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Local Growth Envelopes of Triebel-Lizorkin Spaces of Generalized Smoothness

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ABSTRACT. The concept of local growth envelope $(\mathcal{E}_{LG}A, u)$ *of the quasi-normed function space A is applied to the Triebel-Lizorkin spaces of generalized smoothness* $F^{\sigma,N}_{p,q}(\mathbb{R}^n)$. *In order to achieve this, a standardization result for these and corresponding Besov spaces is derived.*

1. Introduction

In recent years there has been a lot of work on the determination of the local growth envelopes of function spaces of the Besov and Triebel-Lizorkin type (see [17, 27, 5, 4, 2, 3]). These results can be seen as refinements of the famous Sobolev embedding theorem, being related also with the popular topic of searching for sharp embeddings between classes of function spaces (see, for example, [12, 13, 23, 15, 11, 14, 16]). In any case, local growth envelopes identify the greatest possible growth which functions from some given space can stand locally, so they are useful in distinguishing between spaces and have even been used (together with the related concept of continuity envelope) in the proof of the necessity of conditions giving continuous embeddings between spaces.

As the scale of spaces of Besov and Triebel-Lizorkin type comprises nowadays generalized versions, in various directions, which are proving useful in other areas of mathematics (see, for example, [8, 7, 24]), the question naturally arises about how the corresponding growth envelopes look like. For one of such generalizations, in the direction of the so-called spaces of generalized smoothness, there is now, after the work [10], and its forerunner [9], a common framework for several approaches to generalized smoothness scattered in the literature (see the mentioned work for references, and also below for additional remarks). Also, the techniques developed in [10] (and in [9]), namely atomic representations, made

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a broad class of such spaces to become suitable for the investigation of the corresponding growth envelopes.

Accordingly, and by improving techniques used in the determination of the local growth envelopes of spaces which partially generalize in the direction of generalized smoothness, it was finally possible in [3] to solve the problem (apart some borderline cases) for Besov spaces of generalized smoothness in the broad sense of [9] and whenever atomic decompositions are available. Relying on the idea used in previous partial works, Triebel-Lizorkin spaces should then be dealt with on the basis of a comparison with Besov spaces. However, at the time the work [3] was written, such a comparison for such general spaces was not available.

In the present work we start by developing a so-called standardization procedure, which will allow the determination of tight embeddings between Besov and Triebel-Lizorkin spaces of generalized smoothness and open the way to the determination (again, apart borderline cases) of the local growth envelopes for the latter spaces, which we do next.

This standardization procedure might have independent interest, and this and the related spaces of generalized smoothness have even a bit of history which we would like to briefly recall to finish this introduction.

Function spaces of generalized smoothness were introduced and investigated, independently, by M. L. Goldman and G. A. Kalyabin in the middle of the seventies of the last century with the help of differences and general weight functions and on the basis of expansions in series of entire functions, respectively. In both cases the defined function spaces $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$ and $F^{\sigma,N}_{p,q}(\mathbb{R}^n)$ are subspaces of $L_p(\mathbb{R}^n)$. For these spaces a standardization was proved in [18]—see also [20]. In it, the weight sequence $\beta = (2^j)_{j\in\mathbb{N}_0}$ was fixed and a suitable sequence $(M_j)_{j\in\mathbb{N}_0}$ was defined with the help of $(N_k)_{k\in\mathbb{N}_0}$, $(\sigma_k)_{k\in\mathbb{N}_0}$ and $(2^j)_{j\in\mathbb{N}_0}$ such that $B^{\sigma,N}_{p,q} = B^{\beta,M}_{p,q}$, and analogously for F-spaces. This gave a standard weight sequence $(2^j)_{j\in\mathbb{N}_0}$ and a general sequence $(M_j)_{j\in\mathbb{N}_0}$, and is restricted to function spaces which are subspaces of $L_p(\mathbb{R}^n)$ and the Banach space case.

But a more powerful tool would be a standardization which fixed the sequence M by $M = (2^j)_{j\in\mathbb{N}_0}$, because for this standard dyadic resolution of the $\mathbb{R}^n$ a lot of results are meanwhile available. Under a natural restriction to the sequence $(N_k)_{k\in\mathbb{N}_0}$ which excludes the exponential growth of this sequence, such a result seems in some sense straightforward. But it is also a bit technical to prove, and we found no reference of it in the literature, not even in the Banach space case. For this reason, the first thing we prove in this article is actually such a standardization theorem.

In [2] a connection between admissible sequences and special functions was considered. We define here also a class of suitable functions, corresponding to admissible sequences, and describe the standardization result with the help of these functions, too.

2. Notations and Conventions

Definition 1. By an *admissible sequence* we will always mean a sequence $\gamma = (\gamma_k)_{k\in\mathbb{N}_0}$ of positive numbers such that there are two positive constants κ_0 and κ_1 with

$$\kappa_0\,\gamma_k \le \gamma_{k+1} \le \kappa_1\gamma_k \quad \text{for any} \quad k \in \mathbb{N}_0\,. \tag{2.1}$$

We call κ_0 and κ_1 *equivalence constants* associated with γ.

We shall need the following notation with respect to an admissible sequence:

$$\underline{\gamma}_k := \inf_{j\geq 0} \frac{\gamma_{j+k}}{\gamma_j} \quad \text{and} \quad \overline{\gamma}_k := \sup_{j\geq 0} \frac{\gamma_{j+k}}{\gamma_j}, \quad k \in \mathbb{N}_0 . \tag{2.2}$$

Note that, in particular, $\underline{\gamma}_1$ and $\overline{\gamma}_1$ are the best constants κ_0 and κ_1 in (2.1), respectively.

In [2] the upper and lower Boyd indices of the given sequence were introduced, respectively, by

$$\alpha_\gamma := \lim_{k\to\infty} \frac{\log_2 \overline{\gamma}_k}{k} \quad \text{and} \quad \beta_\gamma := \lim_{k\to\infty} \frac{\log_2 \underline{\gamma}_k}{k} .$$

Assumption. *We will denote* $N = (N_k)_{k\in\mathbb{N}_0}$ *a sequence of real positive numbers such that there exist two numbers* $1 < \lambda_0 \leq \lambda_1$ *with*

$$\lambda_0 N_k \leq N_{k+1} \leq \lambda_1 N_k \qquad \textit{for any} \quad k \in \mathbb{N}_0 . \tag{2.3}$$

In particular, N is admissible and is a so-called strongly increasing sequence (cf. [9, Definition 2.2.1]), which in particular guarantees that there exists a number $l_0 \in \mathbb{N}$ such that

$$2N_j \leq N_k \quad \text{for any} \quad j, k \quad \text{such that} \quad j + l_0 \leq k . \tag{2.4}$$

This is true, for instance, if we choose for l_0 *a natural number* such that

$$\lambda_0^{l_0} \geq 2 \tag{2.5}$$

holds. We will fix such an l_0 in the following.

Nevertheless, the assumption concerning λ_0 is not restrictive with regard to the function spaces we are interested in—see [9, Remark 4.1.2].

For a fixed sequence $N = (N_k)_{k\in\mathbb{N}_0}$ as in the Assumption we define the associated covering $\Omega^N = (\Omega_k^N)_{k\in\mathbb{N}_0}$ of $\mathbb{R}^n$ by

$$\Omega_k^N = \left\{\xi \in \mathbb{R}^n \ : \ |\xi| \leq N_{k+l_0}\right\}, \quad k = 0, 1, \cdots l_0 - 1 , \tag{2.6}$$

and

$$\Omega_k^N = \left\{\xi \in \mathbb{R}^n \ : \ N_{k-l_0} \leq |\xi| \leq N_{k+l_0}\right\} \quad \text{if} \quad k \geq l_0 \tag{2.7}$$

with l_0 as defined in (2.5).

