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Poisson Wavelets on the Sphere

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ABSTRACT. In this article we summarize the basic formulas of wavelet analysis with the help of Poisson wavelets on the sphere. These wavelets have the nice property that all basic formulas of wavelet analysis as reproducing kernels, etc. may be expressed simply with the help of higher degree Poisson wavelets. This makes them numerically attractive for applications in geophysical modeling.

1. Definition of Poisson Wavelets

Wavelet analysis on the sphere has by now become a well known technique for the decomposition of arbitrary functions over the sphere into elementary contributions which behave like standard wavelets at least at small scales, see e.g., [1, 2, 6, 7, 8, 10] In this article we want to summarize the principal formulas for continuous wavelet analysis on the sphere with special emphasis to the so-called Poisson wavelets. These functions have found application in the field of geomagnetic modeling as well as in gravity field modeling [4, 5, 12, 11]. In geophysical modeling it is important that the basic functions with respect to which the observable are to be expanded satisfy the specific needs of this community. In particular, the basic functions must have an easy physical interpretation. Moreover, simple algorithms to evaluate the functions are mandatory. Both requirements are met for Poisson wavelets, as we shall see in this article.

For the convenience of the reader and in view of applications in Geophysics we formulate everything on a sphere of radius R instead of simply the unit sphere. For $R > 0$ we denote by Ω_R the sphere of radius R, $\Omega_R = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = R^2\}$. We denote by $\operatorname{Int}\Omega_R$ the interior and by $\operatorname{Ext}\Omega_R$ the exterior

$$\operatorname{Int}(\operatorname{Ext})\Omega_R = \left\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 < (>) R^2\right\}.$$

Consider two points $x, y \in \mathbb{R}^3$ $x \neq 0$, $|x| < R \leq |y|$ and a real number $d \geq 0$. We then define the exterior Poisson wavelet of degree d at pole position x evaluated at y through the

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following absolutely convergent series:

$$W_x^{\mathrm{ext},d}(y) = \frac{R}{|y|} \sum_{l=0}^{\infty} l^d \left(\frac{|x|}{|y|}\right)^l Q_l\big(\hat{x} \cdot \hat{y}\big) \quad x \in \operatorname{Int} \Omega_R,\, y \in \overline{\operatorname{Ext} \Omega_R}\,. \tag{1.1}$$

The name "exterior" is with respect of the harmonic continuation: The exterior wavelets correspond to harmonic functions in $\operatorname{Ext} \Omega_R$ (see Proposition 1 below). For $z \neq 0$ we write $\hat{z} = z/|z|$ for the unit vector in direction z. That x actually is a pole will be shown below. We include $d = 0$ in the definition, although in that case we do obtain a wavelet strictly speaking (see Theorem 4 below). We have introduced the factor R in front to make the wavelets dimensionless. These functions are zonal functions, that is, they are rotational symmetric around the axis $\hat{x}$. As usual, we identify a zonal function with a function $f : [-1, 1] \to \mathbb{C}$ through $y \mapsto f(\hat{x} \cdot \hat{y})$. The function $Q_l : [-1, 1] \to \mathbb{R}$, denotes the reproducing kernel of the space harmonic functions of degree l. We have $Q_l = (2l + 1)P_l$, where P_l is the Legendre Polynomial of order l. The reproducing property of the Q_l reads

$$\int_{\Omega_R} Q_l\big(\hat{x} \cdot \hat{y}\big) Q_{l'}\big(\hat{y} \cdot \hat{z}\big)\, d\omega(y) = 4\pi R^2 \delta_{l,l'} Q_l\big(\hat{x} \cdot \hat{z}\big)\,, \tag{1.2}$$

where $d\omega$ is the surface measure inherited from Euclidean space, so that

$$\int_{\Omega_R} d\omega = 4\pi R^2\,.$$

The interior Poisson wavelet is defined analogously for $|x| > R \geq |y|$ as

$$W_x^{\mathrm{int},d}(y) = \frac{R}{|x|} \sum_{l=0}^{\infty} l^d \left(\frac{|y|}{|x|}\right)^l Q_l\big(\hat{x} \cdot \hat{y}\big) \quad x \in \operatorname{Ext} \Omega_R,\, y \in \overline{\operatorname{Int} \Omega_R}\,.$$

For $d = 0$ we obtain the interior, resp. exterior, Poisson kernel

$$W^{\mathrm{ext},0}(x, y) = P^{\mathrm{int}}(x, y) = \tfrac{R}{|y|} \sum \left(\tfrac{|x|}{|y|}\right)^l Q_l\big(\hat{x} \cdot \hat{y}\big), \quad |x| < |y|\,.$$
$$W^{\mathrm{int},0}(x, y) = P^{\mathrm{ext}}(x, y) = \tfrac{R}{|x|} \sum \left(\tfrac{|y|}{|x|}\right)^l Q_l\big(\hat{x} \cdot \hat{y}\big), \quad |x| > |y|\,.$$

It solves the interior, resp. exterior, Dirichlet problem with boundary value s.

$$P^{\mathrm{int}} s(x) = \frac{1}{4\pi R^2} \int_{\Omega_R} P^{\mathrm{int}}(x, y) s(y)\, d\omega(y), \quad x \in \operatorname{Int} \Omega_R\,.$$

$P^{\mathrm{int}} s$ is the unique function which is harmonic in $\operatorname{Int} \Omega_R$ and which takes as boundary value the function s (see e.g., [3]). We have that $s_\lambda(x) = P^{\mathrm{int}} s(\lambda \hat{x})$ converges in $L^2(\Omega)$ as well as pointwise almost everywhere. In an analogous way, $P^{\mathrm{ext}} s$ is the unique function harmonic in $\operatorname{Ext} \Omega_R$ bounded at ∞ and taking s as boundary value.

