^-_{S,\dim(S)}(\Lambda) > 0$.
 (c) *There exists some $h > 0$ such that* $\inf_{x\in S} \#(\Lambda \cap Q_h(x)) > 0$.
 (d) *Λ contains a subsequence of positive uniform density.*
 (e) *There exists some $h > 0$ such that Λ is h-dense.*

Proof Note that, under the hypotheses of Proposition 2.9, Propositions 2.2 and 2.3 and the proof of Proposition 2.8 (iii) imply that, without loss of generality, we may restrict to the case that $S = \mathbb{R}^d$ and $\Lambda \subset V = \mathbb{R}^n \times \{0, \dots, 0\} \subset \mathbb{R}^d$.

(i): (b), (c), (d), and (e) are equivalent by [6, Lemma 2.3]. To prove (d) $\Leftrightarrow$ (f) observe that

$$\#\big(\Lambda \cap Q_h(x)\big) = \#\{\lambda \in \Lambda : -x \in Q_h - \lambda\} = \#\{\lambda \in \Lambda : x \in Q_h(\lambda)\}.$$

This immediately settles the claim.

The fact that (a) implies (b) is immediate. To show that (b) implies (a), we proceed *a contrario* and assume that $\mathcal{D}^+_{\mathbb{R}^d,n}(\Lambda) = \infty$. First, we note that, as a consequence of the assumption $\Lambda \subset V$, we may write $\Lambda = \Lambda' \times \{0, \dots, 0\} \subset \mathbb{R}^n \times \{0, \dots, 0\} = \mathbb{R}^d$. Hence, $\mathcal{D}^+_{\mathbb{R}^d,n}(\Lambda) = \mathcal{D}^+_{\mathbb{R}^n,n}(\Lambda')$ and $\mathcal{D}^+_{\mathbb{R}^d,d}(\Lambda) = \mathcal{D}^+_{\mathbb{R}^n,d}(\Lambda')$. Next, we again invoke [6, Lemma 2.3] to conclude that $\sup_{x\in\mathbb{R}^n} \#(\Lambda' \cap Q_h(x)) = \infty$, which, in turn, implies that $\mathcal{D}^+_{\mathbb{R}^n,d}(\Lambda') = \infty$.

(ii): Clearly, (b) implies (a). On the other hand, since $\Lambda \subset \mathbb{R}^n \times \{0, \dots, 0\}$, $\mathcal{D}^-_{\mathbb{R}^d,n}(\Lambda) > 0$ only if $n = d$.

The implication (b) $\Rightarrow$ (c) is immediate.

To show (c) $\Rightarrow$ (d), let $h, \delta > 0$ be such that $\#(\Lambda \cap Q_h(x)) > \delta$ for all $x \in \mathbb{R}^d$. Hence, in particular, each set $\Lambda \cap Q_h(x)$ contains at least one element. Thus, for each $k \in \mathbb{Z}^d$, there exists some $y_k \in \Lambda \cap Q_h(2hk)$. Since $(y_k)_{k\in\mathbb{Z}^d}$ is a $2h$-perturbation of $(2h\mathbb{Z})^d$ and $\mathcal{D}_{\mathbb{R}^d,d}((2h\mathbb{Z})^d) = (2h)^{-d}$, Lemma 2.6 implies that $(y_k)_{k\in\mathbb{Z}^d}$ has uniform density equal to $(2h)^{-d}$, which proves the claim.

Next suppose that Λ contains a subsequence Δ of positive uniform density. Since $\mathcal{D}^-_{\mathbb{R}^d,d}(\Lambda) > \mathcal{D}_{\mathbb{R}^d,d}(\Delta) > 0$, (d) implies (b).

The equivalence of (c) and (e) is immediate. □

Theorem 2.10 *Let S, V be affine subspaces of $\mathbb{R}^d$, and suppose that $\Lambda \subset V$ and that there exists z_0 such that $V + z_0 \subset S$. Denote the affine dimension of V by n. Then,*

(i) $\dim^+_S(\Lambda) \in [0, n] \cup \{\infty\}$ *and*
(ii) $\dim^-_S(\Lambda) \in \{0\} \cup [n, \infty]$.

Proof As before, we may assume without loss of generality that $S = \mathbb{R}^d$. Assume that $\dim^+_{\mathbb{R}^d}(\Lambda) =: s > n$ and $s < \infty$. By definition, this implies $\mathcal{D}^+_{\mathbb{R}^d,n}(\Lambda) = \infty$. By Proposition 2.9(i), $\sup_{x\in\mathbb{R}^d} \#(\Lambda \cap Q_h(x)) = \infty$ for all $h > 0$. Hence $\mathcal{D}^+_{\mathbb{R}^d,s+1}(\Lambda) = \infty$, a contradiction to $s = \sup\{r > 0 : \mathcal{D}^+_{\mathbb{R}^d,r}(\Lambda) > 0\}$ and $s < \infty$. This proves (i).

To show (ii) assume that $\dim^-_{\mathbb{R}^d}(\Lambda) =: s \in (0, n)$. This implies $\mathcal{D}^-_{\mathbb{R}^d,n}(\Lambda) = 0$. By Proposition 2.9(ii), $\inf_{x\in\mathbb{R}^d} \#(\Lambda \cap Q_h(x)) = 0$ for all $h > 0$. Hence also $\mathcal{D}^-_{\mathbb{R}^d,\frac{s}{2}}(\Lambda) = 0$, which contradicts to $s = \inf\{r > 0 : \mathcal{D}^-_{\mathbb{R}^d,r}(\Lambda) < \infty\}$ and $s > 0$. □

Theorem 2.10 implies that the upper Beurling dimension with respect to $\mathbb{R}^d$ serves as an extension of the Lebesgue dimension to discrete subsets by assigning a value between 0 and the Lebesgue dimension of the Euclidean space or infinity to each such subset. It further indicates that the lower Beurling dimension provides a subdivision for those sets which have infinite upper Beurling dimension. Moreover, for subsets with uniform Beurling dimension, i.e., for which the lower Beurling dimension coincides with the upper Beurling dimension, it follows that this dimension is either 0, d, or ∞. There are, however, examples of Λ, S, and V not satisfying the conditions of Theorem 2.10 such that the Beurling dimension with respect to S is any nonnegative number, as can be seen by considering the sequence $\Lambda = \{\pm n^\alpha : n \in \mathbb{N}\}$ with $S = \{0\}$ and $V = \mathbb{R}$.

We close this section by showing that Beurling dimension inherits the "intuitive" dimension of a discrete set.