Definition 2. For a fixed $N = (N_k)_{k\in\mathbb{N}_0}$ as in the Assumption and for the associated covering $\Omega^N = (\Omega_k^N)_{k\in\mathbb{N}_0}$ of $\mathbb{R}^n$, a system $\varphi^N = (\varphi_k^N)_{k\in\mathbb{N}_0}$ will be called a (generalized) partition of unity subordinated to Ω^N if:

(i) $\varphi_k^N \in C_0^\infty\left(\mathbb{R}^n\right)$ and $\varphi_k^N(\xi) \geq 0$ if $\xi \in \mathbb{R}^n$ for any $k \in \mathbb{N}_0$; (2.8)

(ii) $\operatorname{supp} \varphi_k^N \subset \Omega_k^N$ for any $k \in \mathbb{N}_0$; (2.9)

(iii) for any $\gamma \in \mathbb{N}_0^n$ there exists a constant $c_\gamma > 0$ such that for any $k \in \mathbb{N}_0$

$$\left|D^\gamma \varphi_k^N(\xi)\right| \leq c_\gamma \left(1 + |\xi|^2\right)^{-|\gamma|/2} \quad \text{for any} \quad \xi \in \mathbb{R}^n ; \tag{2.10}$$

(iv) there exists a constant $c_\varphi > 0$ such that

$$0 < \sum_{k=0}^{\infty} \varphi_k^N(\xi) = c_\varphi < \infty \quad \text{for any} \quad \xi \in \mathbb{R}^n . \tag{2.11}$$

Recalling that $\mathcal{S}$ stands for the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ equipped with the usual topology and $\mathcal{S}'$ denotes its topological dual, the space of all tempered distributions on $\mathbb{R}^n$, we have the following.

Definition 3. Let $(\sigma_k)_{k\in\mathbb{N}_0}$ be an admissible sequence. Let $(N_k)_{k\in\mathbb{N}_0}$ be an admissible sequence satisfying the Assumption and let φ^N be a system of functions as in Definition 2. Let $0 < p < \infty$ and $0 < q \leq \infty$.

The Triebel-Lizorkin space $F^{\sigma,N}_{p,q}$ of generalized smoothness is defined as

$$\left\{ f \in \mathcal{S}' \,:\, \|f \mid F^{\sigma,N}_{p,q}\| := \left\| \left(\sum_{k=0}^{\infty} \sigma_k^q \left| \mathcal{F}^{-1}\left(\varphi_k^N \mathcal{F} f\right)(\cdot)\right|^q \right)^{1/q} \mid L_p\big(\mathbb{R}^n\big) \right\| < \infty \right\}.$$

(usual modification when $q = \infty$) where $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand, respectively, for the Fourier transformation and its inverse.

Note that if $0 < p \leq \infty$ and $0 < q \leq \infty$ then the Besov space of generalized smoothness $B^{\sigma,N}_{p,q}$ is defined in an analogous way, by interchanging the roles of the quasi-norms in $L_p(\mathbb{R}^n)$ and in ℓ_q.

Note also that if $N_k = 2^k$ and $\sigma = (2^{ks})_{k\in\mathbb{N}_0}$ with s real, then the spaces $F^{\sigma,N}_{p,q}$ coincide with the usual Triebel-Lizorkin spaces $F^s_{p,q}$ on $\mathbb{R}^n$, and the spaces $B^{\sigma,N}_{p,q}$ coincide with the usual Besov spaces $B^s_{p,q}$ on $\mathbb{R}^n$. We shall use the simpler notation $F^s_{p,q}$ and $B^s_{p,q}$ in the more classical situation just mentioned. Even for general admissible σ, when $N_k = 2^k$ we shall write simply $F^\sigma_{p,q}$ and $B^\sigma_{p,q}$ instead of $F^{\sigma,N}_{p,q}$ and $B^{\sigma,N}_{p,q}$, respectively.

We use the equivalence "$\sim$" in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1\, a_k \leq b_k \leq c_2\, a_k \quad \text{or} \quad c_1\, \varphi(x) \leq \psi(x) \leq c_2\, \varphi(x)$$

for all admitted values of the discrete variable k or the continuous variable x, where $(a_k)_k$, $(b_k)_k$ are nonnegative sequences and φ, ψ are nonnegative functions.

Definition 4. We say that the function $\varphi : [1,\infty) \to (0,\infty)$ belongs to $\mathcal{V}$ if φ is measurable and satisfies

$$0 < \underline{\varphi}(t) := \inf_{s\in[1,\infty)} \frac{\varphi(ts)}{\varphi(s)} \quad \text{for all } t \in [1,\infty)$$

and

$$\overline{\varphi}(t) := \sup_{s\in[1,\infty)} \frac{\varphi(ts)}{\varphi(s)} < \infty \quad \text{for all } t \in [1,\infty)\,,$$

where we also assume that $\underline{\varphi}$ and $\overline{\varphi}$ are measurable functions.

Remark 1. Conditions of the above type have been used at several places, e.g., in connection with real interpolation with a function parameter or in the theory of function spaces with generalized smoothness. However, the related functions φ are usually defined either on $\mathbb{R}$ or on $(0,\infty)$. The above definition was given first in connection with weighted Besov spaces in [21]—see also there for details and further references.

We can give Definition 3 of the space $F^{\sigma,N}_{p,q}$ and the corresponding one for $B^{\sigma,N}_{p,q}$ just by starting considering appropriate functions Σ and N in $\mathcal{V}$ (note the abuse of notation in the case of N) and defining the sequences σ and N by $\sigma_k = \Sigma(2^k)$ and $N_k = N(2^k)$. A similar, though mixed, approach was followed in [22, 6]: There on the side of N a sequence was considered—even the particular one $(2^k)_k$—and only on the side of the smoothness a function was considered, and in a slightly different class defined on $(0,\infty)$. On the other hand, with appropriate assumptions on the functions Σ and N involved—namely, that $\Sigma, N \in \mathcal{V}$, N is strictly increasing and $\lambda_0 N(t) \leq N(2t)$, for some $\lambda_0 > 1$—the scales of spaces thus defined are the same as the ones defined with the help of sequences, and, as it was shown in [2] in a similar context, one can even choose functions and sequences in such a way that the so-called Boyd indices of both coincide. For this dual way of defining function spaces, we refer also to [1].

As to the relation between functions Σ and N and sequences σ and N defining the same spaces, with the help of [21, Lemma 1] we can even state the following:

- For each admissible sequence $\sigma = (\sigma_k)_{k\in\mathbb{N}_0}$ there exists a *corresponding function* $\Sigma \in \mathcal{V}$, that is, a function such that $\Sigma(t) \sim \sigma_k$ for all $t \in [2^k, 2^{k+1})$. For example, we can define
$$\Sigma(t) = \sigma_k + \left(2^{-k}t - 1\right)(\sigma_{k+1} - \sigma_k) \quad \text{for} \quad t \in \left[2^k, 2^{k+1}\right) .$$
- Vice versa, for each function $\Sigma \in \mathcal{V}$ the sequence σ with $\sigma_k := \Sigma(2^k)$ is an admissible sequence with $\kappa_0 = \underline{\Sigma}(2)$ and $\kappa_1 = \overline{\Sigma}(2)$ to which Σ corresponds.
- If in addition an admissible sequence $(N_k)_{k\in\mathbb{N}_0}$ fulfills the Assumption, then there exists a corresponding function $N \in \mathcal{V}$, strictly increasing and with $\lambda_0 N(t) \leq N(2t)$ for all $t \geq 1$.
- And again, vice versa. If $N \in \mathcal{V}$ is a function with the property described above, then the sequence $N_k := N(2^k)$ is admissible, fulfills the Assumption and N corresponds to it.

In what follows, all unimportant positive constants will be denoted by c, occasionally with additional subscripts within the same formula.

3. A Standardization Result

The following Fourier-multiplier theorem—[26, Theorem 1.6.3]—will be the main tool in the proof of the standardization theorem in the case of the F-spaces.