2. The Wavelets as Multipoles

Since $y \mapsto Q_l(\hat{x} \cdot \hat{y})/|y|^{l+1}$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$, it follows from the definition that $W_x^{\mathrm{ext},d}$ is harmonic for $|y| > |x|$, where the series may be differentiated term by term. Since

$x \mapsto |x|^l Q_l(\hat{x}\cdot\hat{y})$ is harmonic, it follows that $x \mapsto W_x^{\mathrm{ext},d}(y)$ is harmonic too for $|x| < |y|$. For $d \in \mathbb{N}_0$ the Poisson wavelets can actually be continued harmonically in x and y to all of $\mathbb{R}^3 \times \mathbb{R}^3$ with the diagonal $\{x = y\}$ removed. On the diagonal they have a superposition of poles of order $\leq d+1$.

Proposition 1. *The exterior and the interior wavelet with $d \in \mathbb{N}_0$ may be uniquely harmonically continued to functions over $\mathbb{R}^3\times\mathbb{R}^3\backslash\{x = y\}$. Here the exterior wavelet satisfies*

$$\begin{aligned} W_x^{\mathrm{ext},d}(y) &= \sum_{l=0}^{d+1}\left(2\alpha_l^{d+1}+\alpha_l^d\right)\frac{Rl!|x|^l P_l\left(\widehat{y-x}\cdot\hat{x}\right)}{|y-x|^{l+1}} \\ &= (-1)^{d+1}\sum_{l=0}^{d+1}\left(2\beta_l^{d+1}-\beta_l^d\right)\frac{Rl!|y|^l P_l\left(\widehat{x-y}\cdot\hat{y}\right)}{|y-x|^{l+1}} . \end{aligned}$$

The interior Poisson wavelet can be written as

$$\begin{aligned} W_x^{\mathrm{int}\,.d}(y) &= (-1)^{d+1}\sum_{l=1}^{d+1}\left(2\beta_l^{d+1}-\beta_l^d\right)\frac{Rl!|x|^l P_l\left(\widehat{y-x}\cdot\hat{x}\right)}{|y-x|^{l+1}} \\ &= \sum_{l=0}^{d+1}\left(2\alpha_l^{d+1}+\alpha_l^d\right)\frac{Rl!|y|^l P_l\left(\widehat{x-y}\cdot\hat{y}\right)}{|y-x|^{l+1}} . \end{aligned}$$

The coefficients α_l^d are recursively defined through

$$\begin{aligned} \alpha_l^d &= \alpha_{l-1}^{d-1} + l\alpha_l^{d-1}, \quad d \geq 1 \\ \alpha_k^0 &= \delta_{k,0} . \end{aligned}$$

The coefficients β_l^d are recursively defined through

$$\begin{aligned} \beta_l^d &= \beta_{l-1}^{d-1} + (l+1)\beta_l^{d-1}, \quad d \geq 1 \\ \beta_k^0 &= \delta_{k,0} . \end{aligned}$$

Remark 1. This proposition gives us a way for computing these wavelets numerically without summing series of spherical harmonics. Only $d+1$ zonal spherical harmonics around the new point have to be summed.

Proof. Using the relation

$$t\partial_t t^l = lt^l ,$$

we may write for $d \geq 1$ by adding $0 = t\partial_t 1$

$$W_x^{\mathrm{ext},d}(y) = \left(2(|x|\partial_{|x|})^{d+1} + (|x|\partial_{|x|})^d\right)\frac{R}{|y|}\sum_{l=0}^{\infty}\left(\frac{|x|}{|y|}\right)^l P_l\left(\hat{x}\cdot\hat{y}\right) .$$

Here $\partial_{|x|}F(x)$ stands for the derivation in the radial direction: $(d/d\lambda)|_{\lambda=0}\,F(x+\lambda\hat{x})$. Since

$$\sum_{l=0}^{\infty}\left(\frac{|x|}{|y|}\right)^l P_l\left(\hat{x}\cdot\hat{y}\right) = \frac{|y|}{|x-y|}$$

for $|x| < |y|$ we have the following formula

$$W_x^{\text{ext},d}(y) = \big(2(|x|\partial_{|x|})^{d+1} + (|x|\partial_{|x|})^d\big)\frac{R}{|x-y|} \; . \tag{2.1}$$

Now introduce numbers α_k^d through

$$\sum_{k=0}^{\infty} \alpha_k^d t^k \partial_t^k = (t\partial_t)^d \; .$$

Note that the sum is in fact finite. The nonzero numbers can be computed through the recursion as stated in the proposition. Now using

$$\partial_{|x|}^l \frac{1}{|x-y|} = \frac{l!P_l\big(\widehat{y-x}\cdot\hat{x}\big)}{|x-y|^{l+1}} \; ,$$

the first formula of the proposition follows.

For the interior wavelets, we can write in a similar way

$$W_x^{\text{int},d}(y) = (-1)^{d+1}\frac{R}{|x|}\big(2(|x|\partial_{|x|})^{d+1} - (|x|\partial_{|x|})^d\big)\frac{|x|}{|x-y|}, \quad |x| > |y| \; . \tag{2.2}$$

We introduce the coefficients β_l^d through

$$\sum_{l=0}^{\infty} \beta_l^d t^d \partial_t^d = t^{-1}(t\partial_t)^d(t\cdot) \; .$$

They satisfy the recursion relation as in the proposition. As before the first formula for the interior wavelet follows. Concerning the exchange of x and y, we have the following symmetry:

$$W_y^{\text{int},d}(x) = W_x^{\text{ext},d}(y), \quad x \neq y \; . \tag{2.3}$$

This certainly holds for $|x| < |y|$ as can be seen from the defining series. By the uniqueness of the harmonic extension, it holds for $x \neq y$ and hence the last two formulas of the proposition hold true. □

From now on, we consider the functions as given in the above proposition. The representation of the proposition may be used to give an expansion of the exterior wavelets around the origin and of the interior wavelets around ∞.