Theorem 2.11 *Let $\Lambda \subset \mathbb{R}^d$ with $\mathcal{D}^+_d(\Lambda) < \infty$, and let $0 \le n < d$. Suppose that Λ is contained in a translated bounded neighborhood of an n-dimensional subspace V of $\mathbb{R}^d$, i.e., $\Lambda \subset y_0 + \{x \in \mathbb{R}^d : \mathrm{dist}(x, V) \le \epsilon\}$ for $y_0 \in \mathbb{R}^d$ and $\epsilon > 0$. Then, for any affine subspace S such that there exists $z_0 \in \mathbb{R}^d$ satisfying $S + z_0 \supset V$, we have*

$$\dim^+_S(\Lambda) \le n.$$

Proof Let P be the orthogonal projection onto V. Note that $P(\Lambda)$ is an ϵ-perturbation of Λ, so by Theorem 2.7, $\dim_S^+(P(\Lambda)) = \dim_S^+(\Lambda)$. Moreover, since $S + z_0 \supset V$, Theorem 2.10 and Proposition 2.3 imply that $\dim_S^+(P(\Lambda)) \in [0, n] \cup \{\infty\}$, since $P(\Lambda)$ is contained in an n-dimensional subspace. Therefore, since $\dim_S^+(\Lambda) < \infty$, $\dim_S^+(\Lambda) = \dim_S^+(P(\Lambda)) \leq n$. $\square$

2.3 Comparison with Other Dimensions

In this section, we compare Beurling dimensions with other dimensions; namely, the *mass dimensions*, $\dim_{LM}$ and $\dim_{UM}$, and the *discrete Hausdorff dimension*, $\dim_H$. The first can be defined as

$$\dim_{LM}(\Lambda) = \liminf_{n\to\infty} \frac{\ln \#(\Lambda \cap Q_n)}{\ln n}$$

and

$$\dim_{UM}(\Lambda) = \limsup_{n\to\infty} \frac{\ln \#(\Lambda \cap Q_n)}{\ln n},$$

see, e.g., [1]. (We shall see in this section that mass dimensions are special cases of Beurling dimensions.) For the precise definition of the *discrete Hausdorff dimension*, we refer to [1]. For other notions of discrete dimensions and for a discussion of relations between them, we refer to [19] and [20].

First, we present yet another version of the definition of the Beurling dimension, which will facilitate comparisons to the mass dimensions.

Proposition 2.12 *Let $\Lambda \subset \mathbb{R}^d$ and S be an affine subspace of $\mathbb{R}^d$. Then we have*

$$\dim_S^-(\Lambda) = \liminf_{h\to\infty} \inf_{x\in S} \frac{\ln \#(\Lambda \cap Q_h(x))}{\ln h}$$

and

$$\dim_S^+(\Lambda) = \limsup_{h\to\infty} \sup_{x\in S} \frac{\ln \#(\Lambda \cap Q_h(x))}{\ln h}.$$

Proof We only study the upper dimension. The proof for the lower dimension is similar.

First, consider $0 < r < \dim^+(\Lambda)$. In this case, we have $\mathcal{D}_{S,r}^+(\Lambda) = \infty$. Therefore there exists a sequence $(h_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} h_n = \infty$ such that

$$\lim_{n\to\infty} \frac{\sup_{x\in S} \#(\Lambda \cap Q_{h_n}(x))}{h_n^r} = \infty.$$

Without loss of generality we can assume that, for all $n \in \mathbb{N}$,

$$\frac{\sup_{x\in S} \#(\Lambda \cap Q_{h_n}(x))}{h_n^r} > 1.$$

This yields

$$\limsup_{h\to\infty}\sup_{x\in S}\frac{\ln\#(\Lambda\cap Q_h(x))}{\ln h}\geq\limsup_{n\to\infty}\sup_{x\in S}\frac{\ln\#(\Lambda\cap Q_{h_n}(x))}{\ln h_n}$$
$$>\limsup_{n\to\infty}\frac{\ln(h_n^r)}{\ln h_n}=r.$$

Since this holds for every $r<\dim_S^+(\Lambda)$, we obtain

$$\limsup_{h\to\infty}\sup_{x\in S}\frac{\ln\#(\Lambda\cap Q_h(x))}{\ln h}\geq\dim_S^+(\Lambda). \tag{3}$$

Secondly, let $r>\dim_S^+(\Lambda)$, which implies $\mathcal{D}_{S,r}^+(\Lambda)=0$. Fix $\epsilon>0$. Then there exists $H>0$ such that for all $h>H$:

$$\frac{\sup_{x\in S}\#(\Lambda\cap Q_h(x))}{h^r}\leq\epsilon.$$

Therefore we obtain

$$\limsup_{h\to\infty}\sup_{x\in S}\frac{\ln\#(\Lambda\cap Q_h(x))}{\ln h}\leq\limsup_{h\to\infty}\frac{\ln(\epsilon h^r)}{\ln h}=\limsup_{h\to\infty}\frac{\ln\epsilon+r\ln h}{\ln h}=r.$$

Thus

$$\limsup_{h\to\infty}\sup_{x\in S}\frac{\ln\#(\Lambda\cap Q_h(x))}{\ln h}\leq\dim_S^+(\Lambda). \tag{4}$$

Now (3) and (4) yield the claim. □

Proposition 2.12 implies that mass dimension is Beurling dimension with respect to $S=\{0\}$. We state it formally in the following corollary, where $\dim_{LM}$ and $\dim_{UM}$ denote the lower and upper mass dimensions, respectively.

Corollary 2.13 *For every $\Lambda\subset\mathbb{R}^d$ and S an affine subspace of $\mathbb{R}^d$,*

$$\dim_S^-(\Lambda)\leq\dim_{LM}(\Lambda)=\dim_{\{0\}}^-(\Lambda)\leq\dim_{UM}(\Lambda)=\dim_{\{0\}}^+(\Lambda)\leq\dim_S^+(\Lambda).$$

Proof This follows immediately from Proposition 2.12 and the definition of mass dimension. □

We present now several examples which illustrate the differences between Beurling dimension with respect to $S=\mathbb{R}^d$, mass dimensions, and (to a lesser extent) discrete Hausdorff dimension.

Example 2.14 Let $\Lambda=\{(m,n):m\in\mathbb{N},n\in\mathbb{Z}\}\subset\mathbb{R}^2$. Then, $\dim_{\mathbb{R}^d}^-(\Lambda)=0$ and $\dim_{\mathbb{R}^d}^+(\Lambda)=2$, whereas we have $\dim_{LM}(\Lambda)=\dim_{UM}(\Lambda)=2$.

Example 2.15 Define the set $\Lambda \subset \mathbb{Z}^2$ to be the union of sets $\Lambda_n, n \in \mathbb{N}$, where $\Lambda_n = \{(k,l) : 2^n \le k < 2^n + 2^{n-1}, 0 \le l < 2^{n-1}\}$. Then, it is not difficult to observe that $\dim^-_{\mathbb{R}^d}(\Lambda) = 0$ and $\dim^+_{\mathbb{R}^d}(\Lambda) = 2$, since the set Λ contains arbitrarily large pieces of the lattice $\mathbb{Z}^2$. On the other hand, the mass dimension of Λ exists and is equal to 1, and the discrete Hausdorff dimension of Λ is equal to 0.

Example 2.16 Define the set $\Lambda \subset \mathbb{R}^2$ as follows: for each $n \in \mathbb{N}$, define $\Lambda_n = \{(2^n - k/2^n, l/2^n) : 1 \le k, l \le 2^n\}$; let $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$. It follows immediately from the definition that the upper Beurling dimension of Λ with respect to $S = \mathbb{R}^d$ is infinity, because of the increasing concentration of points. On the other hand, it is not difficult to observe that the upper mass dimension of Λ satisfies $\dim_{UM}(\Lambda) = 2$. Moreover, the discrete Hausdorff dimension of any set is bounded from above by the upper mass dimension, and so $\dim_H(\Lambda) \le 2$.

3 Application to Pseudoframes for Subspaces

We start by stating the definition of a pseudoframe for subspaces (see [17]). In the following, given a closed subspace E of a separable Hilbert space, we always denote the orthogonal projection onto this subspace by P_E.

