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Old and New Morrey Spaces with Heat Kernel Bounds

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ABSTRACT. *Given $p \in [1, \infty)$ and $\lambda \in (0, n)$, we study Morrey space $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ of all locally integrable complex-valued functions f on $\mathbb{R}^n$ such that for every open Euclidean ball $B \subset \mathbb{R}^n$ with radius r_B there are numbers $C = C(f)$ (depending on f) and $c = c(f, B)$ (relying upon f and B) satisfying*

$$r_B^{-\lambda} \int_B |f(x) - c|^p \, dx \le C$$

and derive old and new, two essentially different cases arising from either choosing $c = f_B = |B|^{-1} \int_B f(y)\, dy$ or replacing c by $P_{t_B}(x) = \int_{t_B} p_{t_B}(x, y) f(y)\, dy$—where t_B is scaled to r_B and $p_t(\cdot, \cdot)$ is the kernel of the infinitesimal generator L of an analytic semigroup $\{e^{-tL}\}_{t \ge 0}$ on $\mathrm{L}^2(\mathbb{R}^n)$. Consequently, we are led to simultaneously characterize the old and new Morrey spaces, but also to show that for a suitable operator L, the new Morrey space is equivalent to the old one.

1. Introduction

As well-known, a priori estimates mixing L^p and Lip_λ are frequently used in the study of partial differential equations—naturally, the so-called Morrey spaces are brought into play (cf. [24]). A locally integrable complex-valued function f on $\mathbb{R}^n$ is said to be in the Morrey space $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$, $1 \le p < \infty$ and $\lambda \in (0, n + p)$, if for every Euclidean open ball $B \subset \mathbb{R}^n$ with radius r_B there are numbers $C = C(f)$ (depending on f) and $c = c(f, B)$ (relying upon f and B) satisfying

$$r_B^{-\lambda} \int_B |f(x) - c|^p \, dx \le C \, .$$

The space of $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$-functions was introduced by Morrey [18]. Since then, the space has been studied extensively—see, for example, [4, 13, 12, 20, 21, 22, 28].

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We would like to note that as in [20], for $1 \le p < \infty$ and $\lambda = n$, the spaces $\mathrm{L}^{p,n}(\mathbb{R}^n)$ are variants of the classical BMO (bounded mean oscillation) function space of John-Nirenberg. For $1 \le p < \infty$ and $\lambda \in (n, n+p)$, the spaces $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ are variants of the homogeneous Lipschitz spaces $\mathrm{Lip}_{(\lambda-n)/p}(\mathbb{R}^n)$.

Clearly, the remaining cases: $1 \le p < \infty$ and $\lambda \in (0, n)$ are of independent interest, and hence motivate our investigation. The purpose of this article is twofold. First, we explore some new characterizations of $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ through the fact that $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ consists of all locally integrable complex-valued functions f on $\mathbb{R}^n$ satisfying

$$\|f\|_{\mathrm{L}^{p,\lambda}} = \sup_{B\subset\mathbb{R}^n} \left[r_B^{-\lambda} \int_B |f(x) - f_B|^p \, dx \right]^{1/p} < \infty , \tag{1.1}$$

where the supremum is taken over all Euclidean open balls $B = B(x_0, r_B)$ with center x_0 and radius r_B, and f_B stands for the mean value of f over B, i.e.,

$$f_B = |B|^{-1} \int_B f(x)\, dx \, .$$

The second aim is to use those new characterizations as motives of a continuous study of [1, 7, 5, 9] and so to introduce new Morrey spaces $\mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$ associated with operators. Roughly speaking, if L is the infinitesimal generator of an analytic semigroup $\{e^{-tL}\}_{t\ge 0}$ on $\mathrm{L}^2(\mathbb{R}^n)$ with kernel $p_t(x, y)$ which decays fast enough, then we can view $P_t f = e^{-tL} f$ as an average version of f at the scale t and use the quantity

$$P_{t_B} f(x) = \int_{\mathbb{R}^n} p_{t_B}(x, y) f(y)\, dy$$

to replace the mean value f_B in the equivalent semi-norm (1.1) of the original Morrey space, where t_B is scaled to the radius of the ball B. Hence, we say that a function f (with appropriate bound on its size $|f|$) belongs to the space $\mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$ (where $1 \le p < \infty$ and $\lambda \in (0, n)$), provided

$$\|f\|_{\mathrm{L}_L^{p,\lambda}} = \sup_{B\subset\mathbb{R}^n} \left[r_B^{-\lambda} \int_B |f(x) - P_{t_B} f(x)|^p \, dx \right]^{1/p} < \infty \tag{1.2}$$

where $t_B = r_B^m$ for a fixed constant $m > 0$—see the forthcoming Sections 2.2 and 3.1.

We pursue a better understanding of (1.1) and (1.2) through the following aspects:

In Section 2, we collect most useful materials on the bounded holomorphic functional calculus.

In Section 3, we study some characterizations of $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ and $\mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$ and give a criterion for $\mathrm{L}^{p,\lambda}(\mathbb{R}^n) \subseteq \mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$. The later fact illustrates that $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ exists as the minimal Morrey space, and consequently induces a concept of the maximal Morrey space.

In Section 4, we establish an identity formula associated with the operator L. This formula is a key to handle the quadratic features of the old and new Morrey spaces.

As an immediate continuation of Section 4, Section 5 is devoted to Littlewood-Paley type characterizations of $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ and $\mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$ via the predual of $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ (cf. [28]) and a number of important estimates for functions in $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ and $\mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$. Moreover, we show that for a suitable semigroup $\{e^{-tL}\}_{t>0}$, $\mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$ equals $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ with equivalent seminorms—in particular, if L is either $-\Delta$ or $\sqrt{-\Delta}$ on $\mathbb{R}^n$, then $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ coincides with

$L^{p,\lambda}_{\sqrt{-\Delta}}(\mathbb{R}^n)$ and $L^{p,\lambda}_{-\Delta}(\mathbb{R}^n)$, where $\Delta = \Delta_x = \sum_{k=1}^{n} \partial^2/\partial x_k^2$ is the classical Laplace operator in the spatial variable $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Throughout, the letters $c, c_1, c_2, \ldots$ will denote (possibly different) constants that are independent of the essential variables.

2. Preliminaries

2.1 Holomorphic Functional Calculi of Operators

We start with a review of some definitions of holomorphic functional calculi introduced by McIntosh [17]. Let $0 \le \omega < \nu < \pi$. We define the closed sector in the complex plane $\mathbb{C}$

$$S_\omega = \{z \in \mathbb{C} : |\arg z| \le \omega\} \cup \{0\}$$

and denote the interior of S_ω by S_ω^0.

We employ the following subspaces of the space $H(S_\nu^0)$ of all holomorphic functions on S_ν^0:

$$H_\infty\big(S_\nu^0\big) = \big\{b \in H\big(S_\nu^0\big) : \ ||b||_\infty < \infty\big\} ,$$

where

$$||b||_\infty = \sup\big\{|b(z)| : z \in S_\nu^0\big\}$$

and

$$\Psi\big(S_\nu^0\big) = \big\{\psi \in H\big(S_\nu^0\big) : \exists s > 0, \ |\psi(z)| \le c|z|^s\big(1 + |z|^{2s}\big)^{-1}\big\} .$$

Given $0 \le \omega < \pi$ and $\mathcal{I}$ – the identity operator on $L^2(\mathbb{R}^n)$, a closed operator L in $L^2(\mathbb{R}^n)$ is said to be of type ω if its spectrum $\sigma(L) \subset S_\omega$, and for each $\nu > \omega$, there exists a constant c_ν such that

$$\big\|(L - \lambda\mathcal{I})^{-1}\big\|_{2,2} = \big\|(L - \lambda\mathcal{I})^{-1}\big\|_{L^2\to L^2} \le c_\nu|\lambda|^{-1}, \quad \lambda \notin S_\nu .$$

If L is of type ω and $\psi \in \Psi(S_\nu^0)$, we define $\psi(L) \in \mathcal{L}(L^2, L^2)$ by

$$\psi(L) = \frac{1}{2\pi i}\int_\Gamma (L - \lambda\mathcal{I})^{-1}\psi(\lambda)\, d\lambda , \tag{2.1}$$

where Γ is the contour $\{\xi = re^{\pm i\theta} : r \ge 0\}$ parametrised clockwise around S_ω, and $\omega < \theta < \nu$. Clearly, this integral is absolutely convergent in $\mathcal{L}(L^2, L^2)$ (which is the class of all bounded linear operators on L^2), and it is straightforward to show, using Cauchy's theorem, that the definition is independent of the choice of $\theta \in (\omega, \nu)$. If, in addition, L is one-one and has dense range and if $b \in H_\infty(S_\nu^0)$, then $b(L)$ can be defined by

$$b(L) = [\psi(L)]^{-1}(b\psi)(L) \quad \text{where} \quad \psi(z) = z(1 + z)^{-2} .$$

It can be shown that $b(L)$ is a well-defined linear operator in $L^2(\mathbb{R}^n)$.

We say that L has a bounded H_∞ calculus in $L^2(\mathbb{R}^n)$ provided there exists $c_{\nu,2} > 0$ such that $b(L) \in \mathcal{L}(L^2, L^2)$ and

$$\|b(L)\|_{2,2} = \|b(L)\|_{L^2\to L^2} \le c_{\nu,2}||b||_\infty \quad \forall b \in H_\infty\big(S_\nu^0\big) .$$

For the conditions and properties of operators which have holomorphic functional calculi, see [17] and [2] which also contain a proof of the following convergence lemma.

Lemma 1. *Let X be a complex Banach space. Given $0 \le \omega < \nu \le \pi$, let L be an operator of type ω which is one-to-one with dense domain and range. Suppose $\{f_\alpha\}$ is a uniformly bounded net in $H_\infty(S_\nu^0)$, which converges to $f \in H_\infty(S_\nu^0)$ uniformly on compact subsets of S_ν^0, such that $\{f_\alpha(L)\}$ is a uniformly bounded net in the space $\mathcal{L}(X, X)$ of continuous linear operators on X. Then $f(L) \in \mathcal{L}(X, X)$, $f_\alpha(L)u \to f(L)u$ for all $u \in X$ and*

$$\|f(L)\| = \|f(L)\|_{X\to X} \le \sup_\alpha \|f_\alpha(L)\| = \sup_\alpha \|f_\alpha(L)\|_{X\to X} .$$

2.2 Two More Assumptions

Let L be a linear operator of type ω on $\mathrm{L}^2(\mathbb{R}^n)$ with $\omega < \pi/2$, hence, L generates a holomorphic semigroup e^{-zL}, $0 \le |\mathrm{Arg}(z)| < \pi/2 - \omega$. Assume the following two conditions.

Assumption (a): The holomorphic semigroup

$$\{e^{-zL}\}_{0\le|\mathrm{Arg}(z)|<\pi/2-\omega}$$

is represented by kernel $p_z(x, y)$ which satisfies an upper bound

$$|p_z(x, y)| \le c_\theta h_{|z|}(x, y) \quad \forall x, y \in \mathbb{R}^n$$

and

$$|\mathrm{Arg}(z)| < \pi/2 - \theta \quad \text{for} \quad \theta > \omega ,$$

where $h_t(\cdot, \cdot)$ is determined by

$$h_t(x, y) = t^{-n/m} g\left(\frac{|x-y|}{t^{1/m}}\right) , \tag{2.2}$$

in which m is a positive constant and g is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon} g(r) = 0 \quad \text{for some } \epsilon > 0 . \tag{2.3}$$

Assumption (b): The operator L has a bounded H_∞-calculus in $\mathrm{L}^2(\mathbb{R}^n)$.