Proposition 1. *Let $0 < p < \infty, 0 < q \leq \infty$. Let $(\Omega_j)_{j\in\mathbb{N}_0}$ be a sequence of compact subsets of $\mathbb{R}^n$ and $d_j > 0$ be the diameter of Ω_j.*

If $t > n/2 + n/\min(p,q)$, then there exists a constant $c > 0$ such that

$$\left\| \left(\mathcal{F}^{-1}(M_j \mathcal{F} f_j)\right)_{j\in\mathbb{N}_0} \mid L_p(l_q)\right\| \leq c \sup_{j\in\mathbb{N}_0} \left\| M_j(d_j\,\cdot) \mid H^t_2\right\| \; \|(f_j)_{j\in\mathbb{N}_0} \mid L_p(l_q)\|$$

holds for all systems $(f_j)_{j\in\mathbb{N}_0} \in L_p(l_q)$ with $\operatorname{supp}\mathcal{F}f_j \subset \Omega_j$ for all j, and all sequences $(M_j)_{j\in\mathbb{N}_0} \subset H^t_2$.

In the B-case we need a scalar version of Proposition 1, including now $p = \infty$, see [26, 1.5.2, Remark 3].

Proposition 2. *Let $0 < p \le \infty$ and let $\Omega_d = \{x \in \mathbb{R}^n : |x| \le d\}$ with $d > 0$.*

If $t > n/\min(p,1) - n/2$, then there exists a constant $c > 0$, independent of d, such that

$$\left\| \mathcal{F}^{-1}(M\mathcal{F}f) \mid L_p \right\| \le c \left\| M(d\,\cdot) \mid H_2^t \right\| \; \| f \mid L_p \|$$

holds for all $f \in L_p$ with $\operatorname{supp}\mathcal{F}f \subset \Omega_d$ and all $M \in H_2^t$.

Furthermore, an easy computation shows that for $k \ge l_0$ and integers M, we have

$$\left\| \varphi_k^N(N_{k+3l_0}\,\cdot) \mid W_2^M \right\| \le c \left(N_{k+3l_0} N_{k-l_0}^{-1} \right)^M \le c\lambda_1^{4l_0 M} \tag{3.1}$$

where the right-hand side is uniformly bounded with respect to k if $(N_k)_{k\in\mathbb{N}_0}$ satisfies the Assumption.

For $k = 0, \cdots, l_0 - 1$

$$\left\| \varphi_k^N(N_{4l_0}\,\cdot) \mid W_2^M \right\| \le c\, N_{4l_0}^M \tag{3.2}$$

is obvious.

Theorem 1. *Let $N := (N_k)_{k\in\mathbb{N}_0}$ satisfy the Assumption and $\sigma := (\sigma_k)_{k\in\mathbb{N}_0}$ be an admissible sequence with equivalence constants κ_0 and κ_1. Define*

$$\beta_j := \sigma_{k(j)}, \quad \text{with} \quad k(j) := \min\left\{ k \in \mathbb{N}_0 : 2^{j-1} \le N_{k+l_0} \right\},$$

if $j \ge 1$ and with l_0 defined in (2.5), and define $\beta_0 := \sigma_{k(1)}$.

Then we have that

$$\mu_0\beta_j \le \beta_{j+1} \le \mu_1\beta_j, \quad j \in \mathbb{N}_0,$$

with $\mu_0 = \min\{1, \kappa_0^{l_0}\}$, $\mu_1 = \max\{1, \kappa_1^{l_0}\}$.

Let, further, $0 < p, q \le \infty$ (with $p \ne \infty$ in the F-case). Then

$$F_{p,q}^{\sigma,N} = F_{p,q}^{\beta}$$

and

$$B_{p,q}^{\sigma,N} = B_{p,q}^{\beta},$$

where $\beta := (\beta_j)_{j\in\mathbb{N}_0}$.

Proof.

Step 1. We start with some preliminary observations. For simplicity we assume without loss of generality that $N_0 = 1$. Otherwise we would have to consider large enough values for j and k in what follows.

Let $\Omega = (\Omega_j)_{j\in\mathbb{N}_0}$ be the standard dyadic covering of $\mathbb{R}^n$, associated with the sequence $(2^j)_{j\in\mathbb{N}_0}$, i.e.,

$$\Omega_0 = \left\{ \xi \in \mathbb{R}^n : |\xi| \le 2 \right\},$$

and

$$\Omega_j = \left\{ \xi \in \mathbb{R}^n : 2^{j-1} \le |\xi| \le 2^{j+1} \right\} \quad \text{if} \quad j \ge 1.$$

Specify $k_0 \geq l_0$. Then $\Omega^N_{k_0} \cap \Omega_j$ can be nonempty only if we have

$$\frac{1}{2} N_{k_0-l_0} \leq 2^j \leq 2N_{k_0+l_0} \ .$$

Let

$$\mathcal{J}(k_0) = \left\{ j \in \mathbb{N}_0 : N_{k_0-l_0} \leq 2^{j+1} \leq 2^2 N_{k_0+l_0} \right\} . \tag{3.3}$$

The set $\mathcal{J}(k_0)$ is always nonempty—let ξ be an element of $\Omega^N_{k_0}$, then there exist at least one j' with $\xi \in \Omega_{j'}$. But then holds $N_{k_0-l_0} \leq |\xi| \leq 2^{j'+1}$ and $2^{j'-1} \leq |\xi| \leq N_{k_0+l_0}$.

We will denote by $j^*(k_0)$ the smallest element of $\mathcal{J}(k_0)$. Then it is easy to see, that $\Omega^N_{k_0}$ has a nonempty intersection with at most the sets $\Omega_{j^*(k_0)}, \ldots, \Omega_{j^*(k_0)+L}$ where

$$L = [2l_0 \log_2 \lambda_1] + 2 \tag{3.4}$$

is independent of k_0.

Moreover, we have

$$j^*(k_0) < j^*(k_0 + 4l_0 + 1) \ . \tag{3.5}$$

And vice versa, if we specify $j_0 \geq 1$ then $\Omega^N_k \cap \Omega_{j_0}$ can be nonempty only if we have

$$2^{j_0-1} \leq N_{k+l_0} \quad \text{and} \quad N_{k-l_0} \leq 2^{j_0+1} \ .$$

Let now

$$\mathcal{K}(j_0) = \left\{ k \in \mathbb{N}_0 : 2^{j_0-1} \leq N_{k+l_0} \quad \text{and} \quad N_{k-l_0} \leq 2^{j_0+1} \right\} . \tag{3.6}$$

The set $\mathcal{K}(j_0)$ is again always nonempty and we denote by $k^*(j_0)$ the smallest element of it.

$k^*(j_0)$ coincides with $k(j_0)$ from the theorem—$k(j_0) \leq k^*(j_0)$ is obvious and the opposite inequality follows by the monotonicity of the sequence $(N_k)_{k\in\mathbb{N}_0}$ and $l_0 \geq 1$. We have $N_{k(j_0)-l_0} \leq N_{k(j_0)-1+l_0} < 2^{j_0-1} < 2^{j_0+1}$, that means $k(j_0)$ belongs to $\mathcal{K}(j_0)$.

Again it is easy to see, that Ω_{j_0} has a nonempty intersection with at most the sets $\Omega^N_{k^*(j_0)}, \ldots, \Omega^N_{k^*(j_0)+4l_0}$ and we have

$$k^*(j_0) \leq k^*(j_0+1) \leq k^*(j_0) + l_0 \quad \text{but} \quad k^*(j_0) < k^*(j_0 + L + 1) \tag{3.7}$$

with L from (3.4).