Definition 3.1 Let E be a closed subspace of a separable Hilbert space $\mathcal{H}$, and let $\{x_i\}_{i\in I}$ be a sequence in $\mathcal{H}$. Then $\{x_i\}_{i\in I}$ is a *Bessel sequence with respect to E* if

$$\sum_{i\in I} |\langle f, x_i\rangle|^2 < \infty \quad \text{for all } f \in E.$$

A Bessel sequence $\{x_i\}_{i\in I}$ w.r.t. E is called a *pseudoframe for the subspace E (PFFS for E)* if there exists another Bessel sequence $\{x_i^*\}_{i\in I}$ in $\mathcal{H}$ such that

$$f = \sum_{i\in I} \langle f, x_i\rangle x_i^* \quad \text{for all } f \in E.$$

The collection $\{x_i^*\}_{i\in I}$ is called a *dual pseudoframe of $\{x_i\}_{i\in I}$ for the subspace E*.

A Bessel sequence $\{x_i\}_{i\in I}$ w.r.t. E is called a *pseudo-Riesz sequence for the subspace E (PRFS for E)* if $\{P_E x_i\}_{i\in I}$ is a Riesz sequence in E.

In the sequel we will make use of the following characterization of PFFS's.

Theorem 3.2 [17, Theorem 4] *Let E be a closed subspace of a separable Hilbert space $\mathcal{H}$, let $\{x_i\}_{i\in I} \subset \mathcal{H}$ be a Bessel sequence w.r.t. E, and let $\{x_i^*\}_{i\in I}$ be a Bessel sequence in $\mathcal{H}$. Then the following conditions are equivalent*:

(i) *$\{x_i\}_{i\in I}$ is a PFFS for E with dual pseudoframe $\{x_i^*\}_{i\in I}$*;
(ii) *The following conditions hold*:
   (a) *$\{P_E x_i\}_{i\in I}$ is a frame for E with dual frame $\{P_E x_i^*\}_{i\in I}$*,
   (b) *For all $f \in H$, $\sum_{i\in I} \langle f, P_E x_i\rangle (I - P_E) x_i^* = 0$.*

Note that in the case where $\{P_E x_i\}_{i\in I}$ is a frame for the subspace E, one can always find $\{x_i^*\}_{i\in I} \subset E$ satisfying the conditions for PFFS. Thus, for our purposes, we will not refer to the dual frame, and we write briefly that $\{x_i\}_{i\in I}$ is a PFFS for E. Furthermore, we say that the *frame bounds of a PFFS for E* are the frame bounds of the frame $\{P_E x_i : i \in I\}$ for E.

In this section, we consider PFFS's with more structure than the general case. In the following, let $G = \{g_1, \dots, g_K\}$ be some finite collection of functions in $L^2(\mathbb{R}^d)$, let $\Lambda = \{\Lambda_1, \dots, \Lambda_K\}$ be a finite collection of subsets of $\mathbb{R}^{2d}$, and let the associated *Gabor system* be defined by

$$\begin{aligned}\mathcal{G}(G,\Lambda) &= \left\{e^{2\pi i x\cdot t} g_k(t-y) : (x,y) \in \Lambda_k,\ 1 \le k \le K\right\}\\ &= \{M_x T_y g_k : (x,y) \in \Lambda_k,\ 1 \le k \le K\},\end{aligned}$$

where M_x and T_y are the modulation and translation operators, respectively. The discrete set Λ will be referred to as the *set of parameters*. We will consider those Gabor systems which are PFFS's for $L^2(E)$ for some $E \subset \mathbb{R}^d$. When E is, for example, a bounded set, one would expect that the modulations and translations together would need to be "sufficiently dense" in order to form a PFFS for $L^2(E)$. One would also expect that the collection of modulations and translations needs to be "sufficiently sparse" in order to form a PRFS. We make these intuitive notions precise in Theorem 3.3 below.

The sets E we will consider will be of the following form. For $E \subset \mathbb{R}^d$ and $0 \le m \le d$, we say *E contains a tube around an m-dimensional space* if there exists an affine subspace A of $\mathbb{R}^d$ of dimension m and an $\epsilon > 0$ such that $\{z \in \mathbb{R}^d : \operatorname{dist}(z, A) < \epsilon\} \subset E$.

With this notation, we are ready to state our main theorem, whose proof will be given at the end of this section.

Theorem 3.3 *Let $\mathcal{G} = \{g_1, \dots, g_K\} \subset L^2(\mathbb{R}^d) \setminus \{0\}$, and let $\Lambda = \{\Lambda_1, \dots, \Lambda_K\}$ be a finite collection of subsets of $\mathbb{R}^{2d}$. Furthermore, let E be a subset of $\mathbb{R}^d$ which contains a tube around an m-dimensional affine subspace A of $\mathbb{R}^d$.*

(i) *If $\mathcal{G}(G,\Lambda)$ is a PFFS for $L^2(E)$, then either $\dim^-_{\mathbb{R}^d\times A}(\Lambda) \ge d+m$ or Λ is not relatively uniformly separated.*
(ii) *If $\mathcal{G}(G,\Lambda)$ is a PRFS for $L^2(E)$, then $\dim^-_{\mathbb{R}^d\times A}(\Lambda) = \dim^+_{\mathbb{R}^d\times A}(\Lambda) = d+m$.*

Let us briefly compare Theorem 3.3 with Nyquist density results of Gabor systems. For Gabor systems, we have the following result from [6].

Theorem 3.4 [6, Theorem 1.1 and 3.6] *Let $\mathcal{G} = \{g_1, \dots, g_K\} \subset L^2(\mathbb{R}^d) \setminus \{0\}$, and let $\Lambda \subset \mathbb{R}^{2d}$.*

(i) *If $\mathcal{G}(G,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$, then $\mathcal{D}^-_{\mathbb{R}^{2d},d}(\Lambda) \ge 1$.*
(ii) *If $\mathcal{G}(G,\Lambda)$ is a Riesz basis for $L^2(\mathbb{R}^d)$, then $\mathcal{D}^-_{\mathbb{R}^{2d},d}(\Lambda) = \mathcal{D}^+_{\mathbb{R}^{2d},d}(\Lambda) = 1$.*

It can be easily seen that in the situation of Gabor PFFS's Beurling dimension plays the role which Beurling density plays for Gabor frames. It is in this sense that it is justified to regard the Beurling dimension as a *Nyquist dimension*.

Note also that, in particular, Theorem 3.3 implies that if a collection of modulations is a frame for $L^2([0,1])$, then the modulations must have positive upper Beurling density, recovering a result of Christensen, Deng, and Heil [6].

In general, for a PFFS $\mathcal{G}(G,\Lambda)$, if Λ is not relatively separated (in particular, $\mathcal{G}(G,\Lambda)$ is not Bessel in $L^2(\mathbb{R}^d)$), then the Λ can be quite odd. For example, in the case $E\subset\mathbb{R}$, there is no restriction on those $(x,y)\in\Lambda$ for which the support of $T_x g$ is disjoint from E for PFFS's, so the lower dimension can be made as large as desired. It is also possible to construct examples of PFFS's for $L^2(E)$ for which $\dim^-_{\mathbb{R}^d\times A}(\Lambda)$ is less than $d+m$, see Example 3.13. It is thus more surprising that in the case of PRFS's, we are able to obtain a Nyquist dimension as in Theorem 3.3(ii).

