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Toeplitz Operators and Group Representations

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Research Survey

ABSTRACT. Toeplitz operators on the Bergman space of the unit disc can be written as integrals of the symbol against an invariant operator field of rank-one projections. We consider a generalization of the Toeplitz calculus obtained upon taking more general invariant operator fields, and also allowing more general domains than the disc; such calculi were recently introduced and studied by Arazy and Upmeier, but also turn up as localization operators in time-frequency analysis (witnessed by recent articles by Wong and others) and in representation theory and mathematical physics. In particular, we establish basic properties like boundedness or Schatten class membership of the resulting operators. A further generalization to the setting when there is no group action present is also discussed, and the various settings in which similar operator calculi appear are briefly surveyed.

1. Introduction

Let $\mathbf{D}$ be the unit disc in the complex plane, dm the Lebesgue area measure normalized so that $m(\mathbf{D}) = 1$, and $L^2_{\text{hol}}(\mathbf{D})$ the Bergman space on $\mathbf{D}$, i.e., the subspace of all holomorphic functions in $L^2(\mathbf{D})$. It is well known that the function

$$K(x, y) = \frac{1}{\left(1 - x\overline{y}\right)^2}$$

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is the reproducing kernel for $L^2_{\text{hol}}(\mathbf{D})$, that is

$$f(x) = \int_{\mathbf{D}} f(y) K(x, y)\, dm(y) = \langle f, K_x \rangle , \qquad K_x := K(\cdot, x) ,$$

for any $f \in L^2_{\text{hol}}$ and $x \in \mathbf{D}$. Let P be the orthogonal projection in $L^2(\mathbf{D})$ onto $L^2_{\text{hol}}(\mathbf{D})$. For $f \in L^\infty(\mathbf{D})$, the Toeplitz operator T_f with symbol f is the operator on L^2_{hol} defined by

$$T_f u = P(fu) . \tag{1.1}$$

Since for any $u, v \in L^2_{\text{hol}}(\mathbf{D})$,

$$\left\langle T_f u, v \right\rangle = \langle fu, v \rangle = \int_{\mathbf{D}} f(x) \; \langle u, K_x \rangle \; \langle K_x, v \rangle \; dm(x) ,$$

one can rewrite the definition (1.1) as

$$T_f = \int_{\mathbf{D}} f(x) \; T_x \; d\mu(x) , \tag{1.2}$$

where

$$d\mu(x) := \frac{dm(x)}{\left(1 - |x|^2\right)^2} = K(x, x)\, dm(x)$$

is the invariant measure on $\mathbf{D}$, and T_x are the rank-one operators

$$T_x := \frac{\langle \cdot, K_x \rangle\, K_x}{K(x, x)} = \langle \cdot, k_x \rangle\, k_x, \qquad k_x := \frac{K_x}{\|K_x\|} . \tag{1.3}$$

(The unit vectors k_x, the normalized reproducing kernels, are known as "coherent states" in quantum optics.) Here the convergence in (1.2) is meant in the weak operator topology.

The formula (1.2) reveals a very nice feature of Toeplitz operators, namely, their *Möbius invariance*. Recall that for any biholomorphic self-map g of the disc (Möbius transformation), the associated change-of-variable mapping

$$U_g : f \mapsto (f \circ g) \cdot g'$$

is unitary on L^2 and L^2_{hol}; further, $U_{g_1} U_{g_2} = U_{g_2 g_1}$. Thus the operators U_g form a unitary anti-representation of the group G of all Möbius maps (biholomorphic self-maps of $\mathbf{D}$). An easy calculation shows that the Bergman projection P, the Toeplitz operators T_f and the operators T_x from (1.3) satisfy

$$U_g P U_g^* = P ,$$
$$U_g T_f U_g^* = T_{f \circ g} ,$$

and

$$U_g^* T_x U_g = T_{gx} . \tag{1.4}$$

Mappings $x \mapsto T_x$ satisfying the last equality are called *invariant operator fields* on $\mathbf{D}$.

