J Fourier Anal Appl (2009) 15: 179–200
DOI 10.1007/s00041-008-9022-4

The Maximal Operator Associated to a Nonsymmetric Ornstein–Uhlenbeck Semigroup

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Received: 23 May 2007 / Published online: 19 April 2008

Abstract Let $(\mathcal{H}_t)_{t\geq 0}$ be the Ornstein–Uhlenbeck semigroup on $\mathbb{R}^d$ with covariance matrix I and drift matrix $\lambda(R-I)$, where $\lambda>0$ and R is a skew-adjoint matrix, and denote by γ_∞ the invariant measure for $(\mathcal{H}_t)_{t\geq 0}$. Semigroups of this form are the basic building blocks of Ornstein–Uhlenbeck semigroups which are normal on $L^2(\gamma_\infty)$. We prove that if the matrix R generates a one-parameter group of periodic rotations, then the maximal operator $\mathcal{H}_* f(x)=\sup_{t\geq o}|\mathcal{H}_t f(x)|$ is of weak type 1 with respect to the invariant measure γ_∞. We also prove that the maximal operator associated to an arbitrary normal Ornstein–Uhlenbeck semigroup is bounded on $L^p(\gamma_\infty)$ if and only if $1<p\leq\infty$.

Keywords Ornstein–Uhlenbeck semigroup · Maximal operator · Weak type

Mathematics Subject Classification (2000) 42B25 · 47D03

1 Introduction

Let Q be a real, symmetric, and positive definite $d\times d$-matrix, and let B be a nonzero real $d\times d$-matrix whose eigenvalues have negative real part. Then, for every $t\in(0,\infty]$, we can define the family of Gaussian measures γ_t on $\mathbb{R}^d$ with mean zero and covariance operators

$$Q_t=\int_0^t \mathrm{e}^{sB}Q\mathrm{e}^{sB^*}\,\mathrm{d}s,\qquad t\in(0,\infty],\tag{1.1}$$

Communicated by Carlos Kenig.

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i.e., the measures

$$d\gamma_t(x) = (2\pi)^{-d/2}(\det Q_t)^{-1/2} e^{-\frac{1}{2}\langle Q_t^{-1}x,x\rangle}\, d\lambda(x), \quad \forall t \in (0,\infty].$$

The Ornstein–Uhlenbeck semigroup is the family of operators $(\mathcal{H}_t^{Q,B})_{t\geq 0}$ defined by

$$\mathcal{H}_t^{Q,B} f(x) = \int_{\mathbb{R}^d} f\big(e^{tB}x - y\big)\, d\gamma_t(y) \tag{1.2}$$

on the space $C_b(\mathbb{R}^d)$ of bounded continuous functions. The matrices Q and B are called the *covariance* and the *drift* matrix, respectively.

It is well known that γ_∞ is the unique invariant measure for $\mathcal{H}_t^{Q,B}$ and that $(\mathcal{H}_t^{Q,B})_{t\geq 0}$ is a diffusion semigroup on $(\mathbb{R}^d, \gamma_\infty)$ (see, for instance, [2]). Thus, formula (1.2) defines a semigroup of positive contractions on $L^p(\gamma_\infty)$ for every $p \geq 1$, which we shall also denote by $(\mathcal{H}_t^{Q,B})_{t\geq 0}$.

In this paper we are concerned with the boundedness of the maximal operator

$$\mathcal{H}_*^{Q,B} f(x) = \sup_{t\geq 0}\big|\mathcal{H}_t^{Q,B} f(x)\big|.$$

It is well known that by Banach's principle (see [3]) this maximal operator is a key tool to investigate the almost everywhere convergence of $\mathcal{H}_t^{Q,B} f$ to f as t tends to 0 for f in $L^p(\gamma_\infty)$.

If the semigroup $(\mathcal{H}_t^{Q,B})_{t\geq 0}$ is symmetric, i.e., if $\mathcal{H}_t^{Q,B}$ is self-adjoint on $L^2(\gamma_\infty)$ for every $t \geq 0$, then $\mathcal{H}_*^{Q,B}$ is bounded on $L^p(\gamma_\infty)$ for every p in $(1,\infty]$ by the Littlewood–Paley–Stein theory for symmetric semigroups of contractions on all L^p spaces [11]. Is the result still true if we drop the symmetry assumption? In the same monograph [11] Stein says that, for general diffusion semigroups, the condition of self-adjointness cannot be much modified. Indeed, if one considers the semigroup of translations $\mathcal{T}_t f(x) = f(x+t)$ on the one-dimensional torus $\mathbb{T}$, for every p in $[1,\infty]$, it is easy to construct a function f in $L^p(\mathbb{T})$ such that $\sup_{t\geq 0}|T_t f(x)| = \infty$ everywhere. Notice that $(\mathcal{T}_t)_{t\geq 0}$ is a semigroup of normal, actually unitary, operators.

However, in Theorem 4.2 below we show that Stein's proof of the maximal theorem for semigroups of symmetric contractions on all $L^p(\mu)$, $1 \leq p \leq \infty$, can be adapted to semigroups of *normal* contractions such that the generator of the semigroup on $L^2(\mu)$ is a sectorial operator of angle $\phi < \pi/2$. Since the generator of the Ornstein–Uhlenbeck on $L^2(\gamma_\infty)$ is sectorial of angle strictly less than $\pi/2$, this implies that if $(\mathcal{H}_t^{Q,B})_{t\geq 0}$ is normal on $L^2(\gamma_\infty)$, then the maximal operator $\mathcal{H}_*$ is bounded on $L^p(\gamma_\infty)$ for every p in $(1,\infty]$.

It remains to investigate the boundedness of the Ornstein–Uhlenbeck maximal operator $\mathcal{H}_*^{Q,B}$ on $L^1(\gamma_\infty)$. In Sect. 4 we show that $\mathcal{H}_*^{Q,B}$ is always unbounded on $L^1(\gamma_\infty)$. This still leaves open the question of the validity of the weak type 1 estimate

$$\gamma_\infty\Big(\big\{x \in \mathbb{R}^d : \big|\mathcal{H}_*^{Q,B} f(x)\big| > \alpha\big\}\Big) \leq \frac{C\|f\|_1}{\alpha}, \quad \forall f \in L^1(\gamma_\infty),\ \forall \alpha > 0.$$

Even in the symmetric case very little is known about the weak type 1 boundedness of the Ornstein–Uhlenbeck maximal operator. The only result which is known is for the semigroup with covariance matrix $Q = I$ and drift matrix $B = -I$ for which the weak type 1 boundedness of $\mathcal{H}_*^{Q,B}$ is due to B. Muckenhoupt [9] in dimension one and to P. Sjögren [10] in arbitrary dimension. Sjögren's proof was subsequently simplified in [6] and [4]. The arguments in these papers easily extend to the case where $B = -\lambda I$ for some $\lambda > 0$. However, already the case where B is a diagonal matrix with at least two different eigenvalues seems to require new ideas.

In this paper we investigate the weak type 1 estimate for the maximal operator associated to the Ornstein–Uhlenbeck semigroup with covariance matrix $Q = I$ and drift $B = -\lambda(I - R)$, where $\lambda > 0$ and R is a nonzero real $d \times d$ skew-adjoint matrix. The interest of these semigroups is motivated by the fact that they are the basic building blocks of normal Ornstein–Uhlenbeck semigroups. Indeed, in Sect. 2 we show that, after a change of variables, any normal Ornstein–Uhlenbeck semigroup can be written as the product of commuting semigroups of this form.

For these *particular* semigroups, we shall prove two results. First, we shall prove that the "truncated" maximal operator

$$\mathcal{H}_{*,[0,T]}^{Q,B} f(x) = \sup_{t\in[0,T]} \left|\mathcal{H}_t^{Q,B} f(x)\right|$$

is of weak type 1. Second, we shall prove that if the one-parameter group of rotations $(\mathrm{e}^{tR})_{t\in\mathbb{R}}$ generated by R is periodic, then the full maximal operator $\mathcal{H}_*^{Q,B}$ is of weak type 1.

Finally we mention that, by using the results of the present paper, in [5] we have proved that first-order Riesz transforms associated to the generators of these 'periodic' semigroups are of weak type 1.

We now briefly describe the content of the paper. In Sect. 2 we characterize the generators of normal Ornstein–Uhlenbeck semigroups and we show that, after a change of coordinates, normal semigroups are the products of commuting semigroups with covariance matrix $Q = I$ and drift $B = -\lambda(I - R)$ with $\lambda > 0$ and R a real skew-adjoint matrix.

In Sect. 3 we give an explicit representation of the integral kernel of these semigroups with respect to the invariant measure. We show that, modulo an orthogonal change of coordinates, the semigroup kernel is the product of the kernel of a symmetric semigroup and some two-dimensional kernels. Ultimately, this will enable us to reduce the problem of the weak type 1 boundedness of the maximal operator to proving estimates of kernels defined on $\mathbb{R}^2 \times \mathbb{R}^2$.

In Sect. 4 we study the boundedness of the maximal operator $\mathcal{H}_*^{Q,B}$ on $L^p(\gamma_\infty)$, $1 \le p \le \infty$, for *arbitrary* Q and B. We prove that the truncated maximal operator is always unbounded on $L^1(\gamma_\infty)$ and that, when the semigroup is normal, the full maximal operator is bounded on $L^p(\gamma_\infty)$, $1 < p \le \infty$.

Finally, in Sect. 5 we prove the weak type estimate for the truncated and the full maximal operator when $Q = I$ and $B = -\lambda(I - R)$. By the results of Sect. 3 the kernel of the semigroup is a perturbation of the kernel of a symmetric semigroup. When t is close to zero, the perturbation is small, and the kernel of the nonsymmetric semigroup can be controlled by the kernel of the symmetric semigroup. The same thing happens in the periodic case when t is close to an integer multiple of a period.