Our method of proof is inspired by techniques to prove density results given in the recent preprint [11]. We will show that PFFS's for $L^2(E)$ of the type mentioned in Theorem 3.3 satisfy a modified version of the Homogeneous Approximation Property (HAP) of Ramanathan and Steger. Using this modified HAP, we will then show that the set of parameters Λ has lower dimension greater than or equal to $d+m$. We will also use the results from Sect. 2 to show that the upper dimension of a PRFS is bounded above by $d+m$.

3.1 Preliminary Lemmas

We begin by recalling the following definition and lemma from [11].

Definition 3.5 (i) Given a set $U\subset\mathbb{R}^{2d}$, for each $t\geq 0$, define

$$U_t=\left\{x\in\mathbb{R}^{2d}:\mathrm{dist}(x,U)<t\right\}.$$

(ii) The *Fréchet distance* between two closed sets $U,V\subset\mathbb{R}^{2d}$ is

$$[U,V]=\inf\{t\geq 0: U\subset V_t \quad \text{and } V\subset U_t\}.$$

(iii) Given closed sets $U_n\subset\mathbb{R}^{2d}$ and given a closed set $V\subset\mathbb{R}^{2d}$, we say that U_n *converges weakly* to V if

$$\lim_{n\to\infty}[U_n\cap K, V\cap K]=0 \quad \text{for all compact } K\subset\mathbb{R}^{2d}.$$

In this case, we write $U_n\xrightarrow{w} V$.

Lemma 3.6 [11, Lemma 2.10] *Let $\Lambda\subset\mathbb{R}^{2d}$ be a countable sequence which is δ-uniformly separated for some $\delta>0$. Then given any sequence $\{z_n\}_{n\in\mathbb{N}}$ of points in $\mathbb{R}^{2d}$, there exists a subsequence $\{w_n\}_{n\in\mathbb{N}}$ of $\{z_n\}_{n\in\mathbb{N}}$ and a sequence $\Lambda'\subset\mathbb{R}^{2d}$ such that*

$$\Lambda-w_n\xrightarrow{w}\Lambda' \ \ as \ \ j\to\infty, \quad k=1,\dots,N.$$

The following two lemmas will be heavily employed in the proof of Theorem 3.10.

Lemma 3.7 *Let $a\leq d$ be a nonnegative integer, $G=\{g_1,\dots,g_K\}\subset L^2(\mathbb{R}^d)\setminus\{0\}$, $\Lambda=\{\Lambda_1,\dots,\Lambda_K\}$ be a finite collection of subsets of $\mathbb{R}^{2d}$, and $E\subset\mathbb{R}^d$ be such that the following conditions hold*:

(a) *Each Λ_k is 2δ-uniformly separated for some $\delta > 0$;*
(b) *E is a tube around the subspace $\{x \in \mathbb{R}^d : x_{d-a+1} = \cdots = x_d = 0\}$ for $1 \le a \le d$, or E is a convex neighborhood of the origin if $a = 0$;*
(c) *$\mathcal{G}(G, \Lambda)$ is a PFFS for $L^2(E)$ with frame bounds A and B.*

Then, for each $x_0 \in \mathbb{R}^d$, $y_0 \in \mathbb{R}^a$, and $0 \in \mathbb{R}^{d-a}$, $\{M_x T_y g_k : (x, y) \in \Lambda_k + (x_0, y_0, 0) : 1 \le k \le K\}$ is a PFFS for $L^2(E)$ with frame bounds A and B.

Proof Let $h \in L^2(E)$, and define $\tilde{h}(t) = e^{-2\pi i x_0 t} h(t + (y_0, 0))$. Note that $\|h\|_2 = \|\tilde{h}\|_2$, $P_{L^2(E)} h = h$, and $P_{L^2(E)} \tilde{h} = \tilde{h}$. The result follows from the following computation:

$$\begin{aligned}\sum_{k=1}^{K} \sum_{(x,y)\in\Lambda_k+(x_0,y_0,0)} |\langle P_{L^2(E)} M_x T_y g_k, h\rangle|^2 &= \sum_{k=1}^{K} \sum_{(x,y)\in\Lambda_k+(x_0,y_0,0)} |\langle M_x T_y g_k, h\rangle|^2 \\ &= \sum_{k=1}^{K} \sum_{(x,y)\in\Lambda_k} \left|\left\langle M_x T_y g_k, \tilde{h}\right\rangle\right|^2 \\ &= \sum_{k=1}^{K} \sum_{(x,y)\in\Lambda_k} \left|\left\langle P_{L^2(E)} M_x T_y g_k, \tilde{h}\right\rangle\right|^2. \qquad \square\end{aligned}$$

Lemma 3.8 *Let a, E, G, and $\Lambda = \{\Lambda_1, \ldots, \Lambda_K\}$ be as in Lemma 3.7. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$, $\{y_n\}_{n\in\mathbb{N}}$ a sequence in $\mathbb{R}^a$, and, for each n, let $z_n = (x_n, y_n, 0) \in \mathbb{R}^{2d}$. Suppose that, for each $1 \le k \le K$, $\Lambda_k - z_n \xrightarrow{w} \Lambda'_k$. Then, $\mathcal{G}(G, \Lambda')$ is a PFFS for $L^2(E)$ with the same frame bounds, where $\Lambda' = \{\Lambda'_1, \ldots, \Lambda'_K\}$.*

Proof Note that, for each $1 \le k \le K$, a cube of the form $Q_\delta(z)$ can contain at most one point in Λ_k. Using the weak convergence of $\Lambda_k - z_n \xrightarrow{w} \Lambda'_k$, it can also be shown that each cube $Q_\delta(z)$ can contain at most one point of Λ'_k.

Choose $\epsilon > 0$ and let $f \in M^1(\mathbb{R}^d) \cap L^2(E)$. Then, for each $1 \le k \le K$, the short-time Fourier transform $V_{g_k} f$ defined by $V_{g_k} f(x, y) = \langle f, M_x T_y g_k\rangle$, $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, is an element of the amalgam space $W(L^\infty, \ell^2)$. By using an equivalent norm of this space, we can find $m \in \mathbb{N}$ such that, for each $1 \le k \le K$,

$$\sum_{j\in\mathbb{Z}^{2d}\setminus Q_m(0)} \|V_{g_k} f \cdot 1_{Q_\delta(\delta j)}\|_\infty^2 < \frac{\epsilon}{2}.$$

Fix $1 \le k \le K$. Set $R = (2m + 1)\delta$. If $(x, y) \in \mathbb{R}^{2d} \setminus Q_R(0)$, then there is a unique $j \in \mathbb{Z}^{2d} \setminus Q_m(0)$ such that $(x, y) \in Q_\delta(\delta j)$. Hence,

$$\begin{aligned}\sum_{(x,y)\in\Lambda'_k\setminus Q_R(0)} |\langle f, M_x T_y g_k\rangle|^2 &= \sum_{(x,y)\in\Lambda'_k\setminus Q_R(0)} |V_{g_k} f(x, y)|^2 \\ &\le \sum_{j\in\mathbb{Z}^{2d}\setminus Q_m(0)} \sup_{(u,v)\in Q_\delta(\delta j)} |V_{g_k} f(u, v)|^2 < \frac{\epsilon}{2}.\end{aligned}$$

Similarly, for each $n \in \mathbb{N}$, we have

$$\sum_{(x,y)\in(\Lambda_k - z_n)\setminus Q_R(0)} |\langle f, M_x T_y g_k\rangle|^2 < \frac{\epsilon}{2}.$$