Now, we give some consequences of the Assumptions (a) and (b) which will be used later.

First, if $\{e^{-tL}\}_{t>0}$ is a bounded analytic semigroup on $\mathrm{L}^2(\mathbb{R}^n)$ whose kernel $p_t(x, y)$ satisfies the estimates (2.2) and (2.3), then for any $k \in \mathbb{N}$, the time derivatives of p_t satisfy

$$\left|t^k \frac{\partial^k p_t(x, y)}{\partial t^k}\right| \le \frac{c}{t^{n/m}} g\left(\frac{|x-y|}{t^{1/m}}\right) \quad \text{for all } t > 0 \text{ and almost all } x, y \in \mathbb{R}^n . \tag{2.4}$$

For each $k \in \mathbb{N}$, the function g might depend on k but it always satisfies (2.3). See Theorem 6.17 of [19].

Secondly, L has a bounded H_∞-calculus in $\mathrm{L}^2(\mathbb{R}^n)$ if and only if for any nonzero function $\psi \in \Psi(S_\nu^0)$, L satisfies the square function estimate and its reverse

$$c_1\|f\|_{\mathrm{L}^2} \le \left(\int_0^\infty \|\psi_t(L)f\|_{\mathrm{L}^2}^2 \frac{dt}{t}\right)^{1/2} \le c_2\|f\|_{\mathrm{L}^2} \tag{2.5}$$

for some $0 < c_1 \le c_2 < \infty$, where $\psi_t(\xi) = \psi(t\xi)$. Note that different choices of $\nu > \omega$ and $\psi \in \Psi(S_\nu^0)$ lead to equivalent quadratic norms of f.

As noted in [17], positive self-adjoint operators satisfy the quadratic estimate (2.5). So do normal operators with spectra in a sector, and maximal accretive operators. For the definitions of these classes of operators, we refer readers to [27].

The following result, existing as a special case of [6, Theorem 6], tells us the L^2-boundedness of a bounded H_∞-calculus can be extended to L^p-boundedness, $p > 1$.

Lemma 2. *Under the Assumptions (a) and (b), the operator L has a bounded H_∞-calculus in $\mathrm{L}^p(\mathbb{R}^n)$, $p \in (1, \infty)$, that is, $b(L) \in \mathcal{L}(\mathrm{L}^p, \mathrm{L}^p)$ with*

$$\|b(L)\|_{p,p} = \|b(L)\|_{\mathrm{L}^p \to \mathrm{L}^p} \le c_{\nu,p}\|b\|_\infty \quad \forall b \in H_\infty\big(S_\nu^0\big) .$$

Moreover, if $p = 1$ then $b(L)$ is of weak type $(1, 1)$.

Thirdly, the Littlewood-Paley function $\mathcal{G}_L(f)$ associated with an operator L is defined by

$$\mathcal{G}_L(f)(x) = \left(\int_0^\infty |\psi_t(L)f|^2 \, \frac{dt}{t}\right)^{1/2} , \tag{2.6}$$

where again $\psi \in \Psi(S_\nu^0)$, and $\psi_t(\xi) = \psi(t\xi)$. It follows from Theorem 6 of [3] that the function $\mathcal{G}_L(f)$ is bounded on L^p for $1 < p < \infty$. More specifically, there exist constants c_3, c_4 such that $0 < c_3 \le c_4 < \infty$ and

$$c_3\|f\|_{\mathrm{L}^p} \le \|\mathcal{G}_L(f)\|_{\mathrm{L}^p} \le c_4\|f\|_{\mathrm{L}^p} \tag{2.7}$$

for all $f \in \mathrm{L}^p$, $1 < p < \infty$.

By duality, the operator $\mathcal{G}_{L^*}(f)$ also satisfies the estimate (2.7), where L^* is the adjoint operator of L.

2.3 Acting Class of Semigroup $\{e^{-tL}\}_{t>0}$

We now define the class of functions that the operators e^{-tL} act upon. Fix $1 \le p < \infty$. For any $\beta > 0$, a complex-valued function $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be a function of type $(p; \beta)$ if f satisfies

$$\left(\int_{\mathbb{R}^n} \frac{|f(x)|^p}{(1+|x|)^{n+\beta}}\, dx\right)^{1/p} \le c < \infty . \tag{2.8}$$

We denote by $\mathcal{M}_{(p;\beta)}$ the collection of all functions of type $(p; \beta)$. If $f \in \mathcal{M}_{(p;\beta)}$, the norm of $f \in \mathcal{M}_{(p;\beta)}$ is defined by

$$\|f\|_{\mathcal{M}_{(p;\beta)}} = \inf\left\{c \ge 0 : (2.8) \text{ holds}\right\} .$$

It is not hard to see that $\mathcal{M}_{(p;\beta)}$ is a complex Banach space under $\|f\|_{\mathcal{M}_{(p;\beta)}} < \infty$. For any given operator L, let

$$\Theta(L) = \sup\left\{\epsilon > 0 : (2.3) \text{ holds }\right\} \tag{2.9}$$

and write

$$\mathcal{M}_p = \begin{cases} \mathcal{M}_{(p;\Theta(L))} & \text{if} \quad \Theta(L) < \infty ; \\ \bigcup\limits_{\beta:\, 0<\beta<\infty} \mathcal{M}_{(p;\beta)} & \text{if} \quad \Theta(L) = \infty . \end{cases}$$

Note that if $L = -\Delta$ or $L = \sqrt{-\Delta}$ on $\mathbb{R}^n$, then $\Theta(-\Delta) = \infty$ or $\Theta(\sqrt{-\Delta}) = 1$.

For any $(x,t) \in \mathbb{R}^n \times (0, +\infty) = \mathbb{R}^{n+1}_+$ and $f \in \mathcal{M}_p$, define

$$P_t f(x) = e^{-tL} f(x) = \int_{\mathbb{R}^n} p_t(x,y) f(y)\, dy \tag{2.10}$$

and

$$Q_t f(x) = tLe^{-tL} f(x) = \int_{\mathbb{R}^n} -t\Big(\frac{dp_t(x,y)}{dt}\Big) f(y)\, dy\ . \tag{2.11}$$

It follows from the estimate (2.4) that the operators $P_t f$ and $Q_t f$ are well defined. Moreover, the operator Q_t has the following two properties:

(i) For any $t_1, t_2 > 0$ and almost all $x \in \mathbb{R}^n$,

$$Q_{t_1} Q_{t_2} f(x) = t_1 t_2 \Big(\frac{d^2 P_t}{dt^2}\Big|_{t=t_1+t_2} f\Big)(x)\ ;$$

(ii) the kernel $q_{t^m}(x,y)$ of Q_{t^m} satisfies

$$\big|q_{t^m}(x,y)\big| \le ct^{-n} g\Big(\frac{|x-y|}{t}\Big) \tag{2.12}$$

where the function g satisfies the condition (2.3).

3. Basic Properties

3.1 A Comparison of Definitions

Assume that L is an operator which generates a semigroup e^{-tL} with the heat kernel bounds (2.2) and (2.3). In what follows, $B(x,t)$ denotes the ball centered at x and of the radius t. Given $B = B(x,t)$ and $\lambda > 0$, we will write λB for the λ-dilate ball, which is the ball with the same center x and with radius λt.

Definition 1. Let $1 \le p < \infty$ and $\lambda \in (0, n)$. We say that

(i) $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ provided (1.1) holds;

(ii) $f \in \mathcal{M}_p$ associated with an operator L, is in $\mathrm{L}^{p,\lambda}_L(\mathbb{R}^n)$ provided (1.2) holds.

Remark 1.

(i) Note first that $(\mathrm{L}^{p,\lambda}(\mathbb{R}^n), \|\cdot\|_{\mathrm{L}^{p,\lambda}})$ and $(\mathrm{L}^{p,\lambda}_L(\mathbb{R}^n), \|\cdot\|_{\mathrm{L}^{p,\lambda}_L})$ are vector spaces with the seminorms vanishing on constants and

$$\mathcal{K}_{L,p} = \Big\{ f \in \mathcal{M}_p : P_t f(x) = f(x) \text{ for almost all } x \in \mathbb{R}^n \text{ and all } t > 0 \Big\}\ ,$$

respectively. Of course, the spaces $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ and $\mathrm{L}^{p,\lambda}_L(\mathbb{R}^n)$ are understood to be modulo constants and $\mathcal{K}_{L,p}$, respectively. See Section 6 of [8] for a discussion of the dimensions of $\mathcal{K}_{L,2}$ when L is a second order elliptic operator of divergence form or a Schrödinger operator.

(ii) We now give a list of examples of $\mathrm{L}^{p,\lambda}_L(\mathbb{R}^n)$ in different settings.

(α) Define P_t by putting $p_t(x,y)$ to be the heat kernel or the Poisson kernel:

$$(4\pi t)^{-n/2} e^{-|x-y|^2/4t} \quad \text{or} \quad \frac{c_n t}{\big(t^2 + |x-y|^2\big)^{(n+1)/2}} \quad \text{where} \quad c_n = \frac{\Gamma\big(\frac{n+1}{2}\big)}{\pi^{\frac{n+1}{2}}}\ .$$

Then we will show that the corresponding space $\mathrm{L}^{p,\lambda}_L(\mathbb{R}^n)$(modulo $\mathcal{K}_{L,p}$) coincides with the classical $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$(modulo constants).

(β) Consider the Schrödinger operator with a nonnegative potential $V(x)$:

$$L = -\Delta + V(x)\ .$$

To study singular integral operators associated to L such as functional calculi $f(L)$ or Riesz transform $\nabla L^{-1/2}$, it is useful to choose P_t with kernel $p_t(x, y)$ to be the heat kernel of L. By domination, its kernel $p_t(x, y)$ has a Gaussian upper bound.

The following proposition shows that $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ is a subspace of $\mathrm{L}^{p,\lambda}_L(\mathbb{R}^n)$ in many cases.

Proposition 1. *Let $1 \le p < \infty$ and $\lambda \in (0, n)$. Given an operator L which generates a semigroup e^{-tL} with the heat kernel bounds* (2.2) *and* (2.3)*. A necessary and sufficient condition for the classical space $\mathrm{L}^{p,\lambda}(\mathbb{R}^n) \subseteq \mathrm{L}^{p,\lambda}_L(\mathbb{R}^n)$ with*

$$\|f\|_{\mathrm{L}^{p,\lambda}_L} \le c\|f\|_{\mathrm{L}^{p,\lambda}} \tag{3.1}$$

is that for every $t > 0$, $e^{-tL}(1) = 1$ almost everywhere, that is, $\int_{\mathbb{R}^n} p_t(x, y)\,dy = 1$ for almost all $x \in \mathbb{R}^n$.

Proof. Clearly, the condition $e^{-tL}(1) = 1$, a.e. is necessary for $\mathrm{L}^{p,\lambda}(\mathbb{R}^n) \subseteq \mathrm{L}^{p,\lambda}_L(\mathbb{R}^n)$. Indeed, let us take $f = 1$. Then, (3.1) implies $\|1\|_{\mathrm{L}^{p,\lambda}_L} = 0$ and thus for every $t > 0$, $e^{-tL}(1) = 1$ almost everywhere.