This enables us to apply the results of [4] to prove the weak-type estimate for the truncated maximal operator and of the full maximal operator in the periodic case.

2 Preliminaries

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is a core for the infinitesimal generator $\mathcal{L}_{Q,B}$ of the semigroup $(\mathcal{H}_t^{Q,B})_{t\geq 0}$ on $L^p(\gamma_\infty)$ for every p, $1<p<\infty$, and

$$\mathcal{L}_{Q,B} f = \frac{1}{2}\operatorname{tr}\bigl(Q\nabla^2\bigr) f + \langle Bx, \nabla\rangle f, \quad \forall f \in \mathcal{S}\bigl(\mathbb{R}^d\bigr).$$

By a result of G. Metafune, J. Prüss, A. Rhandi, and R. Schnaubelt (see [8, Lemma 2.2]) there exists a linear change of coordinates in $\mathbb{R}^d$ which allows us to reduce the analysis of the operator $\mathcal{L}_{Q,B}$ to the case where $Q=I$ and Q_∞ is a diagonal matrix. Indeed, let M_1 be an invertible real matrix such that $M_1 Q M_1^* = I$ and M_2 an orthogonal matrix such that $M_2 M_1 Q_\infty M_1^* M_2 = \operatorname{diag}(\lambda_1, \dots, \lambda_d) := D_\lambda$ for some $\lambda_j > 0$. Then, if we take $M = M_2 M_1$ and denote by $\Phi_M : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ the similarity transformation defined by $\Phi_M f(x) = f(M^{-1}x)$, we have that $\mathcal{L}_{Q,B} = \Phi_M^{-1} \mathcal{L}_{I,\tilde{B}} \Phi_M$, where

$$\tilde{B} = -\frac{1}{2} D_{1/\lambda} + R, \tag{2.1}$$

and R is a matrix such that

$$R D_\lambda = -D_\lambda R^*. \tag{2.2}$$

The invariant measure for the semigroup generated by $\mathcal{L}_{I,\tilde{B}}$ is

$$\mathrm{d}\tilde{\gamma}_\infty(x) = (2\pi)^{-d/2} (\det D_\lambda)^{-1/2} \mathrm{e}^{-\frac{1}{2}\langle D_\lambda^{-1} x, x\rangle} \,\mathrm{d}\lambda(x).$$

Moreover, $\tilde{\gamma}_\infty(E) = \gamma_\infty(M^{-1}E)$ for every Borel subset E of $\mathbb{R}^d$, and Φ_M extends to an isometry of $L^p(\gamma_\infty)$ onto $L^p(\tilde{\gamma}_\infty)$.

By (2.1) we can write the operator $\mathcal{L}_{I,\tilde{B}}$ as the sum

$$\mathcal{L}_{I,\tilde{B}} = \mathcal{L}^0 + \mathcal{R}, \tag{2.3}$$

where $\mathcal{L}^0 = \frac{1}{2}\Delta - \frac{1}{2}\langle D_{1/\lambda} x, \nabla\rangle$ and $\mathcal{R} = \langle Rx, \nabla\rangle$ are the symmetric and antisymmetric parts of $\mathcal{L}_{I,\tilde{B}}$ on $L^2(\tilde{\gamma}_\infty)$, respectively. Thus, the operator $\mathcal{L}_{Q,B}$ is symmetric on $L^2(\gamma_\infty)$ if and only if $R=0$.

Let $(\mathcal{H}_t^{I,\tilde{B}})_{t\geq 0}$ be the semigroup generated by $\mathcal{L}_{I,\tilde{B}}$ and $\mathcal{H}_*^{I,\tilde{B}}$ the corresponding maximal operator. Clearly, $\mathcal{H}_*^{Q,B}$ is bounded on $L^p(\gamma_\infty)$ or of weak type 1 with respect to γ_∞ if and only if $\mathcal{H}_*^{I,\tilde{B}}$ is bounded on $L^p(\tilde{\gamma}_\infty)$ or of weak type 1 with respect to $\tilde{\gamma}_\infty$. Thus, the analysis of the maximal operator $\mathcal{H}_*^{Q,B}$ can be reduced to the case where $Q=I$ and $\tilde{Q}_\infty = \operatorname{diag}\{\lambda_1, \dots, \lambda_d\}$ for some $\lambda_j > 0$.

Proposition 2.1 *Let $\tilde{B}$, D_λ, and R be the matrices associated to Q and B as in (2.1). Denote by $\mathcal{L}^0$ and $\mathcal{R}$ the symmetric and antisymmetric parts of $\mathcal{L}_{I,\tilde{B}}$ as in (2.3). Then the following properties are equivalent*:

(i) *The semigroup $(\mathcal{H}_t^{Q,B})_{t\geq 0}$ is normal on $L^2(\gamma_\infty)$.*
(ii) *The symmetric and antisymmetric parts of $\mathcal{L}_{I,\tilde{B}}$ commute*; *i.e.*,

$$\big[\mathcal{L}^0, \mathcal{R}\big]\phi = 0, \quad \forall \phi \in \mathcal{S}\big(\mathbb{R}^d\big).$$

(iii) $R + R^* = 0$.
(iv) *D_λ and R commute.*

Proof We claim that $\mathcal{L}^*_{I,\tilde{B}} = \mathcal{L}^0 - \mathcal{R}$. Indeed, on the one hand, $(\mathcal{L}^0)^* = \mathcal{L}^0$, since $\mathcal{L}^0$ is symmetric. On the other hand, integrating by parts, we get that

$$\begin{aligned}\mathcal{R}^* &= -\mathcal{R} + \big\langle Rx, D_\lambda^{-1}x\big\rangle - \operatorname{tr} R\\ &= -\mathcal{R},\end{aligned}$$

since $\operatorname{tr} R = 0$ and that $\langle Rx, D_\lambda^{-1}x\rangle = 0$, since $\langle Rx, D_\lambda^{-1}x\rangle = \langle x, R^* D_\lambda^{-1}x\rangle = -\langle x, D_\lambda^{-1}Rx\rangle = -\langle D_\lambda^{-1}x, Rx\rangle$ by (2.2).

The semigroup $(\mathcal{H}_t^{Q,B})_{t\geq 0}$ is normal if and only if its generator $\mathcal{L}_{Q,B}$ on $L^2(\gamma_\infty)$ is normal, and this happens if and only if $\mathcal{L}_{I,\tilde{B}}$ is normal on $L^2(\tilde{\gamma}_\infty)$, i.e., $[\mathcal{L}_{I,\tilde{B}}, \mathcal{L}^*_{I,\tilde{B}}]\phi = 0$ for all ϕ in $\mathcal{S}(\mathbb{R}^d)$. Now

$$\begin{aligned}\big[\mathcal{L}_{I,\tilde{B}}, \mathcal{L}^*_{I,\tilde{B}}\big] &= \big[\mathcal{L}^0 + \mathcal{R}, \mathcal{L}^0 - \mathcal{R}\big]\\ &= 2\big[\mathcal{R}, \mathcal{L}^0\big].\end{aligned}$$

This shows that (i) and (ii) are equivalent. Next observe that

$$\big[\mathcal{R}, \mathcal{L}^0\big] = -\langle \nabla, R\nabla\rangle + \frac{1}{2}\big\langle (RD_{1/\lambda} - D_{1/\lambda}R)x, \nabla\big\rangle. \tag{2.4}$$

Hence, $[\mathcal{R}, \mathcal{L}^0]$ vanishes if and only if $\langle \nabla, R\nabla\rangle$ and $\langle (RD_{1/\lambda} - D_{1/\lambda}R)x, \nabla\rangle$ both vanish, as can be easily seen by fixing any pair of indices j, k and an arbitrary point x_0 and applying the commutator to a test function ϕ which coincides with $(x - x_0)_j(x - x_0)_k$ in a neighborhood of x_0. Now, $\langle \nabla, R\nabla\rangle$ vanishes if and only if $R + R^* = 0$. Thus, (ii) implies (iii). To prove the converse observe that by (2.2) the identity $R + R^* = 0$ implies that R and D_λ commute. Thus, also $D_{1/\lambda}$ and R commute. Hence, $[\mathcal{R}, \mathcal{L}^0] = 0$ by (2.4). Finally, if (iv) holds, then $R + R^* = 0$ by (2.2). This concludes the proof of the proposition. □

In the last part of this section we show that operators of the form $\mathcal{L}_{Q,B}$ with $Q = I$ and $B = \frac{1}{2\alpha}(R - I)$, where $\alpha > 0$ and R is a $d \times d$ skew-symmetric real matrix, are the basic building blocks of normal Ornstein–Uhlenbeck operators. This motivates the interest in studying the maximal operator associated to semigroups generated by them.

