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Function Spaces Associated with Schrödinger Operators: The Pöschl-Teller Potential

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ABSTRACT. *We address the function space theory associated with the Schrödinger operator $H = -d^2/dx^2 + V$. The discussion is featured with potential $V(x) = -n(n+1)\operatorname{sech}^2 x$, which is called in quantum physics Pöschl-Teller potential. Using a dyadic system, we introduce Triebel-Lizorkin spaces and Besov spaces associated with H. We then use interpolation method to identify these spaces with the classical ones for a certain range of $p, q > 1$. A physical implication is that the corresponding wave function $\psi(t, x) = e^{-itH} f(x)$ admits appropriate time decay in the Besov space scale.*

1. Introduction

Let $H = -d^2/dx^2 + V$ be a Schrödinger operator on $\mathbb{R}$ with real-valued potential function V. In quantum physics, H is the energy operator of a particle having one degree of freedom with potential V. If the potential has certain decay at ∞, then one may expect that asymptotically, as time tends to infinity, the motion of the associated perturbed quantum system resembles the free evolution. Indeed, it is well-known that if $\int_{\mathbb{R}}(1 + |x|)|V(x)|\,dx < \infty$, then the absolute continuous spectrum of H is $[0, \infty)$, the singular continuous spectrum is empty, and there is only finitely many negative eigenvalues. Moreover, the wave operators $W_\pm = s - \lim_{t\to\pm\infty} e^{itH} e^{-itH_0}$ exists and are complete [5, 10, 26].

Recently, several authors have studied function spaces associated with Schrödinger operators [19, 13, 14, 11, 12, 2]. One of the goals has been to develop the associated

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Littlewood-Paley theory, in order to give a unified approach. Motivated by the treatment in [2, 13] for the barrier and Hermite cases, we consider H with the negative potential

$$V_n(x) = -n(n+1)\,\mathrm{sech}^2 x, \quad n \in \mathbb{N}, \tag{1.1}$$

which is called the Pöschl-Teller potential [4, 18]. The study of H with this potential is related to linearization of nonlinear wave and Schrödinger equations. In this article, we are mainly concerned with characterization and identification of the Triebel-Lizorkin spaces and Besov spaces associated with H. Notice that in contrast to the potentials studied in [2, 13, 11, 12], $H = H_0 + V_n$ is not a positive operator and it has a resonance at zero.

Suppose $\{\varphi_j\}_0^\infty \subset C_0^\infty(\mathbb{R})$ satisfy: (i) supp $\varphi_0 \subset \{|x| \le 1\}$, supp $\varphi_j \subset \{2^{j-2} \le |x| \le 2^j\}$, $j \ge 1$; (ii) $|\varphi_j^{(m)}(x)| \le c_m 2^{-mj}$, $\forall j, m \in \mathbb{N}_0$; and (iii)

$$\sum_{j=0}^\infty \varphi_j(x) = 1, \qquad \forall\, x \in \mathbb{R}. \tag{1.2}$$

Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$. The *Triebel-Lizorkin space associated with H,* denoted by $F_p^{\alpha,q}(H)$, is defined to be the completion of the subspace $L_0^2 := \{f \in L^2(\mathbb{R}) : \|f\|_{F_p^{\alpha,q}(H)} < \infty\}$, where the quasi-norm $\|\cdot\|_{F_p^{\alpha,q}(H)}$ is initially defined for $f \in L^2(\mathbb{R})$ as

$$\|f\|_{F_p^{\alpha,q}(H)} = \left\| \left(\sum_{j=0}^\infty 2^{j\alpha q} |\varphi_j(H) f|^q \right)^{1/q} \right\|_{L^p} \tag{1.3}$$

(with usual modification if $q = \infty$). Similarly, the *Besov space associated with H,* denoted by $B_p^{\alpha,q}(H)$, is defined by the quasi-norm

$$\|f\|_{B_p^{\alpha,q}(H)} = \left(\sum_{j=0}^\infty 2^{j\alpha q} \|\varphi_j(H) f\|_{L^p}^q \right)^{1/q}. \tag{1.4}$$

In Section 3 we give a maximal function characterization of $F_p^{\alpha,q}(H)$. We show in Theorem 2 that

$$\|f\|_{F_p^{\alpha,q}(H)} \approx \left\| \left(\sum_{j=0}^\infty \left(2^{j\alpha} \varphi_{j,s}^* f\right)^q \right)^{1/q} \right\|_p, \tag{1.5}$$

where $\varphi_{j,s}^* f$ is the Peetre type maximal function with $s > 1/\min(p,q)$. Therefore the definition of the $F_p^{\alpha,q}(H)$-norm is independent of the choice of $\{\varphi\}_{j\ge 0}$.

The proof of (1.5) essentially depends on the decay estimates in Lemma 3 for the kernel of $\varphi_j(H)$, which can be expressed in terms of continuum and discrete eigenfunctions of H. In Section 2 we solve the eigenfunction Equation (2.1) for $k \in \mathbb{R} \cup \{i, \dots, ni\}$ ($i = \sqrt{-1}$), based on a method suggested in [20]. In Section 4, using the explicit kernel of $\varphi_j(H)$ we give a proof of Lemma 3 for high and local energies. It turns out that for the absolute continuous part of H, the high and local energy analysis is simpler than the barrier potential, although H has a nonempty pure point spectrum.

A natural question arises: What is the relation between the perturbed function spaces and the ordinary ones, namely, $F_p^{\alpha,q}(\mathbb{R})$ and $B_p^{\alpha,q}(\mathbb{R})$? In this regard, we show in Section 5 that $F_p^{0,2}(H)$ is identically the L^p space, $1 < p < \infty$. Furthermore, in Section 6 we obtain the following result (Theorem 5) by means of complex interpolation: If $\alpha > 0$, $1 < p < \infty$ and $2p/(p+1) < q < 2p$, then

$$F_p^{\alpha,q}(H) = F_p^{2\alpha,q}(\mathbb{R}) \tag{1.6}$$

and if $\alpha > 0$, $1 \le p < \infty$, $1 \le q \le \infty$, then

$$B_p^{\alpha,q}(H) = B_p^{2\alpha,q}(\mathbb{R}) .$$

The method in proving $F_p^{0,2}(H) = L^p$ is similar to the Hermite case [13]. However, the identification (1.6) seems new for $\alpha > 0$. It is not difficult to see that the analogue of (1.6) does not hold for the Hermite case, where the potential is x^2.

As an application of the function space method we obtain a global time decay result (Theorem 6) for the solution to the Schrödinger Equation (6.1), namely,

$$\left\| e^{-itH} f \right\|_{L^{p'}} \lesssim \langle t \rangle^{-(\frac{1}{p}-\frac{1}{2})} \| f \|_{B_p^{4\beta,2}(\mathbb{R})}$$

for any f in the continuous subspace, $1 < p \le 2$ and $\beta = |\frac{1}{p} - \frac{1}{2}|$ being the critical exponent, which is a consequence of the local and long time decay estimates from [19] and [17]. Here the perturbed function spaces play an important role in the interpretation of the mapping properties of operators between the abstract and classical spaces. It provides a necessary tool in realizing the above inequality by means of embedding and interpolation.

Finally, we mention that the homogeneous F and B spaces seem to deserve special attention. The crucial reason is that, to our surprise somehow, the decay estimates for the *low energy* ($-\infty < j < 0$) that are required for the derivative of $\varphi_j(H)E_{ac}(x, y)$ does not hold, which leaves open the question on obtaining the homogeneous version of Theorem 2. In a sequel to this article we will consider the homogeneous case and study the spectral multiplier problem on the F and B spaces.

2. The Eigenfunctions of H

Let $V_n = -n(n+1)\,\text{sech}^2 x$ and $H_0 = -d^2/dx^2$. In this section we derive a simple expression for the continuum eigenfunctions of $H = H_0 + V_n$, which are the scattering solutions to the Lippman-Schwinger Equation (2.3). We also show that the bound state eigenfunctions are rapid decaying functions.

