J Fourier Anal Appl (2008) 14: 60–88
DOI 10.1007/s00041-007-9002-0

Semyanistyi's Integrals and Radon Transforms on Matrix Spaces

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Received: 22 September 2006 / Published online: 30 January 2008

Abstract We introduce a new analytic family of intertwining operators which include the Radon transform over matrix planes and its inverse. These operators generalize integral transformations introduced by Semyanistyi (Dokl. Akad. Nauk SSSR 134:536–539, 1960) in his research related to the hyperplane Radon transform in $\mathbb{R}^n$. We obtain an extended version of Fuglede's formula, connecting generalized Semyanistyi's integrals, Radon transforms and Riesz potentials on the space of real rectangular matrices. This result combined with the matrix analog of the Hilbert transform leads to variety of new explicit inversion formulas for the Radon transform of functions of matrix argument.

Keywords Radon transforms · Matrix spaces · Riesz distributions · Inversion formulas

Mathematics Subject Classification (2000) 42B10 · 52A22

Communicated by Fulvio Ricci.

The authors were supported in part by the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany). The first author was also supported by Abraham and Sarah Gelbart Research Institute for Mathematical Sciences. The second author was also supported by the NSF grants EPS-0346411 (Louisiana Board of Regents) and DMS-0556157).

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1 Introduction

Radon transforms play an important role in different branches of mathematics and have many applications [6, 11, 14, 19, 25]. The present article deals with Radon transforms of functions of matrix argument introduced by Petrov [31] in 1967. After first publications on this subject [4, 31, 32, 46, 47], it became clear that Radon transforms on matrix spaces (and also on Grassmann manifolds) have a number of striking distinctive features that do not happen in the classical theory of similar transforms over planes in $\mathbb{R}^n$; see [15–18, 27–30, 32, 33, 40, 52]. Some of these features are still mysterious and require new ideas. The main focus of the present article is a matrix generalization of the important analytic family of integral transforms introduced by Semyanistyi [44] in his study of the hyperplane Radon transform on $\mathbb{R}^n$. Implementation of this family makes inversion problem for the Radon transform conceptually transparent and provides a variety of new explicit inversion formulas.

Let us describe the essence of the matter. Let $\mathfrak{M}_{n,m}$, $n \geq m$, be the space of $n \times m$ real matrices $x = (x_{i,j})$. We fix an integer k, $1 \leq k < n$, and let $V_{n,n-k} = \{\xi \in \mathfrak{M}_{n,m} : \xi'\xi = I_{n-k}\}$ be the Stiefel manifold of orthonormal $(n-k)$-frame in $\mathbb{R}^n$. Here ξ' denotes the transpose of ξ, and I_{n-k} is the identity matrix. Let $\mathfrak{T}$ be the set of matrix planes in $\mathfrak{M}_{n,m}$. Each such plane is defined by

$$\tau \equiv \tau(\xi, t) = \{x \in \mathfrak{M}_{n,m} : \xi' x = t\}, \qquad \xi \in V_{n,n-k}, \quad t \in \mathfrak{M}_{n-k,m} . \tag{1.1}$$

The Radon transform associated to planes (1.1) and the relevant dual transform are defined by

$$\hat{f}(\tau) = \int_{x\in\tau} f(x), \qquad \check{\varphi}(x) = \int_{\tau\ni x} \varphi(\tau) , \tag{1.2}$$

the integration being performed against the corresponding canonical measures. We call (1.2) the *rank-one Radon transforms* if $m = 1$, and the *higher-rank Radon transforms* if $m > 1$. The case $m = 1$, when $\hat{f}(\tau)$ is the usual k-plane transform in $\mathbb{R}^n$, is well investigated; see [14, 19, 39] and references therein.

In 1960, Semyanistyi [44] came up with an interesting idea to regard the hyperplane Radon transform and its dual (the case $m = 1$, $k = n - 1$) as members of suitable analytic families of operators P^α and $\overset{*}{P}{}^{\alpha}$, $\alpha \in \mathbb{C}$, so that $P^0 f = \hat{f}$ and $\overset{*}{P}{}^{0}\varphi = \check{\varphi}$ (we adopt our own notation). This gives a variety of inversion formulas $c_n f = (-\Delta)^{(n+\alpha-1)/2} \overset{*}{P}{}^{\alpha} \hat{f}$, where $c_n = \text{const}$ and $(-\Delta)^{(n+\alpha-1)/2}$ is a power of the Laplace operator that can be realized in different ways. This approach was extended in [39] to k-plane Radon transforms in $\mathbb{R}^n$ for all $1 \leq k < n$. Specifically, let $\mathcal{G}_{n,k}$ be the manifold of nonoriented k-dimensional planes τ in $\mathbb{R}^n$ and let $|x - \tau|$ be the Euclidean distance between the point $x \in \mathbb{R}^n$ and the k-plane $\tau \in \mathcal{G}_{n,k}$. Then the generalized Semyanistyi's integrals are defined by

$$\big(P^\alpha f\big)(\tau) = c_{n,k}(\alpha) \int\limits_{\mathbb{R}^n} f(x)\, |x - \tau|^{\alpha+k-n}\, dx, \qquad \tau \in \mathcal{G}_{n,k} , \tag{1.3}$$

$$\big(\overset{*}{P}{}^{\alpha}\varphi\big)(x)=c_{n,k}(\alpha)\int\limits_{\mathcal{G}_{n,k}}\varphi(\tau)\,|x-\tau|^{\alpha+k-n}\,d\tau,\qquad x\in\mathbb{R}^n\,,\tag{1.4}$$

and coincide with those in [44] when $k=n-1$. It was proved [39] that

$$\overset{*}{P}{}^{\alpha}\hat{f}=c_{n,k}I^{\alpha+k}f\tag{1.5}$$

which implies $c_{n,k}f=(-\Delta)^{(k+\alpha)/2}\overset{*}{P}{}^{\alpha}\hat{f}$ where α can be chosen as we please. All constants in these formulas are explicitly determined. The case $\alpha=0$ in (1.5) is known as Fuglede's formula [9]. This idea was generalized to totally geodesic Radon transforms on arbitrary spaces of constant curvature and proved to be useful in applications [35–38, 41].

Plan of the article and main results. Our aim is to extend Semyanistyi's idea to the higher-rank Radon transforms (1.2). Section 2 contains necessary prerequisites on matrix spaces and higher-rank Radon transforms. In particular, for generic point $x\in\mathfrak{M}_{n,m}$ and a matrix plane $\tau\in\mathfrak{T}$, we define a natural substitute of the Euclidean distance to be a positive semi-definite matrix

$$d(x,\tau)=\big[\big(\xi'x-t\big)'\big(\xi'x-t\big)\big]^{1/2}\,.\tag{1.6}$$

The quantity $|x-\tau|$ in (1.3) and (1.4) is accordingly replaced by

$$|x-\tau|_m=\det(d(x,\tau))\,.\tag{1.7}$$

In Section 3, we study special classes of Schwartz distributions on $\mathfrak{M}_{n,m}$ of the Riesz type. Following terminology in [8] (see also [23, 29, 40]), we call them *zeta distributions*. Additional information about modern theory of zeta distributions can be found in [1–3, 5, 21, 42, 45] where the main focus lies beyond the scope of our article. Section 4 deals with Riesz potentials and Hilbert transforms of functions of matrix argument. In the rank-one case $m=1$, they are indispensable tools for Radon inversion [19]. Our aim here is to obtain explicit representations of Riesz potentials outside of the domain of their absolute convergence. We also define the *generalized Hilbert transform* on the space of square matrices as a pseudo-differential operator with the symbol $\operatorname{sgn}\det(y)$ (up to a constant multiple). This agrees with the usual Hilbert transform on the real line [26, 48] and extends as a linear bounded operator on $L^2(\mathfrak{M}_{n,m})$.

Section 5 is a core of the article and relies on Sections 3 and 4. Here we extend generalized Semyanistyi's integrals (1.3) and (1.4) to the higher-rank case. Specifically, for functions f on $\mathfrak{M}_{n,m}$ and φ on $\mathfrak{T}$, we define $P^{\alpha}f$ and $\overset{*}{P}{}^{\alpha}\varphi$ to be analytic continuations of the integrals

$$\big(P^{\alpha}f\big)(\tau)=\frac{1}{\gamma_{n-k,m}(\alpha)}\int\limits_{\mathfrak{M}_{n,m}}f(x)\,|x-\tau|_m^{\alpha+k-n}\,dx,\qquad\tau\in\mathfrak{T}\,,\tag{1.8}$$

$$\big(\overset{*}{P}{}^{\alpha}\varphi\big)(x)=\frac{1}{\gamma_{n-k,m}(\alpha)}\int\limits_{\mathfrak{T}}\varphi(\tau)\,|x-\tau|_m^{\alpha+k-n}\,d\tau,\qquad x\in\mathfrak{M}_{n,m}\,,\tag{1.9}$$

with the suitable normalizing constant $\gamma_{n-k,m}(\alpha)$. For $\alpha = 0$, we get $P^0 f = \hat{f}$ and $\overset{*}{P}{}^0 \varphi = \check{\varphi}$ up to constant multiples. We prove the generalized Fuglede formula

$$\overset{*}{P}{}^\alpha \hat{f} = c_{n,k,m} I^{\alpha+k} f \,, \tag{1.10}$$

which mimics (1.5) and contains the relevant matrix modification of the Riesz potential on the right-hand side. All constants in (1.8)–(1.10) are determined explicitly. Section 6 contains a series of inversion formulas for the higher-rank Radon transform which follow from (1.10) and have the general structure $c_{n,k,m} f = I^{-\alpha-k} \overset{*}{P}{}^\alpha \hat{f}$. In particular, for $\alpha = -k$, this yields

$$c_{n,k,m} f = \overset{*}{P}{}^{-k} \hat{f} \,.$$

Explicit expressions for $\overset{*}{P}{}^{-k} \hat{f}$ are given by (6.1)–(6.3). Thus we see that the Radon transform is actually a member of the analytic family $\{P^\alpha\}$ and the inverse Radon transform belongs to the dual family $\{\overset{*}{P}{}^\alpha\}$.

Some comments are in order. (1) First results for the Radon transforms (1.2) were obtained by Petrov [31, 32] for $k = n - m$ and extended by Shibasov [47] to all $1 \le k \le n - m$. Basic inversion formulas in these articles contain divergent integrals having been understood in the sense of regularization. Many important calculations in [47] are unfortunately skipped. We note that our presentation is almost self-contained and all integrals in inversion formulas (6.1)–(6.3) are exhibited in explicit and readable form. Our philosophy, eventually based on explicit representation of analytic continuation of zeta integrals, is also different. Moreover, for k odd, when inversion formulas are essentially nonlocal, we reveal the following intriguing difference between the cases $k = n - m$ and $k < n - m$: In the second case our formula (6.2) does not contain the Hilbert transform. One should add that the method of the paper conceptually agrees with the classical Radon-Helgason scheme [19] and has the same nature as decomposition of the delta function in plane waves; cf. [13]. (2) Integrals (1.8) and (1.9) are absolutely convergent for sufficiently good f and φ if and only if $\mathrm{Re}\,\alpha > m - 1$. Otherwise, they have a complicated structure of singularities and must be understood in the sense of analytic continuation. The crux is to obtain explicit and readable formulas for these analytic continuations. (3) For $\alpha = 0$, when $\overset{*}{P}{}^0$ is actually the dual Radon transform, formula (1.10) was obtained in [27]; see also [40]. Letting α vary, we achieve more flexibility which enables us to choose the most effective Radon inversion formula for every specific triple $\{k, m, n\}$. Another advantage of the method is that, playing with α, we provide analytic continuation of complicated integrals with fairly elementary explicit expressions. (4) An alternative approach to inversion of higher-rank Radon transforms on $\mathfrak{M}_{n,m}$ was suggested in [28]. It relies on Gårding-Gindikin fractional integrals over the cone of positive definite matrices and agrees with the previous work by Grinberg, Rubin, and Zhang [18, 53] for Radon transforms on Grassmannians. Another Radon inversion algorithm involving wavelet-like transforms on $\mathfrak{M}_{n,m}$ was developed in [30]. The range of the Radon transform $f \to \hat{f}$ in (1.2) on Schwartz functions was characterized by Gonzalez and Kakehi [16] in group-theoretic terms. The method of the present paper essentially differs from those in cited publications and increases our knowledge of the object.

