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Prevalence of Multifractal Functions in S^ν Spaces

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ABSTRACT. Spaces called S^ν were introduced by Jaffard [16] as spaces of functions characterized by the number $\simeq 2^{\nu(\alpha)j}$ of their wavelet coefficients having a size $\gtrsim 2^{-\alpha j}$ at scale j. They are Polish vector spaces for a natural distance. In those spaces we show that multifractal functions are prevalent (an infinite-dimensional "almost-every"). Their spectrum of singularities can be computed from ν, which justifies a new multifractal formalism, not limited to concave spectra.

1. Introduction

1.1 Multifractal Formalisms

The goal of multifractal analysis is to compute the *spectrum of singularities* d_f of a locally bounded function f, being defined as

$$d_f(h) := \dim_H \left\{x, h_f(x) = h\right\}$$

where $\dim_H$ stands for the Hausdorff dimension and $h_f(x)$ for the pointwise Hölder exponent of f at x (see [8] for definitions). Variants of this definition have been introduced, in particular, replacing the Hölder exponent by some weaker notion, such as the T_u^p exponents of [5] as in [17], but this one is the simplest and most fundamental. However, it is not applicable for the practical computation of d_f given, say, a sampled version of f. Several formulas called *multifractal formalisms* have been proposed to estimate d_f, most of them based on the wavelet coefficients of f. They share the advantage of being easy to compute and relatively stable from a numerical point of view, and the disadvantage of being limited to uniform Hölder functions: This limitation is intrinsic to the theorems relating the pointwise Hölder exponent to the size of the wavelet coefficients, see [18].

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The most widespread of these formulas is the so-called *thermodynamic multifractal formalism* that we briefly recall. Let us suppose for simplicity that f is defined on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and let $\Lambda := \{(j,k),\ j \in \mathbb{N}, 0 \leq k < 2^j\}$. Let $c_{j,k}$, $(j,k) \in \Lambda$ be the wavelet coefficients of f in a L^∞-normalized, periodized wavelet basis $(\psi_{j,k}(x) := \sum_l \psi(2^j(x-l)-k))$. Assuming the mother wavelet ψ to be sufficiently decreasing, localized, and having enough zero moments, then one can compute for all $p > 0$

$$\tau_f(p) := \liminf_{j\to\infty} \frac{\log_2\left(\sum_k |c_{j,k}|^p\right)}{-j} \tag{1.1}$$

and, p_c being the solution of $\tau_f(p) = 0$,

$$d_1(h) := \min\left(\inf_{p \geq p_c} \left(hp - \tau_f(p)\right), 1\right). \tag{1.2}$$

Remark. In the literature about multifractal formalism, the function $\eta_f = 1 + \tau_f$ is sometimes used instead of τ_f.

We refer to [15] for the details and proofs of the following facts. It is known that for any uniform Hölder function the inequality $d_f(h) \leq d_1(h)$ holds for all h; this is the best that can be expected in all generality; it is actually possible to construct uniform Hölder functions f with quite arbitrary d_f and τ_f, as in [14].

It is worth recalling whence the above inequality comes, and the connection with Besov spaces. From the characterization of those spaces in terms of wavelets coefficients (see [4, 22, 7]), (1.1) is equivalent to:

$$\tau_f(p) = \sup\left\{s,\ f \in B_p^{\frac{s+1}{p},\infty}\right\}.$$

So, given a concave function τ, the knowledge that $\tau_f(p) \geq \tau(p)$ for all $p \geq p_c$ amounts to saying that $f \in B^\tau := \bigcap_{p \geq p_c} B_p^{\frac{\tau(p)+1}{p},\infty}$. Using the same wavelet characterization of the Besov spaces, it was then shown in [15] that when $p \geq p_c$, $f \in B_p^{\frac{\tau(p)+1}{p},\infty}$ implies that $d_f(h) \leq hp - \tau(p)$. The upper bound $d_f(h) \leq d_1(h)$ follows.

1.2 Prevalence

The statistical approach to multifractal formalism is not yet very developed, for the lack of a suitable framework (see, for instance, [1] or [10]). Computing the wavelets coefficients of a function gives an information on the largest space B^τ this function belongs to. If nothing else is known, what is, in a statistical sense, the status of formula (1.2)?