Step 2. Let $\sigma := (\sigma_k)_{k\in\mathbb{N}_0}$ be an admissible sequence with equivalence constants κ_0 and κ_1, and define $\beta_j = \sigma_{k^*(j)}$. Then there exist positive constants, independent of $j \geq 1$ and k such that

$$\min\left(1, \kappa_0^{4l_0}\right) \leq \frac{\sigma_k}{\beta_j} = \frac{\sigma_k}{\sigma_{k^*(j)}} \leq \max\left(1, \kappa_1^{4l_0}\right) \quad \text{for all} \quad k \in \mathcal{K}(j) \ . \tag{3.8}$$

The cardinality of $\mathcal{K}(j)$ is not larger than $4l_0 + 1$ and so (3.8) follows immediately. In case of $\kappa_0 < 1$ we have to choose the minimum on the left-hand side, and if $\kappa_1 < 1$ we have to choose the maximum on the right-hand side.

The estimation of the counterpart—there exist positive constants c_0 and c_1, independent of j and $k \geq l_0$, such that

$$c_0 \leq \frac{\beta_j}{\sigma_k} = \frac{\sigma_{k^*(j)}}{\sigma_k} \leq c_1 \quad \text{for all} \quad j \in \mathcal{J}(k) \tag{3.9}$$

holds—is more delicate.

First notice that (3.5) gives

$$j^*(k) + l \le j^*(k + (4l_0 + 1)l) \quad , \quad l \in \mathbb{N}_0 \, .$$

By the construction in Step 1 we have also

$$k - 4l_0 \le k^*\big(j^*(k)\big) \le k$$

at least for $k > 2l_0$; otherwise $j^*(k)$ might be zero, but we can temporarily extend the definition of $\mathcal{K}(j_0)$ in (3.6) to $j_0 = 0$ and see that this remains true for all $k \ge l_0$.

Both together gives for $l = 0, \cdots, L$

$$k - 4l_0 \le k^*\big(j^*(k) + l\big) \le k^*\big(j^*(k + (4l_0 + 1)l)\big) \le k + (4l_0 + 1)L \, . \tag{3.10}$$

Now we can determine c_0 and c_1 from (3.9) by

$$c_0 = \min\big(1, \kappa_1^{-4l_0}, \kappa_0^{(4l_0+1)L}\big) \quad \text{and} \quad c_1 = \max\big(1, \kappa_0^{-4l_0}, \kappa_1^{(4l_0+1)L}\big) \, .$$

Moreover, similar to (3.8), we obtain by (3.7) that $(\beta_j)_{j \in \mathbb{N}_0}$ is an admissible sequence and

$$\min\big(1, \kappa_0^{l_0}\big) \le \frac{\beta_{j+1}}{\beta_j} \le \max\big(1, \kappa_1^{l_0}\big) \quad \text{for all} \quad j \in \mathbb{N}_0 \, . \tag{3.11}$$

Step 3. Denote by $(\varphi_j)_{j \in \mathbb{N}_0}$ a function system, related to the dyadic decomposition $(\Omega_j)_{j \in \mathbb{N}_0}$, and let $c_\varphi = 1$ for this system.

Then we have

$$\mathcal{F}^{-1}\big(\varphi_k^N \mathcal{F} f\big) = \sum_{j=0}^{\infty} \mathcal{F}^{-1}\big(\varphi_j \varphi_k^N \mathcal{F} f\big) = \sum_{j \in \mathcal{J}(k)} \mathcal{F}^{-1}\big(\varphi_j \varphi_k^N \mathcal{F} f\big) \, .$$

We put in Proposition 1

$$M_k = \varphi_k^N$$

and

$$f_k = \sigma_k \sum_{j \in \mathcal{J}(k)} \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \qquad \text{if} \quad k \ge l_0 \, ,$$

$$f_k = \sigma_k \Bigg(\sum_{j=0}^{j^*(l_0)-1} + \sum_{j \in \mathcal{J}(l_0)} \Bigg) \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \quad \text{if} \quad k = 0, \cdots, l_0 - 1 \, .$$

Then of course

$$\mathcal{F}^{-1}(M_k \mathcal{F} f_k) = \sigma_k \mathcal{F}^{-1}\big(\varphi_k^N \mathcal{F} f\big) \, ,$$

and

$$\operatorname{supp} \mathcal{F} f_k \subset \bigcup_{j \in \mathcal{J}(k)} \operatorname{supp} \varphi_j \subset \{\xi : |\xi| \le N_{k+3l_0}\} \quad \text{if} \quad k \ge l_0 \, ,$$

$$\operatorname{supp} \mathcal{F} f_k \subset \{\xi : |\xi| \le N_{4l_0}\} \qquad \text{if} \quad k = 0, \cdots, l_0 - 1 \, .$$

Now Proposition 1 and (3.1), (3.2) give

$$\left\| \left(\sigma_k \mathcal{F}^{-1}\left(\varphi_k^N \mathcal{F} f\right)\right)_{k\in\mathbb{N}_0} \mid L_p(l_q)\right\| \leq c \left\|(f_k)_{k\in\mathbb{N}_0} \mid L_p(l_q)\right\| . \tag{3.12}$$

For the rest of the proof, assume, for simplicity, that $q \neq \infty$; otherwise usual changes have to be made in what follows.

Let $k \geq l_0$; then (the cardinality of $\mathcal{J}(k)$ is not larger than $L+1$)

$$\sigma_k^q \left| \sum_{j\in\mathcal{J}(k)} \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)\right|^q \leq c_{q,L}^q \left(\max_{j\in\mathcal{J}(k)} \frac{\sigma_k}{\beta_j}\right)^q \sum_{j\in\mathcal{J}(k)} \beta_j^q \left|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)\right|^q .$$

We take c_0 from (3.9) and get

$$\sum_{k=l_0}^{\infty} \sigma_k^q \left| \sum_{j\in\mathcal{J}(k)} \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)\right|^q \leq c_{q,L}^q c_0^{-q} \sum_{k=l_0}^{\infty} \sum_{j\in\mathcal{J}(k)} \beta_j^q \left|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)\right|^q .$$

But because of (3.5) each φ_j can occur in the double sum not more than $(L+1)(4l_0+1)$ times.

A similar estimate can be given for the first l_0 summands and each φ_j with $0 \leq j \leq j^*(l_0) + L$ can occur only l_0 times. The counterpart to (3.9) is obvious because of the limited number of β_j , $0 \leq j \leq j^*(l_0) + L$ and σ_k , $0 \leq k \leq l_0 - 1$, which are involved. Together with the key estimate (3.12) this gives

$$\left\| f | F_{p,q}^{\sigma,N} \right\| \leq c \left\| f | F_{p,q}^{\beta} \right\| .$$

We obtain the opposite inequality by changing the roles of φ_k^N and φ_j.

Step 4. In the case of B-spaces we use the scalar multiplier theorem. Again we have

$$\mathcal{F}^{-1}\left(\varphi_k^N \mathcal{F} f\right) = \mathcal{F}^{-1}\left(\varphi_k^N \sum_{j\in\mathcal{J}(k)} \varphi_j \mathcal{F} f\right) .$$

Now we use Proposition 2 with $M = \varphi_k^N$ and the role of f over there being played now by

$$\sum_{j\in\mathcal{J}(k)} \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \quad \text{if} \quad k \geq l_0 .$$

Similar to Step 3 we get

$$\left\|\mathcal{F}^{-1}(\varphi_k^N \mathcal{F} f) \mid L_p\right\| \leq c \left\| \sum_{j\in\mathcal{J}(k)} \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \mid L_p \right\|$$

with a constant $c > 0$ and independent of k. Again, by the considerations of Step 1 and Step 2, we obtain

$$\sum_{k=l_0}^{\infty} \sigma_k^q \left\|\mathcal{F}^{-1}\left(\varphi_k^N \mathcal{F} f\right) | L_p\right\|^q \leq c_{p,L}^{*q} c_0^{-q} c \sum_{j=0}^{\infty} \beta_j^q \left\|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f) | L_p\right\|^q .$$

Together with a similar estimate of the first l_0 summands this gives

$$\left\| f | B_{p,q}^{\sigma,N} \right\| \leq c \left\| f | B_{p,q}^{\beta} \right\| .$$

Again we obtain the opposite inequality by changing the roles of φ_k^N and φ_j. $\square$

Remark 2. Let Σ and N be corresponding functions to the sequences $(\sigma_k)_{k\in\mathbb{N}_0}$ and $(N_k)_{k\in\mathbb{N}_0}$, respectively, with N moreover strictly increasing and continuous. By N^{-1} we denote the inverse function, defined on $[N(1), \infty)$. Define

$$\hat{\beta}_j := \Sigma\big(N^{-1}\big(2^j\big)\big) \quad j \in \mathbb{N}_0, \text{ large enough} .$$

Then $(\hat{\beta}_j)_j$ is an admissible sequence, $\hat{\beta}_j \sim \beta_j$ and consequently $B^{\hat{\beta}}_{p,q} = B^{\beta}_{p,q} = B^{\sigma,N}_{p,q}$ and $F^{\hat{\beta}}_{p,q} = F^{\beta}_{p,q} = F^{\sigma,N}_{p,q}$, respectively.