2.1 Scattering Equation

Consider the eigenvalue problem for $(1 + |x|)V \in L^1$,

$$He(x, k) = k^2 e(x, k), \quad k \in \mathbb{R} , \tag{2.1}$$

with asymptotics

$$e_\pm(x, k) \sim \begin{cases} T_\pm(k)e^{ikx} & \text{if} \quad x \to \pm\infty \\ e^{ikx} + R_\pm(k)e^{-ikx} & \text{if} \quad x \to \mp\infty , \end{cases} \tag{2.2}$$

where $\pm$ indicate the sign of k. We will use the notation

$$e(x,k) = \begin{cases} e_+(x,k) & \text{if} \quad k > 0 \\ e_-(x,k) & \text{if} \quad k < 0 \,. \end{cases}$$

The coefficients $T_\pm(k)$ and $R_\pm(k)$ in (2.2) are called the *transmission coefficients* and *reflection coefficients,* resp. They satisfy the conservation law $|T_\pm(k)|^2 + |R_\pm(k)|^2 = 1$. It is easy to see that (2.1) together with (2.2) is equivalent to the Lippman-Schwinger equation

$$e_\pm(x,k) = e^{ikx} + \frac{1}{2i|k|} \int e^{i|k|\,|x-y|} V(y) e_\pm(y,k)\, dy \,. \tag{2.3}$$

2.2 Inductive Construction of the Solution

Let y_n be the general solution of

$$y_n'' + n(n+1)\,\mathrm{sech}^2 x\; y_n = -k^2 y_n \,.$$

If $n = 0$, $y_0 = Ae^{ikx} + Be^{-ikx}$. If $n \geq 1$, according to [20, Section 2.6] we have by induction

$$y_n(x) = A(k) D_n \cdots D_1\big(e^{ikx}\big) + B(k) D_n \cdots D_1\big(e^{-ikx}\big) \,,$$

where D_n denotes the differential operator

$$D_n = \frac{d}{dx} - n \tanh x, \qquad n \in \mathbb{N} \,. \tag{2.4}$$

Here we observe that since $\frac{d}{dx}(\tanh x) = 1 - \tanh^2 x$,

$$\begin{aligned} D_n \cdots D_1\big(e^{ikx}\big) &= p_n(\tanh x, ik) e^{ikx} \,, \\ D_n \cdots D_1\big(e^{-ikx}\big) &= q_n(\tanh x, ik) e^{-ikx} \,, \end{aligned} \tag{2.5}$$

where $p_n(x,k)$ and $q_n(x,k)$ are polynomials of degree n in x, k and have real coefficients.

Let $e_n(x,k)$ denote the particular solution of (2.3) with $V = V_n$. Using the asymptotics (2.2) we solve $e_n(x,k)$ as in the following lemma.

Lemma 1. *Let $n \in \mathbb{N}$. There exists a polynomial $p_n(x,k)$ of degree n in x, k such that*

$$e_{n,\pm}(x,k) = A_n^\pm(k) p_n(\tanh x, ik) e^{ikx} \,.$$

Furthermore the following hold.

(a) *The constants $A_n^\pm(k)$ are given by*

$$A_n^+(k) = \prod_{j=1}^n \frac{1}{j+ik} \qquad \text{and} \qquad A_n^-(k) = (-1)^n \prod_{j=1}^n \frac{1}{j-ik} \,.$$

(b) *The transmission coefficients $T_{n,\pm}(k)$ are*

$$T_{n,+}(k) = (-1)^n \prod_{j=1}^n \frac{j-ik}{j+ik} \qquad \text{and} \qquad T_{n,-}(k) = (-1)^n \prod_{j=1}^n \frac{j+ik}{j-ik} \,.$$

(c) *The reflection coefficients $R_{n,\pm}(k)$ are all zero.*

Proof. In light of the above discussion we write

$$e_{n,\pm}(x,k) = A_n^{\pm}(k)p_n(\tanh x, ik)e^{ikx} + B_n^{\pm}(k)q_n(\tanh x, ik)e^{-ikx} . \tag{2.6}$$

First we assume $k > 0$. Substituting (2.6) into the (2.2), we obtain that $B_n^{+}(k) = 0 = R_{n,+}(k)$,

$$A_n^{+}(k)p_n(-1, ik) = 1 \tag{2.7}$$

and

$$T_{n,+}(k) = A_n^{+}(k)p_n(1, ik) = \frac{p_n(1, ik)}{p_n(-1, ik)} . \tag{2.8}$$

Thus, (2.6) becomes

$$e_{n,+}(x,k) = A_n^{+}(k)p_n(\tanh x, ik)e^{ikx} .$$

From (2.5) we easily derive the recurrence formula

$$p_n(\tanh x, ik) = \operatorname{sech}^2 x\ p'_{n-1}(\tanh x, ik) + (ik - n\tanh x)p_{n-1}(\tanh x, ik) . \tag{2.9}$$

Since $p'_{n-1}(x,k)$ is a polynomial in x, it follows that

$$\lim_{x\to\pm\infty} p'_{n-1}(\tanh x, ik) = p'_{n-1}(\pm 1, ik)$$

is bounded. Taking the limit in (2.9) as $x \to \pm\infty$ we find

$$p_n(\pm 1, ik) = (ik \mp n)p_{n-1}(\pm 1, ik) .$$

Since $e_0(x,k) = e^{ikx}$, i.e., $p_0 = 1$, $A_0^{+} = 1$, we obtain

$$p_n(1, ik) = (-1)^n \prod_{j=1}^{n}(j - ik)$$

and

$$p_n(-1, ik) = \prod_{j=1}^{n}(j + ik) = (-1)^n \overline{p_n(1,k)} .$$

Now for $k > 0$, (a), (b) in the lemma follow from (2.7), (2.8).

For k negative, similarly it holds that $B_n^{-}(k) = 0 = R_{n,-}(k)$ and instead of (2.7), (2.8), we have

$$A_n^{-}(k)p_n(1, ik) = 1$$

and

$$T_{n,-}(k) = A_n^{-}(k)p_n(-1, ik) .$$

Then the results for A_n^{-}, $T_{n,-}$ and $e_{n,-}(x,k)$ follow. □

From (2.5) we can also see

$$p_n(\tanh x, -ik) = (-1)^n p_n(-\tanh x, ik) \tag{2.10}$$

by simple induction. Thus, we obtain the following formula for the continuum eigenfunctions.

Theorem 1. *Assume $k \in \mathbb{R} \setminus \{0\}$. Then*

$$e_n(x,k) = (\mathrm{sign}(k))^n \left(\prod_{j=1}^{n} \frac{1}{j + i|k|} \right) P_n(x,k) e^{ikx} ,$$

where $P_n(x,k) = p_n(\tanh x, ik)$ is defined by the recursion formula

$$p_n(\tanh x, ik) = \frac{d}{dx}\big(p_{n-1}(\tanh x, ik)\big) + (ik - n\tanh x) p_{n-1}(\tanh x, ik) .$$

In particular, the function

$$\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \ni (x,k) \mapsto e_n(x,k) \in \mathbb{C}$$

is analytic with $e_n(x,-k) = e_n(-x,k)$. Moreover, the function

$$(x,y,k) \mapsto e_n(x,k)\overline{e_n(y,k)} = \left(\prod_{j=1}^{n} \frac{1}{j^2 + k^2} \right) P_n(x,k) P_n(y,-k) e^{ik(x-y)}$$

is real analytic on $\mathbb{R}^3$.

2.3 The Point Spectrum

For $(1+|x|)V \in L^1$, we know that the point spectrum of $H_0 + V$ is given by the simple eigenvalues $-\mu^2$ such that $T_+(k)$ has a (simple) pole at $i\mu$; see e.g., [10, p. 146]. Therefore we have the following.

Lemma 2. *The point spectrum of $H = H_0 + V_n$ consists of*

$$\sigma_{pp} = \big\{ -1, -4, \ldots, -n^2 \big\} .$$

The corresponding eigenfunctions are Schwartz functions that are linear combinations of $\mathrm{sech}^m x \tanh^k x$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$.

Proof. The statement about σ_{pp} follows from the fact that $k = ij$, $j = 1, \ldots, n$, are the poles of $T_{n,+}(k) = (-1)^n \prod_{j=1}^{n} (j - ik)(j + ik)^{-1}$. For $k^2 = -j^2$, let $y_{n,j}$ be the corresponding eigenfunction. By induction we find that

$$\begin{aligned} y_{j,j} &= \mathrm{sech}^j x \\ y_{j+1,j} &= D_{j+1} \mathrm{sech}^j x \\ y_{j+m,j} &= D_{j+m} y_{j+m-1,j} , \qquad m \in \mathbb{N} . \end{aligned}$$

Hence, the bound states are given by

$$y_{n,j}(x) = D_n \cdots D_{j+1} \mathrm{sech}^j x, \quad j = 1, \ldots, n-1 ,$$

and

$$y_{n,n}(x) = \operatorname{sech}^n x\ . \qquad \square$$

Remark 1. There is a continuous extension of V_n when n is replaced by a continuous parameter in $\mathbb{R}$. We can find the scattering solutions of (2.3) by using the two real fundamental solutions given in [15]. However, we do not intend to include them here since the expression (which involves hypergeometric functions) seems quite complicated.