For the sufficiency, we borrow the idea of [16, Proposition 3.1]. To be more specific, suppose $f \in \mathrm{L}^{p,\lambda}(\mathbb{R}^n)$. Then for any Euclidean open ball B with radius r_B, we compute

$$\begin{aligned}
\|f - P_{t_B} f\|_{\mathrm{L}^p(B)} &\le \|f - f_B\|_{\mathrm{L}^p(B)} + \|f_B - P_{t_B} f\|_{\mathrm{L}^p(B)} \\
&\le \|f\|_{\mathrm{L}^{p,\lambda}} r_B^{\lambda/p} + \left(\int_B \left(\int_{\mathbb{R}^n} |f_B - f(y)| P_{t_B}(x, y)\,dy\right)^p dx\right)^{1/p} \\
&= \|f\|_{\mathrm{L}^{p,\lambda}} r_B^{\lambda/p} + \left(\int_B \Big(I(B) + J(B)\Big)^p dx\right)^{1/p} ,
\end{aligned}$$

where

$$I(B) = \int_{2B} |f_B - f(y)| P_{t_B}(x, y)\,dy$$

and

$$J(B) = \sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^k B} |f_B - f(y)| P_{t_B}(x, y)\,dy\ .$$

Next we make further estimates on $I(B)$ and $J(B)$. Thanks to (2.2) and (2.3), we have

$$\|I(B)\|_{\mathrm{L}^p(B)} \le c r_B^{-n} g(0) \|f_B - f\|_{\mathrm{L}^1(B)} \le c r_B^{\lambda/p} \|f\|_{\mathrm{L}^{p,\lambda}}\ .$$

Again, using (2.2) and (2.3), we derive that for $x \in B$ and $y \in 2^{k+1}B \setminus 2^k B$,

$$P_{t_B}(x, y) \le c r_B^{-n} g(2^k) \le c r_B^{-n} 2^{-k(n+\epsilon)}, \quad k = 1, 2, \dots\ ,$$

where $\epsilon > 0$ is a constant. Consequently,

$$\begin{aligned}\|J(B)\|_{\mathrm{L}^p(B)} &\le cr_B^{-n}\left(\int_B\left(\sum_{k=1}^{\infty}g(2^k)\int_{2^{k+1}B\setminus 2^kB}|f_B-f(y)|\,dy\right)^p dx\right)^{1/p}\\ &\le cr_B^{n/p-n}\sum_{k=1}^{\infty}g(2^k)\left(\int_{2^{k+1}B}|f_{2^{k+1}B}-f(y)|\,dy+(2^kr_B)^n|f_{2^{k+1}B}-f_B|\right)\\ &\le cr_B^{\lambda/p}\|f\|_{\mathrm{L}^{p,\lambda}}\left(\sum_{k=1}^{\infty}2^{-k(\epsilon+\frac{n-\lambda}{p})}+\sum_{k=1}^{\infty}k2^{-k\epsilon}\right).\end{aligned}$$

Putting these inequalities together, we find $f\in \mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$. □

3.2 Fundamental Characterizations

In the argument for Proposition 1, we have used the following crucial fact that for any $f\in\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ and a constant $K>1$,

$$|f_B-f_{KB}|\le cr_B^{\frac{\lambda-n}{p}}\|f\|_{\mathrm{L}^{p,\lambda}}.$$

Now, this property can be used to give a characterization of $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ spaces in terms of the Poisson integral. To this end, we observe that if

$$f\in\mathcal{M}_{\sqrt{-\Delta},p}=\left\{f\in\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n):|f(\cdot)|^p\left(1+|\cdot|^{n+1}\right)^{-1}\in\mathrm{L}^1(\mathbb{R}^n)\right\},$$

then we can define the operator $e^{-t\sqrt{-\Delta}}$ by the Poisson integral as follows:

$$e^{-t\sqrt{-\Delta}}f(x)=\int_{\mathbb{R}^n}p_t(x-y)f(y)\,dy,\quad t>0,$$

where

$$p_t(x-y)=\frac{c_nt}{\left(t^2+|x-y|^2\right)^{(n+1)/2}}.$$

Proposition 2. *Let $1\le p<\infty$, $\lambda\in(0,n)$ and $f\in\mathcal{M}_{\sqrt{-\Delta},p}$. Then $f\in\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ if and only if*

$$|||f|||_{\mathrm{L}^{p,\lambda}(\mathbb{R}^n)}=\left(\sup_{(x,t)\in\mathbb{R}^{n+1}_+}t^{n-\lambda}e^{-t\sqrt{-\Delta}}\left(\left|f-e^{-t\sqrt{-\Delta}}f(x)\right|^p\right)(x)\right)^{1/p}<\infty. \quad (3.2)$$

Proof. On the one hand, assume (3.2). Note that $|y-x|<t$ implies

$$\frac{c_nt}{\left(t^2+|y-x|^2\right)^{\frac{n+1}{2}}}\ge ct^{-n}.$$

For a fixed ball $B=B(x,r_B)$ centered at x, we let $t_B=r_B$. We then have

$$\begin{aligned}r_B^{-\lambda}\|f-f_B\|^p_{\mathrm{L}^p(B)} &\le cr_B^{-\lambda}\left\|f-e^{-t_B\sqrt{-\Delta}}f(x)\right\|^p_{\mathrm{L}^p(B)}\\ &\le cr_B^{n-\lambda}\int_B\left|f(y)-e^{-t_B\sqrt{-\Delta}}f(x)\right|^p\frac{c_nt_B}{\left(t_B^2+|y-x|^2\right)^{\frac{n+1}{2}}}\,dy\\ &\le c|||f|||^p_{\mathrm{L}^{p,\lambda}},\end{aligned}$$

whence producing $f \in \mathrm{L}^{p,\lambda}(\mathbb{R}^n)$.

On the other hand, suppose $f \in \mathrm{L}^{p,\lambda}(\mathbb{R}^n)$. In a similar manner to proving the sufficiency part of Proposition 1, we obtain that if $(x,t) \in \mathbb{R}^{n+1}_+$ then

$$e^{-t\sqrt{-\Delta}}\big(\big|f - e^{-t\sqrt{-\Delta}}f(x)\big|^p\big)(x) \le ct^{\lambda-n}\|f\|^p_{\mathrm{L}^{p,\lambda}} + c\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\frac{|f(y)-f_B|^p t}{\big(t^2+|y-x|^2\big)^{\frac{n+1}{2}}}\,dy$$
$$\le ct^{\lambda-n}\|f\|^p_{\mathrm{L}^{p,\lambda}},$$

and hence (3.2) holds. □

Remark 2. Since a simple computation gives

$$\begin{aligned}
&e^{-t\sqrt{-\Delta}}\big(\big|f - e^{-t\sqrt{-\Delta}}f(x)\big|^2\big)(x)\\
&= \int_{\mathbb{R}^n}\big(f(y) - e^{-t\sqrt{-\Delta}}f(x)\big)\overline{\big(f(y) - e^{-t\sqrt{-\Delta}}f(x)\big)}\,p_t(x-y)\,dy\\
&= \int_{\mathbb{R}^n}|f(y)|^2 p_t(x-y)\,dy - e^{-t\sqrt{-\Delta}}f(x)\left(\int_{\mathbb{R}^n}\overline{f(y)}\,p_t(x-y)\,dy\right)\\
&\quad - \overline{e^{-t\sqrt{-\Delta}}f(x)}\left(\int_{\mathbb{R}^n}f(y)p_t(x-y)\,dy\right) + \big|e^{-t\sqrt{-\Delta}}f(x)\big|^2\\
&= e^{-t\sqrt{-\Delta}}|f|^2(x) - \big|e^{-t\sqrt{-\Delta}}f(x)\big|^2,
\end{aligned}$$

we have that if $f \in \mathcal{M}_{\sqrt{-\Delta},2}$ then $f \in \mathrm{L}^{2,\lambda}(\mathbb{R}^n)$ when and only when

$$\sup_{(x,t)\in\mathbb{R}^{n+1}_+} t^{n-\lambda}\Big(e^{-t\sqrt{-\Delta}}|f|^2(x) - \big|e^{-t\sqrt{-\Delta}}f(x)\big|^2\Big) < \infty$$

which is equivalent to (see also [15] for the BMO-setting, i.e., $\lambda = n$)

$$\sup_{(x,t)\in\mathbb{R}^{n+1}_+} t^{n-\lambda}\int_{\mathbb{R}^{n+1}_+} G_{\mathbb{R}^{n+1}_+}\big((x,t),(y,s)\big)\big|\nabla_{y,s}e^{-s\sqrt{-\Delta}}f(y)\big|^2\,dy\,ds < \infty,$$

where $G_{\mathbb{R}^{n+1}_+}((x,t),(y,s))$ is the Green function of $\mathbb{R}^{n+1}_+$ and $\nabla_{y,s}$ is the gradient operator in the space-time variable (y,s).

To find out an $\mathrm{L}^{p,\lambda}_L(\mathbb{R}^n)$ analog of Proposition 2, we take Proposition 2.6 of [7] into account, and establish the following property of the class of operators P_t.

Lemma 3. *Let $1 \le p < \infty$ and $\lambda \in (0,n)$. Suppose $f \in \mathrm{L}^{p,\lambda}_L(\mathbb{R}^n)$. Then:*
(i) *For any $t > 0$ and $K > 1$, there exists a constant $c > 0$ independent of t and K such that*

$$|P_t f(x) - P_{Kt}f(x)| \le ct^{\frac{\lambda-n}{pm}}\|f\|_{\mathrm{L}^{p,\lambda}_L} \tag{3.3}$$

for almost all $x \in \mathbb{R}^n$.
(ii) *For any $\delta > 0$, there exists $c(\delta) > 0$ such that*

$$\int_{\mathbb{R}^n}\frac{t^{\delta/m}}{\big(t^{1/m} + |x-y\big)^{n+\delta}}|(\mathcal{I} - P_t)f(y)|\,dy \le c(\delta)t^{\frac{\lambda-n}{pm}}\|f\|_{\mathrm{L}^{p,\lambda}_L} \tag{3.4}$$

for any $x \in \mathbb{R}^n$.

Proof.