2 Preliminaries

2.1 Matrix Spaces. Notation

In the following, $\mathfrak{M}_{n,m} \sim \mathbb{R}^{nm}$ is the space of real matrices $x = (x_{i,j})$ having n rows and m columns, $n \geq m$, $\mathfrak{M}_m = \mathfrak{M}_{m,m}$, $dx = \prod_{i=1}^{n} \prod_{j=1}^{m} dx_{i,j}$ is the volume element on $\mathfrak{M}_{n,m}$, x' denotes the transpose of x, and I_m is the identity $m \times m$ matrix. Given a square matrix a, we denote by $\det(a)$ the determinant of a, and by $|a|$ the absolute value of $\det(a)$; $\mathrm{tr}(a)$ stands for the trace of a. For $x \in \mathfrak{M}_{n,m}$, we denote

$$|x|_m = \det(x'x)^{1/2} . \tag{2.1}$$

If $m = 1$, then this is the usual Euclidean norm on $\mathbb{R}^n$. For $m > 1$, $|x|_m$ is the volume of the parallelepiped spanned by the column-vectors of x [10, p. 251]. We use standard notations $O(n)$ and $SO(n)$ for the orthogonal group and the special orthogonal group of $\mathbb{R}^n$ with the normalized invariant measure of total mass 1; $M(n, m)$ is the group of motions of $\mathfrak{M}_{n,m}$ acting by the rule $x \to \gamma x \beta + b$, where $\gamma \in O(n)$, $\beta \in O(m)$, and $b \in \mathfrak{M}_{n,m}$.

Let $\mathcal{S}_m \sim \mathbb{R}^{m(m+1)/2}$ be the space of $m \times m$ real symmetric matrices $s = (s_{i,j})$ with the volume element $ds = \prod_{i \leq j} ds_{i,j}$. We denote by $\mathcal{P}_m$ the cone of positive definite matrices in $\mathcal{S}_m$; $\overline{\mathcal{P}}_m$ is the closure of $\mathcal{P}_m$, that is the set of all positive semi-definite $m \times m$ matrices. For $r \in \mathcal{P}_m$ ($r \in \overline{\mathcal{P}}_m$), we write $r > 0$ ($r \geq 0$). Given a and b in S_m, the inequality $a > b$ means $a - b \in \mathcal{P}_m$ and the symbol $\int_a^b f(s)\, ds$ denotes the integral over the set $(a + \mathcal{P}_m) \cap (b - \mathcal{P}_m)$.

The group $G = GL(m, \mathbb{R})$ of real nonsingular $m \times m$ matrices g acts transitively on $\mathcal{P}_m$ by the rule $r \to grg'$. The corresponding G-invariant measure on $\mathcal{P}_m$ is

$$d_* r = |r|^{-d}\, dr, \qquad |r| = \det(r), \qquad d = (m + 1)/2 \tag{2.2}$$

[49, p. 18]. Let T_m be the subgroup of $GL(m, \mathbb{R})$ consisting of upper triangular matrices $(t_{i,j})$ with positive diagonal entries. Each $r \in \mathcal{P}_m$ has a unique representation $r = t't,\ t \in T_m$, so that

$$\begin{gathered}\int\limits_{\mathcal{P}_m} f(r)\, dr = \int\limits_0^\infty t_{1,1}^m\, dt_{1,1} \int\limits_0^\infty t_{2,2}^{m-1}\, dt_{2,2} \ldots \int\limits_0^\infty t_{m,m} \tilde{f}(t_{1,1}, \ldots, t_{m,m})\, dt_{m,m} , \\ \tilde{f}(t_{1,1}, \ldots, t_{m,m}) = 2^m \int\limits_{\mathbb{R}^{m(m-1)/2}} f(t't)\, dt_*, \quad dt_* = \prod_{i<j} dt_{i,j} ,\end{gathered} \tag{2.3}$$

[49, p. 22], [24, p. 592]. In the last integration, the diagonal entries of the matrix t are given by the arguments of $\tilde{f}$, and the strictly upper triangular entries of t are variables of integration.

For $\mathrm{Re}\,\alpha > d-1$, the Siegel gamma function of $\mathcal{P}_m$ is defined by

$$\Gamma_m(\alpha) = \int\limits_{\mathcal{P}_m} \exp(-\mathrm{tr}(r))|r|^{\alpha} d_* r = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(\alpha - j/2) \,, \tag{2.4}$$

[8, 15, 49]. It extends meromorphically to all $\alpha \in \mathbb{C}$ and obeys

$$(-1)^m \frac{\Gamma_m(1-\alpha/2)}{\Gamma_m(-\alpha/2)} = 2^{-m} \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} = 2^{-m}(\alpha, m) \tag{2.5}$$

where $(\alpha, m) = \alpha(\alpha+1)\cdots(\alpha+m-1)$ is the Pochhammer symbol.

The relevant beta function has the form

$$B_m(\alpha, \beta) = \int\limits_0^{I_m} |r|^{\alpha-d} |I_m - r|^{\beta-d}\, dr = \frac{\Gamma_m(\alpha)\Gamma_m(\beta)}{\Gamma_m(\alpha+\beta)}, \quad d = (m+1)/2 \,. \tag{2.6}$$

This integral converges absolutely if and only if $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > d-1$.

In the following, all functions spaces on $\mathfrak{M}_{n,m}$ are identified with the corresponding spaces on $\mathbb{R}^{nm}$. For instance, $\mathcal{S}(\mathfrak{M}_{n,m})$ denotes the Schwartz space of infinitely differentiable rapidly decreasing functions. The Fourier transform of a function $f \in L^1(\mathfrak{M}_{n,m})$ is defined by

$$(\mathcal{F}f)(y) = \int\limits_{\mathfrak{M}_{n,m}} \exp\big(\mathrm{tr}(iy'x)\big) f(x)\, dx, \qquad y \in \mathfrak{M}_{n,m} \,. \tag{2.7}$$

This extends to distributions $f \in \mathcal{S}'(\mathfrak{M}_{n,m})$ by the Parseval formula

$$(\mathcal{F}f, \mathcal{F}\varphi) = (2\pi)^{nm}(f, \varphi), \qquad \varphi \in \mathcal{S}(\mathfrak{M}_{n,m}) \,, \tag{2.8}$$

where for $f \in L^1_{\mathrm{loc}}(\mathfrak{M}_{n,m})$,

$$(f, \varphi) = \int\limits_{\mathfrak{M}_{n,m}} f(x)\overline{\varphi(x)}\, dx \,.$$

For $n \ge m$, let $V_{n,m} = \{v \in \mathfrak{M}_{n,m} : v'v = I_m\}$ be the Stiefel manifold of orthonormal m-frames in $\mathbb{R}^n$. The group $O(n)$ acts transitively on $V_{n,m}$ by the left matrix multiplication. This is also true for $SO(n)$ if $n > m$. We fix an invariant measure dv on $V_{n,m}$ normalized by

$$\sigma_{n,m} \equiv \int\limits_{V_{n,m}} dv = \frac{2^m \pi^{nm/2}}{\Gamma_m(n/2)} \,, \tag{2.9}$$

[24, p. 70], and denote $d_* v = \sigma_{n,m}^{-1}\, dv$.

Lemma 1 (polar decomposition) *Let* $x \in \mathfrak{M}_{n,m}$, $n \geq m$. *If* rank$(x) = m$, *then*

$$x = vr^{1/2}, \qquad v \in V_{n,m}, \qquad r = x'x \in \mathcal{P}_m \ , \tag{2.10}$$

and $dx = 2^{-m}|r|^{(n-m-1)/2}\, dr\, dv$.

This statement can be found, e.g., in [8, 20, 24].

Lemma 2 *Let* $x \in \mathfrak{M}_{n,m}$, $n \geq m$. *If* rank$(x) = m$, *then*

$$x = vt, \qquad v \in V_{n,m}, \qquad t \in T_m \ ,$$

so that

$$dx = \prod_{j=1}^{m} t_{j,j}^{n-j}\, dt_{j,j}\, dt_*\, dv, \qquad dt_* = \prod_{i<j} dt_{i,j} \ .$$

This statement is also well known and has different proofs. For instance, it can be easily derived from Lemma 1 and (2.3); see Lemma 2.7 in [40].

2.2 Differential Operators

The *Cayley-Laplace operator* Δ on $\mathfrak{M}_{n,m}$ is defined by

$$\Delta = \det\big(\partial'\partial\big), \qquad \partial = (\partial/\partial x_{i,j}) \ , \tag{2.11}$$

and yields the Bernstein type equality [40]:

$$\Delta^k |x|_m^{\alpha+2k-n} = B_k(\alpha)|x|_m^{\alpha-n} \ , \tag{2.12}$$

where

$$B_k(\alpha) = \prod_{i=0}^{m-1}\prod_{j=0}^{k-1}(\alpha - i + 2j)(\alpha - n + 2 + 2j + i) = B_k(n - \alpha - 2k) \ . \tag{2.13}$$

In terms of the Fourier transform, the action of Δ represents a multiplication by the homogeneous polynomial $(-1)^m|y|_m^2$ of degree $2m$ of nm variables $y_{i,j}$. For $m > 1$, the operator Δ is neither elliptic nor hyperbolic, although, for some n, m and ℓ, its power Δ^ℓ enjoys the strengthened Huygens' principle; see [23] for details.