2.4 Projection of the Spectral Operator $\phi(H)$

Given $V \in L^1 \cap L^2$, it is known that $H = H_0 + V$ is selfadjoint on the domain $D(H) = D(H_0) = W_2^2(\mathbb{R})$, the usual Sobolev space of order 2 in L^2. We decompose $L^2 = \mathcal{H}_{ac} \oplus \mathcal{H}_{pp}$, where $\mathcal{H}_{ac}$ denotes the absolute continuous subspace and $\mathcal{H}_{pp}$ the pure point subspace. Let E_{ac}, E_{pp} be the corresponding orthogonal projections, respectively. For a measurable function ϕ we define $\phi(H)$ by functional calculus as usual. Then it follows that

$$\phi(H)f = \phi(H)E_{ac}f + \phi(H)E_{pp}f = \phi(H)\big|_{\mathcal{H}_{ac}} f + \phi(H)\big|_{\mathcal{H}_{pp}} f\ .$$

Let $e(x,k)$ be the scattering solution of (2.3) and $e_j(x)$ the eigenfunction of H with (negative) eigenvalue λ_j. If ϕ is continuous and compactly supported, we have the following expression [26]

$$\phi(H)f(x) = \int K_{ac}(x,y)f(y)\,dy + \sum_{\lambda_j \in \sigma_{pp}} \phi(\lambda_j)(f,e_j)e_j, \quad f \in L^1 \cap L^2\ , \tag{2.11}$$

where

$$K_{ac}(x,y) = (2\pi)^{-1} \int \phi\big(k^2\big)e(x,k)\bar{e}(y,k)\,dk \tag{2.12}$$

is the kernel of $\phi(H)E_{ac}$. Note that if $e(x,k)$ is smooth in x, then $K_{ac}(x,y)$ is smooth in x, y. If letting $K_{pp}(x,y) = \sum_j \phi(\lambda_j)e_j(x)e_j(y)$, we can write (2.11) in a more compact form

$$\phi(H)f(x) = \int K(x,y)f(y)\,dy\ , \tag{2.13}$$

where $K = K_{ac} + K_{pp}$. We mention that in the case $(1+|x|)V \in L^1$ the kernel formula (2.12) agrees with the usual one using the Jost functions [17, 10].

3. Maximal Function Characterization

Let $H = H_0 + V_n$. This section is mainly to give a quasi-norm characterization of $F_p^{\alpha,q}(H)$ and $B_p^{\alpha,q}(H)$ using Peetre type maximal function. Consequently, the $F(H)$ and $B(H)$ spaces are well-defined in the sense that different dyadic systems give rise to equivalent quasi-norms.

Let $\{\varphi_j\}_0^\infty$ be a system satisfying conditions (i), (ii) as in Section 1, i.e.,

(i) supp $\varphi_0 \subset [-1,1]$, supp $\varphi_j \subset [-2^j, -2^{j-2}] \cup [2^{j-2}, 2^j]$, $j \geq 1$;

(ii) $|\varphi_j^{(m)}(x)| \le c_m 2^{-mj}, \quad \forall j, m \in \mathbb{N}_0.$

Denote $K_j(x, y) = \varphi_j(H)(x, y)$ the kernel of $\varphi_j(H)$ as given by the formula (2.13). To simplify notation we let

$$w_j(x) := 1 + 2^{j/2}|x| \,. \tag{3.1}$$

Lemma 3. *Let $j \ge 0$. Then for each $m \in \mathbb{N}_0$ there exist constants $C_m, C'_m > 0$ such that*

(a) $|K_j(x, y)| \le C_m 2^{j/2} w_j(x-y)^{-m}$

(b) $|\frac{\partial}{\partial x} K_j(x, y)| \le C'_m 2^j w_j(x-y)^{-m}$.

We postpone the proof till Section 4.

For $s > 0$ define the analogue of Peetre maximal function:

$$\varphi^*_{j,s} f(x) = \sup_{t\in\mathbb{R}} \frac{|\varphi_j(H) f(t)|}{w_j(x-t)^s} \tag{3.2}$$

and

$$\varphi^{**}_{j,s} f(x) = \sup_{t\in\mathbb{R}} \frac{\left|(\varphi_j(H) f)'(t)\right|}{w_j(x-t)^s} \,.$$

Lemma 4. *Let $s > 0$ and $j \in \mathbb{N}_0$. Then there exists a constant $C = C_s > 0$ such that*

$$\varphi^{**}_{j,s} f(x) \le C 2^{j/2} \varphi^*_{j,s} f(x) \,.$$

Before the proof we note the following identity that will be used often later on. Suppose $\{\psi_j\}$ be a dyadic system as in Section 1. Then

$$\varphi_j(H) f = \sum_{\nu=-1}^{1} \psi_{j+\nu}(H)\varphi_j(H) f, \qquad f \in L^2 \,, \tag{3.3}$$

with the convention $\psi_{-1} \equiv 0$, which follows from the equality $\varphi_j(x) = \sum_{\nu=-1}^{1} \psi_{j+\nu}(x)\varphi_j(x)$ for all x.

Proof. By (3.3) we have

$$\frac{d}{dt}(\varphi_j(H) f)(t) = \sum_{\nu=-1}^{1} \int_{\mathbb{R}} \frac{\partial}{\partial t}(\psi_{j+\nu}(H)(t, y))\varphi_j(H) f(y)\, dy \,.$$

Apply Lemma 3 to obtain

$$\frac{\left|\frac{d}{dt}(\varphi_j(H) f)(t)\right|}{w_j(x-t)^s} \le C_m \sum_{\nu=-1}^{1} 2^{j+\nu} \int_{\mathbb{R}} \frac{|\varphi_j(H) f(y)|}{w_{j+\nu}(t-y)^m w_j(x-t)^s}\, dy \,.$$

It follows from the definition of $\varphi^*_{j,s} f$ that

$$\begin{aligned}\frac{\left|\frac{d}{dt}(\varphi_j(H) f)(t)\right|}{w_j(x-t)^s} &\le C_m \sum_{\nu=-1}^{1} 2^{j+\nu} \varphi^*_{j,s} f(x) \int_{\mathbb{R}} \frac{w_j(t-y)^s}{w_{j+\nu}(t-y)^m}\, dy \\ &\le C_s 2^{j/2} \varphi^*_{j,s} f(x) \,,\end{aligned}$$

provided $m - s > 1$. This proves Lemma 4. □

The next lemma (Peetre maximal inequality) follows from Lemma 4 by a standard argument; see [21, p. 16] or [2]. Let M be the Hardy-Littlewood maximal function

$$Mf(x) := \sup_I \frac{1}{|I|} \int_I |f(x+y)|\, dy \,,$$

where the supremum runs over all intervals in $(-\infty, \infty)$.

Lemma 5. *Let $s > 0$ and $j \in \mathbb{N}_0$. There exists a constant $C_s > 0$ such that*

$$\varphi^*_{j,s} f(x) \le C_s \big[M\big(|\varphi_j(H) f|^{1/s}\big)\big]^s (x)\,.$$

Remark 2. It is well known that M is bounded on L^p, $1 < p < \infty$, i.e.,

$$\|Mf\|_p \le C\|f\|_p \,. \tag{3.4}$$

Moreover, if $1 < p < \infty$, $1 < q \le \infty$ and $\{f_j\}$ is a sequence of functions, then

$$\left\| \left(\sum_j |Mf_j|^q \right)^{1/q} \right\|_{L^p} \le C_{p,q} \left\| \left(\sum_j |f_j|^q \right)^{1/q} \right\|_{L^p} \,, \tag{3.5}$$

(usual modification if $q = \infty$) by the Fefferman-Stein vector-valued maximal inequality.

We now state the following theorem on maximal function characterization of $F^{\alpha,q}_p(H)$.

Theorem 2. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$. Let $\{\varphi_j\}_{j \ge 0}$ be a system satisfying (i), (ii), and (iii) as given in Section 1. If $s > 1/\min(p,q)$, then we have for $f \in L^2$*

$$\|f\|_{F^{\alpha,q}_p(H)} \approx \left\| \left(\sum_{j=0}^{\infty} \big(2^{j\alpha} \varphi^*_{j,s} f\big)^q \right)^{1/q} \right\|_p \,. \tag{3.6}$$

Furthermore, $F^{\alpha,q}_p(H)$ is a quasi-Banach space (Banach space if $p \ge 1$, $q \ge 1$) and it is independent of the choice of $\{\varphi_j\}_{j\ge 0}$.