If there were for each τ a natural (probability) measure on B^τ such that almost surely $d_1(h) = d_f(h)$ for all h, then (1.2) would be a consistent estimator for the non parametric problem of recovering τ. Unfortunately no such measure exists because B^τ is not locally compact. Nevertheless, the concept of *prevalence* [6, 12, 13], that we briefly recall below, allows us to define a notion of "almost surely" that is translation invariant and does not depend on any arbitrary measure.

Definition 1. Let X be a complete metric vector space. A Borel set $A \subset X$ is called *shy* (*Haar-null* in [6]) if there exists a Borel measure μ, strictly positive on some compact set $K \subset X$, such that $\forall x \in X$, $\mu(A + x) = 0$. Such a measure is called *transverse* to A. A subset A of E is shy if it is included in a shy Borel set. A set is *prevalent* if its complement is shy.

Following the terminology used in this context, such a measure μ on X will be called a *probe* measure for A.

In finite dimension, the Fubini-Tonelli Theorem shows that a set is prevalent if and only if it has full Lebesgue measure. If X is metric, complete and separable, then the requirement that $\mu(K) > 0$ for some compact set K is automatically fulfilled because all probability measures are tight (there exists a compact set of measure arbitrarily close to 1), see [21], for instance. Other basic properties of prevalence include:

(1) If A is prevalent, then $x + A$ is prevalent for every x in E.
(2) If A is prevalent, then λA is prevalent for every $\lambda \neq 0$.
(3) A prevalent set is dense.
(4) A countable intersection of prevalent sets is prevalent.

Fraysse and Jaffard proved indeed in [9] the following.

Theorem 1. *For f in a prevalent subset of B^τ,*

$$d_f(h) = \begin{cases} \inf_{p \geq p_c} (hp - \tau(p)) & \text{if this } \inf \text{ is } \leq 1 \\ \infty & \text{else} \end{cases} \tag{1.3}$$

(so $d_f(h) = d_1(h)$ when $d_f(h) \neq -\infty$).

We shall say that almost surely (in the sense of prevalence), (1.3) holds. Together with the universal upper bound ($d_f(h) \leq d_1(h)$), this result justifies the *validity* of the thermodynamic formalism.

1.3 S^ν Spaces

However, by the nature of the Legendre transform in (1.2), d_1 will always be a concave function, increasing with slope $\geq p_c$ when $d_1(h) < 1$. A real spectrum of singularities has no reason to satisfy either of these two features (concavity and increasingness), which are indeed limitations to the range of validity of the thermodynamic formalism. The second of these features can be taken care of using *wavelet leaders* instead of wavelet coefficients, see [19]. The object of this article is to address the first one by proposing a new multifractal formalism that can detect nonconcave spectra, and prove its validity in the same sense as for the thermodynamic one. It is clear that the spaces B^τ do not contain more information than the concave hull of the spectrum of singularities (since almost surely in them d_f is given by a Fenchel-Legendre transform) and we are going to replace them by the spaces S^ν that were studied, mainly for that purpose, in [2]. Let us briefly recall their definition and main properties.

As before, we work with a regular, L^∞-normalized, periodized wavelet basis. In the rest of this article, $c_{j,k}$, $(j,k) \in \Lambda$ are the wavelet coefficients of a distribution f on $\mathbb{T}$. If we set

$$E_j(C,\alpha)(f) := \left\{k : |c_{j,k}| \geq C2^{-\alpha j}\right\}, \quad j \geq 0, \alpha \in \mathbb{R}, C \geq 0 \,,$$

then the *wavelet profile* of f is defined as

$$\nu_f(\alpha) := \lim_{\varepsilon \to 0^+} \left(\limsup_{j \to +\infty} \left(\frac{\log_2(\# E_j(1, \alpha + \varepsilon)(f))}{j} \right) \right), \quad \alpha \in \mathbb{R} .$$

The wavelet profile ν_f is a nondecreasing, right-continuous function, nonnegative when not equal to $-\infty$. It is convenient to understand it in the following way: For all $\alpha \in \mathbb{R}$ and $j \in \mathbb{N}$, the number of wavelet coefficients of f that are larger than $\simeq 2^{-\alpha j}$ is about $2^{\nu_f(\alpha) j}$. We can also remark that since f is a distribution on a compact set, it has finite order, and there exists $\alpha_{\min} \in \mathbb{R}$ such that

$$\sup_{j,k} 2^{\alpha_{\min} j} |c_{j,k}| < +\infty \tag{1.4}$$

in other words, $\nu_f(\alpha) = -\infty$ for every $\alpha < \alpha_{\min}$.