To simplify the notation we write

$$\mathcal{L}(\alpha, R) = \mathcal{L}_{I,\frac{1}{2\alpha}(R-I)} = \frac{1}{2}\Delta - \frac{1}{2\alpha}\langle x, \nabla\rangle + \frac{1}{2\alpha}\langle Rx, \nabla\rangle. \tag{2.5}$$

Let $(\mathcal{H}_t^{Q,B})_{t\geq 0}$ be a normal Ornstein–Uhlenbeck semigroup. By (2.1), after a change of variables, we may assume that its generator is of the form

$$\mathcal{L}_{I,\tilde{B}} = \frac{1}{2}\Delta - \frac{1}{2}\langle D_{1/\lambda}x, \nabla\rangle + \langle Rx, \nabla\rangle,$$

where $R + R^* = 0$ and R commutes with D_λ by Proposition 2.1. Let $\alpha_1, \ldots, \alpha_\ell$ be the distinct eigenvalues of D_λ, and let

$$D_\lambda = \alpha_1 P_1 + \cdots + \alpha_\ell P_\ell$$

be the spectral resolution of D_λ. The matrix R commutes with the projections P_j, and if we set $R_j = 2\alpha R P_j$, then $R_j^* = -R^j$ and $R = \frac{1}{2\alpha}\sum_{j=1}^{\ell} R_j$. Thus, denoting by $\Delta_j = \operatorname{tr}(P_j\nabla^2)$ and $\nabla_j = P_j\nabla$ the Laplacian and gradient with respect to the variables in $P_j\mathbb{R}^d$, we have

$$\mathcal{L}_{I,\tilde{B}} = \sum_{j=1}^{\ell} \mathcal{L}(\alpha_j, R_j),$$

where

$$\mathcal{L}(\alpha_j, R_j) = \frac{1}{2}\Delta_j - \frac{1}{2\alpha_j}\langle x, \nabla_j\rangle + \frac{1}{2\alpha_j}\langle R_j x, \nabla_j\rangle.$$

The semigroup generated by $\mathcal{L}_{I,\tilde{B}}$ is the product of the commuting semigroups $(\mathrm{e}^{t\mathcal{L}(\alpha_j,R_j)})_{t\geq 0}$ generated by the operators $\mathcal{L}(\alpha_j, R_j)$, $j = 1, \ldots, \ell$, which are therefore the basic building blocks of normal Ornstein–Uhlenbeck semigroups.

3 The Kernel of the Semigroup with Respect to the Invariant Measure

For our purposes, it is convenient to write the Ornstein–Ulenbeck semigroup as a semigroup of integral operators with respect to the invariant measure γ_∞. We recall that the Gauss measure with mean zero and covariance matrix Q_t on $\mathbb{R}^d$ is the measure

$$\mathrm{d}\gamma_t(x) = (2\pi)^{-d/2}(\det Q_t)^{-1/2}\mathrm{e}^{-\frac{1}{2}\langle Q_t^{-1}x,x\rangle}\,\mathrm{d}\lambda(x), \quad \forall t \in (0, \infty],$$

where λ denotes the Lebesgue measure. In the following, with a slight abuse of notation, we shall also denote by the same symbol γ_t the density of the measure with respect to λ. A simple change of variables in (1.2) yields

$$\mathcal{H}_t^{Q,B} f(x) = \int h_t(x, y) f(y)\,\mathrm{d}\gamma_\infty(y),$$

where

$$h_t(x,y)=\det\bigl(Q_\infty Q_t^{-1}\bigr)^{1/2}\mathrm{e}^{-\frac{1}{2}[\langle Q_t^{-1}(\mathrm{e}^{tB}x-y),(\mathrm{e}^{tB}x-y)\rangle-\langle Q_\infty^{-1}y,y\rangle]}. \tag{3.1}$$

The main result of this section is that, after an orthogonal change of coordinates, the kernel of the semigroup generated by an operator of the form

$$\mathcal{L}(\alpha,R)=\frac{1}{2}\Delta-\frac{1}{2\alpha}\langle x,\nabla\rangle+\frac{1}{2\alpha}\langle Rx,\nabla\rangle$$

with $\alpha>0$ and $R+R^*=0$ can be written as the product of the kernel of the semigroup generated by its symmetric part $\mathcal{L}(\alpha,0)$ and some two-dimensional kernels (see Theorem 3.1 and formula (3.9)). To simplify the notation, for the rest of this section, we write $\mathcal{L}=\mathcal{L}(\alpha,R)$ and $\mathcal{L}^0=\mathcal{L}(\alpha,0)$. Thus,

$$\mathcal{L}^0=\frac{1}{2}\Delta-\frac{1}{2\alpha}\langle x,\nabla\rangle,\qquad \mathcal{L}=\mathcal{L}^0+\frac{1}{2\alpha}\langle Rx,\nabla\rangle.$$

Henceforth, we shall denote by $(\mathrm{e}^{t\mathcal{L}^0})_{t\ge0}$ and by $(\mathrm{e}^{t\mathcal{L}})_{t\ge0}$ the semigroups generated by $\mathcal{L}^0$ and by $\mathcal{L}$, respectively, and by $h_t^0(x,y)$ and $h_t(x,y)$ their kernels with respect to the invariant measure

$$\mathrm{d}\gamma_\infty(x)=(2\pi\alpha)^{-d/2}\mathrm{e}^{-\frac{|x|^2}{2\alpha}}.$$

By the results of the previous section, the operator $\mathcal{L}^0$ is symmetric, and $\mathcal{L}$ is normal.

To avoid having many α's floating around and to be consistent with the notation in [4], we fix $\alpha=1/2$. The formulas for arbitrary $\alpha>0$ can be obtained from this special case by replacing t by $t/2\alpha$ and (x,y) by $(x/\sqrt{2\alpha},y/\sqrt{2\alpha})$ in formulas (3.2) and (3.3) below.

The kernel of the semigroup $(\mathrm{e}^{t\mathcal{L}^0})_{t\ge0}$ is

$$h_t^0(x,y)=\bigl(1-\mathrm{e}^{-2t}\bigr)^{-d/2}\exp\left\{\frac{1}{2}\left[\frac{|x+y|^2}{\mathrm{e}^t+1}-\frac{|x-y|^2}{\mathrm{e}^t-1}\right]\right\}. \tag{3.2}$$

The operator $\mathcal{R}=\langle Rx,\nabla\rangle$ generates the semigroup of isometries $\mathrm{e}^{t\mathcal{R}}f(x)=f(\mathrm{e}^{tR}x)$ of $L^p(\gamma_\infty)$, $1\le p\le\infty$. Since $\mathrm{e}^{t\mathcal{R}}$ commutes with $\mathrm{e}^{t\mathcal{L}_0}$ for every $t\ge0$, the kernel of $(\mathrm{e}^{t\mathcal{L}})_{t\ge0}$ is

$$h_t(x,y)=h_t^0\bigl(\mathrm{e}^{tR}x,y\bigr). \tag{3.3}$$

We shall exploit the facts that the matrix R is skew-adjoint and that the symmetric semigroup $(\mathrm{e}^{t\mathcal{L}^0})_{t\ge0}$ commutes with orthogonal transformations to prove that, after an orthogonal change of coordinates, the operator $\mathcal{L}$ and the kernel $h_t(x,y)$ can be written in a more convenient form.

First, we consider a special two-dimensional case. For every real number θ, we denote by $R(\theta)$ the 2×2 matrix

$$R(\theta)=\begin{pmatrix}0&\theta\\-\theta&0\end{pmatrix}. \tag{3.4}$$

Let $x \wedge y$ denote the skew-symmetric bilinear form on $\mathbb{R}^2$ defined by

$$x \wedge y = x_1 y_2 - x_2 y_1.$$

Then

$$\left|e^{tR(\theta)}x \pm y\right|^2 = |x|^2 + |y|^2 + 2\cos(t\theta)\langle x, y\rangle \pm \sin(t\theta) x \wedge y,\\ \forall x, y \in \mathbb{R}^2. \tag{3.5}$$

Now, consider the Ornstein–Uhlenbeck operator $\mathcal{L}(\frac{1}{2}, R(\theta))$ on $\mathbb{R}^2$. To simplify the notation, henceforth we write $\mathcal{L}_\theta = \mathcal{L}(\frac{1}{2}, R(\theta))$. Thus,

$$\mathcal{L}_\theta = \frac{1}{2}\Delta - \langle x, \nabla\rangle + \left\langle R(\theta)x, \nabla\right\rangle$$

is the operator with covariance matrix $Q = I$ and drift $B = -I + R(\theta)$. By using (3.2), (3.3), and (3.5) it is straightforward to see that the kernel of the semigroup generated by $\mathcal{L}_\theta$ is

$$h_t^\theta(x, y) = h_t^0(x, y) k_{t\theta}(x, y), \tag{3.6}$$

where $h_t^0(x, y)$ is as in (3.2) with $d = 2$ and

$$k_{t\theta}(x, y) = \exp\left\{-\frac{e^{-t}}{1 - e^{-2t}}\left[\left(1 - \cos(t\theta)\right)\langle x, y\rangle + \sin(t\theta) x \wedge y\right]\right\}. \tag{3.7}$$

Next, we consider the case where the matrix R is a $d \times d$ matrix in block diagonal form with 2×2 blocks of the form (3.4). Let $n = [\![d/2]\!]$ be the greatest integer less than or equal to $d/2$. If $\Theta = (\theta_1, \dots, \theta_n)$ is in $\mathbb{R}^n$, we denote by $R(\Theta)$ the $d \times d$ block-diagonal matrix

$$\begin{pmatrix} R(\theta_1) & & \\ & \ddots & \\ & & R(\theta_n) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} R(\theta_1) & & & \\ & \ddots & & \\ & & R(\theta_n) & \\ & & & 0 \end{pmatrix}$$

according to whether d is even or odd, respectively.

Assume first that d is even. Given a vector x in $\mathbb{R}^d \simeq (\mathbb{R}^2)^n$, we write $x = (\xi_1, \dots, \xi_n)$, where $\xi_k = (x_{2k-1}, x_{2k}) \in \mathbb{R}^2$ for $k = 1, \dots, n$. Let $\mathcal{L}_\Theta = \mathcal{L}(\frac{1}{2}, R(\Theta))$ be the Ornstein–Uhlenbeck operator on $\mathbb{R}^d$ of the form

$$\mathcal{L}_\Theta = \frac{1}{2}\Delta - \langle x, \nabla\rangle + \left\langle R(\Theta)x, \nabla\right\rangle. \tag{3.8}$$

Then $\mathcal{L}_\Theta = \mathcal{L}_{\theta_1} + \cdots + \mathcal{L}_{\theta_n}$, where each $\mathcal{L}_{\theta_k}$ for $k = 1, \dots, n$ is a two-dimensional Ornstein–Uhlenbeck operator acting in the variables $\xi_k = (x_{2k-1}, x_{2k})$ of the form

$$\mathcal{L}_{\theta_k} = \frac{1}{2}\Delta_k - \langle \xi_k, \nabla_k\rangle + \left\langle R(\theta_k)\xi_k, \nabla_k\right\rangle.$$

Here Δ_k and ∇_k denote the two-dimensional Laplacian and gradient in the variables (x_{2k-1}, x_{2k}).