(i) For any $t > 0$, we choose s such that $t/4 \le s \le t$. Assume that $f \in \mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$, where $1 \le p < \infty$ and $\lambda \in (0, n)$, we estimate the term $|P_t f(x) - P_{t+s} f(x)|$. Using the commutative property of the semigroup $\{P_t\}_{t>0}$, we can write

$$P_t f(x) - P_{t+s} f(x) = P_t(f - P_s f)(x) .$$

Since $f \in \mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$, one has

$$\begin{aligned}|P_t f(x) - P_{t+s} f(x)| &\le \int_{\mathbb{R}^n} |p_t(x, y)||f(y) - P_s f(y)|\, dy \\ &\le \frac{c}{\big|B\big(x, t^{1/m}\big)\big|} \int_{\mathbb{R}^n} \Big(1 + \frac{|x-y|}{t^{1/m}}\Big)^{-(n+\epsilon)} |f(y) - P_s f(y)|\, dy \\ &\le c\bigg(\frac{1}{\big|B\big(x, s^{1/m}\big)\big|} \int_{B(x,s^{1/m})} |f(y) - P_s f(y)|^p\, dy\bigg)^{1/p} \\ &\quad + \frac{c}{\big|B\big(x, s^{1/m}\big)\big|} \int_{B(x,s^{1/m})^c} \Big(1+\frac{|x-y|}{s^{1/m}}\Big)^{-(n+\epsilon)} |f(y)-P_s f(y)|\, dy \\ &\le cs^{\frac{\lambda-n}{pm}} \|f\|_{\mathrm{L}_L^{p,\lambda}} + \mathrm{I} .\end{aligned}$$

We then decompose $\mathbb{R}^n$ into a geometrically increasing sequence of concentric balls, and obtain

$$\begin{aligned}\mathrm{I} &= c\sum_{k=0}^{\infty} \frac{1}{\big|B\big(x, s^{1/m}\big)\big|} \int_{B(x,2^{k+1}s^{1/m})\setminus B(x,2^k s^{1/m})} \Big(1+\frac{|x-y|}{s^{1/m}}\Big)^{-(n+\epsilon)} |f(y)-P_s f(y)|\, dy \\ &\le c\sum_{k=0}^{\infty} 2^{-k(n+\epsilon)} \frac{1}{\big|B\big(x, s^{1/m}\big)\big|} \int_{B(x,2^{k+1}s^{1/m})} |f(y) - P_s f(y)|\, dy ,\end{aligned}$$

since

$$\big(1 + s^{-1/m}|x-y|\big)^{-n-\epsilon} \le c2^{-k(n+\epsilon)} \quad \forall\ y \in B\big(x, 2^{k+1}s^{1/m}\big)\setminus B\big(x, 2^k s^{1/m}\big) .$$

For a fixed positive integer k, we consider the ball $B(x, 2^k s^{1/m})$. This ball is contained in the cube $Q[x, 2^{k+1}s^{1/m}]$ centered at x and of the side length $2^{k+1}s^{1/m}$. We then divide this cube $Q[x, 2^{k+1}s^{1/m}]$ into $[2^{k+1}([\sqrt{n}\,] + 1)]^n$ small cubes $\{Q_{x_{k_i}}\}_{i=1}^{N_k}$ centered at x_{k_i} and of equal side length $([\sqrt{n}\,] + 1)^{-1}s^{1/m}$, where $N_k = [2^{k+1}([\sqrt{n}\,] + 1)]^n$. For any $i = 1, 2, \cdots, N_k$, each of these small cubes $Q_{x_{k_i}}$ is then contained in the corresponding ball B_{k_i} with the same center x_{k_i} and radius $r = s^{1/m}$, Consequently, for any ball $B(x, 2^k t)$, $k = 1, 2, \cdots$, there exists a corresponding collection of balls $B_{k_1}, B_{k_2}, \cdots, B_{k_{N_k}}$ such that

(i) each ball B_{k_i} is of the radius t;

(ii) $B(x, 2^k s^{1/m}) \subset \bigcup_{i=1}^{N_k} B_{k_i}$;

(iii) there exists a constant $c > 0$ independent of k such that $N_k \le c2^{kn}$;

(iv) each point of $B(x, 2^k s^{1/m})$ is contained in at most a finite number M of the balls B_{k_i}, where M is independent of k.

Applying the properties (i), (ii), (iii), and (iv) above, we obtain

$$
\begin{aligned}
\mathrm{I} &\le c\sum_{k=0}^{\infty} 2^{-k(n+\epsilon)} \frac{1}{\big|B\big(x, s^{1/m}\big)\big|} \int_{\bigcup_{i=1}^{N_{k+1}} B_{k_i}} |f(y) - P_t f(y)|\, dy \\
&\le c\sum_{k=0}^{\infty} 2^{-k(n+\epsilon)} \sum_{i=1}^{N_{k+1}} \frac{1}{|B_{k_i}|} \int_{B_{k_i}} |f(y) - P_s f(y)|\, dy \\
&\le c\sum_{k=0}^{\infty} 2^{-k(n+\epsilon)} N_{k+1} \sup_{i:1\le i\le N_{k+1}} \left(\frac{1}{|B_{k_i}|} \int_{B_{k_i}} |f(y) - P_s f(y)|^p\, dy \right)^{1/p} \\
&\le c\sum_{k=0}^{\infty} 2^{-k(n+\epsilon)} 2^{kn} s^{\frac{\lambda-n}{pm}} \|f\|_{\mathrm{L}_L^{p,\lambda}} \\
&\le c s^{\frac{\lambda-n}{pm}} \|f\|_{\mathrm{L}_L^{p,\lambda}}\ ,
\end{aligned}
$$

which gives (3.3) for the case $t/4 \le s \le t$.

For the case $0 < s < t/4$, we write

$$
P_t f(x) - P_{t+s} f(x) = (P_t f(x) - P_{2t} f(x)) - (P_{t+s}(f - P_{t-s} f)(x)\ .
$$

Noting that $(t+s)/4 \le (t-s) < t+s$, we obtain (3.3) by using the same argument as above. In general, for any $K > 1$, let l be the integer satisfying $2^l \le K < 2^{l+1}$, hence $l \le \log_2 K$. This, together with the fact that $\lambda \in (0, n)$, imply that there exists a constant $c > 0$ independent of t and K such that

$$
\begin{aligned}
|P_t f(x) - P_{Kt} f(x)| &\le \sum_{k=0}^{l-1} |P_{2^k t} f(x) - P_{2^{k+1} t} f(x)| + |P_{2^l t} f(x) - P_{Kt} f(x)| \\
&\le c\sum_{k=0}^{l-1} (2^k t)^{\frac{\lambda-n}{pm}} \|f\|_{\mathrm{L}_L^{p,\lambda}} + c(Kt)^{\frac{\lambda-n}{pm}} \|f\|_{\mathrm{L}_L^{p,\lambda}} \\
&\le c t^{\frac{\lambda-n}{pm}} \|f\|_{\mathrm{L}_L^{p,\lambda}}
\end{aligned}
$$

for almost all $x \in \mathbb{R}^n$.

(ii) Choosing a ball B centered at x and of the radius $r_B = t^{1/m}$, and using (3.3), we have

$$
\begin{aligned}
&\left(\frac{1}{|2^k B|} \int_{2^k B} |f(y) - P_t f(y)|^p\, dy \right)^{1/p} \\
&\le \left(\frac{1}{|2^k B|} \int_{2^k B} |f(y) - P_{t_{2^k B}} f(y)|^p\, dy \right)^{1/p} + \sup_{y\in 2^k B} |P_{t_{2^k B}} f(y) - P_t f(y)| \\
&\le c t^{\frac{\lambda-n}{pm}} \|f\|_{\mathrm{L}_L^{p,\lambda}}
\end{aligned} \tag{3.5}
$$

for all k. Putting $2^{-1}B = \emptyset$, we read off

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \frac{t^{\delta/m}}{\big(t^{1/m}+|x-y\big)^{n+\delta}} |(\mathcal{I}-P_t)f(y)|\,dy \\
&\le \sum_{k=0}^{\infty} \int_{2^kB\backslash 2^{k-1}B} \frac{t^{\delta/m}}{\big(t^{1/m}+|x-y\big)^{n+\delta}} |(\mathcal{I}-P_t)f(y)|\,dy \\
&\le c\sum_{k=0}^{\infty} 2^{kn}2^{-k(n+\delta)} \frac{1}{\big|2^kB\big|}\int_{2^kB} |f(y)-P_tf(y)|\,dy \\
&\le c\sum_{k=0}^{\infty} 2^{-k\delta}\bigg(\frac{1}{\big|2^kB\big|}\int_{2^kB} |f(y)-P_tf(y)|^p\,dy\bigg)^{1/p} \\
&\le c\sum_{k=0}^{\infty} 2^{-k\delta} t^{\frac{\lambda-n}{pm}} \|f\|_{\mathrm{L}_L^{p,\lambda}} \\
&\le ct^{\frac{\lambda-n}{pm}} \|f\|_{\mathrm{L}_L^{p,\lambda}}\,. \qquad \square
\end{aligned}
$$

The above analysis suggests us to introduce the maximal Morrey space as follows.

Definition 2. Let $1 \le p < \infty$ and $\lambda \in (0, n)$. We say that $f \in \mathcal{M}_p$ is in $\mathrm{L}^{p,\lambda}_{L,\max}(\mathbb{R}^n)$ associated with an operator L, if there exists some constant c (depending on f) such that

$$
\big|P_t\big(|f-P_tf|^p\big)(x)\big|^{1/p} \le ct^{\frac{\lambda-n}{pm}} \quad \text{for almost all } x\in\mathbb{R}^n \text{ and } t>0\,. \tag{3.6}
$$

The smallest bound c for which (3.6) holds then taken to be the norm of f in this space, and is denoted by $\|f\|_{\mathrm{L}^{p,\lambda}_{L,\max}}$.

Using Lemma 3, we can derive a characterization in terms of the maximal Morrey space under an extra hypothesis.

Proposition 3. *Let $1 \le p < \infty$ and $\lambda \in (0, n)$. Given an operator L which generates a semigroup e^{-tL} with the heat kernel bounds (2.2) and (2.3). Then $\mathrm{L}^{p,\lambda}_L(\mathbb{R}^n) \subseteq \mathrm{L}^{p,\lambda}_{L,\max}(\mathbb{R}^n)$. Furthermore, if the kernels $p_t(x, y)$ of operators P_t are nonnegative functions when $t > 0$, and satisfy the following lower bounds*

$$
p_t(x,y) \ge \frac{c}{t^{n/m}} \tag{3.7}
$$

for some positive constant c independent of t, x and y, then, $\mathrm{L}^{p,\lambda}_{L,\max}(\mathbb{R}^n) = \mathrm{L}^{p,\lambda}_L(\mathbb{R}^n)$.

Proof. Let us first prove $\mathrm{L}^{p,\lambda}_L(\mathbb{R}^n) \subseteq \mathrm{L}^{p,\lambda}_{L,\max}(\mathbb{R}^n)$. For any fixed $t > 0$ and $x \in \mathbb{R}^n$, we choose a ball B centered at x and of the radius $r_B = t^{1/m}$. Let $f \in \mathrm{L}^{p,\lambda}_L(\mathbb{R}^n)$. To estimate (3.6), we use the decay of function g in (2.3) to get

$$
\begin{aligned}
\big|P_t\big(|f-P_tf|^p\big)(x)\big| &\le \int_{\mathbb{R}^n} |p_t(x,y)||f(y)-P_tf(y)|^p\,dy \\
&\le c\sum_{k=0}^{\infty}\frac{1}{|B|}\int_{2^kB\backslash 2^{k-1}B} g\bigg(\frac{|x-y|}{t^{1/m}}\bigg)|f(y)-P_tf(y)|^p\,dy
\end{aligned}
$$

$$
\begin{aligned}
&\le c\sum_{k=0}^{\infty}2^{kn}g\big(2^{(k-1)}\big)\frac{1}{|2^kB|}\int_{2^kB}|f(y)-P_tf(y)|^p\,dy\\
&\le c\sum_{k=0}^{\infty}2^{kn}g\big(2^{(k-1)}\big)t^{\frac{\lambda-n}{m}}\|f\|^p_{\mathrm{L}_L^{p,\lambda}}\\
&\le ct^{\frac{\lambda-n}{m}}\|f\|^p_{\mathrm{L}_L^{p,\lambda}}\,.
\end{aligned}
$$

This proves $\|f\|_{\mathrm{L}^{p,\lambda}_{L,\max}}\le c\|f\|_{\mathrm{L}_L^{p,\lambda}}$.

