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On Schrödinger Propagator for the Special Hermite Operator

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Abstract We establish a Strichartz type estimate for the Schrödinger propagator $e^{it\mathcal{L}}$ for the special Hermite operator $\mathcal{L}$ on $\mathbb{C}^n$. Our method relies on a regularization technique. We show that no admissibility condition is required on (q, p) when $1 \leq q \leq 2$.

Keywords Schrödinger equation · Oscillatory semi group · Special Hermite expansion · Special Hermite operator

Mathematics Subject Classification (2000) Primary 42C10 · Secondary 35G10 · 47A63

1 Introduction

The quantum harmonic oscillator Hamiltonian $H = -\Delta + |x|^2$ on $\mathbb{R}^n$ has the representation

$$H = \frac{1}{2}\sum_{j=1}^{n}\left(A_j A_j^* + A_j^* A_j\right)$$

in terms of the creation operators $A_j = -\frac{d}{dx_j} + x_j$ and the annihilation operators $A_j^* = \frac{d}{dx_j} + x_j$, $j = 1, 2, \ldots n$. There is an interesting analogous operator $\mathcal{L}$ on $\mathbb{C}^n$,

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given by

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{n}\left(Z_j\overline{Z}_j + \overline{Z}_jZ_j\right)$$

where $Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2}\bar{z}_j$, $\overline{Z}_j = -\frac{\partial}{\partial \bar{z}_j} + \frac{1}{2}z_j$, $j = 1, 2, \ldots n$. Here $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$ denote the complex derivatives $\frac{\partial}{\partial_{x_j}} \mp i\frac{\partial}{\partial_{y_j}}$, respectively. The operator $\mathcal{L}$ was introduced by R. S. Strichartz [12], and is known as the special Hermite operator. In explicit terms it has the form

$$\mathcal{L} = -\Delta + \frac{1}{4}|z|^2 - i\sum_{1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right) \tag{1.1}$$

where Δ is the Laplacian on $\mathbb{C}^n$.

What makes this operator interesting is that, it is associated to certain convolution structure on $\mathbb{C}^n$ (see, Section 2), by virtue of which, the solutions to the initial value problems for basic linear differential equations like heat, wave and Schrödinger equation for $\mathcal{L}$ can be expressed in terms of this convolution structure on $\mathbb{C}^n$.

With any self adjoint differential operator L having the spectral decomposition $L = \int_E \lambda\, dP_\lambda$, we can associate an oscillatory group $\{e^{-itL} : t \in \mathbb{R}\}$ which is defined by

$$e^{-itL} = \int_E e^{-it\lambda} dP_\lambda\ , \tag{1.2}$$

where dP_λ denotes the spectral projection for L; i.e., a projection valued measure on the spectrum E of L. The spectrum may be continuous, discrete or a combination of both, in general. If the spectrum is discrete, then the above integral reduces to an infinite sum. Oscillatory groups of the above form arise as solution operators for the initial value problem for the Schrödinger equation. Hence, we may call the group $\{e^{-itL} : t \in \mathbb{R}\}$ the Schrödinger propagator for L.

Consider the initial value problem for the Schrödinger equation for $\mathcal{L}$:

$$i\partial_t u(z,t) - \mathcal{L}u(z,t) = 0, \quad z \in \mathbb{C}^n,\ t > 0 \tag{1.3}$$

$$u(z,0) = f(z)\ . \tag{1.4}$$

For $f \in L^2(\mathbb{C}^n)$ the solution to this initial value problem is given by

$$u(z,t) = e^{-it\mathcal{L}} f(z)\ .$$

Using the representation for $\mathcal{L}$ given by (1.1), the Schrödinger equation for $\mathcal{L}$ takes the form

$$i\partial_t u + i\sum_{1}^{n}(x_j\partial_{y_j} - y_j\partial_{x_j})u + \Delta u - \frac{1}{4}|z|^2 u = 0\ .$$

This equation may be thought of as a prototype for a generalized Schrödinger equation of the form

$$i\partial u = (-\Delta_x + V(x))u = 0$$

where ∂ denotes a first order linear differential operator on $\mathbb{R}^m \times \mathbb{R}$ of the form

$$a_0(x)\partial_t + \sum_{j=1}^{m} a_j(x)\partial_{x_j} + b(x), \quad x \in \mathbb{R}^m, t \in \mathbb{R} .$$

The study of such general Schrödinger equations is significant because they include as special case, Schrödinger equation with a magnetic vector potential A and scalar potential V: $\sum_{j=1}^{m}(i\partial_j + A_j)^2 u + Vu = 0$.

The present article is concerned with the study of the regularity property of the Schrödinger propagator for the special Hermite operator $\mathcal{L}$. The main reason for the study of Schrödinger propagator for this operator is to indicate that, the analysis of general Schrödinger equation of the above form could be done more effectively using the harmonic analysis of the special functions associated to the differential operator involved.

In the case of Schrödinger equation for the special Hermite operator $\mathcal{L}$, we see that the analysis is closely related to the harmonic analysis on the Heisenberg group. This is not surprising because the special Hermite operator is closely related to the Heisenberg Laplacian and naturally, the tools of the Harmonic analysis on Heisenberg group come into play.

We begin with a brief discussion of the spectral theory of $\mathcal{L}$. References for the materials discussed in this section are the books by Folland [1] or Thangavelu [13].