Thus, the operators $\mathcal{L}_{\theta_k}$, $k = 1, \ldots, n$, commute as do the semigroups generated by them. This implies that the kernel $h_t^\Theta(x, y)$ of the semigroup $(\mathrm{e}^{t\mathcal{L}_\Theta})_{t\ge 0}$ is the product of the kernels of the semigroups $(\mathrm{e}^{t\mathcal{L}_{\theta_k}})_{t\ge 0}$, $k = 1, \ldots, n$; i.e.,

$$h_t^\Theta(x, y) = \prod_{k=1}^n h_t^{\theta_k}(\xi_k, \eta_k)$$

with $\xi_k = (x_{2k-1}, x_{2k})$ and $\eta_k = (y_{2k-1}, y_{2k})$ in $\mathbb{R}^2$, where $h_t^{\theta_k}(\xi_k, \eta_k)$ are as in (3.6).

If d is odd, then $\mathcal{L}_\Theta = \mathcal{L}_{\theta_1} + \cdots + \mathcal{L}_{\theta_n} + \mathcal{L}_{n+1}$, where $\mathcal{L}_{\theta_k}$, $k = 1, \ldots, n$, are as before, and $\mathcal{L}_{n+1}$ is the one-dimensional symmetric Ornstein–Uhlenbeck operator $\frac{1}{2}\partial^2_{x_{n+1}} - x_{n+1}\partial_{x_{n+1}}$ acting in the variable x_{n+1}. Thus, the kernel $h_t(x, y)$ has an additional factor $h_t^0(x_{n+1}, y_{n+1})$, which is the kernel of a one-dimensional symmetric Ornstein–Uhlenbeck semigroup.

In any case, regardless of the parity of d, by (3.6) we may write the kernel of $\mathrm{e}^{t\mathcal{L}_\Theta}$ in the following way

$$\begin{aligned} h_t^\Theta(x, y) &= h_t^0(x, y) \prod_{j=1}^n k_{t\theta_j}(\xi_j, \eta_j) \\ &= h_t^0(x, y) \prod_{\theta_j \neq 0} k_{t\theta_j}(\xi_j, \eta_j), \end{aligned} \tag{3.9}$$

where $h_t^0(x, y)$ is the kernel of the d-dimensional symmetric semigroup generated by $\frac{1}{2}\Delta - \langle x, \nabla\rangle$, and each $k_{t\theta_j}$ is a two-dimensional kernel as in (3.7).

Finally, we show that the analysis of any operator $\mathcal{L} = \frac{1}{2}\Delta - \langle x, \nabla\rangle + \langle Rx, \nabla\rangle$, where R is a skew adjoint matrix, can be reduced to that of an operator of the form $\mathcal{L}_\Theta$. As in Sect. 2, given an invertible real $d \times d$-matrix M, we denote by $\Phi_M : C(\mathbb{R}^d) \to C(\mathbb{R}^d)$ the transformation defined by $\Phi_M u(y) = u(M^{-1}y)$.

Theorem 3.1 *Let $n = [\![d/2]\!]$ be the greatest integer less than or equal to $d/2$, and let $\mathcal{L}$ be the operator $\frac{1}{2}\Delta - \langle x, \nabla\rangle + \langle Rx, \nabla\rangle$, where R is a $d \times d$ real skew-adjoint matrix. Then there exists a $d \times d$ orthogonal matrix g and a vector $\Theta = (\theta_1, \ldots, \theta_n)$ with $\theta_j \ge 0$ such that $\Phi_g \mathcal{L} \Phi_g^{-1} = \mathcal{L}_\Theta$. Moreover, the kernels $h_t(x, y)$ and $h_t^\Theta(x, y)$ of the semigroups generated by $\mathcal{L}$ and $\mathcal{L}_\Theta$, respectively, satisfy the identity*

$$h_t(x, y) = h_t^\Theta(gx, gy), \quad \forall x, y \in \mathbb{R}^d,\ t > 0.$$

Proof The set $\mathfrak{a} = \{R(\Theta) : \Theta \in \mathbb{R}^n\}$ is a maximal abelian subalgebra of the Lie algebra $\mathfrak{so}(d)$ of skew-symmetric $d \times d$ matrices. Since, by a well-known result of Lie algebras (see [1]), every element of $\mathfrak{so}(d)$ is conjugated to an element of $\mathfrak{a}^+ = \{R(\Theta) : \Theta \in \overline{\mathbb{R}_+}^n\}$, given a skew-symmetric matrix R, there exists an orthogonal matrix g and a vector $\Theta = (\theta_1, \ldots, \theta_n)$ with $\theta_j \ge 0$ such that $R = gR(\Theta)g^{-1}$. The identity $\Phi_g \mathcal{L} \Phi_g^{-1} = \mathcal{L}_\Theta$ follows, because the symmetric part $\frac{1}{2}\Delta - \langle x, \nabla\rangle$ of the operator $\mathcal{L}$ commutes with Φ_g.

This implies that $\Phi_g e^{t\mathcal{L}} \Phi_g^{-1} = e^{t\mathcal{L}_\Theta}$ for every $t \geq 0$. From this the identity between the kernels of the semigroups immediately follows. □

4 Strong Type Estimates

In this section we return to consider an Ornstein–Uhlenbeck semigroup $(\mathcal{H}_t^{Q,B})_{t\geq 0}$ with arbitrary covariance Q and drift B. We prove that the truncated Ornstein–Uhlenbeck maximal operator $\mathcal{H}_{*,[0,T]}^{Q,B}$ is always unbounded on $L^1(\gamma_\infty)$ and, when the semigroup is normal, the full maximal operator $\mathcal{H}_*^{Q,B}$ is bounded on $L^p(\gamma_\infty)$, $1 < p \leq \infty$.

Theorem 4.1 *For all $T > 0$, the operator $\mathcal{H}_{*,[0,T]}^{Q,B}$ is unbounded on $L^1(\gamma_\infty)$.*

Proof Suppose, by contradiction, that $\mathcal{H}_{*,[0,T]}^{Q,B}$ is bounded on $L^1(\gamma_\infty)$ for some $T > 0$. Denote by γ_∞ the density of the invariant measure with respect to the Lebesgue measure. Let (f_n) be a sequence of nonnegative functions of norm 1 in $L^1(\gamma_\infty)$ which converges in the sense of distributions to $\gamma_\infty(0)^{-1}\delta_0$. Then there exists a constant C such that $\|\mathcal{H}_{*,[0,T]}^{Q,B} f_n\|_1 \leq C$ for every n. Moreover,

$$\lim_{n\to\infty} \mathcal{H}_t^{Q,B} f_n(x) = \lim_{n\to\infty} \int h_t(x,y) f_n(y)\, d\gamma_\infty(y) = h_t(x,0)$$

uniformly on compact subsets of $\mathbb{R}^d$. Thus, for n sufficiently large,

$$\mathcal{H}_{*,[0,T]}^{Q,B} f_n(x) \geq \mathcal{H}_t^{Q,B} f_n(x) \geq h_t(x,0) - 1, \quad \forall x \in B(0,1),\ \forall t \in [0,T].$$

Hence,

$$\int_{|x|\leq 1} \sup_{t\in[0,T]} h_t(x,0)\, d\gamma_\infty(x) \leq C. \tag{4.1}$$

Now recall the expression of the kernel $h_t(x,y)$ given in (3.1). Since $Q_t \sim tQ$ for $t \to 0^+$, if $t \in (0,\epsilon)$ for some $\epsilon > 0$ sufficiently small, then there exist positive constants c_0, c_1, and c_2 such that

$$\begin{aligned} h_t(x,0) &= \left(\frac{\det Q_\infty}{\det Q_t}\right)^{1/2} \exp\left\{-\frac{1}{4}\langle Q_t^{-1} e^{tB}x, e^{tB}x\rangle\right\} \\ &\geq c_0 t^{-d/2} \exp\left\{-c_1 \frac{|e^{tB}x|^2}{t}\right\} \\ &\geq c_0 t^{-d/2} \exp\left\{-c_2 \frac{|x|^2}{t}\right\}. \end{aligned}$$

Thus, if $|x| \leq 1$, we have

$$\sup_{0<t<\epsilon} h_t(x,0) \geq c_0 \sup_{0<t<\epsilon} t^{-d/2} e^{-c_2 \frac{|x|^2}{t}} \geq c_\epsilon |x|^{-d},$$

which contradicts (4.1). □

The positive result for $L^p(\gamma_\infty)$, $1 < p \le \infty$, for normal Ornstein–Uhlenbeck semigroups follows from a more general result for normal semigroups of contractions on all L^p-spaces, whose generators on L^2 are sectorial. Indeed, we have the following theorem.

Theorem 4.2 *Let (X, μ) be a σ-finite measure space. Let $(T_t)_{t\ge 0}$ be a semigroup of contractions on $L^p(\mu)$ for every p in $[1, \infty]$, which is strongly continuous for $p < \infty$. Suppose that each T_t is normal on $L^2(\mu)$ and that the spectrum of the generator $\mathcal{G}$ on $L^2(\mu)$ is contained in the sector $-\overline{S}_\theta$ for some $\theta \in [0, \pi/2)$. Then the maximal operator*

$$T_* f(x) = \sup_{t>0} \left| T_t f(x) \right|$$

is bounded on $L^p(\mu)$ for $1 < p \le \infty$.