When dealing with square matrices, we will also need the *Cayley differential operator*

$$\mathcal{D} = \det(\partial/\partial x_{i,j}), \quad x = (x_{i,j}) \in \mathfrak{M}_m \ , \tag{2.14}$$

which interacts with the Fourier transform as follows:

$$(\mathcal{F}[\mathcal{D} f])(y) = (-i)^m \det(y)\, (\mathcal{F} f)(y) \ , \tag{2.15}$$

$$\mathcal{D}(\mathcal{F} f)(y) = i^m\, (\mathcal{F}[f(x)\det(x)])(y) \ . \tag{2.16}$$

2.3 Matrix Planes and Radon Transforms

We recall basic facts from [27, 28] about Radon transforms on $\mathfrak{M}_{n,m}$. For $\xi \in V_{n,n-k}$ and $t \in \mathfrak{M}_{n-k,m}$, $0 < k < n$, a *matrix k-plane* in $\mathfrak{M}_{n,m}$ is defined by

$$\tau \equiv \tau(\xi, t) = \left\{x : x \in \mathfrak{M}_{n,m};\ \xi' x = t\right\} . \tag{2.17}$$

The set $\mathfrak{T}$ of all matrix k-planes is a homogeneous space of the group $M(n,m)$ of matrix motions in the framework of the classical double fibration scheme; see [16], Section 3. For $m = 1$, $\mathfrak{T}$ is the Grassmann manifold of all nonoriented k-dimensional planes in $\mathbb{R}^n$. Every matrix plane (2.17) is actually a usual km-dimensional plane in $\mathbb{R}^{nm}$ (but not vice versa). One can regard $\mathfrak{T}$ as a quotient space $(V_{n,n-k} \times \mathfrak{M}_{n-k,m})/O(n-k)$. Functions $\varphi(\tau)$ on $\mathfrak{T}$ are identified with functions $\varphi(\xi, t)$ on $V_{n,n-k} \times \mathfrak{M}_{n-k,m}$ satisfying $\varphi(\xi\theta', \theta t) = \varphi(\xi, t)$ for all $\theta \in O(n-k)$, and the corresponding measure $d\tau$ on $\mathfrak{T}$ is chosen so that

$$\int\limits_{\mathfrak{T}} \varphi(\tau)\, d\tau = \int\limits_{V_{n,n-k} \times \mathfrak{M}_{n-k,m}} \varphi(\xi, t)\, d_*\xi\, dt . \tag{2.18}$$

A *matrix distance* between points x and y in $\mathfrak{M}_{n,m}$ is defined by

$$d(x, y) = \left[(x-y)'(x-y)\right]^{1/2} . \tag{2.19}$$

A matrix distance between $x \in \mathfrak{M}_{n,m}$ and $\tau = \tau(\xi, t) \in \mathfrak{T}$ is defined accordingly as

$$d(x, \tau) = \left[\left(\xi' x - t\right)'\left(\xi' x - t\right)\right]^{1/2} . \tag{2.20}$$

We denote

$$|x - y|_m = \det(d(x, y)), \qquad |x - \tau|_m = \det(d(x, \tau)) = \left|\xi' x - t\right|_m . \tag{2.21}$$

Lemma 3 (i) *The group $M(n,m)$ of matrix motions,*

$$x \longrightarrow \gamma x \beta + b, \qquad \gamma \in O(n), \qquad \beta \in O(m), \qquad b \in \mathfrak{M}_{n,m} ,$$

acts on $\mathfrak{M}_{n,m}$ and $\mathfrak{T}$ transitively. (ii) *The determinants $|x-y|_m$ and $|x-\tau|_m$ are invariant under the action of $M(n,m)$. Namely,*

$$|gx - gy|_m = |x - y|_m, \qquad |gx - g\tau|_m = |x - \tau|_m, \qquad g \in M(n,m) .$$

(iii) *The distances $d(x,y)$ and $d(x,\tau)$ are invariant under the subgroup $M'(n,m)$ of $M(n,m)$, acting by the rule $x \to \gamma x + b$, $\gamma \in O(n)$, $b \in \mathfrak{M}_{n,m}$.*

Proof Both statements follow from the observation, that if $gx = \gamma x \beta + b$, then, for $\tau = \tau(\xi, t)$, we have $g\tau = \tau(\gamma\xi, t\beta + \xi'\gamma' b)$. □

Note that the matrix plane $\tau = \tau(\xi, t)$, $\xi \in V_{n,n-k}$, $t \in \mathfrak{M}_{n-k,m}$, consists of "points"

$$x = g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix},$$

where $\omega \in \mathfrak{M}_{k,m}$, and $g_\xi \in SO(n)$ is a rotation satisfying

$$g_\xi \xi_0 = \xi, \qquad \xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k} \,. \tag{2.22}$$

This observation leads to the following.

Definition 1 The Radon transform of a function f on $\mathfrak{M}_{n,m}$ is defined by

$$\hat{f}(\tau) \equiv \hat{f}(\xi, t) = \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix}\right) d\omega, \qquad (\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m} \,. \tag{2.23}$$

The dual Radon transform of a function $\varphi(\tau) = \varphi(\xi, t)$ on $\mathfrak{T}$ is defined by

$$\check{\varphi}(x) = \int\limits_{V_{n,n-k}} \varphi\big(\xi, \xi' x\big)\, d_*\xi, \qquad x \in \mathfrak{M}_{n,m} \,. \tag{2.24}$$

In the rank-one case $m = 1$, the operators $\hat{f}(\tau)$ and $\check{\varphi}(x)$ are classical k-plane Radon transform and its dual.

It is known [28] that for $f \in L^p(\mathfrak{M}_{n,m})$, the Radon transform $\hat{f}(\xi, t)$ is finite for almost all $(\xi, t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}$ if and only if $1 \le p < (n + m - 1)/(k + m - 1)$. Moreover, it is injective on $\mathcal{S}(\mathfrak{M}_{n,m})$ if and only if $1 \le k \le n - m$. The dual Radon transform $\check{\varphi}(x)$ is finite almost everywhere on $\mathfrak{M}_{n,m}$ for any locally integrable function φ.

The Radon transform and its dual commute with matrix motions $g \in M(n, m)$. Specifically, if $x \in \mathfrak{M}_{n,m}$ and $gx = \gamma x \beta + b$, where $\gamma \in O(n)$, $\beta \in O(m)$, and $b \in \mathfrak{M}_{n,m}$, then

$$(f \circ g)^\wedge(\xi, t) = \big(\hat{f} \circ g\big)(\xi, t) = \hat{f}\big(\gamma\xi, t\beta + \xi'\gamma' b\big) \,, \tag{2.25}$$

and

$$(\varphi \circ g)^\vee(x) = \big(\check{\varphi} \circ g\big)(x) = \check{\varphi}(\gamma x \beta + b) \,. \tag{2.26}$$

Furthermore, if $f_x(y) = f(x + y)$, then

$$\hat{f}_x(\xi, t) = \hat{f}\big(\xi, \xi' x + t\big) \,. \tag{2.27}$$

3 Zeta Distributions

3.1 Analytic Continuation and Functional Equations

In the classical theory of Radon transforms in $\mathbb{R}^n$, one of the basic inversion methods is based on decomposition of the distribution $|x|^{\alpha-n}/\Gamma(\alpha/2)$ in plane waves, [13]. Diverse higher-rank generalizations of $|x|^{\alpha-n}/\Gamma(\alpha/2)$ fall into the scope of the so-called *zeta distributions* or *zeta integrals*; see [3, 5, 8, 21, 29, 42, 45] and references therein. In this section, we study basic properties of zeta distributions (or integrals) which constitute the background of the method of plane waves in integral geometry in the space of rectangular matrices.

Let $\alpha \in \mathbb{C},\ x \in \mathfrak{M}_{n,m},\ n \ge m,\ |x|_m = \det(x'x)^{1/2}$. We denote

$$\zeta_\alpha(x) = \frac{|x|_m^{\alpha-n}}{\Gamma_m(\alpha/2)}\ . \tag{3.1}$$

If $n = m$ and $x \in \mathfrak{M}_m$, we also set

$$\zeta_\alpha^+(x) = \frac{|\det(x)|^{\alpha-m}}{\Gamma_m(\alpha/2)}, \qquad \zeta_\alpha^-(x) = \frac{|\det(x)|^{\alpha-m}\,\mathrm{sgn}\det(x)}{\Gamma_m((\alpha+1)/2)}\ . \tag{3.2}$$

These functions can be regarded as Schwartz distributions according to the formulas

$$(\zeta_\alpha, f) = \int\limits_{\mathfrak{M}_{n,m}} \zeta_\alpha(x)\,\overline{f(x)}\,dx, \quad \left(\zeta_\alpha^\pm, f\right) = \int\limits_{\mathfrak{M}_m} \zeta_\alpha^\pm(x)\,\overline{f(x)}\,dx\ . \tag{3.3}$$

Following traditional terminology (see, e.g., [8]), we call (3.3) *zeta integrals* and the corresponding distributions will be called *zeta distributions*. For (ζ_α^-, f) we also use the name *the conjugate zeta integral* (or distribution) by analogy with the case $\alpha = 0$, $m = 1$, when the convolution of f with the distribution $\zeta_\alpha^- = \mathrm{p.v.}\frac{1}{x}$ is associated with the Hilbert transform [26, 48] which is also called a conjugate function.

It is known that the integrals (3.3) are absolutely convergent for $\mathrm{Re}\,\alpha > m-1$, and extend as entire functions of $\alpha \in \mathbb{C}$; see, e.g., [23, 33, 46]. Below we suggest a relatively simple procedure of analytic continuation of these integrals and give a series of explicit formulas for these continuations.

To perform analytic continuation, it is natural to utilize the corresponding differential operators. In particular, for $(\zeta_\alpha^\pm, f)$, we make use of the Cayley differential operator $\mathcal{D} = \det(\partial/\partial x_{i,j})$.

Lemma 4 *Let* $x \in \mathfrak{M}_m,\ \mathrm{rank}(x) = m$. *For any* $\lambda \in \mathbb{C}$,

$$\mathcal{D}\left[|\det(x)|^\lambda\right] = (\lambda, m)\,|\det(x)|^{\lambda-1}\,\mathrm{sgn}\det(x)\ , \tag{3.4}$$

$$\mathcal{D}\left[|\det(x)|^\lambda\,\mathrm{sgn}\det(x)\right] = (\lambda, m)\,|\det(x)|^{\lambda-1}\ . \tag{3.5}$$

Proof Note that (3.4) and (3.5) follow one from another. Different proofs of these formulas can be found in [34] and [32]; see also [51, p. 114]. All these proofs are very involved. Below we give an alternative proof which is elementary.