Proof. Because $\varphi^*_{j,s} f(x) \ge |\varphi_j(H) f(x)|$, we only need to show

$$\left\| \left(\sum_{j=0}^{\infty} \big(2^{j\alpha} \varphi^*_{j,s} f\big)^q \right)^{1/q} \right\|_p \le C\|f\|_{F^{\alpha,q}_p(H)} \,, \tag{3.7}$$

but this follows from Lemma 5 and (3.5). Indeed, choosing $0 < r = 1/s < \min(p,q)$, we have

$$\begin{aligned}
\big\|\{2^{j\alpha}\varphi^*_{j,s} f\}\big\|_{L^p(\ell^q)} &\le C_s \big\|\{2^{j\alpha}\big[M\big(|\varphi_j(H)f|^r\big)\big]^{1/r}\}\big\|_{L^p(\ell^q)} \\
&= C_s \left\| \left(\sum_0^{\infty} \big[M\big(2^{j\alpha r}|\varphi_j(H)f|^r\big)\big]^{q/r} \right)^{r/q} \right\|_{L^{p/r}}^{1/r} \\
&\le C_{s,p,q} \big\|\{2^{j\alpha}\varphi_j(H)f\}\big\|_{L^p(\ell^q)} \\
&= C_{s,p,q} \|f\|_{F^{\alpha,q}_p(H)} \,,
\end{aligned}$$

which proves (3.7).

To show the second statement let $\psi = \{\psi_j\}$ be another system satisfying the same conditions as $\varphi = \{\varphi_j\}$. We use (3.3) and Lemma 3 (a) to estimate

$$\begin{aligned} |\varphi_j(H)f(x)| &\leq C2^{j/2} \sum_{\nu=-1}^{1} \int_{\mathbb{R}} \frac{|\psi_{j+\nu}(H)f(y)|}{w_j(x-y)^m}\, dy \\ &\leq C \sum_{\nu=-1}^{1} 2^{j/2} \psi^*_{j+\nu,s} f(x) \int_{\mathbb{R}} \frac{w_{j+\nu}(x-y)^s}{w_j(x-y)^m}\, dy \\ &\leq C \sum_{\nu=-1}^{1} \psi^*_{j+\nu,s} f(x)\,, \end{aligned}$$

provided $m - s > 1$. Thus, for $f \in L^2$

$$\|f\|^{\varphi}_{F_p^{\alpha,q}(H)} \leq C_{s,p,q} \left\| \left\{ 2^{j\alpha} \psi^*_{j,s} f \right\} \right\|_{L^p(\ell^q)} \approx \|f\|^{\psi}_{F_p^{\alpha,q}(H)}\,. \tag{3.8}$$

This concludes the proof. □

Remark 3. Note that the statement in Theorem 2 is true for the more general system $\rho = \{\rho_j\}_0^\infty$ satisfying conditions (i), (ii), and (iii)

$$\sum_j \rho_j(x) \approx c > 0\,.$$

In fact, let us fix a system $\{\varphi_j\}_0^\infty$ as given in Theorem 2. Then the same argument in the proof of (3.8) shows

$$\|f\|^{\rho}_{F_p^{\alpha,q}(H)} \leq C\|f\|^{\varphi}_{F_p^{\alpha,q}(H)}\,.$$

To show the other direction, we define

$$\tilde{\varphi}_j(x) = \varphi_j(x) / \left(\sum_j \rho_j(x) \right)\,.$$

Then it is easy to verify that $\{\tilde{\varphi}_j\}$ satisfies (i), (ii), and so, $\tilde{\varphi}_j(H)(x, y)$ satisfies the nice decay in Lemma 3. Now the identity

$$\varphi_j(x) = \sum_{\nu=-1}^{1} \tilde{\varphi}_j(x) \rho_{j+\nu}(x)$$

and the proof of (3.8) yield

$$\|f\|^{\varphi}_{F_p^{\alpha,q}(H)} \leq C\|f\|^{\rho}_{F_p^{\alpha,q}(H)}\,.$$

3.1 Besov Spaces for H

Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$. We define $B_p^{\alpha,q}(H)$, the *Besov space* associated with H to be the completion of the subspace $\{f \in L^2 : \|f\|_{B_p^{\alpha,q}(H)} < \infty\}$ with respect to the norm $\|\cdot\|_{B_p^{\alpha,q}(H)}$, which is given by (1.4). Then $B_p^{\alpha,q}(H)$ is a quasi-Banach space (Banach space if $p, q \geq 1$).

Theorem 3. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$. If $s > 1/p$, then for $f \in L^2$*

$$\|f\|_{B_p^{\alpha,q}(H)} \approx \left(\sum_{j=0}^{\infty} 2^{j\alpha q} \|\varphi_{j,s}^* f\|_{L^p}^q \right)^{1/q} .$$

Furthermore, $B_p^{\alpha,q}(H)$ is well defined and independent of the choice of $\{\varphi_j\}_{j\geq 0}$.

The proof of Theorem 3 is analogous to that of Theorem 2 but we use (3.4) instead of (3.5).

There is an embedding relation between the $F(H)$ and $B(H)$ spaces that can be shown directly from the definitions, namely,

$$B_p^{s,\min(p,q)}(H) \hookrightarrow F_p^{s,q}(H) \hookrightarrow B_p^{s,\max(p,q)}(H) , \tag{3.9}$$

$0 < p < \infty$, $0 < q \leq \infty$, where $X \hookrightarrow Y$ means, as usual, continuous embedding in the sense that $X \subset Y$ and $\|f\|_Y \leq C\|f\|_X$, $\forall f \in X$. The proof of (3.9) is the same as in the Fourier case; see [23, 2.3.2].

3.2 Lifting Properties of $F(H)$ and $B(H)$ Spaces

Let $c_n > -\inf \sigma(H) = -\inf \sigma_{pp}(H) = n^2$. We need the following lemma in Section 6.

Lemma 6. *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Then $(H + c_n)^s$ maps $F_p^{\alpha,q}(H)$ isomorphically and continuously onto $F_p^{\alpha-s,q}(H)$. Moreover, $\|(H + c_n)^s f\|_{F_p^{\alpha-s,q}(H)} \approx \|f\|_{F_p^{\alpha,q}(H)}$. The analogous statement holds for $B_p^{\alpha,q}(H)$.*

Proof. We only give the proof for $F(H)$. The proof for $B(H)$ is similar.

$$\left\|(H + c_n)^s f\right\|_{F_p^{\alpha-s,q}(H)} = \left\|2^{(\alpha-s)j}(H + c_n)^s \varphi_j(H) f\right\|_{L^p(\ell^q)} = \left\|2^{j\alpha}\psi_j(H) f\right\|_{L^p(\ell^q)} ,$$

where $\psi_j(x) = 2^{-sj}(x+c_n)^s \varphi_j(x)$. Since ψ_j satisfies condition (i), (ii), and (iii), according to Remark 3 we have

$$\left\|(H + c_n)^s f\right\|_{F_p^{\alpha-s,q}(H)} \approx \|f\|_{F_p^{\alpha,q}(H)} .$$

Also, it is easy to see that the inverse of $(H + c_n)^s$ is $(H + c_n)^{-s}$. This proves that the mapping $(H + c_n)^s \colon F_p^{\alpha,q}(H) \to F_p^{\alpha-s,q}(H)$ is surjective. □

4. Proof of Lemma 3

From Section 2 we know $K_j = K_{j,ac} + K_{j,pp}$. We need to show that $K_{j,ac}$, $K_{j,pp}$ both satisfy the decay estimates (a), (b) in the lemma. For the pure point kernel, since $\sigma_{pp} = \{-1, -4, \ldots, -n^2\}$ is finite, it amounts to showing for $0 \leq j \leq 2 + 2\log_2 n$

$$\left|\partial_x^\alpha K_{j,pp}(x, y)\right| \leq C_{m,\alpha}(1 + |x - y|)^{-m}, \quad \forall m \in \mathbb{N}_0,\ \alpha = 0, 1 . \tag{4.1}$$

For other $j's$, the p.p. kernel vanish because supp φ_j are disjoint from the set σ_{pp}. But (4.1) follows from the fact that the eigenfunctions $e_j(x)$ are all Schwartz functions according to Lemma 2. So the nontrivial part will be to prove the decay for the a.c. kernel.

4.1 The Kernel of $\varphi_j(H)E_{ac}$

Recall from Theorem 1 that

$$e_n(x,k) = A_n(k)\, P_n(x,k) e^{ikx} \ ,$$

where $A_n(k) = (\mathrm{sign}(k))^n \prod_{j=1}^n (j+i|k|)^{-1}$ and $P_n(x,k) = p_n(\tanh x, ik)$ is a polynomial of real coefficients and of order n in $\tanh x$ and ik.