The formulas above suggest the following generalization. Let K denote the stabilizer of the origin in G, i.e.,

$$K := \{g \in G : \; g0 = 0\} .$$

(Clearly, K consists precisely of the rotations around the origin.) Then (1.4) implies, first of all, that any invariant operator field satisfies

$$U_k^* T_0 U_k = T_0, \qquad \text{or} \qquad T_0 U_k = U_k T_0, \qquad \forall k \in K \ .$$

Further,

$$T_x = U_g^* T_0 U_g \qquad \text{for any } g \in G \text{ such that } \quad g0 = x \ .$$

Let now, conversely, A be any bounded linear operator on $L^2_{\text{hol}}(\mathbf{D})$ which satisfies

$$U_k A = A U_k \qquad \forall k \in K \ ; \tag{1.5}$$

and set

$$A_x := U_g^* A U_g \qquad \text{for any } g \in G \text{ such that } \quad g0 = x \ .$$

Note that the definition of A_x is consistent: If also $g_1 0 = x$, then $g_1 = gk$ for some $k \in K$, so $U_{g_1} = U_k U_g$ and

$$U_{g_1}^* A U_{g_1} = U_g^* U_k^* A U_k U_g = U_g^* A U_g \qquad \text{by (1.5)} \ .$$

For a function f on $\mathbf{D}$, we can then define the *A-Toeplitz operator* with symbol f by

$$A_f := \int_{\mathbf{D}} f(x) \ A_x \ d\mu(x) \tag{1.6}$$

whenever the integral converges in the weak operator topology. The operators A_f thus reduce to the usual Toeplitz operators for $A = T_0 = \langle \cdot, \mathbf{1} \rangle \, \mathbf{1}$. It is a consequence of the definitions that the field A_x is again invariant:

$$U_g^* A_x U_g = A_{gx} \ , \tag{1.7}$$

and that the operators A_f again transform nicely under G:

$$U_g A_f U_g^* = A_{f \circ g} \ . \tag{1.8}$$

Since the last construction uses only the fact that G acts on $\mathbf{D}$ transitively, it can be carried out in situations much more general than we have just described — for instance, for any group G of transformations acting transitively on a manifold Ω and any unitary anti-representation U_g of G on a Hilbert space H. Examples of the latter situations include the Fock space on $\mathbf{C}^n$ (with the action of $\mathbf{C}^n \ltimes U(n)$), or weighted Bergman spaces on bounded symmetric domains in $\mathbf{C}^n$ (with G the group of all biholomorphic self-maps). In this setting, the operators (1.6) have made appearance in some quantization procedures for symplectic manifolds [1] and have recently been studied from the group-theoretic point of view by Arazy and Upmeier [4] under the name of *invariant symbolic calculi*. Of course, in the context of the Fock space (or, equivalently — via the Bargmann transform — of $L^2(\mathbf{R}^{2n})$), a prime example of operators of the form (1.6) is the well-known Weyl calculus of pseudodifferential operators (as well as other correspondences used in ΨDO theory, such as the Kohn-Nirenberg or Unterberger calculi [25, 48]).

One can give one more twist to the above construction by lifting everything from the disc to the group. More specifically, let dg stand for the Haar measure on G. Under the quotient map $g \mapsto g0$ of G onto $\mathbf{D}$, the Haar measure projects precisely to the invariant measure $d\mu$. That is, for any function f on the disc,

$$\int_G f(g0) \, dg = \int_{\mathbf{D}} f(x) \, d\mu(x) \ . \tag{1.9}$$

Since $U_g^* A U_g = A_{g0}$ by definition, we can thus rewrite (1.6) as

$$A_f = \int_G f(g0)\ U_g^* A U_g\ dg\ .$$

Consequently, if we start with an arbitrary bounded linear operator A on $L^2_{\text{hol}}(\mathbf{D})$ [this time it even need not satisfy the commutation relation (1.5)], then we can define an invariant operator field on G by

$$A_g := U_g^* A U_g\ ,$$

and for a function f on G, the A-Toeplitz operator with symbol f by

$$A_f := \int_G f(g)\ A_g\ dg \tag{1.10}$$

whenever the integral exists (as usual, in the weak operator topology). One again checks that the resulting operator field on G is invariant:

$$U_g^* A_{g'} U_g = A_{gg'}\ , \tag{1.11}$$

and that the resulting A-Toeplitz operators transform nicely under G:

$$U_g A_f U_g^* = A_{f(g\cdot)}\ . \tag{1.12}$$

For operators A which satisfy (1.5) and functions f which are right invariant under K (i.e., $f(gk) = f(g)\ \forall g \in G\ \forall k \in K$), we plainly recover the A-Toeplitz operators from (1.6); in particular, for right K-invariant functions f and $A = \langle \cdot, \mathbf{1} \rangle\, \mathbf{1}$ we recover the Toeplitz operators we have started with.