The eigenfunctions of the operator $\mathcal{L}$ are called the special Hermite functions, which are defined in terms of the Fourier-Wigner transform. For a pair of functions $f, g \in L^2(\mathbb{R}^n)$, the Fourier-Wigner transform is defined to be

$$V(f,g)(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f\left(\xi + \frac{y}{2}\right) g\left(\xi - \frac{y}{2}\right) d\xi ,$$

where $z = x + iy \in \mathbb{C}^n$. For any two multi-indices μ, ν the special Hermite functions $\Phi_{\mu\nu}$ are given by

$$\Phi_{\mu\nu}(z) = V(h_\mu, h_\nu)(z)$$

where h_μ and h_ν are Hermite functions on $\mathbb{R}^n$. Recall that for each nonnegative integer k, the one-dimensional Hermite functions h_k are defined by

$$h_k(x) = \frac{(-1)^k}{\sqrt{2^k k! \sqrt{\pi}}} \left(\frac{d^k}{dx^k} e^{-x^2}\right) e^{\frac{x^2}{2}} .$$

Now for each multi index $\nu = (\nu_1, \cdots, \nu_n)$, the n-dimensional Hermite functions are defined by the tensor product:

$$h_\nu(x) = \Pi_{i=1}^n h_{\nu_i}(x_i), \quad x = (x_1, \cdots, x_n) .$$

A direct computation using the relations

$$\left(-\frac{d}{dx}+x\right)h_k(x)=(2k+2)^{\frac{1}{2}}h_{k+1}(x)\ ,$$
$$\left(\frac{d}{dx}+x\right)h_k(x)=(2k)^{\frac{1}{2}}h_{k-1}(x)$$

satisfied by the Hermite functions h_k show that $\mathcal{L}\Phi_{\mu\,\nu}=(2|\nu|+n)\Phi_{\mu\,\nu}$. Hence, $\Phi_{\mu,\,\nu}$ are eigenfunctions of $\mathcal{L}$ with eigenvalue $2|\nu|+n$ and they also form a complete orthonormal system in $L^2(\mathbb{C}^n)$. Thus every $f\in L^2(\mathbb{C}^n)$ has the expansion

$$f=\sum_{\mu,\,\nu}\langle f,\Phi_{\mu\,\nu}\rangle\Phi_{\mu\,\nu} \tag{1.5}$$

in terms of the eigenfunctions of $\mathcal{L}$.

The above expansion may be written as

$$f=\sum_{k=0}^{\infty}P_kf \tag{1.6}$$

where

$$P_kf=\sum_{\mu,|\nu|=k}\langle f,\Phi_{\mu,\nu}\rangle\Phi_{\mu\nu} \tag{1.7}$$

is the spectral projection corresponding to the eigenvalue $2k+n$. Now for any $f\in L^2(\mathbb{C}^n)$ such that $\mathcal{L}f\in L^2(\mathbb{C}^n)$, by self adjointness of $\mathcal{L}$, we have $P_k(\mathcal{L}f)=(2k+n)P_kf$. It follows that for $f\in L^2(\mathbb{C}^n)$ with $\mathcal{L}f\in L^2(\mathbb{C}^n)$

$$\mathcal{L}f=\sum_{k=0}^{\infty}(2k+n)P_kf\ .$$

Thus in view of (1.2), we can define $e^{-it\mathcal{L}}$ as

$$e^{-it\mathcal{L}}f=\sum_{k=0}^{\infty}e^{-it(2k+n)}P_kf\ . \tag{1.8}$$

It is convenient to measure the regularity of $e^{-it\mathcal{L}}f$ in terms of the Sobolev space $W^s_{\mathcal{L}}(\mathbb{C}^n)$, which for $s\geq 0$ are defined by

$$W^s_{\mathcal{L}}\big(\mathbb{C}^n\big)=\big\{f\in L^2\big(\mathbb{C}^n\big):\mathcal{L}^sf\in L^2\big(\mathbb{C}^n\big)\big\}$$

where $\mathcal{L}^s$ is defined using the spectral theory; $\mathcal{L}^sf=\sum_{k=0}^{\infty}(2k+n)^sP_kf$. Notice that for $s<0$, elements of $W^s_{\mathcal{L}}(\mathbb{C}^n)$ are tempered distributions.

From (1.8) it is clear that $\{e^{-it\mathcal{L}}:t\in\mathbb{R}\}$ is a one parameter group of unitary operators on $L^2(\mathbb{C}^n)$. In particular, $e^{-it\mathcal{L}}$ cannot map $L^2(\mathbb{C}^n)$ into $W^s_{\mathcal{L}}(\mathbb{C}^n)$ for $s>0$, the later being a proper subspace of $L^2(\mathbb{C}^n)$. This rules out the possibility of $e^{-it\mathcal{L}}$ having any regularity in terms of the Sobolev spaces $W^s_{\mathcal{L}}(\mathbb{C}^n)$. This is in fact a general feature of such oscillatory groups.

For the free Schrödinger equation on $\mathbb{R}^n$, R. S. Strichartz showed that for L^2 initial data, the solution does however lie in a higher order L^p space over space-time, namely for $p = \frac{2(n+2)}{n}$. He proved the following theorem, known as the Strichartz' estimate (see, [11]).

Theorem (Strichartz) *Let u be the solution to the inhomogeneous problem*

$$i\partial_t u(x,t) + \Delta u(x,t) = g(x,t), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}$$
$$u(x,0) = f(x)\ .$$

with $f \in L^2(\mathbb{R}^n)$, $g \in L^{\frac{2(n+2)}{n+4}}(\mathbb{R}^n \times \mathbb{R})$. Then $u \in L^{\frac{2(n+2)}{n}}(\mathbb{R}^n \times \mathbb{R})$ and moreover satisfies the inequality

$$\left(\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |u(x,t)|^{\frac{2(n+2)}{n}}\,dx\,dt\right)^{\frac{n}{2(n+2)}} \le C\left(\|f\|_2 + \|g\|_{\frac{2(n+2)}{n+4}}\right)\ .$$

This result has been extended to Schrödinger equations with a wide class of bounded potentials V by Journe, Soffer, and Sogge. In fact, in [5], they proved analogous estimates in the case of bounded potentials satisfying certain pointwise decay condition at infinity. There has been a considerable interest, ever since, in establishing Strichartz' type estimate, as a global regularity result, for Schrödinger equations with general bounded potentials. We refer to [3, 4] and the references there in for such results. An interesting case of an unbounded potential can be seen in [8, 9], where the authors consider the quadratic potential $V(x) = |x|^2$.

Since there is no global regularity in terms of Sobolev spaces in our situation, naturally we look for a Strichartz type estimate for $e^{-it\mathcal{L}}f$. A new feature of the one parameter unitary group $\{e^{-it\mathcal{L}} : t \in \mathbb{R}\}$ given by (1.8) is the periodicity in the t variable. It follows that the solution $u(z,t) = e^{-it\mathcal{L}}f(z)$ to the initial value problem (1.3), (1.4) is periodic in t with period 2π, and hence may be regarded as a function on $\mathbb{C}^n \times \mathbb{S}^1$.