Let $D = \sup_{n\in\mathbb{N}} \#((\Lambda_k - z_n) \cap Q_R(0))$. We estimate the difference in the ℓ_2 norm of the inner products of f with $M_x T_y g_k$ in the cases that $(x, y) \in \Lambda'_k \cap Q_R(0)$ and $(x, y) \in \Lambda_k - z_n \cap Q_R(0)$. Clearly, when $D = 0$, the difference is 0, so we consider the case $D > 0$. Using the continuity of the modulation and translation operators and the fact that $Q_R(0)$ is compact, find $\theta > 0$ such that, for all $(x, y) \in Q_R(0)$ and all $|u|, |v| < \theta$, $\|M_u T_v M_x T_y f - M_x T_y f\|_2 < \epsilon(2(ND)^{1/2}\|f\|_2)^{-1}$. Let n be large enough so that

$$\big[(\Lambda_k - z_n) \cap Q_R(0), \Lambda'_k \cap Q_R(0)\big] < \theta.$$

Then, for each $(x, y) \in (\Lambda_k - z_n) \cap Q_R(0)$, there exist unique points $|u(x, y)|$, $|v(x, y)| < \theta$ such that $(x + u(x, y), y + v(x, y)) \in \Lambda'_k$. Moreover, this correspondence yields a bijection between $(\Lambda_k - z_n) \cap Q_R(0)$ and $\Lambda_k \cap Q_R(0)$. For notational ease, we write $\Lambda_{n,R} = (\Lambda_k - z_n) \cap Q_R(0)$, where the dependence on k is suppressed since k is fixed. We compute

$$\begin{aligned}
&\left|\left(\sum_{(x,y)\in\Lambda'_k\cap Q_R(0)} |\langle f, M_x T_y g_k\rangle|^2\right)^{1/2} - \left(\sum_{(x,y)\in\Lambda_{n,R}} |\langle f, M_x T_y g_k\rangle|^2\right)^{1/2}\right| \\
&\quad\le \left(\sum_{(x,y)\in\Lambda_{n,R}} |\langle f, M_{x+u(x,y)} T_{y+v(x,y)} g_k - M_x T_y g_k\rangle|^2\right)^{1/2} \\
&\quad\le \left(\sum_{(x,y)\in\Lambda_{n,R}} \|f\|^2 \|M_{x+u(x,y)} T_{y+v(x,y)} g_k - M_x T_y g_k\|_2^2\right)^{1/2} < \frac{\epsilon}{2}.
\end{aligned}$$

Now let A and B denote the frame bounds of $\mathcal{G}(G, \Lambda)$, and recall that $z_n = (x_n, y_n, 0)$, where $y_n \in \mathbb{R}^a$ and $0 \in \mathbb{R}^{d-a}$. Employing (b), for each $f \in L^2(E)$, we have $M_{x_n} T_{(y_n,0)} f \in L^2(E)$. Therefore, $\langle M_{x_n} T_{(y_n,0)} f, P_{L^2(E)} g_k\rangle = \langle M_{x_n} T_{(y_n,0)}, g_k\rangle$. In particular, $\|P_{L^2(E)} M_{x_n} T_{(y_n,0)} f\|_2 = \|f\|_2$. With this observation, we allow k to vary again and we compute

$$\begin{aligned}
&\left(\sum_{k=1}^{K} \sum_{(x,y)\in\Lambda'_k} |\langle f, M_x T_y g_k\rangle|^2\right)^{1/2} \\
&\quad\le \left(\sum_{k=1}^{K} \sum_{(x,y)\in\Lambda'_k\cap Q_R(0)} |\langle f, M_x T_y g_k\rangle|^2\right)^{1/2} + \left(\sum_{k=1}^{K} \sum_{(x,y)\in\Lambda'_k\setminus Q_R(0)} |\langle f, M_x T_y g_k\rangle|^2\right)^{1/2} \\
&\quad\le \left(\sum_{k=1}^{K} \sum_{(x,y)\in\Lambda'_k\cap Q_R(0)} |\langle f, M_x T_y g_k\rangle|^2\right)^{1/2} + \frac{\epsilon}{2} + \frac{\epsilon}{2}
\end{aligned}$$

$$\leq \left(\sum_{k=1}^{K} \sum_{(x,y)\in\Lambda_k} |\langle M_{x_n} T_{(y_n,0)} f, M_x T_y g_k \rangle|^2 \right)^{1/2} + \epsilon$$
$$\leq B^{1/2} \| M_{x_n} T_{(y_n,0)} f \|_2 + \epsilon$$
$$= B^{1/2} \| f \|_2 + \epsilon.$$

On the other hand,

$$\begin{aligned}
&\left(\sum_{k=1}^{K} \sum_{(x,y)\in\Lambda'_k} |\langle f, M_x T_y g_k \rangle|^2 \right)^{1/2} \\
&\quad\geq \left(\sum_{k=1}^{K} \sum_{(x,y)\in\Lambda'_k \cap Q_R(0)} |\langle f, M_x T_y g_k \rangle|^2 \right)^{1/2} \\
&\quad\geq \left(\sum_{k=1}^{K} \sum_{(x,y)\in\Lambda'_k \cap Q_R(0)} |\langle f, M_x T_y g_k \rangle|^2 \right)^{1/2} \\
&\quad\geq \left(\sum_{k=1}^{K} \sum_{(x,y)\in\Lambda_k - z_n \cap Q_R(0)} |\langle f, M_x T_y g_k \rangle|^2 \right)^{1/2} - \frac{\epsilon}{2} \\
&\quad\geq \left(\sum_{k=1}^{K} \sum_{(x,y)\in\Lambda_k - z_n} |\langle f, M_x T_y g_k \rangle|^2 \right)^{1/2} \\
&\qquad - \left(\sum_{k=1}^{K} \sum_{(x,y)\in\Lambda_k - z_n \setminus Q_R(0)} |\langle f, M_x T_y g_k \rangle|^2 \right)^{1/2} - \frac{\epsilon}{2} \\
&\quad\geq A^{1/2} \| f \|_2 - \frac{\epsilon}{2} - \frac{\epsilon}{2} \\
&\quad= A^{1/2} \| f \|_2 - \epsilon.
\end{aligned}$$

In the last line, we have used Lemma 3.7. Since $\epsilon > 0$ is arbitrary, $\{P_{L^2(E)} M_x T_y g_k : (x, y) \in \Lambda'_k, 1 \leq k \leq K\}$ is a frame for $L^2(E)$, i.e., $\mathcal{G}(G, \Lambda')$ is a PFFS for $L^2(E)$ with frame bounds A and B. □

It is clear that Lemma 3.8 is not true for general z_n. Indeed, consider $E = [0, 1]$, $g = 1_E$, and $\Lambda = \mathbb{Z} \times \{0\}$. Then $\{M_x T_y g : (x, y) \in \Lambda\}$ is a PFFS for $L^2(E)$, but $\Lambda + (0, 1) = \mathbb{Z} \times \{1\}$ is not.

3.2 The A-Ramanathan–Steger Weak Homogeneous Approximation Property

The Homogeneous Approximation Property (HAP) (cf. [6]) is a common tool to study density conditions of Gabor systems. Lately, the HAP has also been proven for wavelet frames [13]. However, it is not difficult to see that PFFS's do not generally satisfy the HAP. Therefore, we now define a weaker notion of the HAP which PFFS's do satisfy, thereby deriving interesting approximation properties of PFFS.