Conversely, given a function $\nu : \mathbb{R} \to \{-\infty\} \cup [0, 1]$, nondecreasing and right-continuous, such that $\nu(\alpha) = -\infty$, $\forall \alpha < \alpha_{\min}$ and $\nu(\alpha) \in [0, 1]$, $\forall \alpha \geq \alpha_{\min}$, we say that a distribution f belongs to the space $S^\nu = S^\nu(\mathbb{T})$ if its wavelet profile satisfies

$$\forall \alpha \in \mathbb{R}, \nu_f(\alpha) \leq \nu(\alpha) . \tag{1.5}$$

Roughly speaking, f is in S^ν if, for all α and j, it has less than $\simeq 2^{\nu(\alpha) j}$ wavelet coefficients at scale j that are larger than $\simeq 2^{-\alpha j}$.

These spaces are robust, meaning that their definition does not depend on the choice of the wavelet basis, see [16]. To study them, it is convenient to introduce the ancillary spaces

$$E_{m,n} := \left\{ f, \exists C, \# E_j(C, \alpha_n) \leq C 2^{(\nu(\alpha_n) + \varepsilon_m) j} \forall j \right\} \tag{1.6}$$

where α_n is any (fixed) dense sequence in $\mathbb{R}$ and ε_m is a decreasing sequence converging to 0. For $f \in E_{m,n}$, the infimum of the constants C satisfying the inequality in (1.6) is noted $\delta_{m,n}(f, 0)$ and then the distance $\delta_{m,n}(f, g) := \delta_{m,n}(f - g, 0)$ makes $E_{m,n}$ a metric space.

It is proved in [2] that $S^\nu = \bigcap_{m,n} E_{m,n}$ (this intersection not depending on the choice of the sequences above) and that with the distance

$$\delta(f, g) := \sum_{m,n \geq 0} 2^{-(m+n)} \frac{\delta_{m,n}(f, g)}{1 + \delta_{m,n}(f, g)} ,$$

S^ν is a metric, complete and separable space. The distance δ may depend on the sequences α_n, ε_m, but the induced topology does not. The Borel σ-algebra relative to this topology will be noted $\mathcal{B}(S^\nu)$. Also note that whenever $\nu(\alpha_n) = -\infty$ and $\alpha_{n'} \leq \alpha_n$, then $E_{m,n} \subset E_{m',n'}$ and $\delta_{m,n} \geq \delta_{m',n'}$, so the sequence α_n can be taken dense in $[\alpha_z, +\infty)$ for an arbitrary $\alpha_z < \alpha_{\min}$ without changing the intersection above nor the topology on S^ν.

In this article, we shall require furthermore that $\alpha_{\min} > 0$ in (1.4) and, to avoid trivial cases, that there exists $\alpha \geq \alpha_0$ such that $\nu(\alpha) > 0$. Let us recall a classical result [20].

Proposition 1. *When $\alpha > 0$, the distribution f belongs to the uniform Hölder-Zygmund space $C^\alpha = C^\alpha(\mathbb{T})$ if and only if there exists $C < +\infty$ such that for all $(j, k) \in \Lambda$, $\left|c_{j,k}\right| \leq C 2^{-\alpha j}$.*

In other words, if $\alpha_n > 0$ and $\nu(\alpha_n) = -\infty$, then $E_{m,n} = C^{\alpha_n}$. It follows that $S^\nu \subset \bigcap_{\alpha < \alpha_{\min}} C^\alpha$; in particular, its elements are uniform Hölder and their pointwise regularity can be related to the moduli of their wavelet coefficients (see [18]). Then the results obtained on wavelet series in [3] apply, notably.