Let $(\Omega, d\mu)$ be any σ-finite measure space. We consider the mixed L^p space on $\Omega \times \mathbb{S}^1$ given by

$$L^q\big(\mathbb{S}^1; L^p(\Omega)\big) = \big\{u : u \text{ measurable on } \Omega \times \mathbb{S}^1,\ \|u\|_{L^q(\mathbb{S}^1;L^p(\Omega))} < \infty\big\}\ ,$$

where $\|\cdot\|_{L^q(\mathbb{S}^1;L^p(\Omega))}$ denotes the norm in $L^q(\mathbb{S}^1; L^p(\Omega))$:

$$\|u\|_{L^q(\mathbb{S}^1;L^p(\Omega))} = \left(\int_{-\pi}^{\pi}\left[\int_\Omega |u(z,t)|^p\,d\mu(z)\right]^{\frac{q}{p}} dt\right)^{\frac{1}{q}} \tag{1.9}$$

for $1 \le p, q < \infty$. Our result for the Schrödinger propagator is the following Strichartz type estimate.

Theorem A *Let $f \in L^2(\mathbb{C}^n)$ and let $u(z,t) = e^{-it\mathcal{L}} f(z)$. Then u is periodic in t and $u \in L^q(\mathbb{S}^1; L^p(\mathbb{C}^n))$, for all pairs (q,p) such that $2 < q < \infty$, $\frac{1}{q} \geq n(\frac{1}{2} - \frac{1}{p})$ or $1 \leq q \leq 2$, $2 \leq p < \frac{2n}{n-1}$, $n \geq 1$. Further u satisfies the inequality*

$$\|u\|_{L^q(\mathbb{S}^1;\, L^p(\mathbb{C}^n))} \leq C_n \|f\|_2 \tag{1.10}$$

for all $f \in L^2(\mathbb{C}^n)$ for the above ranges of p and q.

Remark 1 It is interesting to note that the admissibility condition on the pairs (q,p), i.e., $\frac{1}{q} \geq n(\frac{1}{2} - \frac{1}{p})$ is required only for $q > 2$.

Since $\|f\|_2 < \infty$, from the definition of the mixed L^p norm given by (1.9), we see that the validity of the above inequality implies that, for almost all t, the inner integral in (1.9) is finite. So we conclude that for almost all $t \in [-\pi, \pi)$ (and hence for almost all $t \in \mathbb{R}$), $u(\cdot, t) = e^{-it\mathcal{L}} f$ lies in a higher order L^p space, namely $L^p(\mathbb{C}^n)$ for $2 \leq p < \frac{2n}{n-1}$. This may be regarded as a regularizing effect of the oscillatory group $\{e^{-it\mathcal{L}} : t \in \mathbb{R}\}$.

The famous article of M. Kiel and T. Tao (see, [6]), proves Strichartz estimates for general one parameter family of operators $U(t)$, satisfying certain growth/decay conditions in t. However, the Schrödinger group $\{e^{-it\mathcal{L}} : t \in \mathbb{R}\}$ does not satisfy the required decay condition for all $t \in \mathbb{R}$. It is not clear *a priori*, if the required decay condition holds even for small t.

Our approach involves basically, a regularization technique; We embed the one parameter unitary group $\{e^{-it\mathcal{L}} : t \in \mathbb{R}\}$ into the complex semi group $\{e^{-z\mathcal{L}} : z = r + it, r > 0, t \in \mathbb{R}\}$, which is infinitely smoothing. We first establish the estimate for the complex semi group and the estimate for the original semi group $\{e^{-it\mathcal{L}} : t \in \mathbb{R}\}$ is then deduced by a suitable limiting argument.

2 Integral Representation of $e^{-it\mathcal{L}}$ and the Kernel Estimate

We show that the Schrödinger group is given by a twisted convolution operator. Recall that the twisted convolution of two functions f and g on $\mathbb{C}^n$ is defined by

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-w) g(w) e^{\frac{i}{2}\Im(z\cdot\bar{w})}\, dw$$

where $\Im$ denotes the imaginary part. We start with recalling the well known orthogonality properties of the special Hermite functions.

$$\Phi_{\mu,\nu} \times \Phi_{\alpha,\beta} = (2\pi)^{n/2} \Phi_{\mu,\beta}\, \delta_{\nu,\alpha} \tag{2.1}$$

where $\delta_{\nu,\alpha} = 1$, if $\nu = \alpha$ and 0 otherwise. The reference for all the results we use regarding the special Hermite functions is the monograph [13] or [12].

Let $\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2)\, e^{-\frac{1}{4}|z|^2}$ be the Laguerre function of order $n-1$. Here L_k^α denote the Laguerre polynomial of degree k and order $\alpha > -1$, defined by the generating function identity, [7]

$$\sum_{k=0}^{\infty} L_k^{\alpha}(x)\omega^k = (1-\omega)^{-\alpha-1} e^{-\frac{\omega}{1-\omega}x}, \quad |\omega| < 1 . \tag{2.2}$$

The special Hermite functions $\Phi_{\mu,\nu}$ are related to the Laguerre functions φ_k by the following relation

$$(2\pi)^{n/2} \sum_{|\nu|=k} \Phi_{\nu\nu} = \varphi_k \tag{2.3}$$

(see, [13], p. 22). The condition (2.1) leads to the orthogonality condition

$$\varphi_k \times \varphi_j = (2\pi)^n \varphi_k \, \delta_{k,j} \tag{2.4}$$

for the Laguerre functions. It follows from (2.2), with $\alpha = n-1$ that the functions φ_k satisfy the generating function identity

$$\sum_{k=0}^{\infty} \varphi_k(z)\omega^k = (1-\omega)^{-n} e^{-\frac{1}{2}\frac{1+\omega}{1-\omega}|z|^2}, \quad |\omega| < 1 \tag{2.5}$$

which, by analytic continuation is valid for any complex number ω with $|\omega| < 1$.