Definition 3.9 Let $E \subset \mathbb{R}^d$. For $h > 0$ and $(u, \eta) \in \mathbb{R}^{2d}$, we set $W(h, u, \eta) = \mathrm{span}\{M_x T_y P_{L^2(E)} g : (x, y) \in \Lambda \cap Q_h(u, \eta)\}$. Let $A \subset \mathbb{R}^{2d}$. We say that the PFFS for $L^2(E)$ $\mathcal{G}(G, \Lambda)$ possesses the *Ramanathan–Steger Weak Homogeneous Approximation Property with respect to* A if, for all $f \in L^2(E)$ and for all $\epsilon > 0$, there exists $R > 0$ such that, for all $(u, \eta) \in A$, $\mathrm{dist}(M_\eta T_u f, W(R, u, \eta)) < \epsilon$.

For this paper, we will abbreviate the Ramanathan–Steger Weak Homogeneous Approximation Property with respect to A by simply the *weak HAP with respect to* A.

Theorem 3.10 *Let $a \le d$ be a nonnegative integer, $G = \{g_1, \dots, g_K\} \subset L^2(\mathbb{R}^d) \setminus \{0\}$, $\Lambda \subset \mathbb{R}^{2d}$, and $E \subset \mathbb{R}^d$ be such that*

(a) *There exists $\delta > 0$ such that for $1 \le k \le K$, Λ_k is 2δ-uniformly separated,*
(b) *E is a tube around the subspace $\{x \in \mathbb{R}^d : x_{d-a+1} = \cdots = x_d = 0\}$ for $1 \le a \le d$, or E is a convex neighborhood of the origin if $a = 0$, and*
(c) *$\mathcal{G}(G, \Lambda)$ is a PFFS for $L^2(E)$.*

Let $A \subset \mathbb{R}^d \times \mathbb{R}^{d-a} \times \{0\}^a$. Then, $\mathcal{G}(G, \Lambda)$ possesses the weak HAP with respect to A.

Proof Suppose that the weak HAP with respect to A fails. Then, there exists a function $f \in L^2(E)$ and $\epsilon > 0$ such that, for each $n \in \mathbb{N}$, there exists $z_n = (u_n, \eta_n) \in A$ such that $\mathrm{dist}(M_{\eta_n} T_{u_n} f, W(n, u_n, \eta_n)) > \epsilon$. By Lemma 3.6, there exists a subsequence $\{w_n\}$ of $\{z_n\}$ and Λ'_k such that, for each $1 \le k \le K$, we have

$$\Lambda_k - w_n \xrightarrow{w} \Lambda'_k \text{ as } n \to \infty.$$

Therefore, by Lemma 3.8, $\mathcal{G}(G, \Lambda')$ is a PFFS for $L^2(E)$ with the same frame bounds. We claim now that, for any $R > 0$,

$$\mathrm{dist}\big(f, \mathrm{span}\big\{M_\xi T_x P_{L^2(E)} g_k : (x, \xi) \in \Lambda'_k \cap Q_R(0), k = 1, \dots, K\big\}\big) \ge \frac{\epsilon}{2}. \quad (5)$$

To see this, choose any scalars $\{c_{k,x,\xi} : (x, \xi) \in \Lambda'_k \cap Q_R(0), k = 1, \dots, K\}$. Let

$$D = \sum_{k=1}^{K} \sum_{(x,\xi)\in\Lambda'_k\cap Q_R(0)} |c_{k,x,\xi}|.$$

From above we know that $\|f - 0\|_2 \ge \epsilon$, so we may assume that $D \ne 0$. Since the family of modulation-translation operators is strongly continuous (cf. [11, Lemma 2.1]), there exists $\theta < \delta/2$ such that, whenever $|x|, |\xi| < \theta$, $(u, \eta) \in A$, and $1 \le k \le K$,

$$\|M_{\xi+\eta} T_{x+u} P_{L^2(E)} g_k - M_\eta T_u P_{L^2(E)} g_k\|_2 < \frac{\epsilon}{2D}.$$

As in the proof of Lemma 3.8, we can find n large enough so that each point of $\Lambda'_k \cap Q_R(0)$ is within θ of a unique point in $\Lambda_k - w_n \cap Q_R(0)$, and conversely. So, we can write

$$\begin{aligned}
&\Lambda_k - w_n \cap Q_R(0)\\
&\quad = \big\{(x + u(x,\xi,k), \xi + \eta(x,\xi,k) : (x,\xi) \in \Lambda'_k \cap Q_R(0),\ k = 1,\dots,K\big\}
\end{aligned}$$

with $|u(x,\xi,k)|, |\eta(x,\xi,k)| < \theta$. Hence,

$$\begin{aligned}
&\Bigg\| f - \sum_{k=1}^{K} \sum_{(x,\xi)\in\Lambda'_K \cap Q_R(0)} c_{k,x,\xi} M_\xi T_x P_{L^2(E)} g_k \Bigg\|_2\\
&\quad \geq \Bigg\| f - \sum_{k=1}^{K} \sum_{(x,\xi)\in\Lambda_K - w_n \cap Q_R(0)} c_{k,x,\xi} M_\xi T_x P_{L^2(E)} g_k \Bigg\|_2\\
&\qquad - \Bigg\| \sum_{k=1}^{K} \sum_{(x,\xi)\in\Lambda'_K \cap Q_R(0)} c_{k,x,\xi} (M_{\xi+\eta(x,\xi,k)} T_{x+u(x,\xi,k)} P_{L^2(E)} g_k\\
&\qquad\qquad - M_\xi T_x P_{L^2(E)} g_k) \Bigg\|_2\\
&\quad \geq \mathrm{dist}\big(f, \mathrm{span}\{M_\xi T_x P_{L^2(E)} g_k : (x,\xi) \in \Lambda_k - w_n \cap Q_R(0), k = 1,\dots,K\}\big)\\
&\qquad - \sum_{k=1}^{K} \sum_{(x,\xi)\in\Lambda'_K \cap Q_R(0)} |c_{k,x,\xi}| \| M_{\xi+\eta(x,\xi,k)} T_{x+u(x,\xi,k)} P_{L^2(E)} g_k\\
&\qquad\qquad - M_\xi T_x P_{L^2(E)} g_k \|_2\\
&\quad \geq \epsilon - \sum_{k=1}^{K} \sum_{(x,\xi)\in\Lambda'_K \cap Q_R(0)} |c_{k,x,\xi}| \frac{\epsilon}{2D} = \frac{\epsilon}{2}.
\end{aligned}$$

Since this is true for every choice of scalars, we conclude that (5) holds. But since R is arbitrary, this implies that $f \notin \overline{\mathrm{span}}(P_{L^2(E)}\mathcal{G}(G,\Lambda'))$. Therefore, there exists some $0 \neq h \in L^2(E) \cap (\overline{\mathrm{span}}(P_{L^2(E)}\mathcal{G}(G,\Lambda')))^\perp$. This contradicts the fact that $P_{L^2(E)}\mathcal{G}(G,\Lambda')$ is a frame for $L^2(E)$ (hence complete in $L^2(E)$). □

Employing the weak HAP as a main ingredient for the proof, we will show that sets of parameters of PFFS's are comparable with sets of parameters of Riesz sequences by means of their densities $D^{\pm}_{A,r}$ (recall Definition 2.1). This Comparison Theorem is directly inspired by the double-projection idea of Ramanathan and Steger [21].