High Energy Estimates ($j > 0$)

Let $\varphi_j \in C_0^\infty(\mathbb{R})$ be given as in the beginning of Section 3. By (2.12) the kernel of $\varphi_j(H)E_{ac}$ is given by

$$\begin{aligned} K_{j,ac}(x,y) &= \frac{1}{2\pi}\int \varphi_j(k^2) e_n(x,k)\overline{e_n(y,k)}\, dk \\ &= \int_0^\infty + \int_{-\infty}^0 \varphi_j(k^2) R(x,y,k)\, e^{ik(x-y)}\, dk := K^+(x,y) + K^-(x,y)\ , \end{aligned}$$

where

$$R(x,y,k) = P(x,k)P(y,-k) / \prod_{j=1}^n \left(j^2 + k^2\right) . \tag{4.2}$$

We only need to deal with $K^+(x,y)$ because $K^-(x,y) = K^+(-x,-y)$ in light of the relation $e_n(x,-k) = e_n(-x,k)$. Let $\lambda = 2^{-j/2}$ throughout this section. We have by integration by parts

$$\begin{aligned} 2\pi \left|K^+(x,y)\right| &= \left| \frac{(-1)^m}{i^m (x-y)^m} \int_{2^{j/2-1}}^{2^{j/2}} \frac{d^m}{dk^m}\left[\varphi_j\left(k^2\right) R(x,y,k)\right] e^{ik(x-y)}\, dk \right| \\ &\le C_m \lambda^{m-1}/|x-y|^m, \quad m \ge 0\ , \end{aligned}$$

where we used for $k \sim \lambda^{-1} \to \infty$ as $j \to \infty$,

$$\begin{cases} \frac{d^i}{dk^i}\left[\varphi_j\left(k^2\right)\right] &= O\left(\lambda^i\right) \\ \frac{\partial^j}{\partial k^j} R(x,y,k) &= O\left(\lambda^j\right) \quad \text{uniformly in } x, y\ . \end{cases} \tag{4.3}$$

The same estimate also holds for $K^-(x,y)$. Hence, we obtain

$$|K_{j,ac}(x,y)| \le C_m \lambda^{-1}/\left(1 + \lambda^{-1}|x-y|\right)^m . \tag{4.4}$$

Low Energy Estimates ($-\infty < j < 0$)

If we allow $j < 0$ with φ_j satisfying conditions (i), (ii) in Section 3, then (4.4) also holds for $j < 0$ by the same proof above, except that instead of (4.3) we use the following estimates: If $k \sim \lambda^{-1} \to 0$ as $j \to -\infty$,

$$\begin{cases} \frac{d^i}{dk^i}\left[\varphi_j\left(k^2\right)\right] &= O\left(\lambda^i\right) \le O\left(\lambda^m\right) \quad \text{if } \ 0 \le i \le m \\ \frac{\partial^j}{\partial k^j} R(x,y,k) &= O(1) \quad \text{uniformly in } x, y\ . \end{cases}$$

However, the low energy case will be needed only in the discussion of homogeneous spaces $\dot{F}_p^{\alpha,q}(H)$, $\dot{B}_p^{\alpha,q}(H)$.

Local Energy Estimates

Fix $\Phi := \varphi_0 \in C_0^\infty(\mathbb{R})$ with support $\subset [-1, 1]$.

$$2\pi \Phi(H) E_{ac}(x, y) = \int_{-1}^{1} \Phi\big(k^2\big) R(x, y, k) e^{ik(x-y)}\, dk\ .$$

Using for $k \to 0$,

$$\begin{cases} \frac{d^i}{dk^i}\big[\Phi(k^2)\big] & = & O(1) \\ \frac{\partial^j}{\partial k^j} R(x, y, k) & = & O(1) \quad \text{uniformly in } x, y \end{cases}$$

and integrating by parts on $[-1, 1]$, where we note that $k \mapsto R(x, y, k)$ is analytic at zero, we obtain for each m

$$|\Phi(H) E_{ac}(x, y)| \le C_m (1 + |x - y|)^{-m}\ .$$

4.2 The Derivative of the Kernel

Using the notation in Section 4.1, we proceed

$$\begin{aligned} 2\pi \frac{\partial}{\partial x} K_{j,ac}(x, y) &= \frac{\partial}{\partial x} \int \varphi_j\big(k^2\big) R(x, y, k) e^{ik(x-y)}\, dk \\ &= \int \varphi_j\big(k^2\big) \frac{\partial}{\partial x}\big[R(x, y, k) e^{ik(x-y)}\big]\, dk \\ &= \int \varphi_j\big(k^2\big) |A(k)|^2 \Big[ikP(x, k) + \frac{\partial}{\partial x} P(x, k)\Big] P(y, -k) e^{ik(x-y)}\, dk\ . \end{aligned}$$

The function $\frac{\partial}{\partial x} P(x, k)$ is a polynomial of $\tanh x$ and ik having degrees $n + 1$ and $n - 1$, resp. Note that if $|k| \sim \lambda^{-1} = 2^{j/2}$, $j > 0$,

$$\Big|\frac{d^i}{dk^i}\big(k\varphi_j\big(k^2\big)\big)\Big| = O\big(\lambda^{i-1}\big)\ ,$$

and if $|k| \le 1$,

$$\Big|\frac{d^i}{dk^i}\big(k\Phi\big(k^2\big)\big)\Big| = O(1)\ .$$

We obtain, by similar arguments as in Section 4.1, for each $m \ge 0$

$$\Big|\frac{\partial}{\partial x} K_{j,ac}(x, y)\Big| \le C_m \lambda^{-2}\big(1 + \lambda^{-1}|x - y|\big)^{-m}, \qquad j > 0$$

and

$$\Big|\frac{\partial}{\partial x} \Phi(H) E_{ac}(x, y)\Big| \le C_m (1 + |x - y|)^{-m}\ .$$

This completes the proof of Lemma 3.

Remark 4. For $-\infty < j < 0$, the best estimate is, for each $m \ge 0$

$$\Big|\frac{\partial}{\partial x} K_{j,ac}(x, y)\Big| \lesssim \lambda^{-1} \operatorname{sech}^2 x \tanh y \big(1 + \lambda^{-1}|x - y|\big)^{-m} + \lambda^{-2}\big(1 + \lambda^{-1}|x - y|\big)^{-m}\ . \quad (4.5)$$

We observe that the first term has only a factor of $\lambda^{-1} = O(2^{j/2})$ as $j \to -\infty$, which makes unavailable the Bernstein inequality and Peetre maximal inequality, namely *low energy* cases of Lemma 4 and Lemma 5, resp. Nevertheless, if we work a little harder, using (4.4) and (4.5) we can obtain a weaker form of Peetre maximal inequality and prove the following: If $1 \le p < \infty$, $0 < q < \infty$, $\alpha \in \mathbb{R}$,

$$\|f\|_{\dot{B}_p^{\alpha,q}(H)} \approx \left\|\left\{2^{j\alpha}\varphi_j^*(H)f\right\}_{j\in\mathbb{Z}}\right\|_{\ell^q(L^p)}$$

and if $1 < p < \infty$, $1 < q < \infty$, $\alpha \in \mathbb{R}$,

$$\|f\|_{\dot{F}_p^{\alpha,q}(H)} \approx \left\|\left\{2^{j\alpha}\varphi_j^*(H)f\right\}_{j\in\mathbb{Z}}\right\|_{L^p(\ell^q)} .$$

5. Identification of $F_p^{0,2}(H) = L^p$, $1 < p < \infty$

Let $\{\varphi_j\}_0^\infty$ be as in Section 1. Then there exists $\{\psi_j\}_0^\infty$ satisfying the same conditions (i), (ii) therein such that

$$\sum_{j=0}^{\infty} \varphi_j(x)\psi_j(x) = 1$$

by taking $\psi_j(x) = \overline{\varphi_j(x)}/\sum|\varphi_j(x)|^2$. We may assume that $\|\varphi_j\|_\infty$, $\|\psi_j\|_\infty$ are all ≤ 1. Let $Q_j = \varphi_j(H)$ and $R_j = \psi_j(H)$. Define the operators $Q : L^2 \to L^2(\ell^2)$ and $R : L^2(\ell^2) \to L^2$ as follows.

$$Q: \ f \mapsto \{Q_j(H)f\}_0^\infty$$

and

$$R: \ \{g_j\}_0^\infty \mapsto \sum_{j=0}^{\infty} R_j g_j .$$

It follows from the definition that

$$\|f\|_{F_p^{0,2}(H)} = \|Qf\|_{L^p(\ell^2)} \tag{5.1}$$

and it is easy to see that $RQ = I : L^2 \to L^2$ and $QR \le 3I : L^2(\ell^2) \to L^2(\ell^2)$. We will use Q and R to identify $F_p^{0,2}(H)$ with L^p.