Proposition 2. *Let $h_{\max} := \inf_{h \geq \alpha_{\min}} \frac{h}{\nu(h)}$. For all $f \in S^\nu$ and $h \in \mathbb{R}$,*

$$d_f(h) \leq d_\nu(h) := \begin{cases} h \sup_{h' \in (0,h]} \frac{\nu(h')}{h'} & \textit{if} \quad h \leq h_{\max} \\ 1 & \textit{else} . \end{cases}$$

Since ν_f can easily be computed from the wavelet coefficients of f, and considering the definition of S^ν, the formalism that we propose to replace or complement (1.2) is simply

$$d_2(h) := \begin{cases} h \sup_{h' \in (0,h]} \frac{\nu_f(h')}{h'} & \text{if} \quad h \leq h_{\max} \\ 1 & \text{else} . \end{cases} \tag{1.7}$$

By Proposition 2, this formalism is already guaranteed to yield an upper bound to $d_f(h)$. The almost sure (prevalent in S^ν) equality is now naturally our goal and the following theorem, which is our main result, will be proved in Section 3.

Theorem 2. *The following three sets are prevalent in S^ν:*

(1) $\{g \in S^\nu : \nu_g(\alpha) = \nu(\alpha),\ \forall \alpha \in \mathbb{R}\}$,

(2) $\{g \in S^\nu : d_g(h) = d_\nu(h),\ \forall h \leq h_{\max}$ *and* $d_g(h) = -\infty,\ \forall h > h_{\max}\}$,

(3) *the set of g whose pointwise regularity is almost everywhere $h_{\max}$.*

Assertion number (2) is the counterpart of Theorem 1, and justifies the validity of (1.7) as a multifractal formalism.

Connection with B^τ. It was proved in [2] that if $\tau(p) = \inf_{h \leq h_{\max}} (hp - \nu(h))$, then B^τ is the smallest intersection of Besov spaces that contains S^ν. Furthermore, $S^\nu = B^\tau$ if and only if ν is concave. Theorem 1 can thus be viewed as a particular case of our Theorem 2.

See Figure 1 for an illustration of a typical case.

2. A Probe Measure on S^ν

In order to prove the prevalence of the sets introduced in Theorem 2, we need a Borel probe measure on S^ν. In this section we identify $f \in S^\nu$ to the sequence of its wavelet coefficients $c_{j,k}$, $(j,k) \in \Lambda$.

We start by constructing a probability measure μ on the sequence space $(\mathbb{C}^\Lambda, \mathcal{F})$, where $\mathcal{F}$ is the σ-algebra generated by the cylinders. Then it remains to show that μ is actually supported in S^ν and is Borel relatively to the metric topology of S^ν.

2.1 Measure on the Sequence Space

As before, $\nu : \mathbb{R} \to \{-\infty\} \cup [0,1]$ is nondecreasing and right-continuous, and there exists $\alpha_{\min} > 0$ such that $\nu(\alpha) = -\infty$, $\forall \alpha < \alpha_{\min}$ and $\nu(\alpha) \in [0,1]$, $\forall \alpha \geq \alpha_{\min}$.

For each $j \geq 0$ let

$$F_j(\alpha) := \begin{cases} 0 & \text{if} \quad \alpha < \alpha_{\min} \\ 2^{-j} \sup\left\{ j^2, 2^{j\nu(\alpha)} \right\} & \text{if} \quad \alpha \geq \alpha_{\min} . \end{cases} \tag{2.1}$$