Theorem 3.11 *Let a, E, G, A, and Λ be as in Theorem 3.10. Let $\Delta_1,\dots,\Delta_L \subset A$ and $\phi_1,\dots,\phi_L \in L^2(E)\setminus\{0\}$. Assume that $\Phi = P_{L^2(E)}\mathcal{G}(\phi_1,\dots,\phi_L,\Delta_1,\dots,\Delta_L)$*

is a Riesz sequence in $L^2(E)$. *Let* $\Lambda = \cup_{k=1}^{K} \Lambda_k$ *and* $\Delta = \cup_{k=1}^{L} \Delta_k$. *Then* $D^+_{A,r}(\Lambda) \geq D^+_{A,r}(\Delta)$ *and* $D^-_{A,r}(\Lambda) \geq D^-_{A,r}(\Delta)$ *for all* $0 < r < \infty$.

Proof Note that by Theorem 3.10 we have that $\mathcal{G} = \mathcal{G}(G, \Lambda)$ possesses the weak HAP with respect to A. Let

$$\tilde{\mathcal{G}} = \bigcup_{k=1}^{K} \{\tilde{g}_{x,\xi,k}\}_{(x,\xi)\in\Lambda_k}$$

denote the canonical dual frame of $P_{L^2(E)}\mathcal{G}$ in $L^2(E)$. Let

$$\tilde{\Phi} = \bigcup_{k=1}^{L} \{\tilde{\phi}_{x,\xi,k}\}_{(x,\xi)\in\Delta_k}$$

denote the dual frame within the closed linear span of Φ.

Given $h > 0$ and $(u, \eta) \in A$, set

$$\tilde{W}(h, u, \eta) = \mathrm{span}\{M_\xi T_x \tilde{g}_k : (x, \xi) \in \Lambda_k \cap Q_h(u, \eta), k = 1, \dots, K\},$$

$$V(h, u, \eta) = \mathrm{span}\{M_\xi T_x \phi_k : (x, \xi) \in \Delta_k \cap Q_h(u, \eta), k = 1, \dots, L\}.$$

Since we have assumed that each Λ_k is uniformly separated, we have that $D^+_{\mathbb{R}^d,r}(\Lambda_k) < \infty$. So, $\tilde{W}$ is a finite-dimensional space.

Fix $\epsilon > 0$. Applying the definition of weak HAP with respect to A to the functions $f = \phi_k$, we see that there exists an $R > 0$ such that

$$\mathrm{dist}\big(M_\eta T_u \phi_k, \tilde{W}(R, u, \eta)\big) < \frac{\epsilon}{D}, \quad \text{for all } (u, \eta) \in A, k = 1, \dots, L,$$

where

$$D = \sup\big\{\|\tilde{\phi}_{u,\eta,k}\| : (u, \eta) \in \Delta_k, k = 1, \dots, L\big\}.$$

Fix an $h > 0$ and $(u, \eta) \in \mathbb{R}^{2d}$. For simplicity, set $V = V(h, u, \eta)$ and $W = \tilde{W}(R + h, u, \eta)$. Define $T : V \to V$ by $T = P_V P_W$. Note that T is self-adjoint and W is finite-dimensional, so T has a finite, real trace.

We now estimate the trace of T. An easy upper bound is given by

$$\mathrm{trace}(T) \leq \mathrm{rank}(T) \leq \dim(W) = \#\big(\Lambda \cap Q_{R+h}(u, \eta)\big).$$

For a lower bound, note that $\{P_{L^2(E)} M_\xi T_x \phi_k : (x, \xi) \in \Delta_k \cap Q_h(u, \eta), k = 1, \dots, L\}$ is a basis for the finite-dimensional space V. The dual basis in V is the biorthogonal system in V, which is

$$\big\{P_V \tilde{\phi}_{x,\xi,k} : (x, \xi) \in \Delta_k \cap Q_h(u, \eta), k = 1, \dots, L\big\}.$$

Therefore, using that $TP_{L^2(E)} = T$, we compute

$$\begin{aligned}
\operatorname{trace}(T) &= \sum_{k=1}^{L} \sum_{(x,\xi)\in\Delta_k\cap Q_h(u,\eta)} \big\langle T(P_{L^2(E)}M_\xi T_x\phi_k),\, P_V\tilde{\phi}_{x,\xi,k}\big\rangle \\
&= \sum_{k=1}^{L} \sum_{(x,\xi)\in\Delta_k\cap Q_h(u,\eta)} \Big(\big\langle M_\xi T_x\phi_k,\, P_V\tilde{\phi}_{x,\xi,k}\big\rangle \\
&\qquad\qquad + \big\langle (P_W - I)(M_\xi T_x\phi_k),\, P_V\tilde{\phi}_{x,\xi,k}\big\rangle\Big) \\
&= \sum_{k=1}^{L} \sum_{(x,\xi)\in\Delta_k\cap Q_h(u,\eta)} \Big(1 - \big\langle (P_W - I)(M_\xi T_x\phi_k),\, P_V\tilde{\phi}_{x,\xi,k}\big\rangle\Big).
\end{aligned}$$

Additionally, if $(x,\xi) \in \Delta \cap Q_h(u,\eta)$, then we have $Q_R(x,\xi) \subset Q_{R+h}(u,\eta)$, hence $W(R,x,\xi) \subset W(R+h,u,\eta)$ and therefore

$$\begin{aligned}
\Big|\big\langle (P_W - I)(M_\xi T_x\phi_k),\, P_V\tilde{\phi}_{x,\xi,k}\big\rangle\Big| &\le \|(P_W - I)(M_\xi T_x\phi_k)\|_2 \big\| P_V\tilde{\phi}_{x,\xi,k}\big\| \\
&\le \operatorname{dist}(M_\xi T_x\phi_k, W(R,x,\xi)) \big\|\tilde{\phi}_{x,\xi,k}\big\|_2 \\
&\le \frac{\epsilon}{D} D = \epsilon.
\end{aligned}$$

Therefore, we have that

$$\operatorname{trace}(T) \ge \sum_{k=1}^{L} \sum_{(x,\xi)\in\Delta_k\cap Q_h(x,\xi)} (1-\epsilon) = (1-\epsilon)\#\big(\Delta \cap Q_h(u,\eta)\big).$$

Finally, combining our upper and lower estimates yields

$$(1-\epsilon)\#\big(\Delta \cap Q_h(u,\eta)\big) \le \#\big(\Lambda \cap Q_{R+h}(u,\eta)\big) \quad \text{for all } (u,\eta) \in A,\ h > 0,$$

and so, for each $r > 0$,

$$\begin{aligned}
D^+_{A,r}(\Delta) &= \limsup_{h\to\infty} \inf_{(u,\eta)\in A} \frac{\#(\Delta \cap Q_h(u,\eta))}{h^r} \\
&\le \frac{1}{1-\epsilon} \limsup_{h\to\infty} \sup_{(u,\eta)\in A} \frac{\#(\Lambda \cap Q_{R+h}(u,\eta))}{(R+h)^r} \frac{(R+h)^r}{h^r} \\
&= \frac{1}{1-\epsilon} D^+_{A,r}(\Lambda).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that $D^+_{A,r}(\Delta) \le D^+_{A,r}(\Lambda)$, and a similar calculation shows that $D^-_{A,r}(\Delta) \le D^-_{A,r}(\Lambda)$. □

3.3 Beurling Dimension of Gabor PFFS's and PRFS's

The following result gives a detailed account of the upper and lower dimensions of sets of parameters of PFFS's and PRFS's for an arbitrary subset $A \subset \mathbb{R}^d \times \mathbb{R}^{d-a} \times \{0\}^a$. This will give rise to Theorem 3.3 by choosing an appropriate subset A.