Proposition 2. *For $d \in \mathbb{N}$, the harmonically extended exterior Poisson wavelet admits the following expansion around* 0*:*

$$W_x^{\text{ext},d}(y) = (-1)^{d+1}\frac{R}{|x|}\sum_{l=0}^{\infty}(l+1)^d\left(\frac{|y|}{|x|}\right)^l Q_l\big(\hat{x}\cdot\hat{y}\big), \quad |y| < |x| \; .$$

The harmonically extended interior Poisson wavelet admits the following expansion around ∞:

$$W_x^{\text{int},d}(y) = (-1)^{d+1}\frac{R}{|y|}\sum_{l=0}^{\infty}(l+1)^d\left(\frac{|x|}{|y|}\right)^l Q_l\big(\hat{x}\cdot\hat{y}\big), \quad |x| < |y| \; .$$

Proof. We may expand expression (2.1) for $|y| < |x|$, to obtain

$$W_x^{\mathrm{ext},d}(y) = \big(2(|x|\partial_{|x|})^{d+1} + (|x|\partial_{|x|})^d\big)\frac{R}{|x|}\sum_{l=0}^{\infty}\left(\frac{|y|}{|x|}\right)^l P_l\big(\hat{x}\cdot\hat{y}\big)\,.$$

Now $t\partial_t t^{-l-1} = -(l+1)t^{-l-1}$, and the first expression follows by exchanging the summation with the differentiation.

The second expression follows from the symmetry (2.3). □

This implies, in particular, the following pair of relations linking interior and exterior wavelets of different degrees $d \in \mathbb{N}_0$ for $x \neq y$, as can be verified from the defining series expansion:

$$W_x^{\mathrm{ext},d}(y) = (-1)^d\sum_{k=0}^{d}\binom{d}{k}W_x^{\mathrm{int},k}(y), \quad W_x^{\mathrm{int},d}(y) = (-1)^d\sum_{k=0}^{d}\binom{d}{k}W_x^{\mathrm{ext},k}(y)\,.$$

Consider for $U > 0$ the transformation

$$I_U(y) = \frac{U^2 y}{|y|^2}, \quad y \neq 0\,.$$

We introduce the Kelvin transform as the mapping ($y^* = I_U(y)$)

$$K_U : s(y) \mapsto U^2 s(I_U(y))/|y| = U^2 s\big(U^2 y/|y|^2\big)/|y| = \big|y^*\big|s\big(y^*\big)\,.$$

It maps harmonic functions inside the sphere to harmonic functions outside the sphere and vice versa. We have the following relation between interior and exterior wavelets ($x^* = I_R(x)$, $y^* = I_R(y)$)

$$W_x^{\mathrm{ext},d}(y) = \big|x^*\big|\big|y^*\big|W_{x^*}^{\mathrm{int},d}\big(y^*\big), \quad K_x W^{\mathrm{ext},d}(x,y) = K_y W^{\mathrm{int},d}(x,y)\,,$$

where K_x denotes the Kelvin transform acting on x for fixed y and K_y is acting on y for fixed x. This formula can be verified for $|x| < U < |y|$ from the series expansion, and again, by the uniqueness of the harmonic continuation it is true for $x \neq y$.

3. Wavelets on the Sphere

The restriction of these functions to the sphere of radius R can actually be interpreted as wavelets on the sphere. The definition given above are simply the upward and downward harmonic continuations of these functions to the exterior, resp. interior, of the sphere. More precisely, for $a > 0$ and $b \in \Omega_R$, we define

$$g_{b,a}^d(y) = \sum (al)^d\, e^{-al}\, Q_l\big(\hat{b}\cdot\hat{y}\big)\,.$$

Observe the similitude with the Cauchy wavelet over the real line as introduced by Paul [13]

$$g_{\beta,\alpha}(t) = \int_0^{\infty}(a\omega)^d e^{-\alpha\omega}e^{i\beta\omega}, \quad \beta \in, \alpha > 0\,.$$

In addition, if we consider the construction of wavelets over the circle through periodization as introduced in [9], the analogy is perfect, since now the periodized wavelet on the circle would look as

$$\sum (al)^d e^{-al} e^{ilb} ,$$

and $t \mapsto e^{ilt}$ is the reproducing kernel of the one-dimensional rotational invariant space spanned by the pure oscillations of frequency l.

We identify points in Int Ω_R, resp. in Ext Ω_R, with points in $\Omega_R \times \mathbb{R}_+$ via

$$x = e^{\pm a} b, \quad a = |\log(|x|/R)|, \quad b = R\hat{x} . \tag{3.1}$$

The exterior wavelet $W_x^{\mathrm{ext},d}$, $b \in \Omega_R$, is the unique function for which the exterior Dirichlet problem holds with boundary value $W_x^{\mathrm{ext},d}(y) = a^{-d} g_{b.a}(y)$ The interior wavelet $W_x^{\mathrm{int},d}$, solves the interior Dirichlet problem with boundary value $W_x^{\mathrm{int},d}(y) = a^{-d} g_{b.a}(y)$. Note that the scale of the wavelet on the sphere Ω_R can essentially be seen as the relative distance from the pole position to the surface of the sphere, whereas the position of the wavelet is the projection of the pole onto the sphere

$$a \simeq \frac{||x| - R|}{R} \quad \text{for} \quad a \ll 1 .$$

4. The Euclidean Limit

At small scales the wavelets on the sphere actually "look like" wavelets in the following precise way. Denote by $N = R\hat{e}$ the North Pole of the sphere Ω_R. Then consider the following mapping

$$\Phi(z) = 4|N|^2 \frac{z + 2N}{|z + 2N|^2} - N = I_{2N}(z + 2N) - N .$$

It is a conformal map. It maps bijectively the upper half-space onto the interior of the sphere. Moreover, it maps bijectively the plane $H = \{y | y \cdot N = 0\}$ onto $\Omega_R - \{-N\}$, the sphere with the South Pole removed. This restriction to the plane and sphere is the inverse of the well-known stereographic projection. Although we do not have a dilation operator on the sphere, we may, with the help of Φ, pull back functions on the sphere to functions on H, where a natural dilation exists. The existence of the Euclidean limit can now be stated precisely as follows.