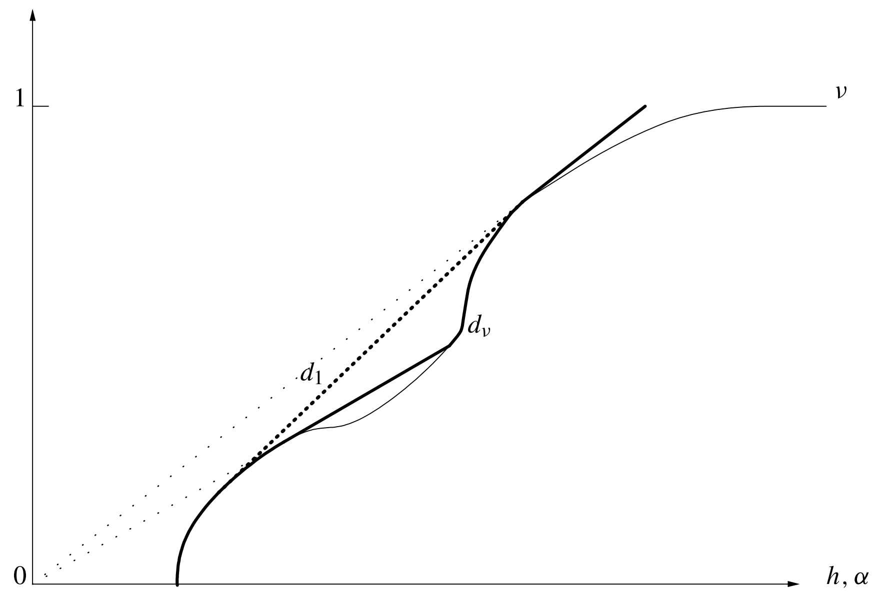

FIGURE 1 A wavelet profile ν, S^ν-prevalent spectrum d_ν, and thermodynamic formalism spectrum d_1.

Because of the conditions imposed on ν, F_j is the repartition function of some probability distribution ρ_j supported on $[\alpha_{\min}, +\infty]$ (where $\rho_j(\{+\infty\}) := 1 - \lim_{\alpha\to+\infty} F_j(\alpha)$ may be > 0). Let $U_{[0,2\pi]}$ denote the uniform measure on $[0, 2\pi]$.

For each index $(j, k) \in \Lambda$ we define the measure $\mu_{j,k}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ to be the image of $\rho_j \otimes U_{[0,2\pi]}$ by the transform $(x, \theta) \mapsto e^{i\theta} 2^{-jx}$. Then

$$\mu := \bigotimes_{(j,k)\in\Lambda} \mu_{j,k}$$

is a probability measure on $(\mathbb{C}^\Lambda, \mathcal{F})$, where $\mathcal{F} := \mathcal{B}(\mathbb{C})^{\otimes\Lambda}$ is also the Borel σ-algebra for the topology of pointwise convergence in the sequence space.

This amounts to drawing random numbers $c_{j,k}$ with independent phases and moduli, such that their phases are uniformly distributed, ρ_j is the law of $-\log_2(|c_{j,k}|)/j$, and it is understood that $\rho_j(\{+\infty\})$ is simply the probability that $c_{j,k} = 0$. Note that for all $\alpha \in \mathbb{R}$,

$$\limsup_{j\to+\infty} \frac{\log_2\left(2^j \rho_j((-\infty, \alpha])\right)}{j} = \nu(\alpha)\ , \tag{2.2}$$

and also that

$$\nu(\alpha) \geq 0 \Rightarrow 2^j \rho_j((-\infty, \alpha]) \geq j^2\ . \tag{2.3}$$

2.2 Random Wavelet Series

Although for the particular choice (2.1) of F_j and ρ_j, the lim sup above is actually a limit, we can from now on suppose only that, along with the hypotheses on ν, (2.2) and (2.3) are satisfied. In that case, $\nu(\frac{\alpha_{\min}}{2}) = -\infty$ implies that, with probability 1, there is only a finite number of $\left|c_{j,k}\right| \geq 2^{-\frac{\alpha_{\min}}{2} j}$. The series

$$f := \sum_{(j,k)\in\Lambda} c_{j,k}\psi_{j,k}$$

is thus uniformly convergent, and its sum is uniform Hölder.

Definition 2. If the wavelet coefficients of f are independent random variables such that the law ρ_j $-\log_2(|c_{j,k}|)/j$ satisfy (2.2) and (2.3), then f is called a *random wavelet series associated to* ν.

The hypothesis on the uniform independent phases, not included in the above definition, will play a role only in Theorem 5.

Let us now recall the main result of [3].

Theorem 3. *If f is a random wavelet series associated to ν, almost surely,*

(1) $f \in S^\nu$
(2) *for all* $\alpha \in \mathbb{R}$, $\nu_f(\alpha) = \nu(\alpha)$
(3) *for all* $h \in \mathbb{R}$, $d_f(h) = d_\nu(h)$ *if* $h \leq h_{\max}$, $-\infty$ *else*
(4) $h_f(x) = h_{\max}$ *almost everywhere.*

Indeed, this result looks very much like Theorem 2 that we are going to prove, except that here "almost surely" refers to the probability, hereafter noted μ, induced by this process. From now on, μ will be restricted to S^ν.