Theorem 4. *Let $1 < p < \infty$. Then $F_p^{0,2}(H)$ and L^p are isomorphic and have equivalent norms.*

To prove the theorem, we will show that $Q\colon L^p \to L^p(\ell^2)$ and $R\colon L^p(\ell^2) \to L^p$, $1 < p < \infty$, that is,

$$\|Qf\|_{L^p(\ell^2)} \lesssim \|f\|_p \quad \text{and} \quad \|Rg\|_p \lesssim \|g\|_{L^p(\ell^2)} \tag{5.2}$$

for $f \in L^2 \cap L^p$ and $g \in L^2(\ell^2) \cap L^p(\ell^2)$, resp. This means that, by a density argument,

$$\|f\|_{F_p^{0,2}(H)} \lesssim \|f\|_p \tag{5.3}$$

and

$$\|f\|_p \lesssim \|f\|_{F_p^{0,2}(H)} . \tag{5.4}$$

Here in view of (5.2), (5.3) follows from (5.1) and (5.4) follows, with $g = Qf$, from the identity $RQ = I$, i.e., $\sum \varphi_j(H)\psi_j(H) = I$. Thus, (5.3) and (5.4) prove Theorem 4.

The remaining part of this section is devoted to showing the boundedness of Q and R in (5.2). In the following, Lemma 7 and Lemma 9 imply that Q is bounded from L^p to $L^p(\ell^2)$, and, Lemma 7 and Lemma 10 imply that R is bounded from $L^p(\ell^2)$ to L^p by interpolation and duality.

Lemma 7. *$Q : L^2 \to L^2(\ell^2)$ and $R : L^2(\ell^2) \to L^2$ are well-defined bounded operators.*

Proof. Let $\{g_j\} \in L^2(\ell^2)$. Note that R_j is bounded on L^2: $\|R_j g\|_2 \le \|\psi_j\|_\infty \|g\|_2 \le \|g\|_2$. Thus,

$$\begin{aligned}\left(\sum_{j=0}^{\infty} R_j g_j, \sum_{j=0}^{\infty} R_j g_j\right) &= \sum_{\nu=-1}^{1} \sum_{j=0}^{\infty} (R_j g_j, R_{j+\nu} g_{j+\nu}) \\ &\le \sum_{\nu=-1}^{1} \sum_{j} \|R_j g_j\|_2 \|R_{j+\nu} g_{j+\nu}\|_2 \\ &\le 3 \sum_j \|g_j\|_2^2 = 3\|g_j\|_{L^2(\ell^2)}^2 .\end{aligned}$$

Similarly, we have $\|Qf\|_{L^2(\ell^2)} \le \sqrt{2}\|f\|_2$ because $\sum_j |\varphi_j(x)|^2 \le 2$ for all x. □

We now derive some necessary estimates for the kernel of $Q_j = \varphi_j(H)$, which is denoted by $Q_j(x, y)$. Define

$$\widetilde{Q}_j(x, y) = \begin{cases} Q_j(x, y) & \text{if} \quad 2^{j/2}|I| \ge 1 \\ Q_j(x, y) - Q_j(x, \bar{y}) & \text{if} \quad 2^{j/2}|I| < 1 . \end{cases}$$

Lemma 8. *Let $I = (\bar{y} - \frac{t}{2}, \bar{y} + \frac{t}{2})$, $t = |I|$ and $I^* = (\bar{y} - t, \bar{y} + t)$. Then there exists a constant C independent of I such that*

(a) *if $2^{j/2}|I| \ge 1$,*

$$\sup_{y \in I} \int_{\mathbb{R}\setminus I^*} |Q_j(x, y)|\, dx \le C\left(2^{j/2}|I|\right)^{-1} .$$

(b) *If $2^{j/2}|I| < 1$,*

$$\sup_{y \in I} \int_{\mathbb{R}\setminus I^*} \left|Q_j(x, y) - Q_j(x, \bar{y})\right| dx \le C 2^{j/2}|I| .$$

In particular, we have

$$\sum_j \int_{\mathbb{R}\setminus I^*} \left|\widetilde{Q}_j(x, y)\right| dx \le \left(2 + \sqrt{2}\right)C . \tag{5.5}$$

Proof. For (a), we let $2^{j/2}|I| \geq 1$ and $y \in I$. Then it follows from Lemma 3 (a) that

$$\begin{aligned}\int_{\mathbb{R}\setminus I^*} |Q_j(x,y)|\,dx &\leq C_m \int_{|x-y|>t/2} \frac{2^{j/2}}{\left(1+2^{j/2}|x-y|\right)^m}\,dx \\ &\leq C\left(2^{j/2}|I|\right)^{-1}, \qquad (m=2)\,.\end{aligned}$$

For (b) we let $2^{j/2}|I| < 1$, $y \in I$ ($\bar{y}$ being the center of I) and apply Lemma 3 (b) to obtain

$$\begin{aligned}\int_{\mathbb{R}\setminus I^*} \left|Q_j(x,y) - Q_j\left(x,\bar{y}\right)\right| dx &= \int_{\mathbb{R}\setminus I^*} \left|\int_{\bar{y}}^{y} \frac{\partial}{\partial z} Q_j(x,z)\,dz\right| dx \\ &\leq C_m\left|y-\bar{y}\right| \int_{|x-\bar{y}|>t} \frac{2^j}{\left(1+2^{j/2-1}\left|x-\bar{y}\right|\right)^m}\,dx \\ &\leq C2^{j/2}|I|, \qquad (m=2)\,.\end{aligned}$$

□

Lemma 9. *Q is bounded from L^1 to weak-$L^1(\ell^2)$, i.e.,*

$$\left|\left\{x : \left(\sum_0^\infty |Q_j f(x)|^2\right)^{1/2} > \lambda\right\}\right| \leq C\lambda^{-1}\|f\|_1, \quad \forall\lambda > 0\,.$$

Proof. Let $f \in L^1$. By the Calderón-Zygmund decomposition, there exists a sequence of disjoint intervals $\{I_k\}$ and functions $\{b_k\}$ with supp $b_k \subset I_k$ such that $f = g + b$ with $g \in L^2$ and $b = \sum_k b_k \in L^1$. Furthermore, for each $\lambda > 0$ the following properties hold

(i) $|g(x)| \leq C\lambda$ a.e.

(ii) $b_k(x) = f(x) - |I_k|^{-1}\int_{I_k} f\,dx,\ x \in I_k$

(iii) $\lambda \leq |I_k|^{-1}\int_{I_k} |f|\,dx \leq 2\lambda$

(iv) $\sum_k |I_k| \leq \lambda^{-1}\|f\|_1$.

From Lemma 7 we know that $Q : L^2 \to L^2(\ell^2)$ is bounded, i.e.,

$$\int \sum_0^\infty |Q_j g(x)|^2\,dx \leq C\|g\|_2^2\,.$$

By Chebyshev inequality we have

$$\left|\left\{x : \left(\sum_0^\infty |Q_j g(x)|^2\right)^{1/2} > \lambda/2\right\}\right| \leq C\lambda^{-2}\|g\|_2^2 \leq C\lambda^{-1}\|f\|_1\,.$$

Now we only need to show

$$\left|\left\{x \notin \cup I_k^* : \left(\sum_j |Q_j b(x)|^2\right)^{1/2} > \lambda/2\right\}\right| \leq C\lambda^{-1}\|f\|_1\,,$$

where $I_k^* = 2I_k$ means the interval of length $2|I_k|$ with the same center as I_k. Note that the left-hand side of the above inequality is bounded by

$$\frac{2}{\lambda}\sum_k \int_{\mathbb{R}\setminus\cup I_k^*} \left(\sum_j |Q_j b_k(x)|^2\right)^{1/2} dx \leq \frac{2}{\lambda}\sum_k \int_{\mathbb{R}\setminus\cup I_k^*} \sum_j |Q_j b_k(x)|\,dx\,. \tag{5.6}$$

For each k, since $\int b_k = 0$, we apply Lemma 8 with $I = I_k$ and estimate above the r.h.s. of (5.6) by

$$\begin{aligned}
&\frac{2}{\lambda}\sum_k \int_{\mathbb{R}\setminus\cup I_k^*} \sum_j \int \left|\widetilde{Q}_j(x,y)\right| |b_k(y)|\, dy\, dx \\
&\leq \frac{2}{\lambda}\sum_k \int_{y\in I_k} |b_k(y)|\, dy \int_{\mathbb{R}\setminus I_k^*} \sum_j \left|\widetilde{Q}_j(x,y)\right| dx \\
&\leq \frac{C}{\lambda}\sum_k \int_{I_k} |b_k(y)|\, dy \leq C\lambda^{-1}\|f\|_1 .
\end{aligned}$$

This completes the proof. □

Lemma 10. *Let $R_j = \psi_j(H)$. Then $R = \{R_j\}$ is bounded from $L^1(\ell^2)$ to weak-L^1.*

Proof. It suffices to show that there exists a constant C such that

$$\left|\left\{x : \left|\sum_0^N R_j f_j(x)\right| > \lambda\right\}\right| \leq C\lambda^{-1}\|\{f_j\}\|_{L^1(\ell^2)} \tag{5.7}$$

for all $N \in \mathbb{N}$, $\{f_j\} \in L^1(\ell^2)$ and $\lambda > 0$. By passing to the limit we see that (5.7) also holds for $N = \infty$ and all $\{f_j\} \in L^1(\ell^2) \cap L^2(\ell^2)$. Then the lemma follows from the fact that $L^1(\ell^2) \cap L^2(\ell^2)$ is dense in $L^1(\ell^2)$.