Again, the construction (1.10) makes sense in much more general settings than the group of holomorphic automorphisms of the disc — in fact, it can be done for any locally compact group G and unitary anti-representation U_g of G on a Hilbert space H. Operators of the form (1.10) have been around at least since Harish-Chandra's work on the representations of semisimple Lie groups (see e.g., the textbook [32] for a condensed overview). Also, for rank-one selfadjoint operators A, i.e., of the form

$$A = \langle \cdot, \phi \rangle\, \phi, \qquad \phi \in H\ ,$$

they are known in wavelet theory as *localization operators*, see, for instance, [17]. Various boundedness and Schatten-class properties of these localization operators have been established in a recent series of articles by Wong and coauthors [29, 52], who also studied the case of the so-called "two-wavelets," which are operator calculi of the form (1.10) for $A = \langle \cdot, \phi \rangle\, \psi$ (i.e., for A still rank-one but not necessarily self-adjoint) [12, 53].

Our main result in this article is the following generalization of Wong's results to the case of an arbitrary operator A.

Theorem 1. *Let G be a unimodular locally compact group and U_g a strongly continuous square-integrable irreducible unitary anti-representation of G in a separable Hilbert space H.*

(a) *If A is bounded and $f \in L^1(G)$, then A_f exists and is bounded:*

$$\|A_f\| \le \|A\|\ \|f\|_1\ .$$

(b) *If A is trace-class and $f \in L^1$, then A_f exists and is trace-class:*

$$\|A_f\|_{\rm tr} \le \|A\|_{\rm tr}\ \|f\|_1 .$$

(c) *If A is trace-class and $f \in L^\infty$, then A_f exists and is bounded:*

$$\|A_f\| \le \|A\|_{\rm tr}\ \|f\|_\infty .$$

(d) *If A belongs to the Schatten class p and $f \in L^q$, where $1 \le p \le \infty$, $1 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} \ge 1$, then A_f exists and belongs to the Schatten class $\mathcal{S}^r$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, and*

$$\|A_f\|_r \le \|A\|_p\ \|f\|_q .$$

All parts of the theorem are, in a sense, optimal — for instance, taking $A = I$ (the identity operator), we have $A_g = U_g^* A U_g = I\ \forall g$, and thus

$$A_f = I \cdot \left(\int f\, dg \right)$$

which evidently does not make sense unless $f \in L^1$. Similarly for (b)-(d).

In particular, if A is trace-class, the theorem shows that the mapping

$$\Gamma :\ f \mapsto A_f$$

maps each L^p into the corresponding $\mathcal{S}^p$. In particular, this holds for the localization-operator case $A = \langle \cdot, \phi \rangle\, \phi$; this is the result of He and Wong [29]. It is worth stating this explicitly for the case which was our point of departure, viz. the Toeplitz operators. (Although the result is not entirely new, cf. the remark before Corollary 4.) We refer to Section 2 below for the definitions of the various terms.

Corollary 1. *For any bounded symmetric domain Ω and any one of the standard weighted Bergman spaces on Ω, the mapping $f \mapsto T_f$ is continuous from L^p (with respect to the invariant measure) into $\mathcal{S}^p$, for any $1 \le p \le \infty$.*

Unfortunately, the converse of this corollary is false — T_f can be in $\mathcal{S}^p$ even for $f \notin L^p$; cf. page 256 below. A necessary and sufficient condition for the membership of T_f in the Schatten ideals seems to be unknown, though such criteria exist for nonnegative symbols f (see Zhu [54, Sections 6.3-6.4], Gheorghe [26]).

Furthermore, as Γ maps $L^2(G)$ into the Hilbert space $\mathcal{S}^2$ of Hilbert-Schmidt operators, we can consider its adjoint $\Gamma^* : \mathcal{S}^2 \to L^2$. A short computation reveals that

$$\Gamma^* T(g) = \mathrm{tr}\big(T A_g^*\big) .$$

By duality, Γ^* is continuous from $\mathcal{S}^p$ into L^p for any $1 < p \le \infty$; it turns out that it is also continuous from $\mathcal{S}^1$ into L^1, and thus the composite mappings $\Gamma^*\Gamma$ and $\Gamma\Gamma^*$ act continuously onto each L^p and $\mathcal{S}^p$, respectively, $1 \le p \le \infty$. The mappings $\Gamma^*\Gamma$ and $\Gamma\Gamma^*$ are called the *Berezin* (or *link*) and the *operator-Berezin* transform; in this generality, they were first studied in [4]. Nontrivial examples of these are the recent m-Berezin transforms of Suarez [44] and Nam, Zheng and Zhong [34], with applications to the structure theory of Toeplitz algebras.