4. Local Growth Envelopes for $F^{\sigma,N}_{p,q}$

We start with a result which will be crucial, allowing to reduce the study of the local growth envelopes for F-spaces to the corresponding problem for B-spaces.

Lemma 1. *Let* $0 < p_1 < p < p_2 \le \infty$, $0 < q \le \infty$, $N := (N_k)_{k\in\mathbb{N}_0}$ *according to the Assumption and* $\sigma := (\sigma_k)_{k\in\mathbb{N}_0}$ *an admissible sequence. Let* σ' *and* σ'' *be the (clearly admissible) sequences defined, respectively, by*

$$\sigma'_k = N_k^{n(\frac{1}{p_1}-\frac{1}{p})}\sigma_k, \quad \sigma''_k = N_k^{n(\frac{1}{p_2}-\frac{1}{p})}\sigma_k, \quad k \in \mathbb{N}_0 .$$

Then

$$B^{\sigma',N}_{p_1,u} \hookrightarrow F^{\sigma,N}_{p,q} \hookrightarrow B^{\sigma'',N}_{p_2,v}$$

if, and only if, $0 < u \le p \le v \le \infty$.

Proof. Note that, by the standardization result (Theorem 1)

$$F^{\sigma,N}_{p,q} = F^{\beta}_{p,q}, \quad \text{with } \beta_j := \sigma_{k(j)} ,$$

where $k(j) := \min\{k \in \mathbb{N}_0 : 2^{j-1} \le N_{k+l_0}\}$, $j \in \mathbb{N}$. We can also state that

$$B^{\sigma',N}_{p_1,u} = B^{\beta'}_{p_1,u} \quad \text{and} \quad B^{\sigma'',N}_{p_2,v} = B^{\beta''}_{p_2,v} ,$$

with $\beta'_j = 2^{n(\frac{1}{p_1}-\frac{1}{p})j}\beta_j$ and $\beta''_j = 2^{n(\frac{1}{p_2}-\frac{1}{p})j}\beta_j$, $j \in \mathbb{N}$. In fact, the standardization result allows us to write

$$B^{\sigma',N}_{p_1,u} = B^{\alpha'}_{p_1,u}, \quad \text{where } \alpha'_j = \sigma'_{k(j)} ,$$

that is,

$$\alpha'_j = N_{k(j)}^{n(\frac{1}{p_1}-\frac{1}{p})}\sigma_{k(j)} \sim 2^{n(\frac{1}{p_1}-\frac{1}{p})j}\sigma_{k(j)} = 2^{n(\frac{1}{p_1}-\frac{1}{p})j}\beta_j = \beta'_j .$$

Therefore,

$$B^{\sigma',N}_{p_1,u} = B^{\alpha'}_{p_1,u} = B^{\beta'}_{p_1,u} .$$

The proof of $B^{\sigma'',N}_{p_2,v} = B^{\beta''}_{p_2,v}$ is similar.

So, our lemma will be proved if we can show that

$$B^{\beta'}_{p_1,u} \hookrightarrow F^{\beta}_{p,q} \hookrightarrow B^{\beta''}_{p_2,v}$$

holds if, and only if, $0 < u \leq p \leq v \leq \infty$.

However, this follows immediately from [2, Proposition 4.7], due to the fact that β is also an admissible sequence (cf. statement of Theorem 1). □

The definition of the local growth envelope function requires that we are dealing with regular distributions. Therefore it is reasonable to get first some idea about the spaces of Triebel-Lizorkin type for which it makes sense to estimate such function. The following result goes in that direction.

Proposition 3. *Let $0 < p < \infty$, $0 < q \leq \infty$. Let σ and N be as in the preceding lemma. If*

$$\begin{cases} \left(\sigma_k^{-1} N_k^{\delta}\right)_{k\in\mathbb{N}_0} \in \ell_{p'}, \ \textit{for some } \delta > 0, & \textit{if} \quad 1 \leq p < \infty \\ \left(\sigma_k^{-1} N_k^{n(\frac{1}{p}-1)}\right)_{k\in\mathbb{N}_0} \in \ell_\infty, & \textit{if} \quad 0 < p < 1\,, \end{cases} \tag{4.1}$$

then

$$F^{\sigma,N}_{p,q} \subset L_1^{\mathrm{loc}}\,.$$

Proof. We take advantage of the preceding lemma (for $u = p = v$) in order to state that

$$F^{\sigma,N}_{p,q} \hookrightarrow B^{\sigma'',N}_{p_2,p}\,,$$

for any $p_2 > p$. So, we just have to prove that, for some suitable such p_2, $B^{\sigma'',N}_{p_2,p}$ is in L_1^{loc}.

From [3, Remark 3.20], this will be the case if $(\sigma_k''^{-1} N_k^{n(\frac{1}{p_2}-1)_+})_{k\in\mathbb{N}_0} \in \ell_{p'}$. When $0 < p < 1$, this follows from (4.1) by choosing $1 \geq p_2 > p$; when $1 \leq p < \infty$, again it follows from (4.1), now by choosing $p_2 > p$ such that $\frac{1}{p} - \frac{1}{p_2} \leq \delta$. □

Remark 3. The hypothesis (4.1) will not be enough for what we want to prove later, so we would like to remark that the condition

$$\left(\underline{\sigma}_l^{-1}\overline{N}_l^{n(\frac{1}{p}-1)_+ +\delta}\right)_{l\in\mathbb{N}_0} \in \ell_{\min\{1,p\}}, \quad \text{for some } \ \delta > 0\,, \tag{4.2}$$

also implies that $F^{\sigma,N}_{p,q} \subset L_1^{\mathrm{loc}}$. Actually, (4.2) implies (4.1), as follows easily from the facts $\sigma_l^{-1} \leq \sigma_0^{-1}\underline{\sigma}_l^{-1}$, $N_l \leq N_0\overline{N}_l$, $l \in \mathbb{N}_0$, and $\min\{1, p\} \leq p'$.

Just to get a feeling of how much of the cases where $F^{\sigma,N}_{p,q} \subset L_1^{\mathrm{loc}}$ we are ignoring when assuming (4.1) or (4.2), let us compare with the classical situation: When $N_k = 2^k$, $\sigma_k = 2^{ks}$, $s \in \mathbb{R}$, $k \in \mathbb{N}_0$, (4.1) is equivalent to

$$\begin{cases} s > 0, & \text{if} \quad 1 \leq p < \infty \\ s \geq n\left(\frac{1}{p} - 1\right)_+, & \text{if} \quad 0 < p < 1 \end{cases}, \tag{4.3}$$

while (4.2) is equivalent to

$$s > n\left(\frac{1}{p} - 1\right)_+\,. \tag{4.4}$$

In this classical setting, both (4.3) and (4.4) are "pretty close" to optimal, as it is well-known that for $s < n(\frac{1}{p} - 1)_+$ one never has $F^{\sigma,N}_{p,q} \subset L^{\mathrm{loc}}_1$.