We start with the formula

$$\mathcal{D}_x[f(ax)] = \det(a)\,(\mathcal{D}f)(ax), \qquad a \in GL(m,\mathbb{R})\ , \tag{3.6}$$

which can be easily checked by applying the Fourier transform to both sides. Indeed, if f is good enough at infinity (otherwise we can multiply f by a smooth cut-off function) then by (2.15), the Fourier transform of the left-hand side of (3.6) has the form

$$(-i)^m \det(y)\,\mathcal{F}[f(ax)](y) = \frac{(-i)^m \det(y)}{|\det(a)|^m}(\mathcal{F}f)\big((a^{-1})'y\big)$$

which coincides with the Fourier transform of the right-hand side. If $f(x) = |\det(x)|^\lambda$ then (3.6) yields

$$|\det(a)|^\lambda\,\mathcal{D}\,|\det(x)|^\lambda = \det(a)\,[\mathcal{D}\,|\det(\cdot)|^\lambda](ax)\ .$$

By setting $a = x^{-1}$ (recall that $\operatorname{rank}(x) = m$ so that x is nonsingular), we obtain

$$\mathcal{D}\,|\det(x)|^\lambda = A|\det(x)|^{\lambda-1}\operatorname{sgn}\det(x), \qquad A = \big[\mathcal{D}\,|\det(x)|^\lambda\big](I_m)\ ,$$

and therefore

$$\mathcal{D}\big[|\det(x)|^\lambda \operatorname{sgn}\det(x)\big] = A|\det(x)|^{\lambda-1}\ . \tag{3.7}$$

In order to evaluate A, we make use of the Gaussian functions

$$e(x) = \exp\big(-\operatorname{tr}(x'x)\big) \qquad \text{and} \quad e_1(x) = e(x)\det(x) = (-2)^{-m}(\mathcal{D}e)(x)\ .$$

Applying Lemma 1 and then using an analytic continuation, we obtain

$$(\zeta_\lambda, e) = \frac{\pi^{nm/2}}{\Gamma_m(n/2)}\ . \tag{3.8}$$

Hence,

$$\big(\zeta_\lambda^-, e_1\big) = \big(\zeta_{\lambda+1}^+, e\big) = \frac{\pi^{m^2/2}}{\Gamma_m(m/2)}\ .$$

On the other hand, by (3.7) and (3.8),

$$\begin{aligned}\big(\zeta_\lambda^-, e_1\big) &= (-2)^{-m}\big(\zeta_\lambda^-, \mathcal{D}e\big) = 2^{-m}\big(e(x), \mathcal{D}\big[|\det(x)|^\lambda \operatorname{sgn}\det(x)\big]\big)\\ &= 2^{-m}A\big(\zeta_{\lambda-1}^+, e\big) = c_m 2^{-m}A\,\Gamma_m((\lambda-1+m)/2)\ .\end{aligned}$$

Hence, owing to (2.5), we obtain

$$A = \frac{2^m\,\Gamma_m((\lambda+1+m)/2)}{\Gamma_m((\lambda-1+m)/2)} = (-1)^m\frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda-m)} = (\lambda, m)\ .$$ □

Corollary 1 *For $f \in \mathcal{S}(\mathfrak{M}_{n,m})$,*

$$\big(\zeta_\alpha^-, f\big) = c_\alpha\,\big(\zeta_{\alpha+1}^+, \mathcal{D}f\big), \qquad \big(\zeta_\alpha^+, f\big) = d_\alpha\,\big(\zeta_{\alpha+1}^-, \mathcal{D}f\big)\ , \tag{3.9}$$

where

$$c_\alpha = (-1)^m \frac{\Gamma(\alpha+1-m)}{\Gamma(\alpha+1)}, \qquad d_\alpha = \frac{\Gamma(\alpha+1-m)\,\Gamma(m-\alpha)}{2^m\,\Gamma(\alpha+1)\,\Gamma(-\alpha)}. \tag{3.10}$$

The Fourier transforms of zeta distributions are traditionally realized through the relevant functional equations in accordance with the Parseval equality (2.8). In our case, these equations have the following form.

Theorem 1 *Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $n \geq m$. Then*

$$(\zeta_\alpha, f) = \pi^{-nm/2}\, 2^{m(\alpha-n)}\, (\zeta_{n-\alpha}, \mathcal{F}f), \tag{3.11}$$

$$\left(\zeta_\alpha^-, f\right) = (-i)^m\, \pi^{-m^2/2}\, 2^{m(\alpha-m)} \left(\zeta_{m-\alpha}^-, \mathcal{F}f\right). \tag{3.12}$$

In particular,

$$\left(\zeta_\alpha^+, f\right) = \pi^{-m^2/2}\, 2^{m(\alpha-m)} \left(\zeta_{m-\alpha}^+, \mathcal{F}f\right). \tag{3.13}$$

Proof First we note that both sides of each equality are understood in the sense of analytic continuation and represent entire functions of α. Moreover, (3.13) is a particular case of (3.11). The equality (3.11) was obtained in [7, 8, 12, 32, 34] in the framework of more general considerations. A self-contained proof of them and detailed discussion can be found in [22, 40]. The equality (3.12) was implicitly presented in [32, p. 289]. In fact, it follows from (3.13) owing to the formulas (3.9) and (2.15). Indeed,

$$\begin{aligned}
\left(\zeta_\alpha^-, f\right) &= c_\alpha \left(\zeta_{\alpha+1}^+, \mathcal{D}f\right) = c_\alpha\, \pi^{-m^2/2}\, 2^{m(\alpha+1-m)} \left(\zeta_{m-\alpha-1}^+, \mathcal{F}[\mathcal{D}f]\right) \\
&= (-i)^m\, c_\alpha\, \pi^{-m^2/2}\, 2^{m(\alpha+1-m)}\, \frac{\left((\mathcal{F}f)(y), \det(y)\,|\det(y)|^{-\alpha-1}\right)}{\Gamma_m((m-\alpha-1)/2)} \\
&= c \left(\zeta_{m-\alpha}^-, \mathcal{F}f\right),
\end{aligned}$$

where [use (3.10) and (2.5)]

$$c = (-i)^m\, c_\alpha\, \pi^{-m^2/2}\, 2^{m(\alpha+1-m)}\, \frac{\Gamma_m((m-\alpha+1)/2)}{\Gamma_m((m-\alpha-1)/2)} = (-i)^m\, \pi^{-m^2/2}\, 2^{m(\alpha-m)}.$$

□

3.2 Decomposition in Plane Waves

The following lemma contains decomposition of the distribution ζ_α in matrix plane waves.

Lemma 5 *Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $1 \leq k \leq n-m$, $\alpha \in \mathbb{C}$. Then*

$$(\zeta_\alpha, f) = \frac{\Gamma_m((n-k)/2)}{\Gamma_m(n/2)} \int\limits_{V_{n,n-k}} \left(\zeta_{\alpha-k}, \hat{f}(\xi, \cdot)\right) d_*\xi \tag{3.14}$$

with the zeta distribution $\zeta_{\alpha-k}$ *acting in* $\mathfrak{M}_{n-k,m}$.

Proof In view of analyticity, it suffices to prove (3.14) for $\mathrm{Re}\,\alpha > k+m-1$, when it can be written in terms of absolutely convergent integrals as

$$\int\limits_{\mathfrak{M}_{n,m}} \overline{f(x)}\, |x|_m^{\alpha-n}\, dx = c(\alpha) \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \overline{\hat{f}(\xi,t)}\, |t|_m^{\alpha-n}\, dt\,, \tag{3.15}$$

$$c(\alpha) = \frac{\Gamma_m((n-k)/2)\,\Gamma_m(\alpha/2)}{\Gamma_m(n/2)\,\Gamma_m((\alpha-k)/2)}\,. \tag{3.16}$$

By Fubini's theorem, (3.15) is a direct consequence of the equality

$$\int\limits_{V_{n,n-k}} \left|\xi' x\right|_m^{\alpha-n} d_*\xi = c(\alpha)^{-1}\, |x|_m^{\alpha-n}\,. \tag{3.17}$$

The validity of (3.17) with *some* constant c on the right-hand side follows immediately from the polar decomposition (2.10). Indeed, if $x = vr^{1/2}$, $v \in V_{n,m}$, $r = x'x \in \mathcal{P}_m$, then, by invariance, for $v = \gamma v_0,\ \gamma \in SO(n),\ v_0 = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in V_{n,m}$, we have

$$\int\limits_{V_{n,n-k}} \left|\xi' x\right|_m^{\alpha-n} d_*\xi = c\, r^{(\alpha-n)/2} = c\, |x|_m^{\alpha-n}\,, \tag{3.18}$$

where $c = \int_{V_{n,n-k}} |\xi' v_0|_m^{\alpha-n}\, d_*\xi$. It remains to show that $c = c(\alpha)^{-1}$. To this end, we multiply both sides of (3.18) by $\exp(-\mathrm{tr}(x'x))$ and integrate in $x \in \mathfrak{M}_{n,m}$. By Lemma 1, the r.h. side becomes

$$c \int\limits_{\mathfrak{M}_{n,m}} |x|_m^{\alpha-n} \exp\big(-\mathrm{tr}\big(x'x\big)\big)\, dx = c\, \sigma_{n,m}\, 2^{-m}\, \Gamma_m(\alpha/2)\,. \tag{3.19}$$

For the l.h. side, by changing the order of integration and setting $x = g_\xi \begin{bmatrix} a \\ b \end{bmatrix}$, where g_ξ is a rotation in (2.22), we obtain

$$\begin{aligned}
&\int\limits_{\mathfrak{M}_{n,m}} \exp\big(-\mathrm{tr}\big(x'x\big)\big)\, dx \int\limits_{V_{n,n-k}} \left|\xi' x\right|_m^{\alpha-n} d_*\xi \\
&\quad = \int\limits_{\mathfrak{M}_{k,m}} \exp\big(-\mathrm{tr}\big(a'a\big)\big)\, da \int\limits_{\mathfrak{M}_{n-k,m}} \exp\big(-\mathrm{tr}\big(b'b\big)\big)\, |b|_m^{\alpha-n}\, db \\
&\quad = 2^{-2m}\, \sigma_{k,m}\, \sigma_{n-k,m}\, \Gamma_m(k/2)\, \Gamma_m((\alpha-k)/2)\,.
\end{aligned}$$

Using (2.9) and comparing calculations, we obtain the result. □

Remark 1 As we can see from the proof, if $\mathrm{Re}\,\alpha > k+m-1$ then (3.15) holds for any locally integrable function f provided that either side exists in the Lebesgue sense.

3.3 Explicit Representations

We have already mentioned that zeta integrals (3.3) are absolutely convergent for $\mathrm{Re}\,\alpha > m-1$. For other α's they must be treated in the framework of the theory of distribution as analytic continuations of integrals (3.3). It turns out that for $\alpha = 1, 2, \dots, m-1$, analytic continuation of (ζ_α, f) still have an integral representation and ζ_α can be regarded as a locally tempered measure on $\mathfrak{M}_{n,m}$ supported on matrices of rank less than m. This striking phenomenon reveals an essential difference between the rank-one case $m=1$ and the higher-rank case $m>1$. Moreover, afore-mentioned representations are also applicable to $\alpha = m, m+1, \dots, n$. These explicit representations will serve as important components of Radon inversion formulas in Section 6.

Theorem 2 *Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$. For $\alpha = k$, $k = 1, 2, \dots, n$,*

$$(\zeta_k, f) = \frac{\pi^{(n-k)m/2}}{\Gamma_m(n/2)} \int\limits_{O(n)} d\gamma \int\limits_{\mathfrak{M}_{k,m}} \overline{f\left(\gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix}\right)}\, d\omega\,. \tag{3.20}$$

Furthermore, for $\alpha = 0$ we have

$$(\zeta_0, f) = \frac{\pi^{nm/2}}{\Gamma_m(n/2)}\, \overline{f(0)}\,. \tag{3.21}$$

For $1 \le k \le \min(m-1, n-m)$, formula (3.20) was obtained in [27, Lemma 3.2] in a slightly different notation. Here we give an alternative proof which covers all $1 \le k \le n$ and might be instructive. The proof consists of a few steps. We first consider the distribution $\mathcal{G}_\alpha$ of Riesz type defined by

$$\mathcal{G}_\alpha(f) = \frac{1}{\Gamma_m(\alpha)} \int\limits_{\mathcal{P}_m} \overline{f(r)}\,|r|^{\alpha-d}\, dr, \qquad d = (m+1)/2\,, \tag{3.22}$$

where f is a Schwartz function on the space $\mathcal{S}_m$ of $m \times m$ symmetric matrices. Integral (3.22) converges absolutely for $\mathrm{Re}\,\alpha > d-1$ and admits analytic continuation as an entire function of α so that

$$\mathcal{G}_0(f) = \overline{f(0)}\,, \tag{3.23}$$

see [8, pp. 132–133].