Proof By examining carefully Stein's proof of the maximal theorem for self-adjoint semigroups of contractions (see [11, pp. 73–81]) one realizes that the self-adjointness plays a rôle only in the proof of the boundedness on $L^2(\mu)$ of the Littlewood–Paley functions

$$g_k(f)(x) = \left(\int_0^\infty \left| t^k D_t^k T_t f(x) \right|^2 \frac{\mathrm{d}t}{t} \right)^{1/2}, \quad k = 1, 2, \dots .$$

However, the same result can also be obtained under the assumptions of the theorem. Indeed, let

$$-\mathcal{G} = \int_{\overline{S}_\theta} z \, \mathrm{d}\mathcal{P}_z$$

be the spectral resolution of $-\mathcal{G}$. By the spectral theorem for normal operators,

$$D_t^k T_t f = (-1)^k \int_{\overline{S}_\theta^+} z^k \mathrm{e}^{-tz} \, \mathrm{d}\mathcal{P}_z f,$$

where $\overline{S}_\theta^+ = \overline{S}_\theta \setminus \{0\}$. Hence,

$$\left\| D_t^k T_t f \right\|_2^2 = \int_{\overline{S}_\theta^+} |z|^{2k} \mathrm{e}^{-2t \operatorname{Re} z} \langle \mathrm{d}\mathcal{P}_z f, f \rangle.$$

Thus,

$$\begin{aligned}
\int_X \left| g_k(f)(x) \right|^2 \mathrm{d}\mu(x) &= \int_X \int_0^\infty \left| t^k D_t^k T_t f(x) \right|^2 \frac{\mathrm{d}t}{t} \, \mathrm{d}\mu(x) \\
&= \int_0^\infty t^{2k} \int_X \left| D_t^k T_t f(x) \right|^2 \mathrm{d}\mu(x) \frac{\mathrm{d}t}{t} \\
&= \int_0^\infty \int_{\overline{S}_\theta^+} |tz|^{2k} \mathrm{e}^{-2t \operatorname{Re} z} \langle \mathrm{d}\mathcal{P}_z f, f \rangle \frac{\mathrm{d}t}{t}
\end{aligned}$$

$$= \int_{\overline{S}_\theta^+} \int_0^\infty |tz|^{2k} \mathrm{e}^{-2t \operatorname{Re} z} \frac{\mathrm{d}t}{t} \langle \mathrm{d}\mathcal{P}_z f, f \rangle$$
$$\leq \frac{\Gamma(2k)}{(2\cos\theta)^{2k}} \int_{\overline{S}_\theta^+} \langle \mathrm{d}\mathcal{P}_z f, f \rangle$$
$$\leq \frac{\Gamma(2k)}{(2\cos\theta)^{2k}} \|f\|_2^2,$$

since $|z| \leq (\cos\theta)^{-1} \operatorname{Re} z$ in $\overline{S}_\theta$. This proves that $f \mapsto g_k(f)$ is bounded on $L^2(\mu)$. The rest of the proof is just as in [11, pp. 76–81]. □

Corollary 4.3 *Let $(\mathcal{H}_t^{Q,B})_{t\geq 0}$ be a normal Ornstein–Uhlenbeck semigroup. Then the maximal operator $\mathcal{H}_*^{Q,B}$ is bounded on $L^p(\gamma_\infty)$ for every p in $(1,\infty)$.*

Proof By [7] the spectrum of the generator of $(\mathcal{H}_t^{Q,B})_{t\geq 0}$ is contained in a sector of angle less than $\pi/2$. Hence, the conclusion follows from Theorem 4.2. □

5 The Weak-Type Estimate

In this section we shall prove the weak type 1 estimate for the maximal operators associated to the normal Ornstein–Uhlenbeck semigroup $(\mathcal{H}_t^{Q,B})_{t\geq 0}$ with covariance $Q = I$ and drift $B = \frac{1}{2\alpha}(R - I)$, where $\alpha > 0$ and R is a skew-symmetric real matrix, i.e., for the semigroup generated by the operator

$$\mathcal{L}(\alpha, R) = \frac{1}{2}\Delta - \frac{1}{2\alpha}\langle x, \nabla\rangle + \frac{1}{2\alpha}\langle Rx, \nabla\rangle.$$

Namely, we shall prove the following theorem.

Theorem 5.1 *For every $T > 0$, the truncated maximal operator*

$$\mathcal{H}_{*,[0,T]} f(x) = \sup_{t\in[0,T]} \left| \mathrm{e}^{t\mathcal{L}(\alpha,R)} f(x) \right|$$

is of weak type 1. If the one-parameter group $(\mathrm{e}^{tR})_{t\in\mathbb{R}}$ is periodic, then the full maximal operator $\mathcal{H}_* f(x) = \sup_{t\geq 0} |\mathrm{e}^{t\mathcal{L}(\alpha,R)} f(x)|$ is of weak type 1.

As we have already remarked in Sect. 3, by a scaling argument we may assume that $2\alpha = 1$.

First, we reduce the problem to proving that two smaller maximal operators are of weak type 1. For every subset A of $\mathbb{R}_+$, denote by $\mathcal{H}_{*,A}$ the maximal operator defined by

$$\mathcal{H}_{*,A} f(x) = \sup_{t\in A} \left| \mathrm{e}^{t\mathcal{L}(1/2,R)} f(x) \right|, \qquad f \in L^1(\gamma_\infty).$$

If I is a closed interval in $\mathbb{R}_+$ and P is a positive number, we denote by $I_P^\sharp$ the union of $P\mathbb{N}$-translates of I, i.e., $I^\sharp = \bigcup_{n\in\mathbb{N}} (I + Pn)$.

Lemma 5.2 *Suppose that, for some* $t_0 > 0$, *the maximal operator* $\mathcal{H}_{*,[0,t_0]}$ *is of weak type* 1. *Then the truncated maximal operator* $\mathcal{H}_{*,[0,T]}$ *is of weak type* 1 *for every* $T > 0$. *If, furthermore, there exists an interval* I *in* $\mathbb{R}_+$ *such that the operator* $\mathcal{H}_{*,I_P^\sharp}$ *is of weak type* 1, *then the full maximal operator* $\mathcal{H}_*$ *is of weak type* 1.

Proof First, we show that if A is a subset of $\mathbb{R}_+$ such that the operator $\mathcal{H}_{*,A}$ is of weak type 1 and $B = \bigcup_{i=1}^N (A + t_i)$ is a finite union of translates of A, then $\mathcal{H}_{*,B}$ is of weak type 1. Indeed,

$$\begin{aligned}\mathcal{H}_{*,B} f(x) &= \sup_{t\in B} \left| e^{t\mathcal{L}(1/2,R)} f(x) \right| = \max_{i=1,\dots,N} \sup_{t\in A} \left| e^{(t+t_i)\mathcal{L}(1/2,R)} f(x) \right| \\ &= \max_{i=1,\dots,N} \sup_{t\in A} \left| e^{t\mathcal{L}(1/2,R)} e^{t_i\mathcal{L}(1/2,R)} f(x) \right| \\ &= \max_{i=1,\dots,N} \mathcal{H}_{*,A} e^{t_i\mathcal{L}(1/2,R)} f(x).\end{aligned}$$

Hence, for $\lambda > 0$ fixed,

$$\begin{aligned}&\gamma_\infty\big(\big\{x \in \mathbb{R}^d : \mathcal{H}_{*,B} f(x) > \lambda\big\}\big) \\ &\quad\le \sum_{i=1}^N \gamma_\infty\big(\big\{x \in \mathbb{R}^d : \mathcal{H}_{*,A} e^{t_i\mathcal{L}(1/2,R)} f(x) > \lambda\big\}\big) \\ &\quad\le \frac{C}{\lambda} \sum_{i=1}^N \big\| e^{t_i\mathcal{L}(1/2,R)} f \big\|_{L^1(\gamma_\infty)} \\ &\quad\le \frac{CN}{\lambda} \|f\|_{L^1(\gamma_\infty)},\end{aligned}$$

since $e^{t_i\mathcal{L}(1/2,R)}$ is a contraction on $L^1(\gamma_\infty)$ for every $i = 1, \dots, N$.

The conclusion follows because the set $[0, T]$ is a finite union of translates of $(0, t_0)$ and $\mathbb{R}_+$ is a finite union of translates of $[0, T]$ and $I_P^\sharp$. □

Thus, we only need to prove the weak type 1 estimate for the operator $\mathcal{H}_{*,A}$ when $A = (0, t_0)$ and $A = I_P^\sharp$ for some $t_0 > 0$ and some closed interval I in $\mathbb{R}_+$. As in the analysis of the maximal operator for the symmetric Ornstein–Uhlenbeck semigroup $(e^{t\mathcal{L}(1/2,0)})_{t\ge 0}$ (see [4]), we shall decompose each of these two maximal operators in a "local" part, given by a kernel living close to the diagonal, and the remaining or "global" part. To this end consider the set

$$L = \big\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \le \min\big(1, |x + y|^{-1}\big)\big\}$$

and denote by G its complement. We shall call L and G the 'local' and 'global' regions, respectively. The local and global parts of the operator $\mathcal{H}_{*,A}$ are defined by

$$\begin{aligned}\mathcal{H}^{\text{loc}}_{*,A} f(x) &= \sup_{t\in A}\left|\int h_t(x,y)\mathbf{1}_L(x,y) f(y)\,\mathrm{d}\gamma(y)\right|,\\ \mathcal{H}^{\text{glob}}_{*,A} f(x) &= \sup_{t\in A}\left|\int h_t(x,y)\mathbf{1}_G(x,y) f(y)\,\mathrm{d}\gamma(y)\right|,\end{aligned} \tag{5.1}$$

where $\mathbf{1}_L$ and $\mathbf{1}_G$ are the characteristic functions of the sets L and G respectively. Clearly,

$$\mathcal{H}_{*,A} f(x) \le \mathcal{H}^{\text{loc}}_{*,A} f(x) + \mathcal{H}^{\text{glob}}_{*,A} f(x).$$

We shall prove separately the weak type 1 estimate for $\mathcal{H}^{\text{loc}}_{*,A}$ and $\mathcal{H}^{\text{glob}}_{*,A}$.