Theorem 1. *The following limit exists pointwise for $d \in \mathbb{N}$ and $y \in H$.*

$$V^d(y) := \lim_{a \to 0} a^{d+2} W^{\mathrm{ext},d}_{e^{-a}N}(\Phi(ay)) \tag{4.1}$$

$$= \lim_{a \to 0} a^{d+2} W^{\mathrm{int},d}_{e^{a}N}(\Phi(ay)) \tag{4.2}$$

$$= \lim_{a \to 0} a^2 g^d_{a,N}(\Phi(ay)) \tag{4.3}$$

$$= 2(d+1)!\, R^{-d-1} \frac{P_{d+1}\big(1/\sqrt{1 + |y/R|^2}\big)}{\big(1 + |y/R|^2\big)^{(d+2)/2}} . \tag{4.4}$$

Since the Legendre Polynomials are of the form

$$P_{2l}(t) = A_l\big(t^2\big), \quad P_{2l+1}(t) = t\, B_l\big(t^2\big)$$

with some polynomials A_l and B_l of degree l and with nonvanishing lowest order coefficient, it follows that V^d has the following structure

$$V^{2d}(Ry) = \frac{D_d\big(|y|^2\big)}{\big(1+|y|^2\big)^{2d+3/2}}, \quad V^{2d-1}(Ry) = \frac{E_d\big(|y|^2\big)}{\big(1+|y|^2\big)^{2d+1/2}}$$

with some polynomials D_d and E_d of degree d. As a corollary we note the following result for the decay of the limit function at ∞.

Corollary 1. *We have*

$$\big|V^d(y)\big| \le O\big(|y|^{-\kappa}\big),\ y \to \infty\,.$$

with $\kappa = 2\lfloor d/2 \rfloor + 3$.

Therefore it is numerically better to use even degree Poisson wavelets since the degree $2d$ and $2d+1$ wavelet have the same asymptotic localization in space. However, the $2d$ wavelet is cheaper to compute numerically.

Proof. We use the expansion of Theorem 1 of Section 2 term by term. First note that for $x = e^{-a}N$ and $y \in H$ we have

$$\begin{aligned}
T^2 &:= |\Phi(ay) - x|^2 \\
&= \left|\frac{4|N|^2(ay+2N)}{|ay+2N|^2} - N - e^{-a}N\right|^2 \\
&= \left|\frac{8|N|^2}{|ay+2N|^2} - 1 - e^{-a}\right|^2 |N|^2 + a^2|y|^2\frac{16|N|^4}{|ay+2N|^4} \\
&= \left|\frac{8|N|^2}{a^2|y|^2+4|N|^2} - 1 - e^{-a}\right|^2 |N|^2 + a^2|y|^2\frac{16|N|^4}{\big(a^2|y|^2+4|N|^2\big)^2} \\
&= a^2\big(R^2+|y|^2\big) + O\big(a^4\big)\,,
\end{aligned}$$

as follows from the fact that $N \cdot y = 0$ and $|N| = R$. Now for the same reason we have

$$\big(\widehat{\Phi(ay)-x}\big)\cdot\hat{x} = \frac{1}{T}\left(\frac{8|N|^3}{|ay+2N|^2} - |N| - e^{-a}|N|\right) = \frac{R}{\sqrt{R^2+|y|^2}} + O(a)\,.$$

Altogether we find

$$\frac{P_l\big(\big(\widehat{\Phi(ay)-e^{-a}N}\big)\cdot\hat{x}\big)}{\big|e^{-a}N - \Phi(ay)\big|^{l+1}} = a^{-l-1}\frac{P_l\big(1/\sqrt{1+|y|^2/R^2}\big)\big)}{\big(R^2+|y|^2\big)^{(l+1)/2}} + O\big(a^{-l}\big)\,.$$

Therefore from the expansion of Theorem 1 and the fact that $\alpha_{d+1}^{d+1} = 1$ and that $\alpha_{d+1}^{d} = 0$ the proof follows. $\square$

5. Quadratic Forms

The basic quadratic form is given by the scalar products of our wavelets over spheres of arbitrary radius. That is, we consider integrals of the following type

$$< W_x^{(\text{ext,int}),d}, W_z^{(\text{ext,int}),f} >_U = \frac{1}{4\pi U^2} \int_{\Omega_U} W_x^{(\text{ext,int}),d}(y) W_z^{(\text{ext,int}),f}(y)\, d\omega(y) .$$

From these expressions more general quadratic forms as those used in geophysical modeling can be derived. Consider a quadratic form of the type

$$B(s) = \frac{1}{4\pi U^2} \int_{\Omega_U} \bar{s} \Lambda s \, d\omega ,$$

with Λ a rotational invariant pseudodifferential operator acting on functions on Ω_R defined through

$$\Lambda : Y_{l,m}(\hat{x}) \mapsto l^q Y_{l,m}(\hat{x}) .$$

Since obviously

$$\Lambda W_x^{\text{ext},d} = W_x^{\text{ext},d+q}, \quad \Lambda W_x^{\text{int},d} = W_x^{\text{int},d+q} ,$$

we may compute any such quadratic forms for the Poisson wavelets with the help of the proposition below.

Another bilinear expression is given by the convolution of two zonal functions. Any function $s : [-1, 1] \to \mathbb{C}$ can be identified with a zonal function on Ω_R by considering for any $x \in \Omega_R$ the function $y \mapsto s(\hat{x} \cdot \hat{y})$. We write $\tau_x s$ for this function. The convolution of s with an arbitrary function u on Ω_R is given by

$$s *_R u(x) = \frac{1}{4\pi R^2} \int_{\Omega_R} s(\hat{x} \cdot \hat{y}) u(y)\, d\omega(y) =< \tau_x s, u >_R .$$

In view of applications, we give the formulas for scalar products over arbitrary radii U. It shows that all these scalar products may be computed by point-evaluation of suitably chosen Poisson wavelets.