We now prove $\mathrm{L}^{p,\lambda}_{L,\max}(\mathbb{R}^n)\subseteq \mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$ under (3.7). For a fixed ball $B=B(x,r_B)$ centered at x, we let $t_B=r_B^m$. For any $f\in \mathrm{L}^{p,\lambda}_{L,\max}(\mathbb{R}^n)$, it follows from (3.7) that one has

$$
\begin{aligned}
\frac{1}{|B|}\int_B|f(y)-P_{t_B}f(y)|^p\,dy &\le c\int_{B(x,t_B^{1/m})}p_{t_B}(x,y)|f(y)-P_{t_B}f(y)|^p\,dy\\
&\le c\int_{\mathbb{R}^n}p_{t_B}(x,y)|f(y)-P_{t_B}f(y)|^p\,dy\\
&\le ct_B^{\frac{\lambda-n}{m}}\|f\|^p_{\mathrm{L}^{p,\lambda}_{L,\max}}\,,
\end{aligned}
$$

which proves $\|f\|_{\mathrm{L}_L^{p,\lambda}}\le c\|f\|_{\mathrm{L}^{p,\lambda}_{L,\max}}$. Hence, the proof of Proposition 3 is complete. □

4. An Identity for the Dual Pairing

4.1 A Dual Inequality and a Reproducing Formula

From now on, we need the following notation. Suppose B is an open ball centered at x_B with radius r_B and $f\in\mathcal{M}_p$. Given an L^q function g supported on a ball B, where $\frac{1}{q}+\frac{1}{p}=1$. For any $(x,t)\in\mathbb{R}^{n+1}_+$, let

$$
F(x,t)=Q_{t^m}(\mathcal{I}-P_{t^m})f(x)\quad\text{and}\quad G(x,t)=Q^*_{t^m}\big(\mathcal{I}-P^*_{r_B^m}\big)g(x)\,, \tag{4.1}
$$

where P_t^* and Q_t^* are the adjoint operators of P_t and Q_t, respectively.

Lemma 4. *Assume that L satisfies the Assumptions (a) and (b) of Section 2.2. Suppose f,g,F,G,p,q are as in* (4.1).

(i) *If f also satisfies*

$$
|||f|||_{\mathrm{L}_L^{p,\lambda}}=\sup_{B\subset\mathbb{R}^n}r_B^{-\frac{\lambda}{p}}\left\|\left\{\int_0^{r_B}|Q_{t^m}(\mathcal{I}-P_{t^m})f(x)|^2\frac{dt}{t}\right\}^{1/2}\right\|_{\mathrm{L}^p(B)}<\infty\,,
$$

where the supremum is taken over all open ball $B\subset\mathbb{R}^n$ with radius r_B, then there exists a constant $c>0$ independent of any open ball B with radius r_B such that

$$
\int_{\mathbb{R}^{n+1}_+}|F(x,t)G(x,t)|\frac{dx\,dt}{t}\le cr_B^{\lambda/p}|||f|||_{\mathrm{L}_L^{p,\lambda}}\|g\|_{\mathrm{L}^q}\,. \tag{4.2}
$$

(ii) *If*

$$h \in \mathrm{L}^q(\mathbb{R}^n), \quad b_m = \frac{36m}{5} \quad \text{and} \quad 1 = b_m \int_0^\infty t^{2m} e^{-2t^m}\big(1 - e^{-t^m}\big)\frac{dt}{t}\ ,$$

then

$$h(x) = b_m \int_0^\infty \big(Q^*_{t^m}\big)^2 \big(\mathcal{I} - P^*_{t^m}\big) h(x) \frac{dt}{t}\ ,$$

where the integral converges strongly in $\mathrm{L}^q(\mathbb{R}^n)$.

Proof.

(i) For any ball $B \subset \mathbb{R}^n$ with radius r_B, we still put

$$T(B) = \big\{(x,t) \in \mathbb{R}^{n+1}_+ : x \in B,\ 0 < t < r_B\big\}\ .$$

We then write

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} |F(x,t)G(x,t)| \frac{dx\,dt}{t} &= \int_{T(2B)} \big|F(x,t)G(x,t)\big| \frac{dx\,dt}{t} \\ &\quad + \sum_{k=1}^\infty \int_{T(2^{k+1}B)\setminus T(2^k B)} \big|F(x,t)G(x,t)\big| \frac{dx\,dt}{t} \\ &= \mathrm{A_1} + \sum_{k=2}^\infty \mathrm{A_k}\ .\end{aligned}$$

Recall that $q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$. Using the Hölder inequality, together with (2.7) (here $\psi(z) = ze^{-z}$), we obtain

$$\begin{aligned}\mathrm{A_1} &\le \left\| \left\{ \int_0^{r_{2B}} \big|Q_{t^m}(\mathcal{I} - P_{t^m}) f(x)\big|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^p(2B)} \\ &\quad \times \left\| \left\{ \int_0^{r_{2B}} \big|Q^*_{t^m}\big(\mathcal{I} - P^*_{r_B^m}\big) g(x)\big|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^q(2B)} \\ &\le \left\| \left\{ \int_0^{r_{2B}} \big|Q_{t^m}(\mathcal{I} - P_{t^m}) f(x)\big|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^p(2B)} \big\|\mathcal{G}_{L^*}\big(\big(\mathcal{I} - P^*_{r_B^m}\big)g\big)\big\|_{\mathrm{L}^q} \\ &\le c r_B^{\frac{\lambda}{p}} \|\!| f |\!\|_{\mathrm{L}^{p,\lambda}_L} \|g\|_{\mathrm{L}^q}\ .\end{aligned}$$

Let us estimate $\mathrm{A_k}$ for $k = 2, 3, \cdots$. Note that for $x \in 2^{k+1}B \setminus 2^k B$ and $y \in B$, we have that $|x - y| \ge 2^{k-1} r_B$. Using (2.4) and the commutative property of $\{P_t\}_{t>0}$, we get

$$\begin{aligned}\big|Q^*_{t^m}\big(\mathcal{I} - P^*_{r_B^m}\big) g(x)\big| &\le \big|Q^*_{t^m} g(x)\big| + c\Big(\frac{t}{t + r_B}\Big)^m \big|Q_{t^m + r_B^m} g(x)\big| \\ &\le c \int_B \frac{t^\epsilon |g(y)|}{(t + |x-y|)^{n+\epsilon}}\, dy \\ &\quad + c\Big(\frac{t}{r_B}\Big)^m \int_B \frac{r_B^\epsilon |g(y)|}{(r_B + |x-y|)^{n+\epsilon}}\, dy \\ &\le \frac{ct^{\epsilon_0}}{\big(2^k r_B\big)^{n+\epsilon_0}} \int_B |g(y)|\, dy \\ &\le \left(\frac{ct^{\epsilon_0}}{\big(2^k r_B\big)^{n+\epsilon_0}}\right) r_B^{\frac{n}{p}} \|g\|_{\mathrm{L}^q}\ ,\end{aligned}$$

where $\epsilon_0 = 2^{-1}\min(m, \epsilon)$ and $q = p/(p-1)$. Consequently,

$$\left\| \left\{ \int_0^{2^k r_B} \left| Q^*_{t^m}\big(\mathcal{I} - P^*_{r_B^m}\big) g(x)\chi_{T(2^{k+1}B)\setminus T(2^k B)} \right|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^q(2^k B)} \le c 2^{kn(\frac{1}{q}-1)} \|g\|_{\mathrm{L}^q} .$$

Therefore,

$$\begin{aligned} \mathrm{A}_k &\le \left\| \left\{ \int_0^{2^k r_B} \left| Q_{t^m}\big(\mathcal{I} - P_{t^m}\big) f(x) \right|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^p(2^k B)} \\ &\quad \times \left\| \left\{ \int_0^{2^k r_B} \left| Q^*_{t^m}\big(\mathcal{I} - P^*_{r_B^m}\big) g(x)\chi_{T(2^{k+1}B)\setminus T(2^k B)} \right|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^q(2^k B)} \\ &\le c\big(2^k r_B\big)^{\frac{\lambda}{p}} 2^{kn(\frac{1}{q}-1)} |||f|||_{\mathrm{L}_L^{p,\lambda}} \|g\|_{\mathrm{L}^q} \\ &\le c 2^{\frac{k(\lambda-n)}{p}} r_B^{\frac{\lambda}{p}} |||f|||_{\mathrm{L}_L^{p,\lambda}} \|g\|_{\mathrm{L}^q} . \end{aligned}$$

Since $\lambda \in (0, n)$, we have

$$\begin{aligned} \int_{\mathbb{R}^{n+1}_+} |F(x,t)G(x,t)| \frac{dx\,dt}{t} &\le c r_B^{\frac{\lambda}{p}} |||f|||_{\mathrm{L}_L^{p,\lambda}} \|g\|_{\mathrm{L}^q} + c \sum_{k=1}^{\infty} 2^{\frac{k(\lambda-n)}{2}} r_B^{\frac{\lambda}{p}} |||f|||_{\mathrm{L}_L^{p,\lambda}} \|g\|_{\mathrm{L}^q} \\ &\le c r_B^{\frac{\lambda}{p}} |||f|||_{\mathrm{L}_L^{p,\lambda}} \|g\|_{\mathrm{L}^q} , \end{aligned}$$

as desired.

(ii) From Lemma 2 we know that L has a bounded H_∞-calculus in L^q for all $q > 1$. This, together with elementary integration, shows that $\{g_{\alpha\beta}(L^*)\}$ is a uniformly bounded net in $\mathcal{L}(\mathrm{L}^q, \mathrm{L}^q)$, where

$$g_{\alpha\beta}(L^*) = b_m \int_\alpha^\beta \big(Q^*_{t^m}\big)^2 \big(\mathcal{I} - P^*_{t^m}\big) \frac{dt}{t}$$

for all $0 < \alpha < \beta < \infty$.

As a consequence of Lemma 1, we have that for any $h \in \mathrm{L}^q(\mathbb{R}^n)$,

$$h(x) = b_m \int_0^\infty \big(Q^*_{t^m}\big)^2 \big(\mathcal{I} - P^*_{t^m}\big) h(x) \frac{dt}{t}$$

where $b_m = \frac{36m}{5}$ and the integral is strongly convergent in $\mathrm{L}^q(\mathbb{R}^n)$. □

4.2 The Desired Dual Identity

Next, we establish the following dual identity associated with the operator L.