Lemma 6 *For $f \in \mathcal{S}(\mathcal{S}_m)$ and $k = 1, 2, \dots, m-1$,*

$$\mathcal{G}_{k/2}(f) = \pi^{-km/2} \int\limits_{\mathfrak{M}_{k,m}} \overline{f(\omega'\omega)}\, d\omega\,. \tag{3.24}$$

Proof For the sake of convenience, we temporarily replace f by $\overline{f}$. Then by (2.3),

$$\mathcal{G}_\alpha(\overline{f}) = \frac{2^m}{\Gamma_m(\alpha)} \int\limits_{T_m} f(t't) \prod_{i=1}^m t_{i,i}^{2\alpha-i} \prod_{i\le j} dt_{i,j} \qquad \operatorname{Re}\alpha > d-1\ .$$

We write $t = a + b$, where $a = (a_{i,j})$ and $b = (b_{i,j})$ are upper triangular matrices so that the lower $n-k$ rows of a and the upper k rows of b consist of zeros. We denote by A and B the sets of all matrices of the form a and b, respectively. Since $t't = a'a + b'b$, then

$$\mathcal{G}_\alpha(\overline{f}) = \frac{2^k \Gamma_{m-k}(\alpha - k/2)}{\Gamma_m(\alpha)} \int\limits_A g_\alpha(a'a) \prod_{i=1}^k a_{i,i}^{2\alpha-i} \prod_{i\le j} da_{i,j}\ , \tag{3.25}$$

where

$$g_\alpha(a'a) = \frac{2^{m-k}}{\Gamma_{m-k}(\alpha - k/2)} \int\limits_B f(a'a + b'b) \prod_{i=1}^{m-k} b_{k+i,k+i}^{2(\alpha-k/2)-i} \prod_{i\le j} db_{k+i,k+j}\ .$$

Note that $g_\alpha(a'a) = \mathcal{G}_{\alpha-k/2}\left(\overline{f\left(\begin{bmatrix} * & * \\ * & \bullet \end{bmatrix}\right)}\right)$ represents the distribution of the same type as $\mathcal{G}_\alpha$ but acting in the $(\bullet)$ matrix variable belonging to $\mathcal{S}_{m-k}$. By (3.25), $\mathcal{G}_\alpha$ is a direct product of two distributions which are analytic in α. Hence, owing to (3.23), $g_{k/2}(a'a) = f(a'a)$, and therefore,

$$\mathcal{G}_{k/2}(\overline{f}) = c \int\limits_A f(a'a) \prod_{i=1}^k a_{i,i}^{k-i} \prod_{i\le j} da_{i,j}\ . \tag{3.26}$$

Here, by (2.4),

$$c = 2^k \lim_{\alpha\to k/2} \frac{2^k \Gamma_{m-k}(\alpha - k/2)}{\Gamma_m(\alpha)} = \frac{2^k \pi^{k(k-m)/2}}{\Gamma_k(k/2)}\ .$$

This representation was actually established in [8, p. 134] and our previous argument follows that work. It remains to show that (3.26) coincides with (3.24) if the latter is written for f replaced by $\bar{f}$. We write ω in (3.24) as $[\eta, \zeta]$, where $\eta \in \mathfrak{M}_{k,k}$, $\zeta \in \mathfrak{M}_{k,m-k}$. Then

$$\pi^{-km/2} \int\limits_{\mathfrak{M}_{k,m}} f(\omega'\omega)\, d\omega = \pi^{-km/2} \int\limits_{\mathfrak{M}_{k,k}} d\eta \int\limits_{\mathfrak{M}_{k,m-k}} f\left(\begin{bmatrix} \eta'\eta & \eta'\zeta \\ \zeta'\eta & \zeta'\zeta \end{bmatrix}\right) d\zeta$$

(set $\eta = \gamma q,\ \gamma \in O(k),\ q \in T_k$ and use Lemma 2)

$$= \pi^{-km/2} \sigma_{k,k} \int\limits_{T_k} \prod_{i=1}^k q_{i,i}^{k-i} \prod_{i\le j} dt_{i,j} \int\limits_{\mathfrak{M}_{k,m-k}} f\left(\begin{bmatrix} q'q & q'y \\ y'q & y'y \end{bmatrix}\right) dy$$

$$\overset{(2.9)}{=} \frac{2^k \pi^{k(k-m)/2}}{\Gamma_k(k/2)} \int\limits_A f\big(a'a\big) \prod_{i=1}^{k} a_{i,i}^{k-i} \prod_{i\le j} da_{i,j}$$

where $a = \begin{bmatrix} q & y \\ 0 & 0 \end{bmatrix}$. This proves the statement. □

Proof of Theorem 2

Step 1. Let first $k > m-1$. In polar coordinates we have

$$(\zeta_k, f) = \frac{1}{\Gamma_m(k/2)} \int\limits_{\mathfrak{M}_{n,m}} \overline{f(x)} |x|_m^{k-n}\, dx$$

$$= \frac{2^{-m}\sigma_{n,m}}{\Gamma_m(k/2)} \int\limits_{\mathcal{P}_m} |r|^{k/2-d}\, dr \int\limits_{O(n)} \overline{f\left(\gamma \begin{bmatrix} r^{1/2} \\ 0 \end{bmatrix}\right)}\, d\gamma\ .$$

Now we replace γ by $\gamma \begin{bmatrix} \beta & 0 \\ 0 & I_{n-k} \end{bmatrix}$, $\beta \in O(k)$, then integrate in $\beta \in O(k)$, and replace the integration over $O(k)$ by that over $V_{k,m}$. We get

$$(\zeta_k, f) = \frac{2^{-m}\sigma_{n,m}}{\sigma_{k,m}\Gamma_m(k/2)} \int\limits_{O(n)} d\gamma \int\limits_{\mathcal{P}_m} |r|^{k/2-d}\, dr \int\limits_{V_{k,m}} \overline{f\left(\gamma \begin{bmatrix} v r^{1/2} \\ 0 \end{bmatrix}\right)}\, dv$$

$$\big(set \quad \omega = v r^{1/2} \in \mathfrak{M}_{k,m}\big)$$

$$= \frac{\sigma_{n,m}}{\sigma_{k,m}\Gamma_m(k/2)} \int\limits_{\mathfrak{M}_{k,m}} d\omega \int\limits_{O(n)} \overline{f\left(\gamma \begin{bmatrix} \omega \\ 0 \end{bmatrix}\right)}\, d\gamma\ .$$

This coincides with (3.20).

Step 2. Our next task is to prove that analytic continuation of (ζ_α, f) at the point $\alpha = k$ ($\le m-1$) has the form (3.20). To this end, we express ζ_α through the distribution (3.22). For $\operatorname{Re}\alpha > m-1$, by passing to polar coordinates, we have $(\zeta_\alpha, f) = 2^{-m}\sigma_{n,m}\mathcal{G}_{\alpha/2}(F)$, where

$$\mathcal{G}_{\alpha/2}(F) = \frac{1}{\Gamma_m(\alpha/2)} \int\limits_{\mathcal{P}_m} \overline{F(r)} |r|^{\alpha/2-d}\, dr, \qquad F(r) = \int\limits_{V_{n,m}} f\big(v r^{1/2}\big)\, d_*v\ .$$

To continue the proof, we need the following.

Lemma 7 *Let $\mathcal{S}(\overline{\mathcal{P}}_m)$ be the space of restrictions onto $\overline{\mathcal{P}}_m$ of the Schwartz functions on $S_m \supset \overline{\mathcal{P}}_m$ with the induced topology. The map*

$$\mathcal{S}(\overline{\mathcal{P}}_m) \ni F \ \to f(x) = F\big(x'x\big)$$

is an isomorphism of $\mathcal{S}(\overline{\mathcal{P}}_m)$ onto the space $\mathcal{S}(\mathfrak{M}_{n,m})^\natural$ of $O(n)$ left-invariant functions on $\mathfrak{M}_{n,m}$.

This important statement, which is well known for $m=1$ (see, e.g., Lemma 5.4 in [50, p. 56]), was presented in a slightly different form by J. Faraut [7, Prop. 3] and derived from the more general result of G. W. Schwarz [43, Theorem 1]. According to (3.24), analytic continuation of $\mathcal{G}_{\alpha/2}(F)$ at $\alpha=k$, $k=1,2,\dots,m-1$, is evaluated as follows:

$$\mathcal{G}_{k/2}(F)=\pi^{-km/2}\int\limits_{\mathfrak{M}_{k,m}}\overline{F\big(\omega'\omega\big)}\,d\omega$$
$$=\pi^{-km/2}\int\limits_{\mathfrak{M}_{k,m}}d\omega\int\limits_{O(n)}\overline{f\left(\gamma\begin{bmatrix}(\omega'\omega)^{1/2}\\ 0\end{bmatrix}\right)}\,d\gamma\ .$$

By making use of the polar coordinates, one can write $\omega'\in\mathfrak{M}_{m,k}$ as

$$\omega'=\beta u_0\big(\omega\omega'\big)^{1/2},\quad \beta\in O(m),\quad u_0=\begin{bmatrix}I_k\\ 0\end{bmatrix}\in V_{m,k}\ .$$

Hence, $\omega=(\omega\omega')^{1/2}u_0'\beta'$, and

$$\big(\omega'\omega\big)^{1/2}=\big(\beta u_0\omega\omega' u_0'\beta'\big)^{1/2}=\beta u_0\big(\omega\omega'\big)^{1/2}u_0'\beta'=\beta u_0\omega\ .$$

By changing variable $\gamma\to\gamma\begin{bmatrix}\beta' & 0\\ 0 & I_{n-m}\end{bmatrix}$, we obtain

$$\mathcal{G}_{k/2}(F)=\pi^{-km/2}\int\limits_{\mathfrak{M}_{k,m}}d\omega\int\limits_{O(n)}\overline{f\left(\gamma\begin{bmatrix}\beta u_0\omega\\ 0\end{bmatrix}\right)}\,d\gamma$$

$$=\pi^{-km/2}\int\limits_{\mathfrak{M}_{k,m}}d\omega\int\limits_{O(n)}\overline{f\left(\gamma\begin{bmatrix}\omega\\ 0\end{bmatrix}\right)}\,d\gamma\ ,$$

and (3.20) follows. For $\alpha=0$, owing to (3.23), we have

$$(\zeta_0,f)=2^{-m}\sigma_{n,m}\overline{F(0)}=\frac{\pi^{nm/2}}{\Gamma_m(n/2)}\,\overline{f(0)}\ .\qquad\square$$