Proposition 3. *Suppose x, z are the poles of two wavelets of degree $d > 0$ and $f > 0$ defined with respect to radius R. For $U > 0$ we consider the inverted points $x^* = I_U(x)$ and $z^* = I_U(z)$. Then their scalar products $< W_x^{\text{ext},d}, W_z^{\text{ext},f} >_U$ over a sphere of radius U may be computed as follows:*

$$\begin{cases} \frac{R}{|x|} W_z^{\text{ext},d+f}(x^*) = \frac{R}{|z|} \big| W_x^{\text{ext},d+f}(z^*), & |x|, |z| < U,\ d, f > 0 \\ (-1)^{f+1} \frac{R}{U} \sum_{k=0}^{f} \binom{f}{k} W_x^{\text{ext}\ ,d+k}(z), & |x| < U < |z|,\ d > 0, f \in \mathbb{N} \\ (-1)^{d+f} \frac{R}{|x|} W_z^{\text{ext},d+f}(x^*) = (-1)^{d+f} \frac{R}{|z|} W_x^{\text{ext},d+f}(z^*), & U < |x|, |z|,\ d, f \in \mathbb{N} . \end{cases}$$

For two interior wavelets, we have for $< W_x^{\text{int},d}, W_z^{\text{int},f} >_U$

$$\begin{cases} \frac{R}{|z|} W_x^{\text{int},d+f}(z^*) = \frac{R}{|x|} W_z^{\text{int},d+f}(x^*), & U < |x|, |z|,\ d, f > 0 \\ (-1)^{d+1} \frac{R}{U} \sum_{k=0}^{d} \binom{d}{k} W_z^{\text{int},f+k}(x), & |x| < U < |z|,\ d \in \mathbb{N}, f > 0 \\ (-1)^{d+f} \frac{R}{|z|} W_x^{\text{int},d+f}(z^*) = (-1)^{d+f} \frac{R}{|x|} W_z^{\text{int},d+f}(x^*), & U < |x|, |z|,\ d, f \in \mathbb{N} . \end{cases}$$

For one interior and one exterior wavelet, we have for $< W_x^{\mathrm{ext},d}, W_z^{\mathrm{int},f} >_U$

$$\begin{cases} (-1)^{d+1} \frac{R}{|z|} \sum_{k=0}^{d} \binom{d}{k} W_x^{\mathrm{int},k+f}(z^*), & U < |x|, |z|,\ d \in \mathbb{N},\ f > 0 \\ \frac{R}{U} W_x^{\mathrm{ext},d+f}(z), & |x| < U < |z|, d, f > 0 \\ (-1)^{d+f} \frac{R}{U} W_x^{\mathrm{ext},d+f}(z), & |z| < U < |x|,\ d, f \in \mathbb{N} \\ (-1)^{f+1} \frac{R}{|z|} \sum_{k=0}^{f} \binom{f}{k} W_x^{\mathrm{ext},d+k}\big(z^*\big), & |x|, |z| < U,\ d > 0, f \in \mathbb{N} . \end{cases}$$

Proof. All formulas may be verified by using the absolutely convergent series expansions of Proposition 2 and of the definitions together with the reproducing formula (1.2). □

In particular, we have the following formula for the L^2-norm of our wavelets:

$$\big\| W_x^{(\mathrm{ext},\mathrm{int}),d} \big\|_{L^2(\Omega_R)}^2 = \frac{R}{|x|} W_x^{(\mathrm{ext},\mathrm{int}),2d}\big(x^*\big) .$$

6. Continuous Wavelet Transform

For any function $s \in L^2(\Omega_R)$ we define its exterior, resp. interior, wavelet transform with respect to a Poisson wavelet of degree $d > 0$ as map from functions on Ω_R to functions in $\mathrm{Ext}\,\Omega_R$, resp. $\mathrm{Int}\,\Omega_R$,

$$\mathcal{W}^{\mathrm{int},d} s(x) = < W_x^{\mathrm{ext},d}, s >_R = \frac{1}{4\pi R^2} \int_{\Omega_R} W_x^{\mathrm{ext},d}(y) s(y)\, d\omega(y), \quad x \in \mathrm{Int}\,\Omega_R ,$$

and

$$\mathcal{W}^{\mathrm{ext},d} s(x) = < W_x^{\mathrm{int},d}, s >_R = \frac{1}{4\pi R^2} \int_{\Omega_R} W_x^{\mathrm{int},d} s(y)\, d\omega(y), \quad x \in \mathrm{Ext}\,\Omega_R .$$

From the properties of the wavelets we have that the interior and the exterior transforms are Kelvin transforms of each other

$$\mathcal{W}^{\mathrm{int},d} s(x) = K_R \mathcal{W}^{\mathrm{ext},d} s(x) = \big|x^*\big| \mathcal{W}^{\mathrm{ext},d} s\big(x^*\big) .$$

In terms of the Fourier coefficients for $x \in \Omega_R$,

$$s(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{s}_{l,m} Y_{l,m}\big(\hat{x}\big), \quad \hat{s}_{l,m} = \frac{1}{4\pi R^2} \int_{\Omega_R} s(x) \bar{Y}_{l,m}\big(\hat{x}\big)\, d\omega(x) ,$$

we have the following formula for the wavelet transform

$$\begin{aligned} \mathcal{W}^{\mathrm{int},d} s(x) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{s}_{l,m} l^d (|x|/R)^l Y_{l,m}\big(\hat{x}\big) , \\ \mathcal{W}^{\mathrm{ext},d} s(x) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{s}_{l,m} l^d (R/|x|)^{l+1} Y_{l,m}\big(\hat{x}\big) , \end{aligned}$$

as can be verified for harmonic polynomials. Since on Ω_R the harmonic polynomials are dense in $L^2(\Omega_R)$ we may conclude by taking limits that the above formula holds for all s. Note that the sum is absolutely convergent since here $|x| < R$, resp. $|x| > R$.