Proposition 4. *Assume that L satisfies the Assumptions (a) and (b) of Section 2.2. Suppose B, f, g, F, G, p, q are defined as in* (4.1). *If $|||f|||_{\mathrm{L}_L^{p,\lambda}} < \infty$ and $b_m = \frac{36m}{5}$, then*

$$\int_{\mathbb{R}^n} f(x)\big(\mathcal{I} - P^*_{r_B^m}\big) g(x)\, dx = b_m \int_{\mathbb{R}^{n+1}_+} F(x,t)G(x,t) \frac{dx\,dt}{t} . \tag{4.3}$$

Proof. From Lemma 4 (i) it turns out that

$$\int_{\mathbb{R}^{n+1}_+} \big|F(x,t)G(x,t)\big| \frac{dx\,dt}{t} < \infty .$$

By the dominated convergence theorem, the following integral converges absolutely and satisfies

$$\int_{\mathbb{R}^{n+1}_+} F(x,t)G(x,t)\frac{dx\,dt}{t} = \lim_{\delta\to 0}\lim_{N\to\infty}\int_\delta^N \int_{\mathbb{R}^n} F(x,t)G(x,t)\frac{dx\,dt}{t}\,.$$

Next, by Fubini's theorem, together with the commutative property of the semigroup $\{e^{-tL}\}_{t>0}$, we have

$$\int_{\mathbb{R}^n} F(x,t)G(x,t)\,dx = \int_{\mathbb{R}^n} f(x)\big(Q^*_{t^m}\big)^2\big(\mathcal{I}-P^*_{t^m}\big)\big(\mathcal{I}-P^*_{r_B^m}\big)g(x)\,dx, \qquad \forall t>0\,.$$

This gives

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} F(x,t)G(x,t)\frac{dx\,dt}{t}\\
&= \lim_{\delta\to 0}\lim_{N\to\infty}\int_\delta^N \left[\int_{\mathbb{R}^n} f(x)\big(Q^*_{t^m}\big)^2\big(\mathcal{I}-P^*_{t^m}\big)\big(\mathcal{I}-P^*_{r_B^m}\big)g(x)\,dx\right]\frac{dt}{t}\\
&= \lim_{\delta\to 0}\lim_{N\to\infty}\int_{\mathbb{R}^n} f(x)\left[\int_\delta^N \big(Q^*_{t^m}\big)^2\big(\mathcal{I}-P^*_{t^m}\big)\big(\mathcal{I}-P^*_{r_B^m}\big)g(x)\frac{dt}{t}\right]dx\\
&= \lim_{\delta\to 0}\lim_{N\to\infty}\int_{\mathbb{R}^n} f_1(x)\left[\int_\delta^N \big(Q^*_{t^m}\big)^2\big(\mathcal{I}-P^*_{t^m}\big)\big(\mathcal{I}-P^*_{r_B^m}\big)g(x)\frac{dt}{t}\right]dx\\
&\quad + \lim_{\delta\to 0}\lim_{N\to\infty}\int_{\mathbb{R}^n} f_2(x)\left[\int_\delta^N \big(Q^*_{t^m}\big)^2\big(\mathcal{I}-P^*_{t^m}\big)\big(\mathcal{I}-P^*_{r_B^m}\big)g(x)\frac{dt}{t}\right]dx\\
&= \mathrm{I}+\mathrm{II}\,, \qquad (4.4)
\end{aligned}$$

where $f_1 = f\chi_{4B}$, $f_2 = f\chi_{(4B)^c}$ and χ_E stands for the characteristic function of $E\subseteq\mathbb{R}^n$.

We first consider the term I. Since $g\in \mathrm{L}^q(B)$, where $q=p/(p-1)$, we conclude $(\mathcal{I}-P^*_{r_B^m})g\in\mathrm{L}^q$. By Lemma 4 (ii), we obtain

$$\big(\mathcal{I}-P^*_{r_B^m}\big)g = \lim_{\delta\to 0}\lim_{N\to\infty} b_m\int_\delta^N \big(Q^*_{t^m}\big)^2\big(\mathcal{I}-P^*_{t^m}\big)\big(\mathcal{I}-P^*_{r_B^m}\big)(g)\frac{dt}{t}$$

in L^q. Hence,

$$\begin{aligned}
\mathrm{I} &= \lim_{\delta\to 0}\lim_{N\to\infty}\int_{\mathbb{R}^n} f_1(x)\left[\int_\delta^N \big(Q^*_{t^m}\big)^2\big(\mathcal{I}-P^*_{t^m}\big)\big(\mathcal{I}-P^*_{r_B^m}\big)(g)(x)\frac{dt}{t}\right]dx\\
&= b_m^{-1}\int_{\mathbb{R}^n} f_1(x)\big(\mathcal{I}-P^*_{r_B^m}\big)g(x)\,dx\,.
\end{aligned}$$

In order to estimate the term II, we need to show that for all $y\notin 4B$, there exists a constant $c=c(g,L)$ such that

$$\sup_{\delta>0,\,N>0}\left|\int_\delta^N \big(Q^*_{t^m}\big)^2\big(\mathcal{I}-P^*_{t^m}\big)\big(\mathcal{I}-P^*_{r_B^m}\big)g(x)\frac{dt}{t}\right| \le c(1+|x-x_0|)^{-(n+\epsilon)}\,. \qquad (4.5)$$

To this end, set

$$\Psi_{t,s}\big(L^*\big)h(y) = \big(2t^m+s^m\big)^3\left(\frac{d^3P^*_r}{dr^3}\bigg|_{r=2t^m+s^m}\big(\mathcal{I}-P^*_{t^m}\big)h\right)(y)\,.$$

Note that

$$\big(\mathcal{I} - P^*_{r_B^m}\big)g = m\int_0^{r_B} Q^*_{s^m}(g)s^{-1}\,ds\ .$$

So, we use (2.3) and (2.4) to deduce

$$\begin{aligned}
&\left|\int_\delta^N \big(Q^*_{t^m}\big)^2\big(\mathcal{I} - P^*_{t^m}\big)\big(\mathcal{I} - P^*_{r_B^m}\big)g(x)\frac{dt}{t}\right| \\
&= \left|\int_\delta^N\int_0^{r_B} \big(Q^*_{t^m}\big)^2 Q^*_{s^m}\big(\mathcal{I} - P^*_{t^m}\big)g(x)\frac{ds\,dt}{st}\right| \\
&\le c\int_\delta^N\int_0^{r_B}\left(\frac{t^{2m}s^m}{\big(t^m+s^m\big)^3}\right)|\Psi_{t,s}(L)g(x)|\frac{ds\,dt}{st} \\
&\le c\int_\delta^N\int_0^{r_B}\left[\int_{B(x_0,r_B)}\left(\frac{t^{2m}s^m}{\big(t^m+s^m\big)^3}\right)\left(\frac{(t+s)^\epsilon}{(t+s+|x-y|)^{n+\epsilon}}\right)|g(y)|\,dy\right]\frac{ds\,dt}{st}\ .
\end{aligned}$$

Because $x \notin 4B$ yields $|x-y| \ge |x-x_0|/2$, the inequality

$$\frac{t^{2m}s^m(t+s)^\epsilon}{\big(t^m+s^m\big)^3} \le c\min\Big\{(ts)^{\epsilon/2}, t^{-\epsilon/2}s^{3\epsilon/2}\Big\}\ ,$$

together with Hölder's inequality and elementary integration, produces a positive constant c independent of $\delta, N > 0$ such that for all $x \notin 4B$,

$$\begin{aligned}
\left|\int_\delta^N Q^2_{t^m}(\mathcal{I} - P_{t^m})g(y)\frac{dt}{t}\right| &\le cr_B^\epsilon|x-x_0|^{-(n+\epsilon)}\|g\|_{\mathrm{L}^1} \\
&\le cr_B^{\epsilon+\frac{n}{2}}\|g\|_{\mathrm{L}^2}|x-x_0|^{-(n+\epsilon)}\ .
\end{aligned}$$

Accordingly, (4.5) follows readily.

We now estimate the term II. For $f \in \mathcal{M}_p$, we derive $f \in \mathrm{L}^p\big((1+|x|)^{-(n+\epsilon_0)}\,dx\big)$. The estimate (4.5) yields a constant $c > 0$ such that

$$\sup_{\delta>0,\,N>0}\int_{\mathbb{R}^n}\left|f_2(x)\int_\delta^N \big(Q^*_{t^m}\big)^2\big(\mathcal{I} - P^*_{t^m}\big)\big(\mathcal{I} - P^*_{r_B^m}\big)(g)(x)\frac{dt}{t}\right|dx \le c\ .$$

This allows us to pass the limit inside the integral of II. Hence,

$$\begin{aligned}
\mathrm{II} &= \lim_{\delta\to 0}\lim_{N\to\infty}\int_{\mathbb{R}^n} f_2(x)\left[\int_\delta^N \big(Q^*_{t^m}\big)^2\big(\mathcal{I} - P^*_{t^m}\big)\big(\mathcal{I} - P^*_{r_B^m}\big)(g)(x)\frac{dt}{t}\right]dx \\
&= \int_{\mathbb{R}^n} f_2(x)\left(\lim_{\delta\to 0}\lim_{N\to\infty}\left[\int_\delta^N \big(Q^*_{t^m}\big)^2\big(\mathcal{I} - P^*_{t^m}\big)\big(\mathcal{I} - P^*_{r_B^m}\big)(g)(x)\frac{dt}{t}\right]\right)dx \\
&= b_m^{-1}\int_{\mathbb{R}^n} f_2(x)\big(\mathcal{I} - P^*_{r_B^m}\big)g(x)\,dx\ .
\end{aligned}$$

Combining the previous formulas for I and II, we obtain the identity (4.3). □

Remark 3. For a background of Proposition 4, see also [8, Proposition 5.1].

5. Description Through Littlewood-Paley Function

5.1 The Space $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ as the Dual of the Atomic Space

Following [28], we give the following definition.

Definition 3. Let $1 < p < \infty$, $q = p/(p-1)$ and $\lambda \in (0, n)$. Then
(i) A complex-valued function a on $\mathbb{R}^n$ is called a (q, λ)-atom provided:
(α) a is supported on an open ball $B \subset \mathbb{R}^n$ with radius r_B;
(β) $\int_{\mathbb{R}^n} a(x)\,dx = 0$;
(γ) $\|a\|_{\mathrm{L}^q} \le r_B^{-\lambda/p}$.
(ii) $\mathrm{H}^{q,\lambda}(\mathbb{R}^n)$ comprises those linear functionals admitting an atomic decomposition $f = \sum_{j=1}^{\infty} \eta_j a_j$, where a_j's are (q, λ)-atoms, and $\sum_j |\eta_j| < \infty$.

The forthcoming result reveals that $\mathrm{H}^{q,\lambda}(\mathbb{R}^n)$ exists as a predual of $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$.

Proposition 5. *Let* $1 < p < \infty$*,* $q = p/(p-1)$ *and* $\lambda \in (0, n)$*. Then* $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ *is the dual* $(\mathrm{H}^{q,\lambda}(\mathbb{R}^n))^*$ *of* $\mathrm{H}^{q,\lambda}(\mathbb{R}^n)$*. More precisely, if* $h = \sum_j \eta_j a_j \in \mathrm{H}^{q,\lambda}(\mathbb{R}^n)$ *then*

$$\langle h, \ell \rangle = \lim_{k\to\infty} \sum_{j=1}^{k} \eta_j \int_{\mathbb{R}^n} a_j(x)\ell(x)\,dx$$

is a well-defined continuous linear functional for each $\ell \in \mathrm{L}^{p,\lambda}(\mathbb{R}^n)$*, whose norm is equivalent to* $\|\ell\|_{\mathrm{L}^{p,\lambda}}$*; moreover, each continuous linear functional on* $\mathrm{H}^{q,\lambda}(\mathbb{R}^n)$ *has this form.*

Proof. See [28, Proposition 5] for a proof of Proposition 5. □

5.2 Characterization of $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ by Means of Littlewood-Paley Function

We now state a full characterization of $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ space for $1 < p < \infty$ and $\lambda \in (0, n)$. For the case $p = 2$, see also [26, Lemma 2.1] as well as [25, Theorem 1 (i)].