The article is organized as follows. Section 2 reviews some preliminaries from representation theory and bounded symmetric domains. The proof of Theorem 1 appears in

Section 3. Section 4 deals with the dual map Γ^* and the Berezin and operator-Berezin transforms. Section 5 discusses extensions to situations when there is no group action present, i.e., to arbitrary weakly measurable operator fields on a domain. Here our results are similar to those obtained recently by Arsu [7], but there is no inclusion between the two. Finally, in the final Section 6 we very briefly survey the various contexts in which Toeplitz operators (as well as the more general calculi studied here) naturally occur, and indicate some open problems and possible directions for future research.

2. Preliminaries

Recall that a topological group is a group G which is also a topological space and such that the mapping $(g_1, g_2) \to g_1^{-1} g_2$ from $G \times G$ into G is continuous. It is known that if G is locally compact, then there exists a unique (up to normalization) regular Borel measure dg on G which is invariant under left translations, i.e., $d(g_1 g) = dg\ \forall g_1 \in G$; dg is called the (left) Haar measure. The measure $d(g^{-1})$ is then invariant under *right* translations; if these two measures coincide, then G is called *unimodular.*

A *unitary representation* of G on a separable Hilbert space H is a mapping $V : g \mapsto V_g$ from G into unitary operators on H which satisfies

$$V_{g_1 g_2} = V_{g_1} V_{g_2} \qquad \forall g_1, g_2 \in G .$$

We will always assume that V is *strongly continuous*, i.e., $g \mapsto V_g u$ is continuous from G into H for any $u \in H$. A unitary representation is called *irreducible* if it is not a direct sum of another two representations; that is, if there is no closed subspace other than $\{0\}$ and H itself which would be invariant under all V_g, $g \in G$.

An irreducible unitary representation is called *square-integrable* if there exists $\psi \in H \setminus \{0\}$ for which

$$\int_G \left| \left\langle V_g \psi, \psi \right\rangle \right|^2 dg < \infty . \tag{2.1}$$

Such elements ψ are called *admissible vectors.* It turns out that if there is one admissible vector, then the set of all admissible vectors is already dense in H; and if G is unimodular, then even any $\psi \in H$, $\psi \neq 0$, is admissible, and the so-called *Schur orthogonality relations*

$$\int_G \left| \left\langle V_g \psi, \phi \right\rangle \right|^2 dg = d_G\ \|\phi\|^2\ \|\psi\|^2 \tag{2.2}$$

hold for any $\phi, \psi \in H$. Here d_G is a constant depending on the normalization of the Haar measure; without loss of generality, we may (and will) assume that the Haar measure has been normalized so that $d_G = 1$.

Passing from V_g to $U_g := V_g^*$, everything that has just been said remains in force also for *anti-representations* in the place of representations (i.e., mappings $g \mapsto U_g$ such that $U_{g_1 g_2} = U_{g_2} U_{g_1}$).

We refer the reader to, for instance, [32] or [3] for the material above and further details.

There is an important class of representations acting on the standard weighted Bergman spaces on bounded symmetric domains. Recall that a domain Ω in $\mathbf{C}^n$ is called *symmetric* if for any $z \in \Omega$ there exists a biholomorphic self-map s_z of Ω which is involutive ($s_z \circ s_z = \text{id}$) and has z as an isolated fixed-point. It turns out that for any two points

$x, y \in \Omega$, one can then find $z \in \Omega$ such that s_z interchanges x and y. In particular, the group G of all biholomorphic self-maps of Ω acts transitively on Ω (so Ω is *homogeneous*). Let $K(x, y)$ be the Bergman kernel of Ω. For $\alpha \geq 0$, the measure

$$d\mu_\alpha(z) := K(z, z)^{-\alpha}\, dm(z)$$

(where dm stands for the Lebesgue measure) is finite, and we have the weighted Bergman space $L^2_{\text{hol}}(\Omega, d\mu_\alpha)$ — the subspace of holomorphic functions in $L^2(\Omega, d\mu_\alpha)$.