Remark. The technical condition (2.3) could be weakened into $\nu(\alpha) \geq 0 \Rightarrow \sum_j 2^j \rho_j((-\infty,\alpha]) = +\infty$ without changing the conclusion of the theorem, but in its present form it conveniently simplifies the proof of Theorem 5. See [3] for details.

2.3 Borel Measurability

So far μ has been defined on $(S^\nu, \mathcal{F}_{|S^\nu})$. The metric topology on S^ν presented in Section 1.3 makes it a Polish space, which is the good framework for prevalence; it remains to show that μ is Borel relatively to this topology. This amounts to show that $\mathcal{B}(S^\nu) \subset \mathcal{F}_{|S^\nu}$. Actually, these σ-algebras are the same.

Theorem 4. *With the notation above,* $\mathcal{B}(S^\nu) = \mathcal{F}_{|S^\nu}$.

Proof. Recall that $\mathcal{F}_{|S^\nu}$ is the Borel σ-algebra for the topology induced on S^ν by the pointwise convergence of sequences. As a consequence of Proposition 3.5 and Theorem 5.7 of [2], any open set for this topology is an open set in the metric topology of S^ν, so $\mathcal{F}_{|S^\nu} \subset \mathcal{B}(S^\nu)$.

Now we just need to prove that there exists a topological basis of S^ν whose elements are in $\mathcal{F}_{|S^\nu}$; since S^ν is separable, it is Lindelöf, that is, any topological basis contains a countable subbasis. Any open set of S^ν, being a union of finite intersections of elements of this subbasis, will thus belong to $\mathcal{F}_{|S^\nu}$, from which we shall deduce that $\mathcal{B}(S^\nu) \subset \mathcal{F}_{|S^\nu}$.

By Proposition 5.3 of [2], a topological basis of S^ν is given by countable intersections of balls of the ancillary spaces $E_{m,n}$ defined by (1.6), endowed with the distance $\delta_{m,n}$. So the proof boils down to studying those balls. Let $m, n \in \mathbb{N}$, $g \in S^\nu$ whose wavelet coefficients will be noted $d_{j,k}$, and $r > 0$ be fixed. The ball of $E_{m,n}$ with radius r and centered on g is the set

$$B_{m,n}(g,r) := \bigcap_{j\geq 0} \left\{ f : \#\big\{k : |c_{j,k} - d_{j,k}| \geq r2^{-\alpha_n j}\big\} \leq r2^{(\nu(\alpha_n)+\varepsilon_m)j} \right\} .$$

If $\nu(\alpha_n) = -\infty$, then

$$B_{m,n}(g,r) = \bigcap_{(j,k)\in\Lambda} \left\{ f, \left|c_{j,k}\right| \leq 2^{-j\alpha_n} \right\}$$

which belongs to $\mathcal{F}_{|S^\nu}$.

If $\nu(\alpha_n) \in \mathbb{R}$, let $\lambda^* = \left\lfloor r2^{j(\nu(\alpha_n)+\varepsilon_m)} \right\rfloor$ and denote by $K_j(\lambda)$ $(1 \leq \lambda \leq \lambda^*)$ the set of the $C^\lambda_{2^{jd}}$ parts of $\{0, \ldots, 2^j - 1\}$ which contain λ coefficients k. For such a $Z \in K_j(\lambda)$, let $C_j(Z)$ be the set of f such that $k \in Z \Leftrightarrow \left|c_{j,k} - d_{j,k}\right| > r2^{-\alpha_n j}$. Now each $C_j(Z)$ belongs to $\mathcal{F}_{|S^\nu}$, and so does

$$B_{m,n}(g,r) = \bigcap_{j\geq 0} \bigcup_{0\leq\lambda\leq\lambda^*} \bigcup_{Z\in K_j(\lambda)} C_j(Z) .$$

□

3. Proof of Theorem 2

The key of the proof consists in showing that if f is the random process previously constructed, and if $g \in S^\nu$ is fixed, then with probability one, $\nu_{f-g} = \nu$. This is done in Theorem 5 below. Because the wavelet coefficients of $f - g$ are independent, this process is also a random wavelet series, and Theorem 3 implies that it is indeed associated to the same ν. In particular, it will have the same almost-sure spectrum of singularities and almost-everywhere regularity (Corollary 1). Then we can conclude about the prevalence of these properties.