Taking twisted convolution on both sides of (1.5) with $\Phi_{\alpha\alpha}$ and using the orthogonality property (2.1), we get

$$f \times \Phi_{\alpha\alpha} = (2\pi)^{n/2} \sum_{\mu} \langle f, \Phi_{\mu\alpha} \rangle \Phi_{\mu\alpha} \, .$$

Now summing both sides with respect to all α such that $|\alpha| = k$, and using (2.3), we see that the spectral projection P_k given by (1.7) has the simpler representation,

$$P_k f = (2\pi)^{-\frac{n}{2}} \sum_{|\alpha|=k} f \times \Phi_{\alpha\alpha}(z) = (2\pi)^{-n} f \times \varphi_k(z) \, . \tag{2.6}$$

Hence, the special Hermite expansion takes the compact form

$$f(z) = (2\pi)^{-n} \sum_{k} f \times \varphi_k(z) \, . \tag{2.7}$$

Moreover, we see that the Schrödinger group $\{e^{-it\mathcal{L}} : t \in \mathbb{R}\}$ given by (1.8), has the representation

$$e^{-it\mathcal{L}} f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-it(2k+n)} f \times \varphi_k(z) \tag{2.8}$$

for $f \in L^2(\mathbb{C}^n)$. We also consider an auxiliary complex semi-group $\{e^{-\eta\mathcal{L}}\}$ for complex parameters $\eta = r + it$ with $\Re(\eta) = r > 0$:

$$e^{-\eta\mathcal{L}} f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-\eta(2k+n)} f \times \varphi_k(z) \, . \tag{2.9}$$

This can be expressed as a twisted convolution $f \times K_\eta(z)$, where

$$K_\eta(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-\eta(2k+n)} \varphi_k(z) \, .$$

The interchange of integral and the infinite sum involved is justified since the above series defining K_η converges in $L^2(\mathbb{C}^n)$.

In view of (2.5), we can write the above equation in the form

$$K_\eta(z) = (2\pi)^{-n} e^{-n\eta} (1-\omega)^{-n} e^{-\frac{1+\omega}{1-\omega}\frac{|z|^2}{4}} \, , \tag{2.10}$$

with $\omega = e^{-2\eta}$, $\Re(\eta) > 0$. Thus we see that the operators in the complex semi group $e^{-\eta\mathcal{L}}$ are twisted convolution operators:

$$e^{-\eta\mathcal{L}} f(z) = \int_{\mathbb{C}^n} f(z-w) K_\eta(w) e^{\frac{i}{2}\Im(z\cdot\bar{w})} \, dw \tag{2.11}$$

with kernel given by (2.10). The following kernel estimate is crucial in our analysis.

Lemma 1 *Let $K_\eta(z)$ be the kernel given by the Equation* (2.10). *Then for $\Re(\eta) > 0$, K_η satisfies the uniform estimate*

$$|K_\eta(z)| \le \frac{2}{|\sin t|^n}, \quad \eta = r + it, \quad z \in \mathbb{C}^n \, . \tag{2.12}$$

Proof Since $\omega = e^{-2(r+it)}$, $1-\omega = 2e^{-(r+it)} \sinh(r+it)$ and hence

$$|1-\omega| \ge 2e^{-r} \, |\sin t| \, \cosh r \, .$$

Also a simple computation shows that

$$\Re\left(\frac{1+\omega}{1-\omega}\right) \ge \frac{1-e^{-2r}}{1+e^{-2r}} \, ,$$

which is positive for $r > 0$. Hence, from (2.10) we see that

$$|K_\eta| \le \frac{2(2\pi)^{-nr} e^{-(n+1)r}}{|\sin t|^n} \le \frac{2}{|\sin t|^n}$$

which proves the lemma. □

We end this section with the following result which is for the convolution on the circle.

Lemma 2 *Let T denote the convolution operator on the circle given by*

$$Tf(t) = \int_{\mathbb{S}^1} K(t-s) f(s) \, ds \, .$$

Assume that K belongs to the weak L^p space $L^\rho_W(\mathbb{S}^1)$, for some $\rho > 1$. Then the inequality

$$\|Tf\|_q \le C_K \|f\|_{q'}$$

is valid for $q = 2\rho$ and also for $1 \le q \le 2$, where C_K is a constant depending only on K.

Proof By the generalized Young's inequality, we have (see, [2]):

$$\|Tf\|_r \le C[K]_\rho \|f\|_{q'}$$

where $[K]_\rho$ denoting the weak $L^\rho(\mathbb{S}^1)$ norm of K, and is valid for all r such that $\frac{1}{q'} + \frac{1}{\rho} = 1 + \frac{1}{r}$, $\rho > 1, q' > 1$. Setting $r = q$, this reads

$$\|Tf\|_q \le C[K]_\rho \|f\|_{q'} \tag{2.13}$$

for $q = 2\rho$. Notice that these arguments are valid for $2 < q < \infty$, since $\rho > 1$ and $q' > 1$ by assumption.

Now we observe that the weak L^p spaces L^ρ_W are in L^1_{loc} for $\rho > 1$: In fact, for any compact set Θ, set $g = f\chi_\Theta$. The distribution function of g is given by $\lambda_g(\alpha) = |\{x : |g(x)| > \alpha\}| = |\{x \in \Theta : |f(x)| > \alpha\}| \le |\Theta|$. Hence, $\lambda_g(\alpha)$ is bounded for $\alpha > 0$. Also $\lambda_g(\alpha) \le \lambda_f(\alpha) \le \frac{C}{\alpha^\rho}$, since $f \in L^\rho_W$. These two inequalities yield $\lambda_g(\alpha) \le \frac{C}{1+\alpha^\rho}$. Thus $\int_\Theta |f| = \|g\|_1 = \int_0^\infty \lambda_g(\alpha) d\alpha \le \int_0^\infty \frac{d\alpha}{1+\alpha^\rho}$. This integral is finite for $\rho > 1$, showing that $f \in L^1_{\text{loc}}$.