We also would like to point out that, as artificial as the role of δ may seem, something stronger than the condition $(\sigma_k^{-1})_{k\in\mathbb{N}_0} \in \ell_\infty$ must in general be assumed in order to have $F^{\sigma,N}_{1,q} \subset L^{\mathrm{loc}}_1$: In the classical setting, [25, Theorem 3.3.2 (i)] tells us that $F^s_{1,q}$ is in L^{loc}_1 if, and only if,

$$\begin{cases} s \geq 0, & \text{when} \quad 0 < q \leq 2 \\ s > 0, & \text{when} \quad 2 < q \leq \infty \end{cases} ;$$

note that the condition $(2^{-ks})_{k\in\mathbb{N}_0} \in \ell_\infty$ is not strong enough to yield the second case above (that is, when $2 < q \leq \infty$).

The estimation of the growth envelope function is of interest only when there are unbounded functions around. So it makes sense to obtain beforehand some information about the spaces of Triebel-Lizorkin type which possess unbounded functions. The next proposition and the following remark go in that direction.

Proposition 4. *Let $0 < p < \infty$, $0 < q \leq \infty$, $N := (N_k)_{k\in\mathbb{N}_0}$ according to the Assumption and $\sigma := (\sigma_k)_{k\in\mathbb{N}_0}$ an admissible sequence. Then*

$$\left(\sigma_k^{-1} N_k^{\frac{n}{p}}\right)_{k\in\mathbb{N}_0} \in \ell_{p'} \quad \textit{if, and only if,} \quad F^{\sigma,N}_{p,q} \hookrightarrow C \,,$$

where C is the space of (complex-valued) bounded and uniformly continuous functions (on $\mathbb{R}^n$) endowed with the sup-norm.

Proof. We rely again on Lemma 1 and on a result for B-spaces corresponding to the assertion to be proved now.

(i) Assume that $(\sigma_k^{-1} N_k^{\frac{n}{p}})_{k\in\mathbb{N}_0} \in \ell_{p'}$. Then also $(\sigma_k''^{-1} N_k^{\frac{n}{p_2}})_{k\in\mathbb{N}_0} \in \ell_{p'}$, for any p_2 with $p < p_2 \leq \infty$, and therefore $F^{\sigma,N}_{p,q} \hookrightarrow B^{\sigma'',N}_{p_2,p} \hookrightarrow C$, see [3, Corollary 3.10].

(ii) Assume now that $F^{\sigma,N}_{p,q} \hookrightarrow C$. Since $B^{\sigma',N}_{p_1,p} \hookrightarrow F^{\sigma,N}_{p,q}$, for any $0 < p_1 < p$, then $B^{\sigma',N}_{p_1,p} \hookrightarrow C$ for all such p_1. By [3, Corollary 4.9], we can then state that $(\sigma_k^{-1} N_k^{\frac{n}{p}})_{k\in\mathbb{N}_0} = (\sigma_k'^{-1} N_k^{\frac{n}{p_1}})_{k\in\mathbb{N}_0} \in \ell_{p'}$. □

Remark 4.

(i) Such type of results was proved first by Kalyabin [19] with the restrictions $1 < p, q < \infty$.

(ii) It's easy to see that we can write L_∞ instead of C in the proposition above.

We recall now the main result for B-spaces which we want to "translate" for F-spaces here, concerning the behavior of the local growth envelope function

$$\mathcal{E}_{\mathrm{LG}}|B^{\sigma,N}_{p,q}(t) := \sup\left\{f^*(t) : \left\| f \mid B^{\sigma,N}_{p,q} \right\| \leq 1\right\} \tag{4.5}$$

near 0, where f^* stands for the decreasing rearrangement of f. In particular, the hypotheses guarantee that, for the parameters involved, (4.5) makes sense.

Proposition 5 ([3, Theorem 4.10]). *Let $0 < p, q \leq \infty$, $N := (N_k)_{k\in\mathbb{N}_0}$ according to the Assumption and $\sigma := (\sigma_k)_{k\in\mathbb{N}_0}$ an admissible sequence. Assume further that*

$$\begin{cases} \left(\underline{\sigma}_l^{-1}\right)_{l\in\mathbb{N}_0} \in \ell_{\min\{q,1\}} & \textit{if} \quad p > 1 \\ \left(\underline{\sigma}_l^{-1} \overline{N}_l^{\,n(\frac{1}{p}-1)+\delta}\right)_{l\in\mathbb{N}_0} \in \ell_{\min\{q,1\}}, \ \textit{for some} \ \ \delta > 0, & \textit{if} \quad 0 < p \leq 1 \,, \end{cases}$$

and

$$\left(\sigma_k^{-1} N_k^{\frac{n}{p}}\right)_{k\in\mathbb{N}_0} \notin \ell_{q'} \,.$$

Let Λ *be any admissible function such that* $\Lambda(z) \sim \sigma_k$*,* $z \in [N_k, N_{k+1}]$*,* $k \in \mathbb{N}_0$*, with equivalence constants independent of k, and let* Φ_u *be defined in* $(0, N_{J_0}^{-n}]$ *by*

$$\Phi_u(t) := \left(\int_{t^{1/n}}^{1} y^{-\frac{n}{p}u} \Lambda\left(y^{-1}\right)^{-u} \frac{dy}{y}\right)^{1/u} \quad \text{if} \quad 0 < u < \infty \,, \tag{4.6}$$

and

$$\Phi_u(t) := \sup_{t^{1/n} \le y \le 1} y^{-\frac{n}{p}} \Lambda\left(y^{-1}\right)^{-1} \quad \text{if} \quad u = \infty \,, \tag{4.7}$$

where $J_0 \in \mathbb{N}$ *is chosen such that* $N_{J_0} > 1$*.*

Then there exists $\varepsilon \in (0, 1)$ *such that*

$$\mathcal{E}_{LG}|B^{\sigma,N}_{p,q}(t) \sim \Phi_{q'}(t), \quad t \in (0, \varepsilon] \,, \tag{4.8}$$

and, considering the range $0 < v \le \infty$*, we have that*

$$\left(\int_0^{\varepsilon} \left(\frac{f^*(t)}{\Phi_{q'}(t)}\right)^{v} \mu_{q'}(dt)\right)^{1/v} \le c \left\| f | B^{\sigma,N}_{p,q} \right\| \tag{4.9}$$

(with the understanding

$$\sup_{t\in(0,\varepsilon]} \frac{f^*(t)}{\Phi_{q'}(t)} \le c \left\| f | B^{\sigma,N}_{p,q} \right\| \tag{4.10}$$

when $v = \infty$*) holds for some* $\varepsilon \in (0, 1)$*,* $c > 0$ *and all* $f \in B^{\sigma,N}_{p,q}$ *if, and only if,* $q \le v \le \infty$*. In* (4.9)*,* $\mu_{q'}$ *means the Borel measure associated with* $-\log_2 \Phi_{q'}$ *in* $(0, \varepsilon]$*.*

We recall the meaning of admissible function, used in the preceding assertion.

Definition 5. A function $\Lambda : (0, \infty) \to (0, \infty)$ is called admissible if it is continuous and if for any $b > 0$ satisfies

$$\Lambda(bz) \sim \Lambda(z) \quad \text{for any } z > 0$$

(where the equivalence constants may depend on b).

Remark 5. There is at least one admissible function related with the sequences σ and N as required in Proposition 5. This follows from [3, Example 2.3].