Remark 2 So far we considered zeta distributions of the form (ζ_k,f). Clearly, for $n=m$ distributions (ζ_k^+,f) fall into the scope of this consideration. Unfortunately, we cannot obtain a simple explicit representation of the conjugate zeta distributions (ζ_k^-,f), $k=0,1,\dots,m-1$. At the first glance, it would be natural to use the formula

$$\big(\zeta_\alpha^-,f\big)=c_\alpha\,\big(\zeta_{\alpha+1}^+,\mathcal{D}f\big),\qquad c_\alpha=(-1)^m\frac{\Gamma(\alpha+1-m)}{\Gamma(\alpha+1)}\ ,$$

see (3.9), in which $(\zeta^+_{\alpha+1}, \mathcal{D} f)$ can be evaluated for $\alpha = k$ by Theorem 2. We cannot do this because $c_\alpha = \infty$ for such α . On the other hand, (ζ^-_k, f) is well defined, and by Lemma 2 we have

$$\left(\zeta^-_k, f\right) = 2^{-m}\,\pi^{-m(m-1)/4}\,(\omega_k, \Phi)\ , \tag{3.27}$$

where

$$\omega_k(t_{1,1}, \dots, t_{m,m}) = \prod_{i=1}^{m} \frac{|t_{i,i}|^{\alpha-i}\,\mathrm{sgn}\,(t_{i,i})}{\Gamma((\alpha-i)/2+1)}\bigg|_{\alpha=k}\ ,$$

$$\Phi(t_{1,1}, \dots, t_{m,m}) = \int\limits_{\mathbb{R}^{m(m-1)/2}} dt_* \int\limits_{O(m)} f(vt)\,\mathrm{sgn}\det(v)\,dv\ .$$

In particular, for $k = 0$,

$$\omega_0(t_{1,1}, \dots, t_{m,m}) = \prod_{i=1}^{m} \frac{|t_{i,i}|^{\lambda}\,\mathrm{sgn}\,(t_{i,i})}{\Gamma(\lambda/2+1)}\bigg|_{\lambda=-i}\ , \tag{3.28}$$

where the generalized functions

$$\frac{|s|^{\lambda}\,\mathrm{sgn}\,(s)}{\Gamma(\lambda/2+1)}\bigg|_{\lambda=-i}\ , \qquad i = 1, 2, \dots, m\ ,$$

are defined as follows. For i odd:

$$\left(\frac{|s|^{\lambda}\,\mathrm{sgn}\,(s)}{\Gamma(\lambda/2+1)}\bigg|_{\lambda=-i}, \varphi\right) = \frac{1}{\Gamma(1-i/2)}\left(s^{-i}, \varphi\right)$$

$$= \int\limits_0^\infty s^{-i}\bigg\{\varphi(s) - \varphi(-s)$$

$$- 2\left[s\varphi'(0) + \frac{s^3}{3!}\varphi'''(0) + \dots + \frac{s^{i-2}}{(i-2)!}\varphi^{(i-2)}(0)\right]\bigg\}\,ds\ .$$

For i even:

$$\left(\frac{|s|^{\lambda}\,\mathrm{sgn}\,(s)}{\Gamma(\lambda/2+1)}\bigg|_{\lambda=-i}, \varphi\right) = \frac{(-1)^{i/2}\,\varphi^{(i-1)}(0)\,(i/2-1)!}{(i-1)!}\,;$$

see [13, Chapter 1, Section 3.5] . Note that the Fourier transform of the distribution ζ^-_0 has the form

$$\left(\mathcal{F}\zeta^-_0\right)(y) = \frac{(-i)^m\,\pi^{m^2/2}}{\Gamma_m((m+1)/2)}\,\mathrm{sgn}\det(y)\ . \tag{3.29}$$

This follows immediately from (3.12) and the Parseval formula (2.8).

In the following, the expression (ζ^-_0, f) will be understood in the sense of regularization according to (3.27), (3.28).

4 Riesz Potentials and the Generalized Hilbert Transform

Riesz potentials on matrix spaces arise in different contexts in integral geometry and representation theory; see [12, 27, 40, 52], and references therein. We recall basic definitions. The *Riesz distribution* $h_\alpha \in \mathcal{S}'(\mathfrak{M}_{n,m})$ is defined as meromorphic continuation of the integral

$$(h_\alpha, f) = \frac{1}{\gamma_{n,m}(\alpha)} \int\limits_{\mathfrak{M}_{n,m}} |x|_m^{\alpha-n} \overline{f(x)}\, dx, \qquad f \in \mathcal{S}(\mathfrak{M}_{n,m}), \tag{4.1}$$

$$\gamma_{n,m}(\alpha) = \frac{2^{\alpha m} \pi^{nm/2} \Gamma_m(\alpha/2)}{\Gamma_m((n-\alpha)/2)}, \qquad \alpha \neq n-m+1, n-m+2, \dots . \tag{4.2}$$

This is just the renormalized version of the zeta distribution so that

$$h_\alpha = \frac{\Gamma_m((n-\alpha)/2)}{2^{\alpha m} \pi^{nm/2}} \zeta_\alpha \tag{4.3}$$

cf. (3.1). The normalizing constant $\gamma_{n,m}(\alpha)$ is chosen according to the Fourier transform formula (4.6) below. For $\mathrm{Re}\,\alpha > m-1$, the distribution h_α is regular and agrees with the function $h_\alpha(x) = |x|_m^{\alpha-n}/\gamma_{n,m}(\alpha)$. The *Riesz potential* of a function $f \in \mathcal{S}(\mathfrak{M}_{n,m})$ is defined as a convolution

$$\big(I^\alpha f\big)(x) = (f * h_\alpha)(x) = \big(h_\alpha, \overline{f_x}\big) = \frac{2^{\alpha m} \pi^{nm/2}}{\Gamma_m((n-\alpha)/2)} \big(\zeta_\alpha, \overline{f_x}\big), \tag{4.4}$$

where $f_x(\cdot) = f(x-\cdot)$, $\alpha \in \mathbb{C}$, $\alpha \neq n-m+1, n-m+2, \dots$. For $\mathrm{Re}\,\alpha > m-1$, (4.4) has the classical form

$$\big(I^\alpha f\big)(x) = \frac{1}{\gamma_{n,m}(\alpha)} \int\limits_{\mathfrak{M}_{n,m}} f(x-y)|y|_m^{\alpha-n}\, dy, \tag{4.5}$$

where the integral on the right-hand side is absolutely convergent.

The following properties of Riesz distributions are inherited from those for the zeta integrals in Section 3.

Lemma 8 *Let* $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $n \geq m$, $\alpha \in \mathbb{C}$, $\alpha \neq n-m+1, n-m+2, \dots$.
(i) *The Fourier transform of the Riesz distribution* h_α *is evaluated by the formula* $(\mathcal{F} h_\alpha)(y) = |y|_m^{-\alpha}$, *the precise meaning of which is*

$$(h_\alpha, f) = (2\pi)^{-nm} \big(|y|_m^{-\alpha}, (\mathcal{F} f)(y)\big). \tag{4.6}$$

(ii) *If* $k = 0, 1, \dots$, *and* Δ *is the Cayley-Laplace operator, then*

$$h_\alpha = (-1)^{mk} \Delta^k h_{\alpha+2k}, \quad \text{i.e.,} \quad (h_\alpha, f) = (-1)^{mk} \big(h_{\alpha+2k}, \Delta^k f\big). \tag{4.7}$$

(iii) *If* $n = m$, $k = 1, 2, \dots$, *and* $\mathcal{D}$ *is the Cayley operator, then*

$$h_\alpha = c\, \mathcal{D}^{2k-1} \zeta^-_{\alpha+2k-1}, \quad \text{i.e.,} \quad (h_\alpha, f) = c\,(-1)^m \big(\zeta^-_{\alpha+2k-1}, \mathcal{D}^{2k-1} f\big), \tag{4.8}$$

$$c=\frac{(-1)^{m(k+1)}\,\Gamma_m(1+(m-\alpha-2k)/2)}{2^{(\alpha+2k-1)m}\,\pi^{m^2/2}}\;.$$

Proof (i) Follows immediately from (4.3) and (3.11). To prove (4.7), according to (2.12), we have

$$\Delta^k h_{\alpha+2k}(x)=\frac{1}{\gamma_{n,m}(\alpha+2k)}\Delta^k|x|_m^{\alpha+2k-n}=\frac{B_k(\alpha)}{\gamma_{n,m}(\alpha+2k)}|x|_m^{\alpha-n}=c\,h_\alpha(x)\;,$$

where by (2.13) and (2.5),

$$c=\frac{B_k(\alpha)\,\gamma_{n,m}(\alpha)}{\gamma_{n,m}(\alpha+2k)}=\frac{B_k(\alpha)\,\Gamma_m(\alpha/2)\,\Gamma_m((n-\alpha)/2-k)}{4^{mk}\,\Gamma_m(\alpha/2+k)\,\Gamma_m((n-\alpha)/2)}=(-1)^{mk}\;.$$

Let us prove (iii). Owing to (4.7), $h_\alpha=(-1)^{mk}\,\mathcal{D}^{2k-1}\,\mathcal{D}h_{\alpha+2k}$. Since by (4.3) and (3.9),

$$h_{\alpha+2k}=\frac{\Gamma_m((m-\alpha-2k)/2)}{2^{(\alpha+2k)m}\,\pi^{m^2/2}}\;\zeta^+_{\alpha+2k},\quad\text{and}\quad\mathcal{D}\zeta^+_{\alpha+2k}=\frac{\Gamma(\alpha+2k)}{\Gamma(\alpha+2k-m)}\;\zeta^-_{\alpha+2k-1}\;,$$

then (4.8) follows after simple calculation using (2.5). □

We will need explicit representation of Riesz potentials $I^\alpha f$ for integral values of α. The case $\alpha=k\le m-1$, when representation (4.5) is inapplicable, is especially important. We start with the case when k is a nonnegative integer.

Theorem 3 *Let* $f\in\mathcal{S}(\mathfrak{M}_{n,m})$, $n\ge m$. *If* $k>0$ *and* $k\ne n-m+1, n-m+2,\dots,$ *then*

$$\big(I^k f\big)(x)=\gamma_1\int\limits_{\mathfrak{M}_{k,m}}d\omega\int\limits_{O(n)}f\left(x-\gamma\begin{bmatrix}\omega\\0\end{bmatrix}\right)d\gamma\;,\tag{4.9}$$

$$=\gamma_2\int\limits_{V_{n,k}}dv\int\limits_{\mathfrak{M}_{k,m}}f(x-v\omega)\,d\omega\;,\tag{4.10}$$

where

$$\gamma_1=2^{-km}\,\pi^{-km/2}\,\Gamma_m\left(\frac{n-k}{2}\right)/\,\Gamma_m\left(\frac{n}{2}\right),\tag{4.11}$$

$$\gamma_2=2^{-k(m+1)}\,\pi^{-k(m+n)/2}\,\Gamma_k\left(\frac{n-m}{2}\right)\;.\tag{4.12}$$

If $k=0$, *then*

$$\big(I^0 f\big)(x)=f(x)\;.\tag{4.13}$$

Proof Equality (4.9) follows from Theorem 2, owing to connection (4.3); (4.10) is a consequence of (4.9), (2.9), and a simple formula

$$\frac{\Gamma_k\left(\frac{n-m}{2}\right)}{\Gamma_k\left(\frac{n}{2}\right)}=\frac{\Gamma_m\left(\frac{n-k}{2}\right)}{\Gamma_m\left(\frac{n}{2}\right)}\,. \tag{4.14}$$

Equality (4.13) follows from (3.21). □

In order to obtain explicit representation of $I^\alpha f$ when α is a negative integer, we introduce *the generalized Hilbert transform*

$$(\mathcal{H}f)(x)=\left(\zeta_0^-,f_x\right)=\left(\zeta_0^- * f\right)(x)\,. \tag{4.15}$$

By (3.29), this is a pseudo-differential operator with the symbol

$$\left(\mathcal{F}\zeta_0^-\right)(y)=\frac{(-i)^m\,\pi^{m^2/2}}{\Gamma_m((m+1)/2)}\,\mathrm{sgn}\det(y)\,. \tag{4.16}$$

Clearly, $\mathcal{H}$ extends as a linear bounded operator on $L^2(\mathfrak{M}_{n,m})$. For $m=1$, it coincides (up to a constant multiple) with the usual Hilbert transform on the real line [26, 48]. Lemma 8 implies the following.