The image of the wavelet transform can be understood in terms of the Hardy space of the ball. The Hardy space of the ball is the Hilbert space $H^{\text{int}}(\Omega_R)$ of Harmonic functions in $\operatorname{Int}\Omega_R$ for which

$$\|s\|_{H^{\text{int}}} = \limsup_{r\to R^-} \|s|_{\Omega_r}\|_{L^2(\Omega_r)} < \infty \, .$$

In terms of the Fourier coefficients

$$s(x) = \sum_l \sum_{m=-l}^{l} \hat{s}_{l,m} Y_{l,m}\big(\hat{x}\cdot\hat{e}\big)|x/R|^l, \quad x \in \operatorname{Int}\Omega_R \, ,$$

the norm of a function in $H^{\text{int}}(\Omega_R)$ reads more easily

$$\|s\|^2_{H^{\text{int}}} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left|\hat{s}_{l,m}\right|^2 .$$

The functions in $H^{\text{int}}(\Omega_R)$ have a well defined boundary behavior. Let $s_r(y) = s(r\hat{y})$, then in the limit s_r converges in quadratic mean to some function f. From this, the boundary function s may be recovered by means of the Poisson integral, $s(x) = P^{\text{int}} f(x)$. The exterior Hardy space $H^{\text{ext}}(\Omega_R)$ is defined as the Kelvin transform of the interior one.

Proposition 4. *Let $U = P^{\text{int}}s$ be the harmonic extension of s to* $\operatorname{Int}\Omega_R$. *Then, for $d \in \mathbb{N}_0$, we have*

$$\mathcal{W}^{\text{int},d}s(x) \quad = \quad (|x|\partial_{|x|})^d U(x) = \partial_\lambda^d U(\lambda x)|_{\lambda=1}, \quad x \in \operatorname{Int}\Omega_R \, .$$

Let $S = P^{\text{ext}}s$ be the harmonic extension of $s \in L^2(\Omega_R)$ to $\operatorname{Ext}\Omega_R$. *We then have*

$$\mathcal{W}^{\text{ext},d}s(x) \quad = \quad |x|^{-1}(-|x|\partial_{|x|})^d(|x|S(x)), \quad x \in \operatorname{Ext}\Omega_R \, .$$

Proof. We may write for $|x| < |y|$

$$W_x^{\text{ext},d}(y)(|x|\partial_{|x|})^d P^{\text{int}}(x,y) \, .$$

Differentiating under the integral the proposition follows. For the interior wavelet we have instead

$$W_x^{\text{int},d}(y)|x|^{-1}(-|x|\partial_{|x|})^d\big(|x|P^{\text{ext}}(x,y)\big)$$

and the proof follows. □

Accordingly, the function s is in the image of the interior wavelet transform of degree $d \in \mathbb{N}$ of $L^2(\Omega_R)$ if and only if for some function $f \in H^{\text{int}}(\Omega_R)$, we have $s = (|x|\partial_{|x|})^d f$. This function is uniquely defined up to some constant. Conversely, a harmonic function s is the wavelet transform of integral degree d of some function in $L^2(\Omega_R)$ if and only if the integrals

$$f\big(r\hat{y}\big) = \int_0^r \frac{dt_d}{t_d}\int_0^{t_d}\frac{dt_{d-1}}{t_{d-1}}\dots\int_0^{t_1}\frac{dt_1}{t_1}s\big(t_1\hat{y}\big) = \frac{1}{\Gamma(d)}\int_0^r (\log(r/\lambda))^{d-1}s\big(\lambda\hat{y}\big)\frac{d\lambda}{\lambda}$$

converge in $L^2(\Omega_R)$ as $r \to R^-$.

We now prove two inversion formulas for the wavelet transform. The following one-dimensional inversion formula holds for the wavelet transform. We consider the approximant for $r < R$ and $\mathcal{W}^{\text{int}}$

$$s_r(y) = \frac{1}{\Gamma(d)} \int_0^r \mathcal{W}^{\text{int},d} s\big(\lambda \hat{y}\big)(\log(r/\lambda))^{d-1} \frac{d\lambda}{\lambda} ,$$

resp. for $r > R$ and $\mathcal{W}^{\text{ext}}$,

$$s_r(y) = \frac{1}{\Gamma(d)} \int_r^\infty \mathcal{W}^{\text{ext},d} s\big(\lambda \hat{y}\big)(\log(\lambda/r))^{d-1} \frac{d\lambda}{r} .$$

Theorem 2. *Let $s \in L^2(\Omega_R)$ be of zero mean, $\int s = 0$. Then for $d > 0$ the approximant converges in quadratic mean*

$$\lim_{r \to R} \|s_r - s\|_{L^2(\Omega_R)} = 0 .$$

Moreover, the convergence is pointwise almost everywhere: For almost all $y \in \Omega_R$ we have

$$\lim_{r \to R} s_r(y) = s(y) .$$

Proof. Using the formula

$$\int_0^1 dt \log^{k-1}(1/t) t^{p-1} = k^{-p} \Gamma(k) ,$$

we see that for $s(y) = Y_{l,m}(\hat{y})$, $l \neq 0$ we have $s_r(y) = (r/R)^l Y_{l,m}(\hat{y})$. It follows that $s_r(y) = P^{\text{int}}(r\hat{y})$ for any $s \in L^2(\Omega_R)$ with $\int s = 0$. The theorem now follows from the approximating properties of the Poisson integral (see e.g., [3]). The proof is the same for the exterior wavelet transform. $\square$

We consider now the approximant

$$s_r(y) = \frac{1}{4\pi \Gamma(d+f)} \int_{\text{Int}\,\Omega_r} \frac{dx}{|x|^3} \log^{d+f-1} \big(r^2/|x|^2\big) \mathcal{W}^{\text{int},d}(x) W_x^{\text{int},f}(y) ,$$

respectively,

$$s_r(y) = \frac{1}{4\pi \Gamma(d+f)} \int_{\text{Ext}\,\Omega_r} \frac{dx}{R^3} \log^{d+f-1} \big(|x|^2/r^2\big) \mathcal{W}^{\text{ext},d}(x) W_x^{\text{ext},f}(y) .$$

Here we approximate a function as superposition of wavelets.