Proposition 6. *Let* $1 < p < \infty$*,* $\lambda \in (0, n)$ *and* $f \in \mathcal{M}_{\sqrt{-\Delta},p}$*. Then the following two conditions are equivalent:*
(i) $f \in \mathrm{L}^{p,\lambda}(\mathbb{R}^n)$*;*
(ii)

$$I(f,p) = \sup_{B\subset\mathbb{R}^n} r_B^{-\frac{\lambda}{p}} \left\| \left\{ \int_0^{r_B} \left| t\frac{\partial}{\partial t} e^{-t\sqrt{-\Delta}} f(x)\right|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^p(B)} < \infty\,,$$

where the supremum is taken over all Euclidean open ball $B \subset \mathbb{R}^n$ *with radius* r_B*.*

Proof. It suffices to verify (ii)⇒(i) for which the reverse implication follows readily from [11, Theorem 2.1]. Suppose (ii) holds. Proposition 5 suggests us to show $f \in (\mathrm{H}^{\frac{p}{p-1},\lambda}(\mathbb{R}^n))^*$ in order to verify (i). Now, let g be a $(\frac{p}{p-1}, \lambda)$-atom and

$$p_t(x) = \frac{c_n t}{\left(t^2 + |x|^2\right)^{\frac{n+1}{2}}}\,.$$

Then for any open ball $B \subset \mathbb{R}^n$ with radius r_B and its tent

$$T(B) = \left\{(x,t) \in \mathbb{R}^{n+1}_+ : \quad x \in B,\ t \in (0, r_B)\right\},$$

we have (cf. [23, p. 183])

$$\begin{aligned}
|\langle f, g\rangle| &= \left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| \\
&= 4\left|\int_{\mathbb{R}^n}\int_0^\infty \Big(t\frac{\partial}{\partial t}p_t * f(x)\Big)\Big(t\frac{\partial}{\partial t}p_t * g(x)\Big)\frac{dt\,dx}{t}\right| \\
&\le 4\big(I(B)+J(B)\big).
\end{aligned}$$

Here,

$$\begin{aligned}
I(B) &= \int_{4B}\int_0^{r_{4B}} \Big|t\frac{\partial}{\partial t}p_t * f(x)\Big|\Big|t\frac{\partial}{\partial t}p_t * g(x)\Big|\frac{dt\,dx}{t} \\
&\le \left(\int_{4B}\left(\int_0^{r_{4B}}\Big|t\frac{\partial}{\partial t}p_t * f(x)\Big|^2\frac{dt}{t}\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_{4B}\left(\int_0^{r_{4B}}\Big|t\frac{\partial}{\partial t}p_t * g(x)\Big|^2\frac{dt}{t}\right)^{\frac{p}{2(p-1)}}dx\right)^{\frac{p-1}{p}} \\
&\le c r_B^{\frac{\lambda}{p}} I(f,p)\|g\|_{\mathrm{L}^{\frac{p}{p-1}}(\mathbb{R}^n)} \\
&\le cI(f,p),
\end{aligned}$$

due to Hölder's inequality, the $\mathrm{L}^{\frac{p}{p-1}}$-boundedness of the Littlewood-Paley $\mathcal{G}$-function, and g being a $(\frac{p}{p-1}, \lambda)$-atom.

Meanwhile,

$$\begin{aligned}
J(B) &= \sum_{k=1}^{\infty}\int_{T(2^{k+1}B)\setminus T(2^kB)} \Big|t\frac{\partial}{\partial t}p_t * f(x)\Big|\Big|t\frac{\partial}{\partial t}p_t * g(x)\Big|\frac{dt\,dx}{t} \\
&\le c\sum_{k=1}^{\infty}\left\|\left\{\int_0^{2^{k+1}r_B}\Big|t\frac{\partial}{\partial t}p_t * f(x)\Big|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\right\|_{\mathrm{L}^p(2^{k+1}B)} \\
&\quad\times\left\|\left\{\int_0^{2^{k+1}r_B}\Big|t\frac{\partial}{\partial t}p_t * g(x)\Big|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\right\|_{\mathrm{L}^{\frac{p}{p-1}}(2^{k+1}B)} \\
&\le c\sum_{k=1}^{\infty}\big(2^k r_B\big)^{\frac{\lambda}{p}} I(f,p) 2^{-\frac{kn}{p}} r_B^{-\frac{\lambda}{p}} \\
&\le cI(f,p),
\end{aligned}$$

for which we have used the Hölder inequality and the fact that if $|y-x| \ge 2^k r_B$ then

$$\Big|t\frac{\partial}{\partial t}p_t * g(x)\Big| \le \frac{ct^3\|g\|_{\mathrm{L}^1(B)}}{\big(2^k r_B\big)^{3+n}} \le \frac{ct^3 r_B^{\frac{n-\lambda}{p}}}{\big(2^k r_B\big)^{3+n}}$$

for the $(\frac{p}{p-1}, \lambda)$-atom g. Accordingly, $f \in \mathrm{L}^{p,\lambda}(\mathbb{R}^n)$. □

5.3 Characterization of $\mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$ by Means of Littlewood-Paley Function

Of course, it is natural to explore a characterization of $\mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$ similar to Proposition 6.

Proposition 7. *Let $1 < p < \infty$, $\lambda \in (0, n)$ and $f \in \mathcal{M}_p$. Assume that L satisfies the Assumptions (a) and (b) of Section 2.2. Then the following two conditions are equivalent:*
(i) $f \in \mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$;
(ii)

$$\||f\||_{\mathrm{L}_L^{p,\lambda}} = \sup_{B\subset\mathbb{R}^n} r_B^{-\frac{\lambda}{p}} \left\| \left\{ \int_0^{r_B} |Q_{t^m}(\mathcal{I} - P_{t^m}) f(x)|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^p(B)} < \infty ,$$

where the supremum is taken over all Euclidean open ball $B \subset \mathbb{R}^n$ with radius r_B.

Proof.
(i)⇒(ii). Suppose $f \in \mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$. Note that

$$Q_{t^m}(\mathcal{I} - P_{t^m}) = Q_{t^m}(\mathcal{I} - P_{t^m})(\mathcal{I} - P_{r_B^m}) + Q_{t^m}(\mathcal{I} - P_{t^m})P_{r_B^m} .$$

So, we turn to verify both

$$\left\| \left\{ \int_0^{r_B} \left|Q_{t^m}(\mathcal{I} - P_{t^m})\left(\mathcal{I} - P_{r_B^m}\right) f(x)\right|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^p(B)} \le c r_B^{\frac{\lambda}{p}} \|f\|_{\mathrm{L}_L^{p,\lambda}} \tag{5.1}$$

and

$$\left\| \left\{ \int_0^{r_B} \left|Q_{t^m}\left(\mathcal{I} - P_{t^m}\right) P_{r_B^m} f(x)\right|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^p(B)} \le c r_B^{\frac{\lambda}{p}} \|f\|_{\mathrm{L}_L^{p,\lambda}} , \tag{5.2}$$

thereby proving (ii). To do so, we will adapt the argument on pp. 85–86 of [14] to present situation—see also p. 955 of [8]. To prove (5.1), let us consider the square function $\mathcal{G}(h)$ given by

$$\mathcal{G}(h)(x) = \left(\int_0^{\infty} \left|Q_{t^m}(\mathcal{I} - P_{t^m})h(x)\right|^2 \frac{dt}{t} \right)^{1/2} .$$

From (2.7), the function $\mathcal{G}(h)$ is bounded on $\mathrm{L}^p(\mathbb{R}^n)$ for $1 < p < \infty$. Let $b = b_1 + b_2$, where $b_1 = (\mathcal{I} - P_{r_B^m}) f \chi_{2B}$, and $b_2 = (\mathcal{I} - P_{r_B^m}) f \chi_{(2B)^c}$. Using Lemma 3, we obtain

$$\begin{aligned}
&\left\| \left\{ \int_0^{r_B} |Q_{t^m}(\mathcal{I} - P_{t^m})b_1(x)|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^p(B)} \\
&\le \left\| \left\{ \int_0^{\infty} |Q_{t^m}(\mathcal{I} - P_{t^m})b_1(x)|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^p} \\
&\le c\|\mathcal{G}(b_1)\|_{\mathrm{L}^p} \\
&\le c\|b_1\|_{\mathrm{L}^p} \\
&= c\left(\int_{2B} \left|\left(\mathcal{I} - P_{r_B^m}\right) f(x)\right|^p dx \right)^{1/p} \\
&\le c\left(\int_{2B} \left|\left(\mathcal{I} - P_{r_{2B}^m}\right) f(x)\right|^p dx \right)^{1/p} + c r_B^{n/p} \sup_{x\in 2B} \left|P_{r_B^m} f(x) - P_{r_{2B}^m} f(x)\right|^p \\
&\le c r_B^{\frac{\lambda}{p}} \|f\|_{\mathrm{L}_L^{p,\lambda}} .
\end{aligned} \tag{5.3}$$

On the other hand, for any $x \in B$ and $y \in (2B)^c$, one has $|x-y| \geq r_B$. From Proposition 2, we obtain

$$\begin{aligned} |Q_{t^m}(\mathcal{I}-P_{t^m})b_2(x)| &\leq c\int_{\mathbb{R}^n\setminus 2B} \frac{t^{\epsilon}}{(t+|x-y|)^{n+\epsilon}}\left|\left(\mathcal{I}-P_{r_B^m}\right)f(y)\right| dy \\ &\leq c\Big(\frac{t}{r_B}\Big)^{\epsilon}\int_{\mathbb{R}^n} \frac{r_B^{\epsilon}}{(r_B+|x-y|)^{n+\epsilon}}\left|\left(\mathcal{I}-P_{r_B^m}\right)f(y)\right| dy \\ &\leq c\Big(\frac{t}{r_B}\Big)^{\epsilon} r_B^{\frac{\lambda-n}{p}}\,\|f\|_{\mathrm{L}_L^{p,\lambda}}\,, \end{aligned}$$

which implies

$$\left\|\left\{\int_0^{r_B}|Q_{t^m}(\mathcal{I}-P_{t^m})b_2(x)|^2\frac{dt}{t}\right\}^{1/2}\right\|_{\mathrm{L}^p(B)} \leq c r_B^{\frac{\lambda}{p}}\|f\|_{\mathrm{L}_L^{p,\lambda}}\,.$$

This, together with (5.3), gives (5.1).

Next, let us check (5.2). This time, we have $0 < t < r_B$, whence getting from Lemma 3 that for any $x \in \mathbb{R}^n$,

$$\left|P_{\frac{1}{2}r_B^m}f(x)-P_{(t^m+\frac{1}{2}r_B^m)}f(x)\right| \leq c r_B^{\frac{\lambda-n}{p}}\,\|f\|_{\mathrm{L}_L^{p,\lambda}}\,.$$

By (2.4), the kernel $K_{t,r_B}(x,y)$ of the operator

$$Q_{t^m}P_{\frac{1}{2}r_B^m} = \frac{t^m}{t^m+\frac{1}{2}r_B^m}Q_{(t^m+\frac{1}{2}r_B^m)}$$

satisfies

$$|K_{t,r_B}(x,y)| \leq c\Big(\frac{t}{r_B}\Big)^m\frac{r_B^{\epsilon}}{(r_B+|x-y|)^{n+\epsilon}}\,.$$

Using the commutative property of the semigroup $\{e^{-tL}\}_{t>0}$ and the estimate (2.4), we deduce

$$\begin{aligned} \left|Q_{t^m}\big(\mathcal{I}-P_{t^m}\big)P_{r_B^m}f(x)\right| &= \left|Q_{t^m}P_{\frac{1}{2}r_B^m}\big(P_{\frac{1}{2}r_B^m}-P_{(t^m+\frac{1}{2}r_B^m)}\big)f(x)\right| \\ &\leq c\Big(\frac{t}{r_B}\Big)^m\int_{\mathbb{R}^n}\frac{r_B^{\epsilon}}{(r_B+|x-y|)^{n+\epsilon}}\left|\big(P_{\frac{1}{2}r_B^m}-P_{(t^m+\frac{1}{2}r_B^m)}\big)f(y)\right| dy \\ &\leq c\Big(\frac{t}{r_B}\Big)^m r_B^{\frac{\lambda-n}{p}}\,\|f\|_{\mathrm{L}_L^{p,\lambda}}\,, \end{aligned}$$

whence deriving

$$\left\|\left\{\int_0^{r_B}\left|Q_{t^m}(\mathcal{I}-P_{t^m})P_{r_B^m}f(x)\right|^2\frac{dt}{t}\right\}^{1/2}\right\|_{\mathrm{L}^p(B)} \leq c r_B^{\frac{\lambda}{p}}\|f\|_{\mathrm{L}_L^{p,\lambda}}\,.$$

This gives (5.2) and consequently (ii).