Theorem 4 *Let $f\in\mathcal{S}(\mathfrak{M}_{n,m})$, $n\geq m$.* (i) *If $k=0,1,2,\dots,$ then*

$$\left(I^{-2k}f\right)(x)=(-1)^{mk}\left(\Delta^k f\right)(x)\,. \tag{4.17}$$

(ii) *If $k=1,2,\dots,$ and $n>m$, then*

$$\left(I^{1-2k}f\right)(x)=c_1\int\limits_{S^{n-1}}dv\int\limits_{\mathbb{R}^m}\left(\Delta^k f\right)\left(x-vy'\right)dy,\qquad c_1=\frac{(-1)^{mk}\,\Gamma((n-m)/2)}{2^{m+1}\,\pi^{(m+n)/2}}\,. \tag{4.18}$$

(iii) *If $k=1,2,\dots,$ and $n=m$, then*

$$\left(I^{1-2k}f\right)(x)=c_2\left(\mathcal{H}\mathcal{D}^{2k-1}f\right)(x),\qquad c_2=\frac{(-1)^{m(k+1)}\,\Gamma_m((m+1)/2)}{\pi^{m^2/2}}\,, \tag{4.19}$$

$\mathcal{H}$ being the generalized Hilbert transform (4.15).

Proof The equality (4.17) is a consequence of (4.4), (4.13), and (4.7):

$$\begin{aligned}\left(I^{-2k}f\right)(x)&=(h_{-2k},f_x)=(-1)^{mk}\left(h_0,\ \Delta^k f_x\right)\\&=(-1)^{mk}\left(\Delta^k f_x\right)(0)=(-1)^{mk}\left(\Delta^k f\right)(x)\,.\end{aligned}$$

Similarly, since $(I^{1-2k}f)(x)=(h_{1-2k},f_x)$, then for $n>m$ we have

$$(h_{1-2k},f_x)=(-1)^{mk}\left(h_1,\ \Delta^k f_x\right)=(-1)^{mk}\left(h_1,\ \left(\Delta^k f\right)_x\right)=(-1)^{mk}\left(I^1\Delta^k f\right)(x)\,,$$

and it remains to apply (4.10) (with $k=1$). If $n=m$, then we apply (4.8) with $\alpha = 1-2k$ and get

$$\begin{aligned}(h_{1-2k}, f_x) &= (-1)^m c_2 \left(\zeta_0^-, \mathcal{D}^{2k-1} f_x\right) = c_2 \left(\zeta_0^-, \left(\mathcal{D}^{2k-1} f\right)_x\right) \\ &= c_2 \left(\mathcal{H}\mathcal{D}^{2k-1} f\right)(x).\end{aligned}$$

□

Remark 3 Formula (4.17) has a local structure, unlike (4.18) and (4.19), which are nonlocal. We would like to draw reader's attention to the fact (4.18) relies on the important Theorem 3 giving representation of the Riesz potential I^1 as a convolution with positive measure.

5 Generalized Semyanistyi Integrals

Let $x \in \mathfrak{M}_{n,m}$, $\tau \equiv \tau(\xi, t)$ is the matrix plane (2.17), and $|x-\tau|_m = |\xi' x - t|_m$ denotes the determinant of the matrix distance between x and τ; see (2.21). This section is the core of the paper. We introduce intertwining operators (with respect to the group $M(n,m)$ of matrix motions) which generalize Semyanistyi's integrals (1.3), (1.4) to the higher-rank case. The main building blocks are Radon transforms, dual Radon transforms, and Riesz potentials on matrix spaces. To avoid possible confusion, we shall discriminate between operators acting on $\mathfrak{M}_{n,m}$ and the similar operators on $\mathfrak{M}_{n-k,m}$. In the following, $\tilde{I}^\alpha$, $\tilde{\Delta}$, $\tilde{\mathcal{D}}$, and $\tilde{\mathcal{H}}$ stand for the Riesz potential, the Cayley-Laplace operator, the Cayley operator, and the generalized Hilbert transform on $\mathfrak{M}_{n-k,m}$. These will be applied to functions $\hat{f}(\xi, t)$ and $\varphi(\xi, t)$ in the t-variable. We assume $1 \le k \le n-m$, and denote by $\mathcal{S}(\mathfrak{T})$ the space of functions $\varphi(\xi, t)$ which are infinitely differentiable in the ξ-variable and belong to the Schwartz space $\mathfrak{M}_{n-k,m}$ in the t-variable uniformly in $\xi \in V_{n,n-k}$.

Definition 2 Let $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $\varphi \in \mathcal{S}(\mathfrak{T})$. The generalized Semyanistyi integrals are defined by

$$P^\alpha f = \tilde{I}^\alpha \hat{f}, \qquad \overset{*}{P}{}^\alpha \varphi = \left(\tilde{I}^\alpha \varphi\right)^\vee, \tag{5.1}$$

where

$$\alpha \in \mathbb{C}, \quad \alpha \ne n-k-m+1, n-k-m+2, \dots . \tag{5.2}$$

Expressions in (5.1) are, in general, understood in the sense of analytic continuation. By (4.13),

$$\left(P^0 f\right)(\xi, t) = \hat{f}(\xi, t), \qquad \left(\overset{*}{P}{}^0 \varphi\right)(x) = \check{\varphi}(x) . \tag{5.3}$$

Our aim is to obtain explicit representation of this analytic continuation for some important values of α, including those that will be needed in the next section.

Lemma 9 *Let* $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $\varphi \in \mathcal{S}(\mathfrak{T})$, $1 \le k \le n-m$. *If* $Re\,\alpha > m-1$ *then* P^α *and* $\overset{*}{P}{}^\alpha$ *are represented by absolutely convergent integrals*

$$\big(P^\alpha f\big)(\xi,t) = \frac{1}{\gamma_{n-k,m}(\alpha)} \int\limits_{\mathfrak{M}_{n,m}} f(x)\,\big|\xi' x - t\big|_m^{\alpha+k-n}\,dx\ , \tag{5.4}$$

$$\big(\overset{*}{P}{}^\alpha\varphi\big)(x) = \frac{1}{\gamma_{n-k,m}(\alpha)} \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi,t)\,\big|\xi' x - t\big|_m^{\alpha+k-n}\,dt\ , \tag{5.5}$$

where $\gamma_{n-k,m}(\alpha)$ *is the normalized constant for the Riesz potential on* $\mathfrak{M}_{n-k,m}$; *cf.* (4.2).

Proof Formula (5.5) follows from (5.1), (2.24), and (4.5). By taking into account (2.23), we also have

$$\begin{aligned}\big(P^\alpha f\big)(\xi,t) &= \frac{1}{\gamma_{n-k,m}(\alpha)} \int\limits_{\mathfrak{M}_{n-k,m}} |y|_m^{\alpha+k-n}\,dy \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi\begin{bmatrix}\omega\\ t-y\end{bmatrix}\right) d\omega \\ &= \frac{1}{\gamma_{n-k,m}(\alpha)} \int\limits_{\mathfrak{M}_{n,m}} f(x)\,\big|\xi' x - t\big|_m^{\alpha+k-n}\,dx\ .\end{aligned}$$

□

Lemma 10 *Let* $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, $\varphi \in \mathcal{S}(\mathfrak{T})$, $1 \le k \le n-m$. *If* ℓ *is a positive integer,* $\ell \le n-k-m$, *then*

$$\big(P^\ell f\big)(\xi,t) = c_\ell \int\limits_{\mathfrak{M}_{k+\ell,m}} dz \int\limits_{O(n-k)} f\left(\xi t - g_\xi \begin{bmatrix} I_k & 0\\ 0 & \gamma\end{bmatrix}\begin{bmatrix} z\\ 0\end{bmatrix}\right) d\gamma\ , \tag{5.6}$$

$$\big(\overset{*}{P}{}^\ell\varphi\big)(x) = c_\ell \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{\ell,m}} dz \int\limits_{O(n-k)} \varphi\left(\xi,\ \xi' x - \gamma\begin{bmatrix} z\\ 0\end{bmatrix}\right) d\gamma\ , \tag{5.7}$$

where

$$c_\ell = 2^{-\ell m}\,\pi^{-\ell m/2}\,\Gamma_m\Big(\frac{n-k-\ell}{2}\Big)\big/\,\Gamma_m\Big(\frac{n-k}{2}\Big)\,. \tag{5.8}$$

Proof We first note that the condition $\ell \le n-k-m$ is motivated by (5.2) and analytic continuations $P^\ell f = \tilde{I}^\ell \hat{f}$, $\overset{*}{P}{}^\ell\varphi = (\tilde{I}^\ell\varphi)^\vee$ of $P^\alpha f$ and $\overset{*}{P}{}^\alpha\varphi$ reduce to that of the Riesz potential on $\mathfrak{M}_{n-k,m}$. One can readily see that $\hat{f}(\xi,t)$, defined by (2.23), is a Schwartz function in the t-variable. Hence, (5.7) follows from (4.9). Furthermore, by (4.9) and (2.23),

$$\big(P^\ell f\big)(\xi,t) = c_\ell \int\limits_{\mathfrak{M}_{\ell,m}} dz \int\limits_{O(n-k)} \hat{f}\left(\xi, t+\gamma\begin{bmatrix} z\\ 0\end{bmatrix}\right) d\gamma$$

$$= c_\ell \int\limits_{\mathfrak{M}_{\ell,m}} dz \int\limits_{O(n-k)} d\gamma \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi\left[\begin{matrix}\omega\\ t+\gamma\begin{bmatrix} z\\ 0\end{bmatrix}\end{matrix}\right]\right) d\omega\,.$$

Since

$$g_\xi\left[\begin{matrix}\omega\\ t+\gamma\begin{bmatrix} z\\ 0\end{bmatrix}\end{matrix}\right] = \xi t + g_\xi \begin{bmatrix} I_k & 0\\ 0 & \gamma\end{bmatrix}\begin{bmatrix}\omega\\ z\\ 0\end{bmatrix},$$

then (5.6) follows if we change the notation $\begin{bmatrix}\omega\\ z\end{bmatrix} \to z$. □

The next two lemmas provide representation of $P^\alpha f$, $f \in \mathcal{S}(\mathfrak{M}_{n,m})$, and $\overset{*}{P}{}^\alpha \varphi$, $\varphi \in \mathcal{S}(\mathfrak{T})$, when α is a negative integer. We emphasize an essential difference between "$-\alpha$ even" and "$-\alpha$ odd," and, in the second case, between $k < n-m$ and $k = n-m$.