Theorem 3. *Let $s \in L^2(\Omega_R)$ be of zero mean, $\int s = 0$. Then for $d, f \geq 0$, $d + f > 0$ the approximant s_r converges in quadratic mean*

$$\lim_{r \to R^-} \|s_r - s\|_{L^2(\Omega_R)} = 0 .$$

Moreover, the convergence is pointwise almost everywhere. For almost all $y \in \Omega_R$ we have

$$\lim_{r \to R^-} s_r(y) = s(y) .$$

Proof. Again we may verify using Proposition 3 that for $s(x) = Y_{l,m}(\hat{x})$, we have $s_r(x) = P^{\text{int}} s(\sqrt{r/R} R\hat{x})$. By density, this holds for arbitrary $s \in L^2(\Omega, d\omega) \ominus \{1\}$. Again, the convergence properties of the Poisson integral allow us to conclude. The proof for the exterior transform is analogous. $\square$

We now want to characterize the image of the wavelet transform not in terms of derivatives of Hardy space functions, but in terms of weighted Bergman spaces. Let $\mathcal{H}^{\text{int},d}(\Omega_R)$ and $\mathcal{H}^{\text{ext},d}(\Omega_R)$ denote the homogeneous weighted Bergman space of harmonic functions in $\text{Int}\,\Omega_R$, resp. $\text{Ext}\,\Omega_R$, which are square integrable with respect to some weight, so that they satisfy

$$\|s\|^2_{\mathcal{H}^{\text{int},d}} = \frac{1}{4\pi\Gamma(2d)} \int_{\text{Int}\,\Omega_R} |s(x)|^2 \log^{2d-1}\left(R^2/|x|^2\right) \frac{dx}{|x|^3} < \infty .$$

and

$$\|s\|^2_{\mathcal{H}^{\text{ext},d}} = \frac{1}{4\pi\Gamma(2d)} \int_{\text{Ext}\,\Omega_R} |s(x)|^2 \log^{2d-1}\left(|x|^2/R^2\right) \frac{dx}{R^3} < \infty .$$

Clearly one space is simply the Kelvin transform of the other. In terms of the Fourier coefficients the norm of $\mathcal{H}^{\text{int},d}(\Omega_R)$ can be expressed as follows

$$\|s\|^2_{\mathcal{H}^{\text{int},d}} = \sum_{l>0} l^{-2d} \sum_{m=-l}^{l} \left|\hat{s}_{l,m}\right|^2 ,$$

as can be shown by integration term by term of the Fourier series. This shows that these spaces are actually Hilbert spaces. Note that the classical Bergman space $B(\Omega_R)$, which consists of all harmonic functions which are square summable over the ball, has norm

$$\|s\|^2_{B(\Omega_R)} = \frac{1}{4\pi R^3} \int_{\text{Int}\,\Omega_R} d\omega(x) |s(x)|^2 = \sum_{l=0}^{\infty} \frac{1}{2l+3} \sum_{m=-l}^{l} \left|\hat{s}_{l,m}\right|^2 .$$

Therefore, for functions of zero mean, $s(0) = 0$, the Bergman norm and the norm in $\mathcal{H}^{\text{int},1/2}$ are equivalent:

$$s(0) = 0 \quad \Rightarrow \quad \frac{1}{5}\|s\|^2_{\mathcal{H}^{\text{int},1/2}(\Omega_R)} \le \|s\|^2_{B(\Omega_R)} \le \frac{1}{2}\|s\|^2_{\mathcal{H}^{\text{int},1/2}(\Omega_R)} .$$

Theorem 4. *For $d > 0$, the interior wavelet transform is a one to one isometry from $L^2(\Omega_R) \ominus \{1\}$ to $\mathcal{H}^{\text{int},d}(\Omega_R)$ and we have*

$$\frac{1}{4\pi\Gamma(2d)} \int_{\text{Int}\,\Omega_R} \left|W^{\text{int},d} s(x)\right|^2 \log^{2d-1}\left(R^2/|x|^2\right) \frac{dx}{|x|^3} = \frac{1}{4\pi R^2} \int_{\Omega_R} |s(y)|^2\, d\omega(y) .$$

For $d > 0$, the interior wavelet transform is a one to one isometry from $L^2(\Omega_R) \ominus \{1\}$ to $\mathcal{H}^{\text{ext},d}(\Omega_R)$ and we have

$$\frac{1}{4\pi\Gamma(2d)} \int_{\text{Ext}\,\Omega_R} \left|W^{\text{int},d} s(x)\right|^2 \log^{2d-1}\left(|x|^2/R^2\right) \frac{dx}{R^3} = \frac{1}{4\pi R^2} \int_{\Omega_R} |s(y)|^2\, d\omega(y) .$$

Proof. The formula holds for any harmonic polynomial s with $s(0) \neq 0$. For arbitrary $f \in L^2(\Omega_R) \ominus \{1\}$, we take a polynomial approximation $s_n \to f$ as we may by density.

Then $\mathcal{W}s_n$ is a Cauchy sequence in $\mathcal{H}^{\text{int},d}(\Omega_R)$ and has a limit F and we have $\|F\| = \|s\|$. By Egorov's theorem, there is a subsequence that converges pointwise almost everywhere. Since $\mathcal{W}s_n(x) \to \mathcal{W}f(x)$ for all x, we have $F(x) = \mathcal{W}f(x)$ for almost all x. Since both functions are continuous we have $F = W$ and the theorem follows. □

For any function $r : \operatorname{Int}\Omega_R \to \mathbb{C}$, resp. $r : \operatorname{Ext}\Omega_R \to \mathbb{C}$, we define the synthesis with exterior, resp. interior, wavelets as

$$\mathcal{M}^{\text{ext},d} r(y) = \frac{1}{4\pi\Gamma(2d)} \int_{\operatorname{Int}\Omega_R} \frac{dx}{|x|^3} \log^{2d-1}\big(R^2/|x|^2\big) r(x) W_x^{\text{ext},d}(y) ,$$

or for any $r : \operatorname{Ext}\Omega_R \to \mathbb{C}$

$$\mathcal{M}^{\text{int},d} r(y) = \frac{1}{4\pi\Gamma(2d)} \int_{\operatorname{Ext}\Omega_R} \frac{dx}{R^3} \log^{2d-1}\big(|x|^2/R^2\big) r(x) W_x^{\text{int},d}(y) ,$$

whenever this integral makes sense. A simple exchange of integrations shows that this operator coincides with the adjoint operator of the wavelet analysis at least on the set of nonconstant polynomials. Therefore we have a unique extension of $\mathcal{M}^{\text{int},d}$ to all of $\mathcal{H}^{\text{int},d}(\Omega_R)$ which we call again $\mathcal{M}^{\text{int},d}$. It satisfies

$$\mathcal{M}^{\text{ext},d} = \big(\mathcal{W}^{\text{int},d}\big)^*, \quad \mathcal{M}^{\text{int},d} = \big(\mathcal{W}^{\text{ext},d}\big)^* .$$

The image of the wavelet transform may now be characterized through the reproducing kernel property.