First, we deal with the local part. We shall actually prove that for all Ornstein–Uhlenbeck semigroups $(\mathcal{H}_t)_{t\ge 0}$, without restrictions on covariance and drift, the local maximal operator $\mathcal{H}^{\text{loc}}_{*} = \mathcal{H}^{\text{loc}}_{*,\mathbb{R}_+}$ is of weak type 1.

Lemma 5.3 *Let $(\mathcal{H}_t)_{t\ge 0}$ be an Ornstein–Uhlenbeck semigroup with arbitrary covariance and drift. Then there exist positive constants c and C such that, for all (x,y) in the local region L,*

$$h_t(x,y) \le C\big(1-\mathrm{e}^{-t}\big)^{-d/2}\gamma_\infty(y)^{-1}\exp\left(-c\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right), \quad \forall t>0. \tag{5.2}$$

Proof Since the real part of the eigenvalues of B is negative, there exist positive constants $\alpha\le\beta$ and C_0 such that $C_0^{-1}\mathrm{e}^{2\alpha s}|x|^2 \le C_0|\mathrm{e}^{sB^*}| \le \mathrm{e}^{2\beta s}|x|^2$ for all $x\in\mathbb{R}^d$ and all $s\in\mathbb{R}$. Thus, by (1.1) there exists a positive constant C such that

$$C^{-1}\big(1-\mathrm{e}^{-t}\big)I \le Q_t \le C\big(1-\mathrm{e}^{-t}\big)I, \quad \forall t\in(0,\infty],$$

and, by (3.1), there exist two positive constants c and C such that

$$h_t(x,y) \le C\big(1-\mathrm{e}^{-t}\big)^{-d/2}\gamma_\infty(y)^{-1}\exp\left(-c\frac{|\mathrm{e}^{tB}x-y|^2}{1-\mathrm{e}^{-t}}\right). \tag{5.3}$$

Now, for all (x,y) in the local region L,

$$\begin{aligned}\big|\mathrm{e}^{tB}x-y\big|^2 &= \big|x-y+\big(\mathrm{e}^{tB}-I\big)x\big|^2\\ &= |x-y|^2 + \big|\big(\mathrm{e}^{tB}-I\big)x\big|^2 + 2\big\langle x-y,\big(\mathrm{e}^{tB}-I\big)x\big\rangle\\ &\ge |x-y|^2 - 2\big\|\mathrm{e}^{tB}-I\big\|\,|x-y||x|\\ &\ge |x-y|^2 - C\big(1-\mathrm{e}^{-t}\big),\end{aligned}$$

since $\|\mathrm{e}^{tB}-I\| \le C(1-\mathrm{e}^{-t})$ and $|x-y||x|\le C$ in the local region L. □

Proposition 5.4 *Let $(\mathcal{H}_t)_{t\ge 0}$ be an Ornstein–Uhlenbeck semigroup with arbitrary covariance and drift. Then the maximal operator $\mathcal{H}^{\text{loc}}_{*}$ is of weak type 1.*

Proof By Lemma 5.3 one has that, for each $f \geq 0$,

$$\begin{aligned}\mathcal{H}_*^{\mathrm{loc}} f(x) &\leq C \sup_{0<s\leq 1} s^{-d/2} \int \mathrm{e}^{-c\frac{|x-y|^2}{s}} \mathbf{1}_L(x,y) f(y)\,\mathrm{d}\lambda(y) \\ &= \mathcal{W} f(x),\end{aligned}$$

say. Since the operator $\mathcal{W}$ is of weak type 1 with respect to the Lebesgue measure and its kernel is supported in the local region L, the conclusion follows by well-known arguments (see for instance [4, Sect. 3]). □

Now we turn to the proof of the weak-type estimate for the global part of the maximal operator associated to the semigroup generated by the special Ornstein–Uhlenbeck operator

$$\mathcal{L}(1/2, R) = \frac{1}{2}\Delta - \langle x, \nabla\rangle + \langle Rx, \nabla\rangle,$$

where R is a skew-symmetric real matrix.

As in Sect. 3, we denote by $h_t(x,y)$ and $h_t^0(x,y)$ the kernels with respect to the invariant measure of the semigroups generated by $\mathcal{L}(1/2, R)$ and its symmetric part

$$\mathcal{L}^0 = \frac{1}{2}\Delta - \langle x, \nabla\rangle,$$

respectively (see (3.2) and (3.3)).

To estimate the semigroup kernel in the global region, it is convenient to simplify the expression of $h_t^0(x,y)$ by means of the change of variables in the parameter t introduced in [4]. We denote by τ the function defined by

$$\tau(s) = \log\frac{1+s}{1-s}, \quad s \in (0,1). \tag{5.4}$$

Notice that τ maps $(0,1)$ onto $\mathbb{R}_+$. It is straightforward to check (see [4]) that, for all s in $(0,1)$,

$$h^0_{\tau(s)}(x,y) = (4s)^{-d/2}(1+s)^d \mathrm{e}^{\frac{|x|^2+|y|^2}{2} - \frac{1}{4}(s|x+y|^2 + \frac{1}{s}|x-y|^2)}. \tag{5.5}$$

Next, as in [4], we introduce the quadratic form

$$\mathcal{Q}_s(x,y) = \left|(1+s)x - (1-s)y\right|^2, \quad x, y \in \mathbb{R}^d. \tag{5.6}$$

Thus,

$$s|x+y|^2 + \frac{1}{s}|x-y|^2 = \frac{1}{s}\mathcal{Q}_s(x,y) - 2|x|^2 + 2|y|^2,$$

and

$$h^0_{\tau(s)}(x,y) = s^{-d/2}\exp\left\{|x|^2 - \frac{1}{4s}\mathcal{Q}_s(x,y)\right\}, \quad \forall s \in (0,1). \tag{5.7}$$

Lemma 5.5 *If $t_0 > 0$ is sufficiently small, there exists a positive constant C such that, for all s in $(0, \tau^{-1}(t_0))$ and all (x, y) in $\mathbb{R}^d \times \mathbb{R}^d$,*

$$h_{\tau(s)}(x, y) \le C s^{-\frac{d}{2}} \mathrm{e}^{|x|^2 - \frac{1}{40s} \mathcal{Q}_s(x,y)}. \tag{5.8}$$

Proof Let $n = [\![d/2]\!]$. The right-hand side of the inequality to prove is invariant under orthogonal transformations. Hence, by Theorem 3.1, it is enough to prove the inequality for the kernel $h_t^{\Theta}(x, y)$ with $\Theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, $\theta_j \ge 0$.

By (3.9) and (5.7),

$$h^{\Theta}_{\tau(s)}(x, y) \le s^{-d/2} \exp\left\{ |x|^2 - \frac{1}{4s} \mathcal{Q}_s(x, y) \right\} \prod_{\theta_j > 0} k_{\tau(s)\theta_j}(\xi_j, \eta_j),$$
$$\forall s \in (0, 1),$$

where $\xi_j = (x_{2j-1}, x_{2j})$ and $\eta_k = (y_{2j-1}, y_{2j})$ are in $\mathbb{R}^2$, and each $k_{t\theta_j}$ is a two-dimensional kernel as in (3.7).

Define

$$M_s(x, y) = \exp\left\{ -\frac{9}{40s} \mathcal{Q}_s(x, y) \right\} \prod_{\theta_j > 0} k_{\tau(s)\theta_j}(\xi_j, \eta_j). \tag{5.9}$$

Then

$$h^{\Theta}_{\tau(s)}(x, y) \le s^{-d/2} \exp\left\{ |x|^2 - \frac{1}{40s} \mathcal{Q}_s(x, y) \right\} M_s(x, y),$$

and to conclude the proof of the lemma, all we need to show is that there exist $s_0 > 0$ sufficiently small and a constant C such that

$$M_s(x, y) \le C, \quad \forall s \in (0, s_0), \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{5.10}$$

Let us denote by $\mathcal{Q}_s^{(m)}$ the quadratic form defined in (5.6) when considered as a function on $\mathbb{R}^m \times \mathbb{R}^m$. Then

$$\mathcal{Q}_s^{(d)}(x, y) = \begin{cases} \sum_{j=1}^n \mathcal{Q}_s^{(2)}(\xi_j, \eta_j) & \text{if } d \text{ is even}, \\ \sum_{j=1}^n \mathcal{Q}_s^{(2)}(\xi_j, \eta_j) + \mathcal{Q}_s^{(1)}(x_{n+1}, y_{n+1}) & \text{if } d \text{ is odd}. \end{cases}$$

Thus, since $\mathcal{Q}_s^{(m)}$ is nonnegative,

$$M_s(x, y) \le \prod_{\theta_j > 0} \exp\left\{ -\frac{9}{40s} \mathcal{Q}_s^{(2)}(\xi_j, \eta_j) \right\} k_{\tau(s)\theta_j}(\xi_j, \eta_j)$$

regardless of the parity of d. Hence, we only need to show that each factor is bounded, i.e., that for every $\theta > 0$, there exist $s_0 \in (0, 1)$ and a constant C such that, for all $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$\exp\left\{ -\frac{9}{40s} \mathcal{Q}_s(x, y) \right\} k_{\tau(s)\theta}(x, y) \le C, \quad \forall s \in (0, s_0), \tag{5.11}$$

where now $\mathcal{Q}_s = \mathcal{Q}_s^{(2)}$, for the sake of brevity.