Lemma 2. *Let* Λ *be an admissible function. Let* $n \in \mathbb{N}$ *and* $0 < p_1, p, p_2 \le \infty$*. Then* Λ' *and* Λ'' *given by*

$$\Lambda'(z) = z^{n(\frac{1}{p_1} - \frac{1}{p})} \Lambda(z), \quad z \in (0, \infty) \,, \tag{4.11}$$

$$\Lambda''(z) = z^{n(\frac{1}{p_2} - \frac{1}{p})} \Lambda(z), \quad z \in (0, \infty) \,, \tag{4.12}$$

are both admissible. If moreover,

$$\Lambda(z) \sim \sigma_k, \quad z \in [N_k, N_{k+1}], \quad k \in \mathbb{N}_0 \,,$$

with equivalence constants independent of k, for given admissible sequences σ and N, then also

$$\Lambda'(z) \sim \sigma'_k, \quad z \in [N_k, N_{k+1}], \quad k \in \mathbb{N}_0 ,$$
$$\Lambda''(z) \sim \sigma''_k, \quad z \in [N_k, N_{k+1}], \quad k \in \mathbb{N}_0 ,$$

in both cases with equivalence constants independent of k, where σ' and σ'' are related to σ as in Lemma 1.

The proof is straightforward.

Remark 6. Recall now the definition (4.6) and (4.7) of Φ_u, which, in particular, depend on the p and Λ given. If we use, instead, p_1 and Λ', then we would write Φ'_u instead of Φ_u, and similarly if we use p_2 and Λ'', in which case we would write Φ''_u. Note, however, that

$$\Phi'_u = \Phi_u = \Phi''_u ,$$

as can easily be seen.

As a consequence, though we would define μ'_u and μ''_u as the Borel measures, respectively associated with $-\log_2 \Phi'_u$ and $-\log_2 \Phi''_u$ in $(0, \varepsilon]$, we see that, in fact,

$$\mu'_u = \mu_u = \mu''_u .$$

We recall that the Borel measure μ_u is the only measure defined on the Borel sets (of $(0, \varepsilon]$) such that $\mu_u([a, b]) = -\log_2 \Phi_u(b) + \log_2 \Phi_u(a) = \log_2 \frac{\Phi_u(a)}{\Phi_u(b)}$, $\forall [a, b] \subset (0, \varepsilon]$.

We are now ready to prove our main result for F-spaces, concerning the behavior of the local growth envelope function

$$\mathcal{E}_{\text{LG}}|F^{\sigma,N}_{p,q}(t) := \sup \left\{ f^*(t) : \| f \mid F^{\sigma,N}_{p,q} \| \le 1 \right\} \tag{4.13}$$

near 0, where—we recall—f^* stands for the decreasing rearrangement of f. We can, in particular, see that the hypotheses we are going to take guarantee—due to Remark 3—that (4.13) makes sense. We note also that the condition

$$\left(\sigma_k^{-1} N_k^{\frac{n}{p}} \right)_{k \in \mathbb{N}_0} \notin \ell_{p'} \tag{4.14}$$

will be assumed because—due to Proposition 4—otherwise (4.13) is bounded, and therefore the outcome has no interest.

Theorem 2. *Let $0 < p < \infty$, $0 < q \le \infty$, $N := (N_k)_{k \in \mathbb{N}_0}$ according to the Assumption and $\sigma := (\sigma_k)_{k \in \mathbb{N}_0}$ an admissible sequence. Assume further that (4.2) and (4.14) hold true. Let Λ be any admissible function such that $\Lambda(z) \sim \sigma_k$, $z \in [N_k, N_{k+1}]$, $k \in \mathbb{N}_0$, with equivalence constants independent of k, and let Φ_u be defined by (4.6) and (4.7) in $(0, N_{J_0}^{-n}]$, where $J_0 \in \mathbb{N}$ is chosen such that $N_{J_0} > 1$.*

Then there exists $\varepsilon \in (0, 1)$ such that

$$\mathcal{E}_{\text{LG}}|F^{\sigma,N}_{p,q}(t) \sim \Phi_{p'}(t), \quad t \in (0, \varepsilon] , \tag{4.15}$$

and, considering the range $0 < v \le \infty$, we have that

$$\left(\int_0^{\varepsilon} \left(\frac{f^*(t)}{\Phi_{p'}(t)} \right)^v \mu_{p'}(dt) \right)^{1/v} \le c \, \| f | F^{\sigma,N}_{p,q} \| \tag{4.16}$$

(with the understanding

$$\sup_{t\in(0,\varepsilon]} \frac{f^*(t)}{\Phi_{p'}(t)} \le c \, \| f | F^{\sigma,N}_{p,q} \| \tag{4.17}$$

when $v = \infty$) holds for some $\varepsilon \in (0,1)$, $c > 0$ and all $f \in F^{\sigma,N}_{p,q}$ if, and only if, $p \le v \le \infty$. As before, in (4.16) $\mu_{p'}$ stands for the Borel measure associated with $-\log_2 \Phi_{p'}$ in $(0,\varepsilon]$.

Proof. We take advantage of Lemma 1. In particular, in what follows we adhere to the notation σ' and σ'' as considered there

Step 1. Note that the hypothesis (4.14) guarantee that

$$\left(\sigma_k''^{-1} N_k^{\frac{n}{p_2}}\right)_{k\in\mathbb{N}_0} \notin \ell_{p'} \ , \tag{4.18}$$

for any p_2 with $p < p_2 \le \infty$, and that the hypothesis (4.2) guarantees that

$$\begin{cases} \left(\underline{\sigma}_l''^{-1}\right)_{l\in\mathbb{N}_0} \in \ell_1 & \text{if} \quad p \ge 1 \\ \left(\underline{\sigma}_l''^{-1} \overline{N}_l^{\,n(\frac{1}{p_2}-1)+\delta}\right)_{l\in\mathbb{N}_0} \in \ell_p, \ \text{for some} \ \delta > 0, & \text{if} \quad 0 < p < 1 \ , \end{cases} \tag{4.19}$$

for any $p_2 \in (p, \infty]$ close enough to p.

Since (4.18) is clear, we just prove (4.19).

Start by observing that

$$\underline{\sigma}_l'' \ge \underline{\sigma}_l \overline{N}_l^{\,n(\frac{1}{p_2}-\frac{1}{p})}, \quad l \in \mathbb{N}_0 \ .$$

Take $p \ge 1$. Then $\sum_{l=0}^{\infty} \underline{\sigma}_l''^{-1} \le \sum_{l=0}^{\infty} \underline{\sigma}_l^{-1} \overline{N}_l^{\,n(\frac{1}{p}-\frac{1}{p_2})} < \infty$, if $p_2 > p$ is chosen such that $n(\frac{1}{p}-\frac{1}{p_2}) \le \delta$, with δ given in (4.2).

Take now $0 < p < 1$. Then

$$\sum_{l=0}^{\infty} \left(\underline{\sigma}_l''^{-1} \overline{N}_l^{\,n(\frac{1}{p_2}-1)+\delta}\right)^p \le \sum_{l=0}^{\infty} \left(\underline{\sigma}_l^{-1} \overline{N}_l^{\,n(\frac{1}{p}-1)+\delta}\right)^p < \infty \ ,$$

using the same δ as in (4.2).

Note that from (4.19) it also follows that, for $p \ge 1$, we can choose $p_2 > p$ close enough to p such that we have both

$$\left(\underline{\sigma}_l''^{-1}\right)_{l\in\mathbb{N}_0} \in \ell_{\min\{p,1\}} \quad \text{and} \quad p_2 > 1 \ ,$$

while for $0 < p < 1$ we can choose $p_2 > p$ close enough to p such that we have both

$$\left(\underline{\sigma}_l''^{-1} \overline{N}_l^{\,n(\frac{1}{p_2}-1)+\delta}\right)_{l\in\mathbb{N}_0} \in \ell_{\min\{p,1\}} \quad \text{and} \quad 0 < p_2 \le 1 \ .$$

Using this, (4.18), Lemma 2, and Remark 6, from Proposition 5 we immediately get that there exists $\varepsilon \in (0,1)$, $c > 0$ such that

$$\mathcal{E}_{\text{LG}} | B^{\sigma'',N}_{p_2,p}(t) \le c \, \Phi_{p'}(t), \quad t \in (0,\varepsilon] \ , \tag{4.20}$$

and also that for any $v \in [p, \infty]$ there is $\varepsilon \in (0, 1)$, $c > 0$ such that

$$\left(\int_0^{\varepsilon} \Big(\frac{f^*(t)}{\Phi_{p'}(t)} \Big)^v \mu_{p'}(dt) \right)^{1/v} \leq c \, \| f | B^{\sigma'',N}_{p_2,p} \| \tag{4.21}$$

(modification if $v = \infty$) for all $f \in B^{\sigma'',N}_{p_2,p}$.