The wavelet coefficients of f satisfy a concentration lemma that we recall for further use. See [3] again and Lemma 2.4 in [23] for a proof.

Lemma 1. *There exist $C_1, C_2 > 0$ such that, if $-\infty \leq a < b$ and $j \geq 0$ are such that $2^j\rho_j((a,b]) \geq j^2$, then*

$$\mu\left(\left\{ f, \frac{1}{2} \leq \frac{\#\left\{k, 2^{-bj} \leq |c_{j,k}| < 2^{-aj}\right\}}{2^j\rho_j((a,b])} \leq 2 \right\}\right) \geq 1 - C_2\frac{2^j}{j^2}e^{-C_1 j^2} .$$

Since we supposed (2.3), $2^j\rho_j((-\infty,\alpha]) \geq j^2$ as soon as $\alpha \geq \alpha_{\min}$.

Theorem 5. *Let f be a random wavelet series associated to ν, with uniform independent phases, such as in Section 2.1, and let $g \in S^\nu$. Then*

$$\mu\big(\big\{f, \nu_{f-g}(\alpha) = \nu(\alpha), \forall\alpha\big\}\big) = 1 .$$

Note that the hypothesis that the phases of $c_{j,k}$ are uniformly distributed, though it could be weakened (any bounded density would do), cannot be completely omitted, as the example of a determinist f and $g = f$ shows (nothing in (2.2) prevents the $\left|c_{j,k}\right|$ from being determinist).

Proof. Let $g \in S^\nu$ (as before, we shall call $d_{j,k}$ its wavelet coefficients). Almost surely $f \in S^\nu$, which is a vector space, so $f - g \in S^\nu$. But by the definition (1.5), this is exactly saying that for all α, $\nu_{f-g}(\alpha) \le \nu(\alpha)$. When $\alpha < \alpha_{\min}$, $\nu_{f-g}(\alpha) = \nu(\alpha) = -\infty$.

It remains to prove that

$$\mu\big(\big\{f, \nu_{f-g}(\alpha) \ge \nu(\alpha), \forall \alpha \ge \alpha_{\min}\big\}\big) = 1 \,.$$

Because of the uniform phase hypothesis on $c_{j,k}$, for any given (j,k), one has with probability $\ge \frac{1}{2}$ that $\Re(c_{j,k}\overline{d_{j,k}}) \le 0$ ($c_{j,k}$ is in the complex half-plane opposite to $d_{j,k}$), so

$$\mu\left(\left\{f, |c_{j,k} - d_{j,k}| \ge |c_{j,k}|\right\}\right) \ge \frac{1}{2} \,.$$

Let $j \ge 1$ and $1 \le N \le 2^j$. Let $K_N \subset \{0, \dots, 2^j - 1\}$ be of cardinal N and $B(j, K_N) := \left\{f, \#\left\{k \in K_N, |c_{j,k} - d_{j,k}| \ge |c_{j,k}|\right\} \ge \frac{N}{3}\right\}$. If we note

$$X_{j,k} := \begin{cases} 1 \text{ if } & |c_{j,k} - d_{j,k}| \ge |c_{j,k}| \\ 0 \text{ else} & \end{cases} \qquad \text{and} \quad S_N = \sum_{k \in K_N} X_{j,k}$$

then $\mathbb{E}(S_N) \ge \frac{N}{2}$ and

$$\begin{aligned} \mu(B(j, K_N)) &= \mu\left(\left\{f, S_N \ge \frac{N}{3}\right\}\right) \\ &\ge 1 - \mu\left(\left\{f, \mathbb{E}(S_N) - S_N > \frac{N}{6}\right\}\right) \\ &\ge 1 - e^{-\frac{N}{18}} \end{aligned} \tag{3.1}$$

by the Hoeffding inequality [11].