To this end we fix β in $(0,1)$, we let δ be a constant in $(0,1)$ to be chosen later, and we denote by $\vartheta = \vartheta(x,y)$ the angle between the two vectors x and y. The set $\mathbb{R}^2 \times \mathbb{R}^2$ is the disjoint union of the five sets

$$
\begin{aligned}
R_1 &= \big\{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \langle x,y\rangle < 0\big\},\\
R_2 &= \big\{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \langle x,y\rangle \ge 0,\ x \wedge y \ge 0\big\},\\
R_3 &= \big\{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \langle x,y\rangle \ge 0,\ x \wedge y < 0,\ |x-y| \ge \beta|y|\big\},\\
R_4 &= \big\{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \langle x,y\rangle \ge 0,\ x \wedge y < 0,\ |x-y| < \beta|y|,\ |\sin\vartheta| \ge \delta\big\},\\
R_5 &= \big\{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \langle x,y\rangle \ge 0,\ x \wedge y < 0,\ |x-y| < \beta|y|,\ |\sin\vartheta| < \delta\big\}.
\end{aligned}
$$

We shall prove that (5.11) holds in each region R_j, $j = 1,\dots,5$. Note that by (3.7) and (5.4)

$$k_{\tau(s)\theta}(x,y) = \mathrm{e}^{-\frac{1-s^2}{4s}[(1-\cos(\tau(s)\theta))\langle x,y\rangle + \sin(\tau(s)\theta) x\wedge y]} \tag{5.12}$$

and that the function $s \mapsto \tau(s)$ is positive and increasing in $(0,1)$ and $\tau(s) \sim 2s$ as $s \to 0^+$. To prove the estimate in R_1, we observe that there exists a constant C_1 such that

$$k_{\tau(s)\theta}(x,y) \le \exp\big\{C_1|x||y|\big\}, \quad \forall x,y \in \mathbb{R}^2,\ \forall s \in (0,1). \tag{5.13}$$

Since $\mathcal{Q}_s(x,y) \ge (1-s^2)(|x|^2+|y|^2)$, because $\langle x,y\rangle < 0$ in R_1, we have that if s_0 is sufficiently small,

$$-\frac{9}{40s}\mathcal{Q}_s(x,y) + C_1|x|\,|y| < 0, \quad \forall (x,y) \in R_1,\ \forall t \in (0,s_0). \tag{5.14}$$

Together (5.13) and (5.14) imply (5.11) in R_1.

The proof of (5.11) in R_2 is straightforward, because in this region $\mathcal{Q}_s(x,y) \ge 0$ and $k_{\tau(s)\theta}(x,y) \le 1$.

Next, suppose that (x,y) is in R_3. Since $\langle x,y\rangle \ge 0$, there exists a constant C_2 such that

$$
\begin{aligned}
k_{\tau(s)\theta}(x,y) &\le \exp\big(C_2|x\wedge y|\big)\\
&= \exp\big(C_2|x||y||\sin\vartheta|\big), \quad \forall s \in (0,1).
\end{aligned} \tag{5.15}
$$

We claim that there exists $s_0 \in (0,1)$ such that

$$-\frac{9}{40s}\mathcal{Q}_s(x,y) + C_2|x||y||\sin\vartheta| \le 0, \quad \forall s \in (0,s_0). \tag{5.16}$$

To prove the claim, first consider the case where $|x| \ge |y|$. Then $\mathcal{Q}_s(x,y) \ge |x-y|^2$ and hence, since $|x-y| \ge |\sin\vartheta||x|$ and $|x-y| \ge \beta|y|$,

$$-\frac{9}{40s}\mathcal{Q}_s(x,y) + C_2|x||y||\sin\vartheta| \le \bigg(-\frac{9}{40s}\beta + C_2\bigg)|x||y||\sin\vartheta| \le 0,$$

provided that $s < \frac{9\beta}{40C_2}$.

Next consider the case where $|x| < |y|$. In this case we have that $\mathcal{Q}_s(x, y) \ge |x - y|^2 - 2s|y|^2$. Thus, since $|x| < |y|$ and $|x - y| \ge \beta|y|$,

$$-\frac{9}{40s}\mathcal{Q}_s(x, y) + C_2|x||y||\sin\vartheta| \le \left(-\frac{9}{40s}\beta^2 + \frac{9}{20} + C_2\right)|y|^2 \le 0,$$

provided that $s < \frac{9\beta^2}{40C_2+18}$.

Thus, (5.16) holds for all (x, y) in R^3 with $s_0 \le \min\{\frac{9\beta}{40C_2}, \frac{9\beta^2}{40C_2+18}\}$. Together (5.15) and (5.16) imply (5.11) in R_3.

The proof of estimate (5.11) in R_4 is similar. Indeed, first of all, (5.15) holds in R_4, because here also $\langle x, y\rangle > 0$. Moreover, arguing much as before, one can show that (5.16) also holds for all (x, y) in R_4 with $s_0 \le \min\{\frac{9\delta^2}{40C_2}, \frac{9\delta^2}{40C_2+18}\}$. The only difference is that one uses the estimates

$$\mathcal{Q}_s(x, y) \ge |x - y|^2 \ge (\sin\vartheta)^2|x|^2 \ge \delta^2|x||y|$$

when $|x| \ge |y|$ and

$$\mathcal{Q}_s(x, y) \ge |x - y|^2 - 2s|y|^2 \ge (\sin\vartheta)^2|y|^2 - 2s|y^2| \ge \left(\delta^2 - 2s\right)|y|^2$$

when $|x| < |y|$. We omit the details. Notice that, so far, we did not need to impose any restriction on δ, which therefore could be any number in $(0, 1)$.

It remains to estimate $h_t(x, y)$ in R_5. We observe that since $\tau(s) \sim 2s$ as $s \to 0^+$ and $\langle x, y\rangle \ge 0$ and $x \wedge y < 0$ in R_5, by (5.12) there exist $s_0 > 0$ and two positive constants $c_0 < 2 < c_1$ such that

$$k_{\tau(s)\theta}(x, y) \le \exp\left\{-c_0\frac{\theta^2}{4}s\langle x, y\rangle - c_1\frac{\theta}{4}x \wedge y\right\}, \quad \forall s \in (0, s_0). \tag{5.17}$$

Moreover, we can choose c_0 and c_1 as close to 2 as we want, provided that we choose s_0 sufficiently small; in particular, we may take

$$c_1^2/c_0 < 18/5. \tag{5.18}$$

Now we are ready to prove estimate (5.11) in R_5. Define

$$E_s(x, y) = -\frac{9}{10}\mathcal{Q}_s(x, y) - c_0\theta^2 s^2\langle x, y\rangle - c_1\theta s x \wedge y.$$

By (5.17),

$$\exp\left\{-\frac{9}{40s}\mathcal{Q}_s(x, y)\right\}k_{\tau(s)\theta}(x, y) \le \exp\left\{\frac{1}{4s}E_s(x, y)\right\}.$$

Thus, to prove (5.11) in R_5 it is enough to show that

$$E_s(x, y) \le 0, \quad \forall s \in (0, s_0),\ \forall (x, y) \in R_5, \tag{5.19}$$

provided that s_0, β, and δ are sufficiently small.

Observe that

$$E_s(x,y) = \lambda(x,y)s^2 + \mu(x,y)s + \nu(x,y),$$

where

$$\begin{aligned}\lambda(x,y) &= -\frac{9}{10}|x+y|^2 - c_0\theta^2\langle x,y\rangle,\\ \mu(x,y) &= \frac{18}{10}\big(|y|^2 - |x|^2\big) - c_1\theta x\wedge y,\\ \nu(x,y) &= -\frac{9}{10}|x-y|^2.\end{aligned}$$

It turns out that, instead of $E_s(x,y)$, it is more convenient to consider the function $|x|^{-1}|y|^{-1}E_s(x,y)$, because the latter function depends only on the variables s, $X = |x|/|y|$, and $\vartheta = \widehat{xy}$. Indeed, if we denote by Ψ the function defined by $\Psi(s,x,y) = (s,X,\vartheta)$, then

$$|x|^{-1}|y|^{-1}E_s(x,y) = F\big(\Psi(s,x,y)\big), \tag{5.20}$$

where

$$F(s,X,\vartheta) = \tilde\lambda(X,\vartheta)s^2 + \tilde\mu(X,\vartheta)s + \tilde\nu(X,\vartheta) \tag{5.21}$$

and

$$\begin{aligned}\tilde\lambda(x,y) &= -\frac{9}{10}\big(X + X^{-1} + 2\cos\vartheta\big) - c_0\theta^2\cos\vartheta,\\ \tilde\mu(x,y) &= \frac{18}{10}\big(X^{-1} - X\big) - c_1\theta\sin\vartheta,\\ \tilde\nu(x,y) &= -\frac{9}{10}\big(X + X^{-1} - 2\cos\vartheta\big).\end{aligned}$$

It is easy to see that $(0,1,0)$ is a critical point of F and that the Hessian $\nabla^2F(0,1,0)$ is definite negative, since $c_1^2 - \frac{18}{5}c_0 < 0$ by (5.18). Thus, $(0,1,0)$ is a local maximum of F and, since $F(0,1,0) = 0$, there exists a neighborhood U of $(0,1,0)$ in which F is ≤ 0. Now, since

$$\Psi\big((0,s_0)\times R_5\big) \subset \big\{(s,X,\vartheta) : s\in(0,s_0),\ |X-1|<\beta,\ -\delta<\sin(\vartheta)\leq 0\big\},$$

we can choose s_0, β, and δ so small that $\Psi((0,s_0)\times R_5)\subset U$. Hence, $F\circ\Psi\leq 0$ in $(0,s_0)\times R_5$. Thus, (5.19) is satisfied, and the proof of the lemma is complete. □

To prove the boundedness of the nontruncated maximal operator we need to assume that the one-parameter group $(\mathrm{e}^{tR})_{t\in\mathbb{R}}$ generated by the skew-adjoint matrix R is periodic. We recall that if I is an interval contained in $\mathbb{R}_+$ and $P>0$, we denote by $I_P^\sharp$ the set $\bigcup_{n\in\mathbb{N}}(I+nP)$.