Let $F(x) = (\sum_{j=0}^{\infty} |f_j(x)|^2)^{1/2} \in L^1$. By the Calderón-Zygmund decomposition there exists a sequence of disjoint open intervals $\{I_k\}$ such that

(i) $|F(x)| \leq C\lambda$, a.e. $x \in \mathbb{R}\setminus \cup_k I_k$

(ii) $\lambda \leq |I_k|^{-1}\int_{I_k} |F(x)|\, dx \leq 2\lambda$, $\forall k$.

Define

$$g_j(x) = \begin{cases} |I_k|^{-1}\int_{I_k} f_j\, dy, & x \in I_k \\ f_j(x) & \text{otherwise}, \end{cases} \qquad b_j(x) = \begin{cases} f_j - g_j, & x \in I_k \\ 0 & \text{otherwise}. \end{cases}$$

Then, if $x \in \mathbb{R} \setminus \cup_k I_k$, $(\sum_{j=0}^{\infty} |g_j(x)|^2)^{1/2} = (\sum_{j=0}^{\infty} |f_j(x)|^2)^{1/2}$, and, if $x \in I_k$

$$\begin{aligned}
\left(\sum_{j=0}^{\infty} |g_j(x)|^2\right)^{1/2} &= \left(\sum_{j=0}^{\infty} |I_k|^{-2}\left|\int_{I_k} f_j(y)\, dy\right|^2\right)^{1/2} \\
&\leq |I_k|^{-1}\int_{I_k}\left(\sum_{j=0}^{\infty} |f_j(y)|^2\right)^{1/2} dy \leq 2\lambda
\end{aligned}$$

by Minkowski inequality. It follows that

$$\begin{aligned}
\|\{g_j(x)\}\|_{L^2(\ell^2)}^2 &= \sum_k \int_{I_k}\left(\sum_j |g_j(x)|^2\right) dx + \int_{\mathbb{R}\setminus\cup I_k}\left(\sum_j |g_j(x)|^2\right) dx \\
&\leq (2\lambda)^2 \sum_k |I_k| + 2\lambda \int_{\mathbb{R}\setminus\cup I_k}\left(\sum_j |f_j|^2\right)^{1/2} dx \\
&\leq C\lambda\|F\|_1 .
\end{aligned}$$

Now by Lemma 7 we obtain

$$\left|\left\{x : \left|\sum_0^N R_j g_j(x)\right| > \lambda/2\right\}\right| \leq C\lambda^{-2} \left\|\sum_0^N R_j g_j\right\|_2^2$$
$$\leq C'\lambda^{-2} \|\{g_j\}\|^2_{L^2(\ell^2)} \leq C\lambda^{-1}\|F\|_1 .$$

It remains to show

$$\left|\left\{x \notin \cup I_k^* : \left|\sum_0^N R_j b_j(x)\right| > \lambda/2\right\}\right| \leq C\lambda^{-1}\|F\|_1 .$$

The left-hand side is not exceeding $\frac{2}{\lambda}\sum_k \int_{\mathbb{R}\setminus\cup I_k^*} |\sum_{j=0}^N R_j b_{j,k}(x)|\, dx$, where $b_{j,k} = b_j \chi_{I_k}$, χ_{I_k} the characteristic function of I_k. For each k, define

$$\widetilde{R}_j^k(x, y) = \begin{cases} R_j(x, y) & \text{if} \quad 2^{j/2}|I_k| \geq 1 \\ R_j(x, y) - R_j\big(x, \bar{y}_k\big) & \text{if} \quad 2^{j/2}|I_k| < 1 , \end{cases}$$

where $\bar{y}_k$ is the center of I_k. Then it follows from Lemma 8 with $I = I_k$ and Q_j replaced by R_j that

$$\int_{\mathbb{R}\setminus I_k^*} \left(\sum_{j=0}^N \big|\widetilde{R}_j^k(x, y)\big|^2\right)^{1/2} dx \leq \int_{\mathbb{R}\setminus I_k^*} \sum_{j=0}^N \big|\widetilde{R}_j^k(x, y)\big|\, dx \leq C, \quad \forall y \in I_k,\ N .$$

Thus, we obtain, using $\int b_{j,k} = 0$,

$$\begin{aligned}\int_{\mathbb{R}\setminus I_k^*} \left|\sum_{j=0}^N R_j b_{j,k}(x)\right| dx &= \int_{\mathbb{R}\setminus I_k^*} \left|\sum_{j=0}^N \int_{I_k} \widetilde{R}_j^k(x, y) b_{j,k}(y)\, dy\right| dx \\ &\leq \int_{I_k} \left(\sum_{j=0}^N |b_{j,k}|^2(y)\right)^{1/2} dy \int_{\mathbb{R}\setminus I_k^*} \left(\sum_{j=0}^N \big|\widetilde{R}_j^k(x, y)\big|^2\right)^{1/2} dx \\ &\leq C \int_{I_k} \left(\sum_{j=0}^N |b_{j,k}|^2\right)^{1/2} dy \\ &\leq 2C \int_{I_k} \left(\sum_{j=0}^\infty |f_j|^2\right)^{1/2} dy .\end{aligned}$$

Hence,

$$\begin{aligned}\left|\left\{x \notin \cup I_k^* : \left|\sum_0^N R_j b_j(x)\right| > \lambda/2\right\}\right| &\leq \frac{4C}{\lambda} \sum_k \int_{I_k} \left(\sum_j |f_j|^2\right)^{1/2} dy \\ &\leq \frac{4C}{\lambda} \left\|\left(\sum_j |f_j|^2\right)^{1/2}\right\|_1 ,\end{aligned}$$

as desired. This completes the proof. □

6. Remarks on Boundedness of the Wave Function

We conclude the article with a boundedness result on the wave function $\psi(t,x) = e^{-itH} f$ which is the solution to the Schrödinger equation

$$i\,\partial_t \psi = H\psi, \qquad \psi(0,x) = f(x)\,. \tag{6.1}$$

We will see that using the $B(H)$ and $F(H)$ space one can obtain a global time decay for $\psi(t,x)$ (Theorem 6). The perturbed Besov space method has been considered in [19, 24, 6, 7] and more recently, [2, 9, 8] involving Schrödinger and wave equations.

By [2, Theorem 7.1] or [19, Theorem 5.1] we know that if V is in the Kato class $\mathcal{K}_d$ and if $\mathcal{D}(H^m) = W_p^{2m}(\mathbb{R}^d)$ for some $m \in \mathbb{N}$, $1 \le p < \infty$, then for $1 \le q \le \infty$, $0 < \alpha < m$, $B_p^{\alpha,q}(H) = B_p^{2\alpha,q}(\mathbb{R}^d)$. It is easy to see that if V is C^∞ with all derivatives bounded, then the domain condition on H is verified for all $m \in \mathbb{N}$.

In the following we assume $H = -d^2/dx^2 + V_n$ and restrict our discussion to the P-T potential, although results here have extensions to general potentials on $\mathbb{R}^d$.

Since $V_n \sim \operatorname{sech}^2 x$ is in the Schwartz class, we have

$$B_p^{\alpha,q}(H) = B_p^{2\alpha,q}(\mathbb{R})$$

for all $\alpha > 0$. In particular, $F_p^{\alpha,p}(H) = F_p^{2\alpha,p}(\mathbb{R})$ since it always holds that $F_p^{\alpha,p} = B_p^{\alpha,p}$ by the definitions [see (1.3), (1.4)]. On the other hand, by Theorem 4, $F_p^{0,2}(H) = L^p = F_p^{0,2}(\mathbb{R})$. Thus, we obtain the following theorem using complex interpolation method; consult [23, 21] or [3] for details.