Theorem 5. *The image of the interior wavelet analysis* Ran $\mathcal{W}^{\text{int},d}$ *is* $\mathcal{H}^{\text{int},d}(\Omega_R)$*. It is a reproducing kernel Hilbert space with reproducing kernel*

$$K^{\text{int},d}(x,z) = \frac{R}{|z|}\big| W^{\text{int},2d}\big(x,z^*\big), \quad z^* = R^2|z|^{-1}\hat{z} ,$$

and we have explicitly with an absolutely convergent integral for each $x \in \operatorname{Int}\Omega_R$

$$r(x) = \frac{1}{4\pi\Gamma(2d)} \int_{\operatorname{Int}\Omega_R} \frac{dz}{|z|^3} \log^{2d-1}\big(R^2/|z|^2\big) K^{\text{int},d}(x,z) r(z) .$$

The image of the exterior wavelet analysis Ran $\mathcal{W}^{\text{ext},d}$ *is* $\mathcal{H}^{\text{ext},d}(\Omega_R)$*. It is the reproducing kernel Hilbert space with reproducing kernel*

$$K^{\text{ext},d}(x,z) = \frac{R}{|z|} W^{\text{ext},2d}\big(x,z^*\big), \quad z^* = R^2|z|^{-1}\hat{z} ,$$

and we have explicitly, with an absolutely convergent integral, for each $x \in \operatorname{Int}\Omega_R$

$$r(x) = \frac{1}{4\pi\Gamma(2d)} \int_{\operatorname{Ext}\Omega_R} \frac{dz}{R^3} \log^{2d-1}\big(|z|^2/R^2\big) K^{\text{ext},d}(x,z) r(z) .$$

Proof. For any polynomial s with $s(0) \neq 0$, we have for $r = \mathcal{W}^{\text{int},d}s$ the identity $r = \mathcal{W}^{\text{int},d}\mathcal{M}^{\text{ext},d}r$ which after an exchange of integrations is the equation of the theorem. For arbitrary $r = \mathcal{W}^{\text{int},d}s$ we take an approximating sequence $s_n \to s$ of polynomials. Since for fixed x the function $z \mapsto K^{\text{int},d}(x,z)$ is in $\mathcal{H}^{\text{int},d}(\Omega_R)$, and since $r_n = \mathcal{W}^{\text{int},d}s_n$ goes to r in $\mathcal{H}^{\text{int},d}(\Omega_R)$, we may conclude. □

In applications, it is sometimes useful to use the b, a variables instead. We then write

$$\mathcal{W}s(b,a) = < g^d_{b,a}, s >_R = \frac{1}{4\pi R^2} \int_{\Omega_R} g^d_{b,a}(y) s(y)\, d\omega(y) \ .$$

The wavelet transform maps functions over Ω_R to functions over the position-scale wavelet phase space on the sphere. We may identify the wavelet phase (=position-scale) space $\mathbb{H} = \Omega_R \times \mathbb{R}_+$ with the interior or exterior of the sphere using the mapping (3.1). We have

$$\mathcal{W}s(b,a) = a^{-d} \mathcal{W}^{\mathrm{int},d} s\big(e^{-a} b\big) = a^{-d} \mathcal{W}^{\mathrm{ext},d} s\big(e^{a} b\big), \quad b \in \Omega_R, a > 0 \ ,$$

and all formulas may be converted accordingly. We therefore have that the wavelet transform is an isometry from $L^2(\Omega_R) \ominus \mathbb{R}$ to $L^2(\Omega_R \times \mathbb{R}_+, d\omega(x)\, da/a)$ and we have

$$\frac{1}{4\pi R^2} \int_{\Omega_R} |s(x)|^2\, d\omega(x) = \frac{1}{4\pi R^2 \Gamma(2d)} \int_{\Omega_R} d\omega(b) \int_0^\infty \frac{da}{a} |\mathcal{W}s(b,a)|^2 \ .$$

The image is the closed subspace of functions that are reproduced by the kernel

$$\Pi\big(b, a; b', a'\big) = \frac{\big(aa'\big)^d}{\big(a+a'\big)^{2d}}\, g^{2d}_{b,aa'}\big(b'\big) = \frac{\big(aa'\big)^d}{\big(a+a'\big)^{2d}}\, g^{2d}_{b',aa'}(b) \ .$$

The reproducing kernel equation reads explicitly, for $r = \mathcal{W}s$,

$$r(b,a) = \frac{1}{4\pi R^2} \int_0^\infty \frac{da'}{a'} \int_{\Omega_R} d\omega\big(b'\big) \Pi\big(b, a; b', a'\big) r\big(b', a'\big) \ .$$

The approximant for the first inversion formula reads

$$s_\epsilon(x) = \frac{1}{\Gamma(d)} \int_\epsilon^\infty \mathcal{W}s(x,a) \left(1 - \frac{\epsilon}{a}\right)^{d-1} \frac{da}{a} \ ,$$

whereas the one for the second inversion formula reads

$$s_\epsilon(x) = \frac{1}{4\pi \Gamma(d+f)} \int_\epsilon^\infty \frac{da}{a} \int_{\Omega_R} d\omega(b) \left(1 - \frac{\epsilon}{a}\right)^{d+f-1} \mathcal{W}^d s(b,a)\, g^f_{b,a}(x) \ .$$

Both approximants converge in $L^2(\Omega_R)$ and pointwise almost everywhere. They have in fact the same convergence properties as the Poisson integral.

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