Using the embedding $F^{\sigma,N}_{p,q} \hookrightarrow B^{\sigma'',N}_{p_2,p}$ given by Lemma 1, it is easily seen that (4.20) and (4.21) also hold true with $F^{\sigma,N}_{p,q}$ instead of $B^{\sigma'',N}_{p_2,p}$.

Step 2. Instead of Proposition 5, we will use now more precise partial results contained in [3].

Recall that, by Remark 3, the hypothesis (4.2) guarantees that $F^{\sigma,N}_{p,q} \subset L_1^{\mathrm{loc}}$. Thus, using Lemma 1, we can also assert that $B^{\sigma',N}_{p_1,p} \subset L_1^{\mathrm{loc}}$, for any $0 < p_1 < p$, and therefore [3, Proposition 4.8] assures us that there exists $\varepsilon \in (0, 1)$, $c > 0$ such that

$$\mathcal{E}_{\mathrm{LG}} | B^{\sigma',N}_{p_1,p}(t) \geq c \, \Phi'_{p'}(t), \quad t \in (0, \varepsilon] \, .$$

Hence, Lemmas 1, 2, and Remark 6 allow us to write also

$$\mathcal{E}_{\mathrm{LG}} | F^{\sigma,N}_{p,q}(t) \geq c \, \Phi_{p'}(t), \quad t \in (0, \varepsilon] \, ,$$

for some $\varepsilon \in (0, 1)$, $c > 0$.

To see that (4.16) cannot hold for $0 < v < p$, assume, on the contrary, that, for such v, there existed $\varepsilon \in (0, 1)$, $c > 0$ such that

$$\left(\int_0^{\varepsilon} \Big(\frac{f^*(t)}{\Phi_{p'}(t)} \Big)^v \mu_{p'}(dt) \right)^{1/v} \leq c \, \| f | F^{\sigma,N}_{p,q} \|$$

for all $f \in F^{\sigma,N}_{p,q}$. Then, using Lemmas 1, 2, and Remark 6, there would also exist $\varepsilon \in (0, 1)$, $c > 0$ such that

$$\left(\int_0^{\varepsilon} \Big(\frac{f^*(t)}{\Phi'_{p'}(t)} \Big)^v \mu'_{p'}(dt) \right)^{1/v} \leq c \, \| f | B^{\sigma',N}_{p_1,p} \| \, . \tag{4.22}$$

for all $f \in B^{\sigma',N}_{p_1,p}$. But since, as we have already seen above, $B^{\sigma',N}_{p_1,p} \subset L_1^{\mathrm{loc}}$ and (4.14) trivially implies that $(\sigma_k'^{-1} N_k^{\frac{n}{p_1}})_{k \in \mathbb{N}_0} \notin \ell_{p'}$, then this would contradict [3, Remark 4.11] [(4.22) can only hold for $v \geq p$]. □

Remark 7.

(1) It is possible to choose ε independent of v in (4.16). In fact, the ε taken for $v = p$ can also be taken for all $v \geq p$—see [27, Proposition 12.2 (i)].

(2) As noted in [3, Remark 4.15], when $1 < p < \infty$, the measure $\mu_{p'}(dt)$ in (4.16) can be replaced by

$$\frac{dt}{\Phi_{p'}(t)^{p'} \, t^{p'} \, \Lambda\big(t^{-1/n}\big)^{p'}} \, .$$

In the particular case when $N_k = 2^k$ and $\sigma_k = 2^{ks} \Psi(2^{-k})$, $k \in \mathbb{N}_0$, with $n(\frac{1}{p}-1)_+ < s < \frac{n}{p}$ and Ψ a "log-type" function (cf. also comments after this remark)—the so-called subcritical case in the setting of [4]—when, again, $1 < p < \infty$, $\mu_{p'}(dt)$ can be further simplified to $\frac{dt}{t}$—cf. [4, Remark 4.8]. Actually, in that subcritical case, and independently of the value of p, $\Phi_{p'}(t)$ itself can be simplified to $t^{\frac{s}{n}-\frac{1}{p}}\Psi(t)^{-1}$.

For other cases where simpler expressions, both for $\Phi_{p'}$ and for $\mu_{p'}$, can be taken instead of our general ones, see the work of Bricchi and Moura [2].

We would like to point out that Theorem 2 covers and extends the results previously obtained by Caetano and Moura [5, 4], which already covered the general statements of Haroske [17] and Triebel [27], as far as growth envelopes for Triebel-Lizorkin-type spaces are concerned. To see this, one just has to note that the spaces considered in [4, Corollary 4.5] are included in our main theorem. It is, in fact, an easy exercise to show that the spaces $F^{\sigma,N}_{p,q}$ considered there, namely with $N_k = 2^k$ and $\sigma_k = 2^{ks}\Psi(2^{-k})$, $k \in \mathbb{N}_0$, where $n(\frac{1}{p} - 1)_+ < s \le \frac{n}{p}$ and Ψ a so-called admissible function (in the context of that article) satisfying $(\Psi(2^{-k})^{-1})_{k\in\mathbb{N}_0} \notin \ell_{p'}$ when $s = \frac{n}{p}$, are such that $(\sigma_k)_{k\in\mathbb{N}_0}$ and $(N_k)_{k\in\mathbb{N}_0}$ are admissible (in our sense), with the latter satisfying also the Assumption and, moreover, (4.2) and (4.14) hold.

Our Theorem 2 also covers and extends Theorem 6.3 (ii) of [2]: It is again an easy exercise to see that the spaces $F^{\sigma,N}_{p,q}$ considered there, namely with $N_k = 2^k$, $k \in \mathbb{N}_0$, and σ an admissible sequence satisfying $n(\frac{1}{p} - 1)_+ < \lim_{l\to\infty} \frac{\log_2 \underline{\sigma}_l}{l} \le \lim_{l\to\infty} \frac{\log_2 \overline{\sigma}_l}{l} < \frac{n}{p}$, verify all the requirements of our Theorem 2.

We finish with a corollary of Theorem 2 which gives more precise information about the sharpness of our results. This is similar to what happens with Besov-type spaces. A proof can be obtained with the help of [27, Proposition 12.2].

Corollary 1. *Consider the same hypothesis of Theorem 2 and* $0 < \varepsilon \le N_{J_0}^{-n}$. *Let* κ *be a positive monotonically decreasing function on* $(0, \varepsilon]$ *and let* $0 < u \le \infty$. *Then*

$$\left(\int_0^{\varepsilon} \left(\kappa(t)\frac{f^*(t)}{\Phi_{p'}(t)}\right)^u \mu_{p'}(dt)\right)^{1/u} \le c \left\| f | F^{\sigma,N}_{p,q} \right\| ,$$

for some $c > 0$ *and all* $f \in F^{\sigma,N}_{p,q}$ *if, and only if,* κ *is bounded and* $p \le u \le \infty$, *with the modification*

$$\sup_{t\in(0,\varepsilon]} \kappa(t)\frac{f^*(t)}{\Phi_{p'}(t)} \le c \left\| f | F^{\sigma,N}_{p,q} \right\| \tag{4.23}$$

if $u = \infty$. *Moreover, if* κ *is an arbitrary nonnegative function on* $(0, \varepsilon]$, *then* (4.23) *above holds if, and only if,* κ *is bounded.*

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