Theorem 5. *If $\alpha > 0$, $1 < p < \infty$ and $2p/(p+1) < q < 2p$, then*

$$F_p^{\alpha,q}(H) = F_p^{2\alpha,q}(\mathbb{R})\,.$$

If $\alpha > 0$, $1 \le p < \infty$ and $1 \le q \le \infty$, then

$$B_p^{\alpha,q}(H) = B_p^{2\alpha,q}(\mathbb{R})\,.$$

From Theorem 5 and [19, Theorem 4.6, Remark 4.7] we obtain the boundedness of $\psi(t,x)$ on ordinary Besov spaces. Let $\langle t\rangle = (1+t^2)^{1/2}$ and let $\beta = \beta(p) = |\frac{1}{2} - \frac{1}{p}|$ be the critical exponent.

Proposition 1. *Let $\alpha > 0$, $1 \le p < \infty$, $1 \le q \le \infty$. Then*

$$\left\| e^{-itH} f \right\|_{B_p^{\alpha,q}(\mathbb{R})} \lesssim \langle t\rangle^{|\frac{1}{p}-\frac{1}{2}|} \|f\|_{B_p^{\alpha+2\beta,q}(\mathbb{R})}\,. \tag{6.2}$$

Moreover, if $2 \le p < \infty$,

$$\left\| e^{-itH} f \right\|_{L^p} \lesssim \langle t\rangle^{|\frac{1}{p}-\frac{1}{2}|} \|f\|_{B_p^{2\beta,2}(\mathbb{R})}$$

and if $1 \le p < 2$,

$$\left\| e^{-itH} f \right\|_{L^p} \lesssim \langle t\rangle^{|\frac{1}{p}-\frac{1}{2}|} \|f\|_{B_p^{2\beta,1}(\mathbb{R})}\,. \tag{6.3}$$

Proof. Let $\{\varphi_j\}_0^\infty$ be a smooth dyadic system. From the proof of [19, Theorem 4.6] we see that

$$\left\| e^{-itH} \varphi_j(H) f \right\|_p \lesssim 2^{j\beta} \langle t\rangle^{|\frac{1}{2}-\frac{1}{p}|} \|\varphi_j(H) f\|_p, \qquad j \ge 0\,.$$

This implies (6.2) by Theorem 5 and

$$\left\| e^{-itH} f \right\|_{B_p^{0,q}(H)} \lesssim \langle t\rangle^{|\frac{1}{2}-\frac{1}{p}|} \| f \|_{B_p^{\beta,q}(H)} . \tag{6.4}$$

Now if $p \geq 2$, then $B_p^{0,2}(H) \hookrightarrow F_p^{0,2}(H)$ according to (3.9). We have

$$\left\| e^{-itH} f \right\|_{L^p} \approx \left\| e^{-itH} f \right\|_{F_p^{0,2}(H)} \lesssim \langle t\rangle^{|\frac{1}{2}-\frac{1}{p}|} \| f \|_{B_p^{\beta,2}(H)} .$$

For $1 \leq p < 2$, because

$$\| f \|_p \leq \sum_{j=0}^{\infty} \| \varphi_j(H) f \|_p = \| f \|_{B_p^{0,1}(H)} ,$$

we see $B_p^{0,1}(H) \hookrightarrow L^p$, which implies (6.3) in light of (6.4). □

One is also interested in understanding the long time behavior of $\psi(t,x)$. From [17] and [8] we know that if $(1+x^2)V \in L^1(\mathbb{R})$, then

$$\left\| e^{-itH} E_{ac} f \right\|_{L^{p'}} \lesssim t^{-(\frac{1}{p}-\frac{1}{2})} \| f \|_{L^p} , \qquad \forall t > 0,\ 1 \leq p \leq 2 , \tag{6.5}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. So Proposition 1 and (6.5) yield

$$\left\| e^{-itH} E_{ac} f \right\|_{L^{p'}} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})} \| f \|_{B_{p'}^{2\beta,2}(\mathbb{R}) \cap L^p} , \qquad 1 < p \leq 2 , \tag{6.6}$$

where we note that E_{ac} is bounded on L^p because E_{pp}, which has the kernel $\sum_{j=1}^{n} e_j(x) e_j(y)$, is bounded on L^p (see the discussion at the beginning of Section 4).

Theorem 6. *Let* $1 < p \leq 2$. *Then*

$$\left\| e^{-itH} E_{ac} f \right\|_{L^{p'}} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})} \| f \|_{B_p^{4\beta,2}(\mathbb{R})} . \tag{6.7}$$

$$\left\| e^{-itH} E_{ac} f \right\|_{L^{p'}} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})} \| f \|_{F_p^{4\beta,2}(\mathbb{R})} . \tag{6.8}$$

Proof. Since $B_p^{4\beta,2}(\mathbb{R}) \hookrightarrow B_{p'}^{2\beta,2}(\mathbb{R})$ (Besov embedding; see e.g., [21, 2.7.1]) and $B_p^{\epsilon,2}(\mathbb{R}) \hookrightarrow L^p$ if $\epsilon > 0$, it follows from (6.6) that

$$\left\| e^{-itH} E_{ac} f \right\|_{L^{p'}} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})} \| f \|_{B_p^{4\beta,2}(\mathbb{R})}$$

provided $1 < p \leq 2$. The second inequality follows from (6.7) and the embedding $F_p^{s,2}(\mathbb{R}) \hookrightarrow B_p^{s,2}(\mathbb{R})$ in light of (3.9). □

Remark 5. For (6.8), if alternatively starting with (6.6) [rather than (6.7)] and using an embedding of Jawerth [21, 2.7.1], we can obtain an improved result: If $1 < p < 2$, $0 < q \leq \infty$, then

$$\left\| e^{-itH} E_{ac} f \right\|_{L^{p'}} \lesssim \langle t\rangle^{-(\frac{1}{p}-\frac{1}{2})} \| f \|_{F_p^{4\beta,q}(\mathbb{R})} .$$

As a consequence we also obtain the following regularity result by the identification in Theorem 5.

Corollary 1. *Let $\alpha > 0$. If $1 < p \le 2$, $1 \le q \le \infty$, then*

$$\left\| e^{-itH} E_{ac} f \right\|_{B_{p'}^{\alpha,q}(\mathbb{R})} \lesssim \langle t \rangle^{-(\frac{1}{p}-\frac{1}{2})} \| f \|_{B_p^{\alpha+4\beta,q}(\mathbb{R})} . \tag{6.9}$$

If $1 < p \le 2$, $p \le q \le 2$, then

$$\left\| e^{-itH} E_{ac} f \right\|_{F_{p'}^{\alpha,q}(\mathbb{R})} \lesssim \langle t \rangle^{-(\frac{1}{p}-\frac{1}{2})} \| f \|_{F_p^{\alpha+4\beta,q}(\mathbb{R})} . \tag{6.10}$$

Proof. Since $B_p^{2\beta,2}(H) = B_p^{4\beta,2}(\mathbb{R})$ by Theorem 5, we can write (6.7) as

$$\left\| e^{-itH} E_{ac} f \right\|_{L^{p'}} \lesssim \langle t \rangle^{-(\frac{1}{p}-\frac{1}{2})} \| f \|_{B_p^{2\beta,2}(H)} . \tag{6.11}$$

Replace f with $\varphi_j(H) f$ in (6.11). Then the B-inequality (6.9) follows from the simple observation that

$$\left(\sum_j 2^{j\alpha q} \| \varphi_j(H) f \|^q_{B_p^{\gamma,2}(H)} \right)^{1/q} \approx \| f \|_{B_p^{\alpha+\gamma,q}(H)} .$$

To show the F-inequality, substitute $f = (H + c_n)^{-\alpha} f$ into (6.8) but use the $F_p^{2\beta,2}(H)$-norm instead. Then by the lifting property in Lemma 6 and Theorem 5, we have

$$\left\| e^{-itH} E_{ac} f \right\|_{F_{p'}^{\alpha,2}(\mathbb{R})} \lesssim \langle t \rangle^{-(\frac{1}{p}-\frac{1}{2})} \| f \|_{F_p^{\alpha+4\beta,2}(\mathbb{R})} . \tag{6.12}$$

Now (6.10) follows from the interpolation between (6.12) and (6.9) with $p = q$, where we note that $B_p^{\alpha,p}(\mathbb{R}) = F_p^{\alpha,p}(\mathbb{R})$. □

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