Lemma 5.6 *Suppose that the skew-adjoint matrix R generates a one-parameter group $(e^{tR})_{t\in\mathbb{R}}$ which is periodic of period P. Then there exist an interval I and a constant C such that, for all s in $\tau^{-1}(I_P^\sharp)$ and all (x,y) in $\mathbb{R}^d\times\mathbb{R}^d$,*

$$h_{\tau(s)}(x,y)\le Cs^{-\frac{d}{2}}e^{|x|^2-\frac{1}{40s}\mathcal{Q}_s(x,y)}.$$

Proof As in the proof of Lemma 5.5, it is enough to prove the inequality for the kernel $h_t^\Theta(x,y)$ with $\Theta=(\theta_1,\dots,\theta_d)\in\mathbb{R}^d$, $\theta_j\ge 0$. Let $\{\theta_1,\dots,\theta_m\}$ be the nonzero components of Θ, i.e., the absolute values of the nonzero eigenvalues of R. Denote by $\theta_{\max}$ the maximum of $\{\theta_1,\dots,\theta_m\}$.

Fix $\delta=\min\{\theta_{\max}^{-1},1/10\}$, and let ϵ be a small positive constant ($\epsilon\le 1/10$ will do). Define $I=[\delta,(1+\epsilon)\delta]$. For all $\theta\in\{\theta_1,\dots,\theta_m\}$, the functions $t\mapsto\cos(\theta t)$ and $t\mapsto\sin(\theta t)$ are periodic of period P, and by considering their Taylor expansions at zero it is easy to see that, for all $\theta\in\{\theta_1,\dots,\theta_m\}$,

$$c_0\le 1-\cos(\theta t)\le c_2,\qquad \sin(\theta t)\le c_1,\quad \forall t\in I_P^\sharp, \tag{5.22}$$

where

$$c_0=\frac{5}{12}\theta^2\delta^2,\qquad c_1=(1+\epsilon)\delta\theta,\quad\text{and}\quad c_2=\frac{(1+\epsilon)^2\delta^2\theta^2}{2}. \tag{5.23}$$

Arguing as in the proof of Lemma 5.5, we can reduce matters to showing that if $\theta\in\{\theta_1,\dots,\theta_m\}$, then there exists a constant C such that

$$e^{-\frac{9}{40s}\mathcal{Q}_s(x,y)}k_{\tau(s)\theta}(x,y)\le C,\quad \forall s\in\tau^{-1}\big(I_P^\sharp\big),\ \forall(x,y)\in\mathbb{R}^2\times\mathbb{R}^2. \tag{5.24}$$

For the sake of the reader, we recall that

$$k_{\tau(s)}(x,y)=\big\{e^{-\frac{e^{-t}}{1-e^{-2t}}[(1-\cos(t\theta))\langle x,y\rangle+\sin(t\theta)x\wedge y]}\big\}_{t=\tau(s)} \tag{5.25}$$

$$=e^{-\frac{1-s^2}{4s}[(1-\cos(\tau(s)\theta))\langle x,y\rangle+\sin(\tau(s)\theta)x\wedge y]}. \tag{5.26}$$

The set $\mathbb{R}^2\times\mathbb{R}^2$ is the disjoint union of the three sets

$$R_1=\big\{(x,y)\in\mathbb{R}^2\times\mathbb{R}^2:\langle x,y\rangle\ge 0,\ x\wedge y\ge 0\big\},$$
$$R_2=\big\{(x,y)\in\mathbb{R}^2\times\mathbb{R}^2:\langle x,y\rangle\ge 0,\ x\wedge y< 0\big\},$$
$$R_3=\big\{(x,y)\in\mathbb{R}^2\times\mathbb{R}^2:\langle x,y\rangle< 0\big\}.$$

We shall prove that (5.24) holds in each region R_j, $j=1,2,3$.

To prove (5.24) in R_1 it is enough to observe that here $k_{t\theta}(x,y)\le 1$ for all t in $\mathbb{R}_+$.

Now suppose that (x,y) is in R_2. Then, by (5.22) and (5.25) we have that

$$k_{\tau(s)\theta}(x,y)\le e^{-\frac{1-s^2}{4s}(c_0\langle x,y\rangle+c_1x\wedge y)},\quad \forall s\in\tau^{-1}\big(I_P^\sharp\big). \tag{5.27}$$

Thus,

$$\exp\left\{-\frac{9}{40s}\mathcal{Q}_s(x,y)\right\}k_{\tau(s)\theta}(x,y)\leq\exp\left\{\frac{1}{4s}F_s(x,y)\right\},$$

where

$$\begin{aligned}F_s(x,y)&=p(x,y)\,s^2+q(x,y)\,s+r(x,y),\\ p(x,y)&=-\frac{9}{10}|x+y|^2+c_0\langle x,y\rangle+c_1\,x\wedge y,\\ q(x,y)&=\frac{18}{10}\big(|y|^2-|x|^2\big),\\ r(x,y)&=-\frac{9}{10}|x-y|^2-c_0\langle x,y\rangle-c_1\,x\wedge y.\end{aligned}$$

Thus, to prove (5.24) in R_2, we only need to show that $F_s(x,y)\leq 0$ for all $(x,y)\in R_2$.

It is an easy matter to see that, with c_0 and c_1 as in (5.23), the leading coefficient $p(x,y)$ and the constant term $r(x,y)$ are negative for all (x,y) in R_2. Thus, it suffices to show that the discriminant q^2-4pr is nonpositive in R_2. If $|y|=|x|$, this is obvious, since then $q(x,y)=0$. If $|y|\neq|x|$, after some simple algebra, using the identity

$$|x+y|^2|x-y|^2=\big(|y|^2-|x|^2\big)^2+4\sin^2(\vartheta)|x|^2|y|^2,$$

we see that $(q^2-4pr)|x|^{-2}|y|^{-2}$ is only a function of the angle ϑ between x and y. Thus, its sign does not change if we rescale in x. In particular, we may reduce matters to the case $|y|=|x|$, where $q=0$. This proves that $F_s(x,y)\leq 0$ for all (x,y) in R_2 and s in $\mathbb{R}$. By (5.27) this implies that (5.24) holds in R_2.

Finally, suppose that (x,y) is in R_3. We have that

$$\exp\left\{-\frac{9}{40s}\mathcal{Q}_s(x,y)\right\}k_{\tau(s)\theta}(x,y)\leq\exp\left\{\frac{1}{4s}G_s(x,y)\right\},$$

where

$$\begin{aligned}G_s(x,y)&=\tilde{p}(x,y)s^2+q(x,y)s+\tilde{r}(x,y),\\ \tilde{p}(x,y)&=-\frac{9}{10}|x+y|^2-c_2\big|\langle x,y\rangle\big|+c_1|x\wedge y|,\\ q(x,y)&=\frac{18}{10}\big(|y|^2-|x|^2\big),\\ \tilde{r}(x,y)&=-\frac{9}{10}|x-y|^2+c_2\big|\langle x,y\rangle\big|-c_1|x\wedge y|,\end{aligned}$$

and c_1, c_2 are as in (5.23). Thus, to prove the desired inequality (5.24), we only need to show that $G_s(x,y)\leq 0$ in R_3. Since it is easy to see that both $\tilde{p}$ and $\tilde{r}$ are negative in R_3, as before, we only need to prove that $q^2-4\tilde{p}\tilde{r}\leq 0$ in R_3. This can be proved by an argument similar to that used in R_2. We omit the details.

Hence, (5.24) holds for all (x, y) in $\mathbb{R}^2 \times \mathbb{R}^2$. This concludes the proof of the lemma. □

We recall two lemmas from [4].

Lemma 5.7 *Let $\vartheta = \vartheta(x, y)$ denote the angle between the nonzero vectors x and y. There exists a constant C such that, for all (x, y) in the global region G,*

$$\sup_{0<s\leq 1} s^{-d/2} \mathrm{e}^{-\frac{1}{40s}\mathcal{Q}_s(x,y)} \leq C \min\left\{\left(1+|x|\right)^d, \left(|x|\sin\vartheta\right)^{-d}\right\}.$$

Lemma 5.8 *The operator*

$$\mathcal{T} f(x) = \mathrm{e}^{|x|^2} \int \min\left\{\left(1+|x|\right)^d, \left(|x|\sin\vartheta\right)^{-d}\right\} f(y)\,\mathrm{d}\gamma_\infty(y)$$

is of weak type 1.

We are now ready to conclude the proof of Theorem 5.1

Proof Let A denote either the set $[0, t_0]$ or $I_P^\sharp$. By Proposition 5.4 the local part of the operator $\mathcal{H}_{*,A}$ is of weak type 1. Thus, it remains only to prove that the global part is of weak type 1. By (5.1), Lemmas 5.5, 5.6, and 5.7 the global part of the operator $\mathcal{H}_{*,A}$ is controlled by the operator $\mathcal{T}$, which is of weak type 1 by Lemma 5.8. The conclusion follows by Lemma 5.2. □

Acknowledgements The authors have received support by the Italian MIUR-PRIN 2005 project "Harmonic Analysis" and by the EU IHP 2002–2006 project "HARP."

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