By the above observation, we have $K \in L^1(\mathbb{S}^1)$. Hence, by Minkowski's inequality for integrals,

$$\|Tf\|_q \le \|K\|_1 \|f\|_q \quad \text{for} \quad 1 \le q \le \infty\,.$$

Integrating this inequality for $q = \infty$ over $\mathbb{S}^1$ yields

$$\|Tf\|_1 \le 2\pi \|K\|_1 \|f\|_\infty\,.$$

Interpolating this with the above L^q estimate for $q = 2$, we get

$$\|Tf\|_q \le C\|K\|_1 \|f\|_{q'} \;\; \text{for} \quad 1 \le q \le 2\,. \qquad \square$$

3 Estimates for the Complex Semi Group

In this section we establish the estimate for the auxiliary complex semi group and deduce the proof of Theorem A. We follow the standard T^*T method to prove the estimate for the complex semi group. We start with the following.

Proposition 1 *Let $\eta = r + it$, $r > 0$ and $|t| > 0$. Then the operators $e^{-\eta\mathcal{L}}$ in the complex semi group satisfy the inequality*

$$\left\|e^{-\eta\mathcal{L}} f\right\|_p < 2|\sin t|^{-2n(\frac{1}{p'} - \frac{1}{2})} \|f\|_{p'}\,,$$

for $1 \le p' \le 2$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof In view of the integral representation (2.11) for $e^{-\eta\mathcal{L}}$ and the uniform estimate for the kernel given by Lemma 1, we get the obvious $L^1 - L^\infty$ estimate:

$$\left\|e^{-\eta\mathcal{L}} f\right\|_\infty \le \frac{2}{|\sin t|^n}\|f\|_1$$

valid for all $\eta = r + it$ such that $r = \Re(\eta) > 0$, $|t| > 0$. Also since $r > 0$, it is clear from (2.9), in view of (2.6), that $e^{-\eta\mathcal{L}}$ are contractions on $L^2(\mathbb{C}^n)$:

$$\left\|e^{-\eta\mathcal{L}} f\right\|_2 < e^{-nr}\|f\|_2 .$$

By Riesz-Thorin interpolation theorem, (see, [10]) these two inequalities yield

$$\|u_r(\cdot, t)\|_p \le C_n(\theta)\|f\|_{p'} \tag{3.1}$$

where $\frac{1}{p'} = \frac{\theta}{1} + \frac{1-\theta}{2}$, $0 < \theta < 1$ and

$$C_n(\theta) = \frac{2^\theta e^{-nr(1-\theta)}}{|\sin t|^{n\theta}} < \frac{2}{|\sin t|^{n\theta}} .$$

This completes the proof since $\frac{\theta}{2} = \frac{1}{p'} - \frac{1}{2}$. □

The next lemma is a technical result.

Lemma 3 *Let* $h(\cdot,\cdot) \in L^{q'}([-\pi,\pi]; L^2(\mathbb{C}^n))$, $1 \le q' \le \infty$. *Then for* $r > 0$, $e^{-(r+it)\mathcal{L}}h(z,t)\,\overline{e^{-(r+is)\mathcal{L}}h(z,s)}$ *considered as a function of* $z \in \mathbb{C}^n$, $t, s \in [-\pi,\pi]$ *belongs to* $L^1(\mathbb{C}^n \times [-\pi,\pi] \times [-\pi,\pi])$.

Proof Since $h(\cdot,\cdot) \in L^{q'}([-\pi,\pi]; L^2(\mathbb{C}^n))$, except for t in a set of measure zero, $h(\cdot,t) \in L^2(\mathbb{C}^n)$. Hence, except for those t, $e^{-(r+it)\mathcal{L}}h(\cdot,t) \in L^2(\mathbb{C}^n)$ for $r > 0$. It follows that $e^{-\eta\mathcal{L}}h(\cdot,t)\,\overline{e^{-\lambda\mathcal{L}}h(\cdot,s)} \in L^1(\mathbb{C}^n)$ for a.e., $s, t \in [-\pi,\pi]$, where we set $\eta = r + it$, $\lambda = r + is$ for $r > 0$.

By Cauchy-Schwarz inequality and the fact that $e^{-\eta\mathcal{L}}$ is bounded on L^2 for any η with $\Re(\eta) > 0$, we see that for almost all s, t

$$\int_{\mathbb{C}^n} \left|e^{-\eta\mathcal{L}}h(\cdot,t)\, e^{-\lambda\mathcal{L}}h(\cdot,s)\right| dz \le \|h(\cdot,t)\|_2\|h(\cdot,s)\|_2 .$$

It follows that

$$\begin{aligned}&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{\mathbb{C}^n} \left|e^{-\eta\mathcal{L}}h(\cdot,t)\, e^{-\lambda\mathcal{L}}h(\cdot,s)\right| dz\, ds\, dt \\ &\quad\le \left(\int_{-\pi}^{\pi}\|h(\cdot,t)\|_2\, dt\right)^2 \\ &\quad\le (2\pi)^{2/q}\|h\|^2_{L^{q'}([-\pi,\pi];\, L^2(\mathbb{C}^n))} < \infty\end{aligned}$$

by holder's inequality and the fact that $h(\cdot,\cdot)\in L^{q'}([-\pi,\pi];L^2(\mathbb{C}^n))$. This proves the lemma. □

Proposition 2 *The estimate*

$$\left\|\int_{\mathbb{S}^1} e^{-(r+it)\mathcal{L}}h(\cdot,t)\,dt\right\|_2 \le C\|h\|_{L^{q'}(\mathbb{S}^1;\,L^{p'}(\mathbb{C}^n))}$$

for $r>0$, *holds for all* $h(\cdot,\cdot)\in L^{q'}(\mathbb{S}^1;\,L^{p'}\cap L^2\ (\mathbb{C}^n))$ *for pairs* (q,p) *such that* $1\le q\le 2,\ 2\le p<\frac{2n}{n-1}$ *or* $2<q\le\infty,\ \frac{1}{q}\ge n(\frac{1}{2}-\frac{1}{p})$.