Lemma 11 *Let $\ell = 1, 2, \ldots$, $1 \le k \le n-m$. Then*

$$\big(P^{-2\ell} f\big)(\xi, t) = (-1)^{m\ell} \tilde{\Delta}^\ell \hat{f}(\xi, t)\,, \tag{5.9}$$

$$\big(\overset{*}{P}{}^{-2\ell}\varphi\big)(x) = (-1)^{m\ell} \int\limits_{V_{n,n-k}} \big(\tilde{\Delta}^\ell \varphi\big)\big(\xi, \xi' x\big)\, d_*\xi\,. \tag{5.10}$$

Proof The statement follows from (5.1) and (4.17). □

Lemma 12 *Let $\ell = 1, 2, \ldots$.* (i) *If $1 \le k < n-m$ then*

$$\big(P^{1-2\ell} f\big)(\xi, t) = c_1 \int\limits_{S^{n-k-1}} dv \int\limits_{\mathbb{R}^m} \big(\tilde{\Delta}^\ell \hat{f}\big)\big(\xi, t - vy'\big)\, dy\,, \tag{5.11}$$

$$\big(\overset{*}{P}{}^{1-2\ell}\varphi\big)(x) = c_1 \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{S^{n-k-1}} dv \int\limits_{\mathbb{R}^m} \big(\tilde{\Delta}^\ell \varphi\big)\big(\xi, \xi' x - vy'\big)\, dy\,, \tag{5.12}$$

where

$$c_1 = \frac{(-1)^{m\ell}\, \Gamma((n-k-m)/2)}{2^{m+1}\, \pi^{(m+n-k)/2}}\,.$$

(ii) *If $k = n-m$, then*

$$\big(P^{1-2\ell} f\big)(\xi, t) = c_2 \big(\tilde{\mathcal{H}}\tilde{\mathcal{D}}^{2\ell-1} \hat{f}(\xi, \cdot)\big)(t)\,, \tag{5.13}$$

$$\big(\overset{*}{P}{}^{1-2\ell}\varphi\big)(x) = c_2 \int\limits_{V_{n,n-k}} \big(\tilde{\mathcal{H}}\tilde{\mathcal{D}}^{2\ell-1} \varphi(\xi, \cdot)\big)\big(\xi' x\big)\, d_*\xi\,, \tag{5.14}$$

where

$$c_2 = \frac{(-1)^{m(\ell+1)}\, \Gamma_m((m+1)/2)}{\pi^{m^2/2}}$$

and $\tilde{\mathcal{H}}$ *denotes the generalized Hilbert transform* (4.15).

Proof (i) follows from (5.1) and (4.18); (ii) is a consequence of (5.1) and (4.19). □

Formulas (5.4)–(5.7) can serve as definitions of $P^\alpha f$ and $\overset{*}{P}{}^\alpha\varphi$ if f and φ are arbitrary locally integrable functions provided the corresponding integrals converge. The following statement, which extends the generalized Fuglede formula (1.5) to the matrix case, is the main result of this section and a core of our Radon inversion method. For the sake of completeness, we present the result both for smooth functions and for arbitrary locally integrable functions under appropriate conditions. As above, we assume $1 \le k \le n-m$.

Theorem 5 (i) *Let* $\alpha \in \mathbb{C}$; $\alpha \ne n-k-m+1, n-k-m+2, \dots$. *If* $f \in \mathcal{S}(\mathcal{S}_m)$ *then*

$$\big(\overset{*}{P}{}^\alpha \hat{f}\big)(x) = c_{n,k,m}\big(I^{\alpha+k} f\big)(x), \qquad c_{n,k,m} = \frac{2^{km}\pi^{km/2}\Gamma_m(n/2)}{\Gamma_m((n-k)/2)}\,. \tag{5.15}$$

(ii) *Let* $\ell_0 = \min\{m-1, n-k-m\}$,

$$\mathfrak{A} = \{0, 1, 2, \dots, \ell_0\} \cup \{\alpha : Re\,\alpha > m-1;\ \alpha \ne n-k-m+1, n-k-m+2, \dots\}\,.$$

If $f \in L^1_{loc}(\mathfrak{M}_{n,m})$ *and* $\alpha \in \mathfrak{A}$ *then* (5.15) *holds provided that the Riesz potential* $(I^{\alpha+k} f)(x)$ *is finite a.e. for* f *replaced by* $|f|$. *In particular, this is true for* $f \in L^p(\mathfrak{M}_{n,m})$ *if*

$$1 \le p < \frac{n}{Re\,\alpha + k + m - 1}\,. \tag{5.16}$$

Proof (i) Let $f \in \mathcal{S}(\mathcal{S}_m)$. We make use of the equality (3.15) with α replaced by $\alpha + k$, $\mathrm{Re}\,\alpha > m-1$. This gives

$$\begin{aligned} &\int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi, t)\, |t|_m^{\alpha+k-n}\, dt \\ &\quad = \frac{\Gamma_m(n/2)\,\Gamma_m(\alpha/2)}{\Gamma_m((\alpha+k)/2)\,\Gamma_m((n-k)/2)} \int\limits_{\mathfrak{M}_{n,m}} f(y)\, |y|_m^{\alpha+k-n}\, dy\,. \end{aligned} \tag{5.17}$$

Replacing $f(y)$ by the shifted function $f_x(y) = f(x+y)$ and taking into account (2.27), we get

$$\begin{aligned} &\int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}\big(\xi, \xi' x + t\big)\, |t|_m^{\alpha+k-n}\, dt \\ &\quad = \frac{\Gamma_m(n/2)\,\Gamma_m(\alpha/2)}{\Gamma_m((\alpha+k)/2)\,\Gamma_m((n-k)/2)} \int\limits_{\mathfrak{M}_{n,m}} f(x+y)\, |y|_m^{\alpha+k-n}\, dy\,, \end{aligned} \tag{5.18}$$

cf. (5.5) and (4.5). Hence, (5.15) follows with the constant

$$c_{n,k,m} = \frac{\Gamma_m(n/2)\,\Gamma_m(\alpha/2)\,\gamma_{n,m}(\alpha+k)}{\Gamma_m((\alpha+k)/2)\,\Gamma_m((n-k)/2)\,\gamma_{n-k,m}(\alpha)} \overset{(4.2)}{=} \frac{2^{km}\pi^{km/2}\Gamma_m(n/2)}{\Gamma_m((n-k)/2)} .$$

By analytic continuation, the result holds for all $\alpha \in \mathbb{C}$, $\alpha \neq n-k-m+1, n-k-m+2, \ldots$. (ii) Suppose $f \in L^1_{\mathrm{loc}}(\mathfrak{M}_{n,m})$. For $\operatorname{Re}\alpha > m-1$, (5.15) follows from (5.18) by taking into account that (5.18) was derived from (3.15), and the latter is also true for locally integrable functions, see Remark 1 with α replaced by $\alpha+k$. For $\alpha = \ell$, $\ell = 1, 2, \ldots \ell_0$, we have

$$\begin{aligned}(\overset{*}{P}{}^{\ell} \hat{f})(x) &= \frac{c_\ell}{\sigma_{n,n-k}} \int\limits_{V_{n,n-k}} d\xi \int\limits_{\mathfrak{M}_{\ell,m}} dz \int\limits_{O(n-k)} d\gamma \int\limits_{\mathfrak{M}_{k,m}} f\left(x - g_\xi \begin{bmatrix} \omega \\ \gamma \begin{bmatrix} z \\ 0 \end{bmatrix} \end{bmatrix}\right) d\omega \\ &= c_\ell \int\limits_{O(n)} d\beta \int\limits_{\mathfrak{M}_{\ell,m}} dz \int\limits_{O(n-k)} d\gamma \int\limits_{\mathfrak{M}_{k,m}} f\left(x - \beta \begin{bmatrix} \omega \\ \gamma \begin{bmatrix} z \\ 0 \end{bmatrix} \end{bmatrix}\right) d\omega ,\end{aligned}$$

where c_ℓ is the constant (5.8). We write

$$\begin{bmatrix} \omega \\ \gamma \begin{bmatrix} z \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} \omega \\ z \\ 0 \end{bmatrix} .$$

Then the change of variables $\beta \begin{bmatrix} I_k & 0 \\ 0 & \gamma \end{bmatrix} \to \beta$ gives

$$\begin{aligned}(\overset{*}{P}{}^{\ell} \hat{f})(x) &= c_\ell \int\limits_{\mathfrak{M}_{\ell,m}} dz \int\limits_{\mathfrak{M}_{k,m}} d\omega \int\limits_{O(n)} f\left(x - \beta \begin{bmatrix} \omega \\ z \\ 0 \end{bmatrix}\right) d\beta \\ &= c_\ell \int\limits_{\mathfrak{M}_{\ell+k,m}} dy \int\limits_{O(n)} f\left(x - \beta \begin{bmatrix} y \\ 0 \end{bmatrix}\right) d\beta = c_{n,k,m} (I^{\ell+k} f)(x) ,\end{aligned}$$

where

$$c_{n,k,m} = \frac{c_\ell\, 2^{(\ell+k)m}\, \pi^{(\ell+k)m/2}\, \Gamma_m(n/2)}{\Gamma_m((n-\ell-k)/2)} = \frac{2^{km}\pi^{km/2}\,\Gamma_m(n/2)}{\Gamma_m((n-k)/2)} . \tag{5.19}$$

The last statement for L^p-functions follows from [40, Theorems 5.10, 5.13]. □

6 Inversion of the Radon Transform

The generalized Fuglede formula (5.15) implies the following inversion result for the Radon transform.

Theorem 6 *Let* $1 \le k \le n-m$, $f \in \mathcal{S}(\mathcal{S}_m)$. *The Radon transform* $\varphi(\xi,t) = \hat{f}(\xi,t)$ *can be inverted by the following formulas*. (i) *For k even*,

$$f(x) = (-1)^{mk/2} c_{n,k,m}^{-1} \int\limits_{V_{n,n-k}} \tilde{\Delta}^{k/2} \varphi(\xi,t)\Big|_{t=\xi' x} d_*\xi \,, \tag{6.1}$$

where $c_{n,k,m}$ *has the form* (5.19). (ii) *For k odd and* $k < n-m$,

$$f(x) = c_1 \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{S^{n-k-1}} dv \int\limits_{\mathbb{R}^m} \big(\tilde{\Delta}^{(k+1)/2}\varphi\big)\big(\xi, \xi' x - v y'\big)\, dy \,, \tag{6.2}$$

$$c_1 = (-1)^{m(k+1)/2} 2^{-km-m-1} \pi^{(k-n)/2 - m(k/2+1)} \Gamma_{m+1}\left(\frac{n-k}{2}\right) / \, \Gamma_m\left(\frac{n}{2}\right).$$

(iii) *For k odd and* $k = n-m$,

$$f(x) = c_2 \int\limits_{V_{n,n-k}} \big(\tilde{\mathcal{H}}\tilde{\mathcal{D}}^k \varphi(\xi,\cdot)\big)\big(\xi' x\big)\, d_*\xi \,, \tag{6.3}$$

$$c_2 = (-1)^{m(k+3)/2} 2^{-km} \pi^{-m(k+m)/2} \Gamma_m\left(\frac{m}{2}\right) \Gamma_m\left(\frac{m+1}{2}\right) / \, \Gamma_m\left(\frac{n}{2}\right) .$$

Proof We write (5.15) with $\alpha = -k$ so that

$$f(x) = c_{n,k,m}^{-1} \big(\overset{*}{P}{}^{-k} \varphi\big)(x) \,, \tag{6.4}$$

where $\overset{*}{P}{}^{-k}$ is the operator (5.1). Now it remains to apply formulas (5.10), (5.12), and (5.14). □

Acknowledgements We are grateful to Prof. E.E. Petrov and Dr. S.P. Khekalo for useful discussions and correspondence.

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