Theorem 3.12 *Let $1 \le a \le d$ be an integer, $G = \{g_1, \ldots, g_K\} \subset L^2(\mathbb{R}^d) \setminus \{0\}$, $\{\Lambda_1, \ldots, \Lambda_K\}$ be a collection of subsets of $\mathbb{R}^{2d}$, and $E \subset \mathbb{R}^d$ be such that*

(a) *E is a tube around the subspace $\{x \in \mathbb{R}^d : x_{d-a+1} = \cdots = x_d = 0\}$, and*
(b) *$\mathcal{G}(G, \Lambda)$ is a PFFS for $L^2(E)$.*

Further, let $S = \mathbb{R}^d \times \mathbb{R}^{d-a} \times \{0\}^a$. Then the following conditions hold:

(i) *Let F contain E. If $\mathcal{G}(G, \Lambda)$ is a PFFS for $L^2(F)$, then for $\Lambda = \cup_{k=1}^K \Lambda_k$,*

$$\dim_S^-(\Lambda) \ge 2d - a \text{ or } \dim_{\mathbb{R}^d}^+(\Lambda) = \infty.$$

In particular, if $\mathcal{G}(G, \Lambda)$ is a Bessel sequence in $L^2(\mathbb{R}^d)$, then $\dim_S^-(\Lambda) \ge 2d - a$.

(ii) *Let H be contained in E. If $\mathcal{G}(G, \Lambda)$ is a PRFS for $L^2(H)$, then*

$$\dim_S^-(\Lambda) = \dim_S^+(\Lambda) = 2d - a.$$

Proof (i): Suppose that, for each $1 \le k \le K$, $\dim_{\mathbb{R}^d}^+(\Lambda_k) < \infty$. Then, by Proposition 2.9 and repeating some Λ_k's if necessary, we may assume that each Λ_k is uniformly separated. Note that if $E = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : |x_i| \le \epsilon \text{ for } d - a + 1 \le i \le d \text{ and some } \epsilon > 0\}$, then let $\phi = 1_{[-\epsilon,\epsilon]}$ and $\Delta = \frac{1}{2\epsilon}\mathbb{Z}^d \times 2\epsilon\mathbb{Z}^{d-a} \times \{0\}^a$ and observe that $\{M_x T_y \phi : (x, y) \in \Delta\}$ is an ONB for $L^2(E)$. Moreover,

$$D_{S,r}^+(\Delta) = D_{S,r}^-(\Delta) = \begin{cases} \infty & : 0 \le r < 2d - a, \\ \frac{(2\epsilon)^{d-a}}{(2\epsilon)^d} & : r = 2d - a, \\ 0 & : r > 2d - a. \end{cases}$$

Therefore, by Theorem 3.11, if $\mathcal{G}(G, \Lambda)$ is a PFFS for $L^2(E)$, then $\dim_S^-(\Lambda) \ge 2d - a$.

Moreover, if F merely contains $E = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : |x_i| \le \epsilon \text{ for } d - a + 1 \le i \le d \text{ and some } \epsilon > 0\}$, then since the projection of a frame is a frame, the result follows from the case considered in the first part of this proof.

(ii): Clearly, it suffices to prove (ii) in the case $K = 1$. First, note that by applying Theorem 3.11 to compare the PRFS given in the statement of Theorem 3.12 to the frame for $L^2(\mathbb{R}^d)$ given by $\mathcal{G}(1_{[0,1]^d}, \mathbb{Z}^{2d})$, it follows that, for all $0 < r < \infty$, $D_{\mathbb{R}^d,r}^+(\Lambda) \le D_{\mathbb{R}^d,r}^-(\mathbb{Z}^{2d})$. In particular, by part (i), this implies that $\dim_{\mathbb{R}^{2d}}(\Lambda) < \infty$, so $\dim_S^-(\Lambda) \ge 2d - a$.

Now, we will show that $\dim_S^+(\Lambda) \le 2d - a$. We will actually show the stronger result that Λ is contained in some tube around $\mathbb{R}^d \times Y$, where $Y = \{x \in \mathbb{R}^d : x_{d-a+1} = \cdots = x_d = 0\}$. Once we have shown this, Theorem 2.11 implies the theorem.

In the case that $g_1 = g$ is compactly supported, since E is a tube around the subspace Y and $P_{L^2(H)}M_xT_yg \neq 0$ for all $(x, y) \in \Lambda$, it follows that Λ must also be contained in a tube around $\mathbb{R}^d \times Y$.

For the general case, note that if $\{P_{L^2(H)}M_xT_yg : (x, y) \in \Lambda\}$ is a Riesz sequence, then it is in particular true that, for some $\epsilon > 0$, $\inf\{\|P_{L^2(H)}M_xT_yg\| : (x, y) \in \Lambda\} > \epsilon > 0$. Now, approximate g with a compactly supported function f so that $\|f - g\| < \epsilon$. Moreover,

$$\inf_{(x,y)\in\Lambda} \|P_{L^2(H)}M_xT_yf\| \leq \inf_{(x,y)\in\Lambda} \|P_{L^2(H)}M_xT_y(f-g)\| + \inf_{(x,y)\in\Lambda} \|P_{L^2(H)}M_xT_yg\|$$
$$\leq \epsilon + \inf_{(x,y)\in\Lambda} \|P_{L^2(H)}M_xT_yg\|.$$

This implies that $\inf_{(x,y)\in\Lambda} \|P_{L^2(H)}M_xT_yg\| > 0$, which in turn implies that Λ is contained in some tube around $\mathbb{R}^d \times Y$, as desired. □

Finally, we can give the proof of Theorem 3.3.

Proof of Theorem 3.3 (i): By a change of basis, we may assume that E is of the form given in Theorem 3.12 with $m = d - a$. Let A also be of the form given in Theorem 3.12. Then, one simply notes that $\dim^+_{\mathbb{R}^d}(\Lambda) \geq \dim^+_A(\Lambda) \geq 2d - (d - m) = d + m$, as desired.

(ii): This claim follows from Theorem 3.12 by similar arguments, we only use part (ii) instead. □

Example 3.13 There exists a PFFS satisfying the conditions given in Theorem 3.12 and such that $\dim^-_S(\Lambda_k) < 2d - a$ for all $1 \leq k \leq K$.

Proof Let $a = 1$, $K = 1$, and $E = \mathbb{R} \times [-1/2, 1/2]$ so that $S = \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$. We construct Λ and g such that $\mathcal{G}(G, \Lambda)$ is a tight PFFS for $L^2(E)$. Let

$$K_0 = \cup_{n=-\infty}^{\infty}[n, n+1] \times \left(\left[0, 2^{-|n|}\right] + 2^{2^{|n|}}\right).$$

Let

$$H_0 = \{0\} \times \left\{-2^{2^{|n|}} + j2^{-|n|} : n \in \mathbb{N}, 0 \leq j \leq 2^{|n|} - 1\right\}.$$

See that $\{(K + h) \cap E : h \in H\}$ is a partition of E. It follows that setting $g = 1_K$ and $\Lambda = \mathbb{Z}^2 \times H$ yields that $\mathcal{G}(G, \Lambda)$ is a PFFS for $L^2(E)$. However, $\dim^-_S(\Lambda) = 2$. It is clear in this case that $\dim^+_{\mathbb{R}^{2d}}(\Lambda) = \infty$, as Theorem 3.12 predicts. □

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