Proof We have

$$\begin{aligned}\left\|\int_{\mathbb{S}^1} e^{-\eta\mathcal{L}}h(\cdot,t)\,dt\right\|_2^2 &= \int_{\mathbb{C}^n}\left(\int_{\mathbb{S}^1} e^{-\eta\mathcal{L}}h(\cdot,t)\,dt\,\overline{\int_{\mathbb{S}^1} e^{-\lambda\mathcal{L}}h(\cdot,s)\,ds}\right)dz\\ &= \int_{\mathbb{C}^n}\left(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} e^{-\eta\mathcal{L}}h(\cdot,t)\,\overline{e^{-\lambda\mathcal{L}}h(\cdot,s)}\,ds\,dt\right)dz\,.\end{aligned}$$

Next we interchange the integrals over $\mathbb{C}^n$ and over $\mathbb{S}^1\times\mathbb{S}^1$ which is justified by Fubini's theorem in view of Lemma 3. Writing down the series expansion for $e^{-\eta\mathcal{L}}h(\cdot,t)$ and $e^{-\lambda\mathcal{L}}h(\cdot,s)$, in terms of the spectral projections and using the orthogonality of the spectral projections, we see that the above integral is same as

$$\begin{aligned}&\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}\int_{\mathbb{C}^n} e^{-(\eta+\bar\lambda)\mathcal{L}}\,\overline{h(\cdot,s)}\,dz\,h(\cdot,t)\,ds\,dt\\ &\quad=\int_{\mathbb{S}^1}\int_{\mathbb{C}^n}\left(\int_{\mathbb{S}^1} e^{-(\eta+\bar\lambda)\mathcal{L}}\,\overline{h(\cdot,s)}\,ds\right)h(\cdot,t)\,dz\,dt\,.\end{aligned}$$

Now applying the Hölder's inequality for the mixed L^p spaces on the RHS of the above equation, we see that

$$\begin{aligned}\left\|\int_{\mathbb{S}^1} e^{-\eta\mathcal{L}}h(\cdot,t)\,dt\right\|_2^2 &\le \left\|\int_{\mathbb{S}^1} e^{-(\eta+\bar\lambda)\mathcal{L}}\overline{h(\cdot,s)}\,ds\right\|_{L^q(\mathbb{S}^1;\,L^p(\mathbb{C}^n))}\\ &\qquad\times\|h(\cdot,\cdot)\|_{L^{q'}(\mathbb{S}^1;\,L^{p'}(\mathbb{C}^n))}\,.\end{aligned}$$

Thus to complete the proof, it is enough to establish the inequality

$$\left\|\int_{\mathbb{S}^1} e^{-(\eta+\bar\lambda)\mathcal{L}}\overline{h(\cdot,s)}\,ds\right\|_{L^q(\mathbb{S}^1;\,L^p(\mathbb{C}^n))} \le C\|h(\cdot,\cdot)\|_{L^{q'}(\mathbb{S}^1;\,L^{p'}(\mathbb{C}^n))}\,.$$

By Minkowski's inequality for integrals and Proposition 1, we see that

$$\begin{aligned}\left\|\int_{\mathbb{S}^1} e^{-(\eta+\bar\lambda)\mathcal{L}}\overline{h(\cdot,s)}\,ds\right\|_p &\le \int_{\mathbb{S}^1}\|e^{-(\eta+\bar\lambda)\mathcal{L}}\overline{h(\cdot,s)}\|_p\,ds \qquad (3.2)\\ &\le \int_{\mathbb{S}^1}\frac{\|h(\cdot,s)\|_{p'}}{|\sin([t-s])|^{2n(\frac{1}{p'}-\frac{1}{2})}}\,ds\end{aligned}$$

for $1 \le p' \le 2$. The last expression defines a convolution operator on the circle with kernel $K(s) = |\sin s|^{-2n(\frac{1}{p'}-\frac{1}{2})} = |\sin s|^{-2n(\frac{1}{2}-\frac{1}{p})}$. Observe that K belongs to the weak L^p space $L^\rho_W(\mathbb{S}^1)$ for all $\rho \le \frac{1}{2n(\frac{1}{2}-\frac{1}{p})}$. Thus taking $L^q(\mathbb{S}^1)$ norm on both sides of (3.2), and using Lemma 2 on RHS, we get,

$$\left\| \int_{\mathbb{S}^1} e^{-(\eta+\bar\lambda)\mathcal{L}}\overline{h(\cdot,s)}\,ds \right\|_{L^q(\mathbb{S}^1;\,L^p(\mathbb{C}^n))} \le C_K \|h(\cdot,\cdot)\|_{L^{q'}(\mathbb{S}^1;\,L^{p'}(\mathbb{C}^n))}$$

valid for $q = 2\rho$ and also for $1 \le q \le 2$. Note that $\rho > 1$ for $q > 2$ and hence $1 < \rho \le \frac{1}{2n(\frac{1}{2}-\frac{1}{p})}$. Thus the above inequality is valid for all pairs (p,q) such that $\frac{1}{q} \ge n(\frac{1}{2}-\frac{1}{p})$, $2 < q \le \infty$ and also for all pairs (q,p) such that $1 \le q \le 2,\ 2 \le p < \frac{2n}{n-1}$. This gives the required estimates and completes the proof. □

The next proposition establishes the mixed norm estimate for the complex semi group $e^{-\eta\mathcal{L}}$.

Proposition 3 *Let $\eta = r + it$ with $r > 0$ and $0 < |t| \le \pi$. If $f \in L^2(\mathbb{C}^n)$, then $e^{-\eta\mathcal{L}} f \in L^q(\mathbb{S}^1; L^p(\mathbb{C}^n))$ for all pairs (q,p) such that $1 \le q \le 2,\ 2 \le p < \frac{2n}{n-1}$ or $2 < q < \infty,\ \frac{1}{q} \ge n(\frac{1}{2}-\frac{1}{p})$. Moreover, $e^{-\eta\mathcal{L}}$ satisfies the inequality*

$$\left\|e^{-\eta\mathcal{L}} f\right\|_{L^q(\mathbb{S}^1;\,L^p(\mathbb{C}^n))} \le C_n \|f\|_2 \tag{3.3}$$

for some constant C_n independent of f and r, for the above ranges of p and q.