Let $\alpha \ge \alpha_{\min}$ be fixed and

$$A(j, \alpha) := \left\{f, \#\left\{k, |c_{j,k}| \ge 2^{-\alpha j}\right\} \ge \frac{1}{2} \sup\left\{j^2, 2^{j\nu(\alpha)}\right\}\right\} \,.$$

By Lemma 1, there exists $C_1, C_2 > 0$ such that, if $j \ge 1$,

$$\mu(A(j, \alpha)) \ge 1 - C_2 \frac{2^j}{j^2} e^{-C_1 j^2} \,.$$

Conditionally to $f \in A(j, \alpha)$, we apply (3.1) to the $N := \frac{1}{2} \max\left(j^2, 2^{j\nu(\alpha)}\right)$ first wavelet coefficients of f satisfying $|c_{j,k}| \ge 2^{-\alpha j}$. Now remark that the set $B(j, K_N)$ in (3.1), depending only on the phases of the wavelet coefficients of f, is independent from $A(j, \alpha)$, which depends only on their moduli. So we have

$$\begin{aligned} \mu\left(\left\{f, \#\left\{k, |c_{j,k} - d_{j,k}| \ge 2^{-\alpha j}\right\} \ge \frac{N}{3}\right\}\right) &\ge \mu(A(j, \alpha))\mu(B(j, K_N)) \\ &\ge 1 - C_2 \frac{2^j}{j^2} e^{-C_1 j^2} - e^{-\frac{N}{18}} \end{aligned}$$

hence

$$\mu\left(\left\{f, \#\left\{k, |c_{j,k} - d_{j,k}| \ge 2^{-\alpha j}\right\} \ge \frac{2^{j\nu(\alpha)}}{6}\right\}\right) \ge 1 - C_2 \frac{2^j}{j^2} e^{-C_1 j^2} - e^{-\frac{j^2}{36}} \,. \tag{3.2}$$

For $n \geq 1$, let $\zeta(n)$ be the unique natural number such that $2^{\zeta(n)}(\zeta(n)-2) < n-2 \leq 2^{\zeta(n)+1}(\zeta(n)-1)$. The $j2^j$ points α_n such that $j = \zeta(n)$ are then chosen to be equispaced in $[0, j-2^{-j}]$. This sequence is dense in $[0, +\infty)$. Furthermore, by construction, it is fully recurrent (meaning that for every $n \geq 1$, there exists n' such that $\zeta(n') = \zeta(n)+1$ and $\alpha_{n'} = \alpha_n$).

We apply (3.2) to the (at most) $j2^j$ numbers $\alpha_n \geq \alpha_{\min}$ such that $n \in \zeta^{-1}(\{j\})$. The series $\sum_j j2^j\left(C_2\frac{2^j}{j^2}e^{-C_1 j^2} + e^{-\frac{j^2}{36}}\right)$ converges, so by the Borel-Cantelli Lemma, with probability one there exists J' such that for each $j \geq J'$, $\forall n \in \zeta^{-1}(\{j\})$,

$$\#\left\{k, |c_{j,k} - d_{j,k}| \geq 2^{-\alpha j}\right\} \geq \frac{2^{j\nu(\alpha)}}{6}.$$

With the full recurrence property stated above, this implies that with probability one, for all n such that $\alpha_n \geq \alpha_{\min}$, $\nu_{f-g}(\alpha_n) \geq \nu(\alpha_n)$, and then by density of this sequence, $\nu_{f-g}(\alpha) \geq \nu(\alpha)$ for all $\alpha \geq \alpha_{\min}$. □

Corollary 1. *Let g and f be as above. Then with probability one*

(1) $d_{f-g}(h) = \begin{cases} d_\nu(h) & \textit{if} \quad h \leq h_{\max} \\ -\infty & \textit{if} \quad h > h_{\max} \end{cases}$

(2) *for almost every x, $h_{f-g}(x) = h_{\max}$.*

Proof. We have established that $f-g$ is indeed a random wavelet series associated to ν. The announced properties all follow from Theorem 3. □

Proof of Theorem 2. If $A \subset S^\nu$ is the set of functions h whose ν_h does not coincide with ν, then given any $g \in S^\nu$, $\mu(g+A) = \mu(f-g \in A) = 0$ by Theorem 5, so μ is transverse to A. The same reasoning with Corollary 1 provides the other assertions. □

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