(ii)⇒(i). Suppose (ii) holds. The duality argument for L^p shows that for any open ball $B \subset \mathbb{R}^n$ with radius r_B,

$$\begin{aligned} \left(r_B^{-\lambda}\int_B\left|f(x)-P_{r_B^m}f(x)\right|^p dx\right)^{1/p} &= \sup_{\|g\|_{\mathrm{L}^q(B)\leq 1}} r_B^{-\lambda/p}\left|\int_{\mathbb{R}^n}\big(\mathcal{I}-P_{r_B^m}\big)f(x)g(x)\,dx\right| \\ &= \sup_{\|g\|_{\mathrm{L}^q(B)\leq 1}} r_B^{-\lambda/p}\left|\int_{\mathbb{R}^n}f(x)\big(\mathcal{I}-P_{r_B^m}^*\big)g(x)\,dx\right|. \qquad (5.4) \end{aligned}$$

Using the identity (4.3), the estimate (4.2) and the Hölder inequality, we have

$$\left|\int_{\mathbb{R}^n} f(x)\big(\mathcal{I}-P^*_{r_B^m}\big)g(x)\,dx\right| \le c\int_{\mathbb{R}^{n+1}_+}\left|Q_{t^m}(\mathcal{I}-P_{t^m})f(x)\,Q^*_{t^m}\big(\mathcal{I}-P^*_{r_B^m}\big)g(x)\right|\frac{dx\,dt}{t}$$
$$\le c r_B^{\lambda/p}\|\!|f|\!\|_{\mathrm{L}^{p,\lambda}_L}\|g\|_{\mathrm{L}^q}\,. \tag{5.5}$$

Substituting (5.5) back to (5.4), by Definition 1 we find a constant $c>0$ such that

$$\|f\|_{\mathrm{L}^{p,\lambda}_L}\le c\|\!|f|\!\|_{\mathrm{L}^{p,\lambda}_L}<\infty\,.$$

This just proves $f\in\mathrm{L}^{p,\lambda}_L(\mathbb{R}^n)$, thereby yielding (i). □

Remark 4. In the case of $p=2$, we can interpret Proposition 7 as a measure-theoretic characterization, namely, $f\in\mathrm{L}^{2,\lambda}_L(\mathbb{R}^n)$ when and only when

$$d\mu_f(x,t)=|Q_{t^m}(\mathcal{I}-P_{t^m})f(x)|^2\frac{dx\,dt}{t}$$

is a λ-Carleson measure on $\mathbb{R}^{n+1}_+$. According to [10, Lemma 4.1], we find further that $f\in\mathrm{L}^{2,\lambda}_L(\mathbb{R}^n)$ is equivalent to

$$\sup_{(y,s)\in\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{n+1}_+}\left(\frac{s}{\big(|x-y|^2+(t+s)^2\big)^{\frac{n+1}{2}}}\right)^{\lambda}d\mu_f(x,t)<\infty\,.$$

5.4 A Sufficient Condition for $\mathrm{L}^{p,\lambda}_L(\mathbb{R}^n)=\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$

In what follows, we assume that L is a linear operator of type ω on $\mathrm{L}^2(\mathbb{R}^n)$ with $\omega<\pi/2$—hence L generates an analytic semigroup e^{-zL}, $0\le|\mathrm{Arg}(z)|<\pi/2-\omega$. We also assume that for each $t>0$, the kernel $p_t(x,y)$ of e^{-tL} is Hölder continuous in both variables x, y and there exist positive constants m, $\beta>0$ and $0<\gamma\le1$ such that for all $t>0$, and $x,y,h\in\mathbb{R}^n$,

$$|p_t(x,y)|\le\frac{ct^{\beta/m}}{\big(t^{1/m}+|x-y|\big)^{n+\beta}}\quad\forall\,t>0,\ x,y\in\mathbb{R}^n\,, \tag{5.6}$$

$$|p_t(x+h,y)-p_t(x,y)|+|p_t(x,y+h)-p_t(x,y)|$$
$$\le\frac{c|h|^{\gamma}t^{\beta/m}}{\big(t^{1/m}+|x-y|\big)^{n+\beta+\gamma}}\quad\forall\,h\in\mathbb{R}^n\quad\text{with}\quad2|h|\le t^{1/m}+|x-y|\,, \tag{5.7}$$

and

$$\int_{\mathbb{R}^n}p_t(x,y)\,dx=\int_{\mathbb{R}^n}p_t(x,y)\,dy=1\quad\forall\,t>0\,. \tag{5.8}$$

Proposition 8. *Let $1<p<\infty$ and $\lambda\in(0,n)$. Given an operator L which generates a semigroup e^{-tL} with the heat kernel bounds* (2.2) *and* (2.3)*. Assume that L satisfies the conditions* (5.6), (5.7)*, and* (5.8)*. Then $\mathrm{L}^{p,\lambda}_L(\mathbb{R}^n)$ and $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ coincide, and their norms are equivalent.*

Proof. Since Proposition 1 tells us that $\mathrm{L}^{p,\lambda}(\mathbb{R}^n) \subseteq \mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$ under the above-given conditions, we only need to check $\mathrm{L}_L^{p,\lambda}(\mathbb{R}^n) \subseteq \mathrm{L}^{p,\lambda}(\mathbb{R}^n)$. Note that $\mathrm{L}^{p,\lambda}(\mathbb{R}^n)$ is the dual of $\mathrm{H}^{q,\lambda}(\mathbb{R}^n)$, $q = p/(p-1)$. It reduces to prove that if $f \in \mathrm{L}_L^{p,\lambda}(\mathbb{R}^n)$, then $f \in (\mathrm{H}^{q,\lambda}(\mathbb{R}^n))^*$. Let g be a (q,λ)-atom. Using the conditions (5.6), (5.7), and (5.8) of the operator L, together with the properties of (q,λ)-atom of g, we can follow the argument for Lemma 4 (ii) to verify

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx = b_m \int_{\mathbb{R}_+^{n+1}} Q_{t^m}(\mathcal{I} - P_{t^m})f(x)Q_{t^m}^* g(x)\frac{dx\,dt}{t} \quad \text{where} \quad b_m = \frac{36m}{5}\ .$$

Consequently,

$$\begin{aligned}
|\langle f, g\rangle| &= \left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| \\
&= \left|\int_{\mathbb{R}_+^{n+1}} Q_{t^m}(\mathcal{I} - P_{t^m})f(x)Q_{t^m}^* g(x)\frac{dx\,dt}{t}\right| \\
&\leq \int_{T(2B)} \left|Q_{t^m}(\mathcal{I} - P_{t^m})f(x)Q_{t^m}^* g(x)\right|\frac{dx\,dt}{t} \\
&\quad + \sum_{k=1}^{\infty}\int_{T(2^{k+1}B)\setminus T(2^k B)} \left|Q_{t^m}(\mathcal{I} - P_{t^m})f(x)\ Q_{t^m}^* g(x)\right|\frac{dx\,dt}{t} \\
&= \mathrm{D_1} + \sum_{k=2}^{\infty}\mathrm{D_k}\ .
\end{aligned}$$

Define the Littlewood-Paley function $\mathcal{G}h$ by

$$\mathcal{G}(h)(x) = \left[\int_0^\infty \left|Q_{t^m}^* h(x)\right|^2 \frac{dt}{t}\right]^{1/2}.$$

By (2.7), $\mathcal{G}$ is bounded on $\mathrm{L}^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Following the proof of Lemma 4 (i), together with the property (γ) of (q,λ)-atom g, we derive

$$\begin{aligned}
\mathrm{D_1} &\leq \left\|\left\{\int_0^{r_{2B}} \left|Q_{t^m}(\mathcal{I} - P_{t^m})f(x)\right|^2\frac{dt}{t}\right\}^{1/2}\right\|_{\mathrm{L}^p(2B)} \left\|\left\{\int_0^{r_{2B}} \left|Q_{t^m}^* g(x)\right|^2\frac{dt}{t}\right\}^{1/2}\right\|_{\mathrm{L}^q(2B)} \\
&\leq \left\|\left\{\int_0^{r_{2B}} \left|Q_{t^m}(\mathcal{I} - P_{t^m})f(x)\right|^2\frac{dt}{t}\right\}^{1/2}\right\|_{\mathrm{L}^p(2B)} \|\mathcal{G}(g)\|_{\mathrm{L}^q} \\
&\leq c r_B^{\frac{\lambda}{p}} \|f\|_{\mathrm{L}_L^{p,\lambda}} \|g\|_{\mathrm{L}^q} \leq c\|f\|_{\mathrm{L}_L^{p,\lambda}}\ .
\end{aligned}$$

On the other hand, we note that for $x \in 2^{k+1}B\setminus 2^k B$ and $y \in B$, we have that $|x-y| \geq 2^{k-1}r_B$. Using the estimate (2.4) and the properties ((α) and (γ) of (q,λ)-atom g, we obtain

$$\begin{aligned}
\left|Q_{t^m}^* g(x)\right| &\leq c\int_B \frac{t^\epsilon}{(t+|x-y|)^{n+\epsilon}}|g(y)|\,dy \\
&\leq \frac{ct^\epsilon}{\left(2^k r_B\right)^{n+\epsilon}}\int_B |g(y)|\,dy \\
&\leq \left(\frac{ct^\epsilon}{\left(2^k r_B\right)^{n+\epsilon}}\right) r_B^{\frac{n-\lambda}{p}},
\end{aligned}$$

which implies

$$\left\| \left\{ \int_0^{2^k r_B} \left| Q_{t^m}^* g(x) \chi_{T(2^{k+1}B)\setminus T(2^k B)} \right|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^q(2^k B)} \le c 2^{kn(\frac{1}{q}-1)} r_B^{-\frac{\lambda}{p}} .$$

Therefore,

$$\begin{aligned} \mathrm{D}_k \le & \left\| \left\{ \int_0^{2^k r_B} |Q_{t^m}(\mathcal{I} - P_{t^m}) f(x)|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^p(2^k B)} \\ & \times \left\| \left\{ \int_0^{2^k r_B} \left| Q_{t^m}^* g(x) \chi_{T(2^{k+1}B)\setminus T(2^k B)} \right|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{\mathrm{L}^q(2^k B)} \\ \le & \; c \big(2^k r_B\big)^{\frac{\lambda}{p}} 2^{kn(\frac{1}{q}-1)} r_B^{-\frac{\lambda}{p}} \| f \|_{\mathrm{L}_L^{p,\lambda}} \\ \le & \; c 2^{\frac{k(\lambda-n)}{p}} \| f \|_{\mathrm{L}_L^{p,\lambda}} . \end{aligned}$$

Since $\lambda \in (0, n)$, we have

$$|\langle f, g \rangle| \le c \| f \|_{\mathrm{L}_L^{p,\lambda}} + c \sum_{k=1}^{\infty} 2^{\frac{k(\lambda-n)}{p}} \| f \|_{\mathrm{L}_L^{p,\lambda}} \le c \| f \|_{\mathrm{L}_L^{p,\lambda}} .$$

This, together with Proposition 5, implies $f \in (\mathrm{H}^{q,\lambda}(\mathbb{R}^n))^* = \mathrm{L}^{p,\lambda}(\mathbb{R}^n)$. □

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