Proof We use the duality argument to estimate the mixed L^p norm. Observe that the dual of $L^q(\mathbb{S}^1;\ L^p(\mathbb{C}^n))$ is $L^{q'}(\mathbb{S}^1;\ L^{p'}(\mathbb{C}^n))$ for $1 \le p,q < \infty$ and that $L^{q'}(\mathbb{S}^1;\ L^{p'}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n))$ is dense in $L^{q'}(\mathbb{S}^1;\ L^{p'}(\mathbb{C}^n))$ for $1 \le q', p' < \infty$.

Let $h \in L^{q'}(\mathbb{S}^1;\ L^{p'}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n))$. Notice that the adjoint of the operator $e^{-\eta\mathcal{L}}$ in $L^2(\mathbb{C}^n)$ is $e^{-\bar\eta\mathcal{L}}$, hence,

$$\int_{\mathbb{S}^1}\int_{\mathbb{C}^n} \left[e^{-\eta\mathcal{L}} f\right](z)\overline{h(z,t)}\,dz\,dt = \int_{\mathbb{S}^1}\int_{\mathbb{C}^n} f(z)\overline{e^{-\bar\eta\mathcal{L}}h(z,t)}\,dz\,dt\ .$$

An interchange of integrals on the RHS, followed by Cauchy-Schwarz inequality yields

$$\left|\int_{\mathbb{S}^1}\int_{\mathbb{C}^n} \left[e^{-\eta\mathcal{L}} f\right](z)\overline{h(z,t)}\,dz\,dt\right| \le \|f\|_2 \left\|\int_{\mathbb{S}^1} e^{-\bar\eta\mathcal{L}}h(\cdot,t)\,dt\right\|_2\ .$$

In view of Proposition 2, this gives the inequality

$$\left|\int_{\mathbb{S}^1}\int_{\mathbb{C}^n} e^{-\eta\mathcal{L}} f(z)\overline{h(z,t)}\,dz\,dt\right| \ \le\ C\|f\|_2\|h\|_{L^{q'}(\mathbb{S}^1;\,L^{p'}(\mathbb{C}^n))}\ .$$

The required estimate follows, by density of $L^{q'}(\mathbb{S}^1;\ L^{p'}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n))$ in $L^{q'}(\mathbb{S}^1;\ L^{p'}(\mathbb{C}^n))$. □

Now we prove the regularity result for the original problem as stated in Theorem A.

Proof of Theorem A We deduce this result from the inequality (3.3) by finding a sequence $\eta_n = r_n + it$ such that $e^{-\eta_n \mathcal{L}} f \to e^{-it\mathcal{L}} f$ in $L^q(\mathbb{S}^1; L^p(\mathbb{C}^n))$ as $r_n \to 0$.

First we observe that for each fixed t, $e^{-\eta\mathcal{L}} f \to e^{-it\mathcal{L}} f$ in $L^2(\mathbb{C}^n)$ as $\Re(\eta) = r \to 0$. In fact, for any sequence $\eta_j = r_j + it$ with $r_j \to 0$, by (2.9), (2.6) we have

$$\left\| e^{-\eta_j \mathcal{L}} f - e^{-it\mathcal{L}} f \right\|_2^2 = \sum_k \left| e^{-(r_j+it)(2k+n)} - e^{-it(2k+n)} \right|^2 \|P_k f\|_2^2 . \tag{3.4}$$

Clearly for each $k \geq 0$, $\left| e^{-(r_j+it)(2k+n)} - e^{-it(2k+n)} \right|$ tends to zero as $r_j \to 0$. Now a dominated convergence argument applied to the sum on the RHS of (3.4) shows that the RHS tends to zero as $r_j \to 0$.

Integrating (3.4) with respect to t, the above argument shows that

$$\int_{\mathbb{S}^1} \int_{\mathbb{C}^n} \left| e^{-(r_j+it)\mathcal{L}} f(z) - e^{-it\mathcal{L}} f(z) \right|^2 dz\, dt \to 0$$

as $r_j \to 0$. In other words $e^{-\eta_j \mathcal{L}} f$ converges to $e^{-it\mathcal{L}} f$ in $L^2(\mathbb{C}^n \times \mathbb{S}^1)$. Thus we can extract a subsequence of this sequence, denoted by $e^{-\eta_n \mathcal{L}} f(z)$, for which

$$\lim_{\Re(\eta_n)\to 0} e^{-\eta_n \mathcal{L}} f(z) = e^{-it\mathcal{L}} f(z) \tag{3.5}$$

for a.e. $(z,t) \in \mathbb{C}^n \times [-\pi, \pi)$.

Now we show that the sequence $e^{-\eta_n \mathcal{L}} f(z)$ is Cauchy in $L^q(\mathbb{S}^1; L^p(\mathbb{C}^n))$. For $0 < r_m < r_n$, let $\eta_m = r_m + it$, and set $\rho = \rho_{n,m} = \eta_n - \eta_m$. Since $e^{-(r+it)\mathcal{L}} = e^{-r\mathcal{L}} e^{-it\mathcal{L}}$, which follows from the definition, we see that

$$\begin{aligned}
&\left\| e^{-\eta_n \mathcal{L}} f - e^{-\eta_m \mathcal{L}} f \right\|_{L^q(\mathbb{S}^1; L^p(\mathbb{C}^n))} \\
&\quad = \left\| e^{-\eta_m \mathcal{L}} \left(e^{-\rho \mathcal{L}} - I \right) f \right\|_{L^q(\mathbb{S}^1; L^p(\mathbb{C}^n))} \\
&\quad \leq C \left\| \left(e^{-\rho \mathcal{L}} - I \right) f \right\|_2
\end{aligned}$$

where I is the identity operator. The last inequality follows from Proposition 3 applied to the L^2 function $\left(e^{-\rho\mathcal{L}} - I \right) f$. Notice that $e^{-\rho\mathcal{L}}$ is bounded on L^2 since $\rho > 0$ and hence $\left(e^{-\rho\mathcal{L}} - I \right) f \in L^2(\mathbb{C}^n)$. Since $\rho \to 0$ as $r_n \to 0$, by a dominated convergence argument as before shows that $\|(e^{-\rho\mathcal{L}} - I) f\|_2 \to 0$ as $r_n \to 0$. This proves that the sequence $e^{-\eta_n \mathcal{L}} f$ is Cauchy in $L^q(\mathbb{S}^1; L^p(\mathbb{C}^n))$ and hence has a limit in $L^q(\mathbb{S}^1; L^p(\mathbb{C}^n))$. By (3.5), this limit has to be $e^{-it\mathcal{L}} f(z)$. □

Remark 2 The proof also follows directly from (3.5) by application of Fatou's lemma twice; with respect to t and z variable separately. This proves that $\|e^{-\eta_n \mathcal{L}} f(z)\|_{L^q(\mathbb{S}^1; L^p(\mathbb{C}^n))} \to \|e^{-it\mathcal{L}} f(z)\|_{L^q(\mathbb{S}^1; L^p(\mathbb{C}^n))}$ as $n \to \infty$. However, the arguments in the above proof shows that $e^{-\eta_n \mathcal{L}} f(z)$ actually converges to $e^{-it\mathcal{L}} f(z)$ in the Banach space $L^q(\mathbb{S}^1; L^p(\mathbb{C}^n))$.

4 Applications to Schrödinger Equation

Now we consider the initial value problem for the Schrödinger equation for $\mathcal{L}$:

$$i\partial_t u(z,t) - \mathcal{L}u(z,t) = 0, \quad z \in \mathbb{C}^n, \ t \in \mathbb{R} \tag{4.1}$$

$$u(z,0) = f(z) . \tag{4.2}$$

It is easy to see that for $f \in L^2(\mathbb{C}^n)$ the function u given by

$$u(z,t) = e^{-it\mathcal{L}} f(z)$$

satisfies the above Schrödinger equation in the distribution sense. Moreover, by the dominated convergence argument discussed above, we can also see that the solution $u(\cdot,t) \to f$ in the L^2 sense as $t \to 0$. Thus the above initial value problem is well posed in $L^2(\mathbb{C}^n)$ and the oscillatory group $\{e^{-it\mathcal{L}} : t \in \mathbb{R}\}$ gives the solution at any time $t \neq 0$, for any given initial data $f \in L^2(\mathbb{C}^n)$.

Theorem A is a regularity theorem for the solution to the above initial value problem. It says that for L^2 initial data, the solution $u(\cdot,\cdot)$ is actually a function that is periodic in t with period 2π and for almost all t, the function $u(t,\cdot)$ belongs to $L^p(\mathbb{C}^n)$ for $2 \le p < \frac{2n}{n-1}$.

Now let us consider the inhomogeneous problem:

$$i\partial_t u(z,t) - \mathcal{L}u(z,t) = g(z,t), \quad z \in \mathbb{C}^n, \ t \in \mathbb{R} \tag{4.3}$$

$$u(z,0) = f(z) . \tag{4.4}$$

In this case, the solution is given by the Duhamel's formula:

$$u(z,t) = e^{-it\mathcal{L}} f(z) - i \int_0^t e^{-i(t-s)\mathcal{L}} g(z,s)\, ds . \tag{4.5}$$

For the inhomogeneous problem we prove the following.

Theorem 2 *Let $f \in L^2(\mathbb{C}^n)$ and $g(z,t) \in L^{q'}(\mathbb{S}^1;\ L^{p'}(\mathbb{C}^n))$ then the solution $u(z,t)$ to the problem (4.3), (4.4) lies in $L^q(\mathbb{S}^1;\ L^p(\mathbb{C}^n))$, for all pairs (q,p) such that $1 \le q \le 2,\ 2 \le p < \frac{2n}{2n-1}$ or $2 < q < \infty,\ \frac{1}{q} \ge n(\frac{1}{2} - \frac{1}{p})$. Further $u(z,t)$ satisfies the inequality*

$$\|u(z,t)\|_{L^q(\mathbb{S}^1;\,L^p(\mathbb{C}^n))} \le C_n \big(\|f\|_2 + \|g\|_{L^{q'}(\mathbb{S}^1;\,L^{p'}(\mathbb{C}^n))} \big)$$

for the above pairs (p,q) with some constant C_n independent of f and g.

Proof By Theorem A, the $L^q(\mathbb{S}^1;\ L^p(\mathbb{C}^n))$ norm of the first term in (4.5) is bounded by $C_n\|f\|_2$ for $1 \le q \le 2,\ 2 \le p < \frac{2n}{2n-1}$ or $2 < q < \infty,\ \frac{1}{q} \ge n(\frac{1}{2} - \frac{1}{p})$.

Also the second term is bounded by

$$\int_0^{2\pi} \left| e^{-i(t-s)\mathcal{L}} g(x,s) \right| ds .$$

Thus an application of Minkowski's inequality for integrals followed by Proposition 1, we see that

$$\left\| \int_0^t e^{-i(t-s)\mathcal{L}} g(\cdot, s)\, ds \right\|_p \leq C_n \int_0^{2\pi} \frac{\|g(\cdot, s)\|_{p'}}{|\sin(t-s)|^{2n(\frac{1}{p'}-\frac{1}{2})}}\, ds\,. \tag{4.6}$$

Notice that RHS of the above equation is a convolution on the circle and $\|g(\cdot, s)\|_{p'} \in L^{q'}(\mathbb{S}^1, ds)$ as a function of s, by hypothesis. Hence, by arguments using Lemma 2 as in Proposition 2, we get

$$\left\| \int_0^{2\pi} \frac{\|g(\cdot, s)\|_{p'}}{|\sin(t-s)|^{2n(\frac{1}{p'}-\frac{1}{2})}}\, ds \right\|_{L^q(\mathbb{S}^1, ds)} \leq C \|g\|_{L^{q'}(\mathbb{S}^1;\, L^{p'}(\mathbb{C}^n))}\,.$$

In view of (4.6), this shows that the second term of (4.5) satisfies

$$\left\| i \int_0^t e^{-i(t-s)\mathcal{L}} g(x, s)\, ds \right\|_{L^q(\mathbb{S}^1;\, L^p(\mathbb{C}^n))} \leq C \|g\|_{L^{q'}(\mathbb{S}^1;\, L^{p'}(\mathbb{C}^n))}\,.$$

This completes the proof. □

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