From the familiar transformation property of the Bergman kernel

$$K(x, y) = K(gx, gy) \cdot Jg(x) \cdot \overline{Jg(y)}, \qquad \forall x, y \in \Omega,\ \forall g \in G\,, \tag{2.3}$$

where Jg denotes the complex Jacobian of g, it follows that

$$d\mu_\alpha(gz) = |Jg(z)|^{2\alpha+2}\, d\mu_\alpha(z)\,. \tag{2.4}$$

Consequently, for any $g \in G$, the map

$$U_g^{(\alpha)} : f \mapsto (f \circ g) \cdot (Jg)^{1+\alpha} \tag{2.5}$$

(with some choice of the branch for the power if $\alpha \notin \mathbf{Z}$) is a unitary operator on $L^2(\Omega, d\mu_\alpha)$ and $L^2_{\text{hol}}(\Omega, d\mu_\alpha)$. Finally, from the chain rule for the Jacobians $J_{g_1 g_2}(z) = J_{g_1}(g_2 z) J_{g_2}(z)$ it transpires that, at least for integer α,

$$U_{g_1}^{(\alpha)} U_{g_2}^{(\alpha)} = U_{g_2 g_1}^{(\alpha)}\,, \tag{2.6}$$

i.e., $U^{(\alpha)}$ is a unitary anti-representation of G on $L^2(d\mu_\alpha)$ and $L^2_{\text{hol}}(d\mu_\alpha)$. For α noninteger, (2.6) still holds but only up to a unimodular constant factor [arising from the arbitrariness of the choice of the power in (2.5)], so that $U^{(\alpha)}$ is only a *projective* anti-representation.

The notable fact about $U^{(\alpha)}$ is that on $L^2_{\text{hol}}(\Omega, d\mu_\alpha)$ these representations are *irreducible* and *square-integrable*. Further, the group G is always *unimodular.* Thus, in particular, every nonzero element in $L^2_{\text{hol}}(\Omega, d\mu_\alpha)$ is an admissible vector, and the Schur orthogonality relations (2.2) hold.

Let $K^{(\alpha)}(x, y)$ be the reproducing kernel of the space $L^2_{\text{hol}}(\Omega, d\mu_\alpha)$. It can be shown that $K^{(\alpha)}(x, y)$ coincides, up to a constant factor, with a power of the unweighted Bergman kernel:

$$K^{(\alpha)}(x, y) = c_\alpha K(x, y)^{1+\alpha}\,. \tag{2.7}$$

Denote $K_x^{(\alpha)} := K^{(\alpha)}(\cdot, x)$ and $k_x^{(\alpha)} := K_x^{(\alpha)}/\|K_x^{(\alpha)}\|$. Consider the operator field

$$T_x^{(\alpha)} := c_\alpha \left\langle \cdot, k_x^{(\alpha)} \right\rangle k_x^{(\alpha)}\,.$$

We claim that this field is invariant with respect to the representation $U^{(\alpha)}$. Indeed, for any function $f \in L^2_{\text{hol}}(\Omega, d\mu_\alpha)$, we have from the reproducing property of the kernel and by the definition of $U^{(\alpha)}$,

$$\begin{aligned}\left\langle f, U_{g^{-1}}^{(\alpha)} K_x^{(\alpha)} \right\rangle &= \left\langle U_g^{(\alpha)} f, K_x^{(\alpha)} \right\rangle = \left(U_g^{(\alpha)} f\right)(x)\\ &= Jg(x)^{1+\alpha} f(gx) = Jg(x)^{1+\alpha} \left\langle f, K_{gx}^{(\alpha)} \right\rangle\,,\end{aligned}$$

so $U_{g^{-1}}^{(\alpha)} K_x^{(\alpha)} = \overline{Jg(x)^{1+\alpha}} K_{gx}^{(\alpha)}$. Hence, $U_{g^{-1}}^{(\alpha)} k_x^{(\alpha)} = \epsilon_{x,g} k_{gx}^{(\alpha)}$ for a complex number $\epsilon_{x,g}$. As both $k_x^{(\alpha)}$ and $k_{gx}^{(\alpha)}$ are unit vectors and $U_{g^{-1}}^{(\alpha)}$ is unitary, necessarily $|\epsilon_{x,g}| = 1$. Consequently,

$$\begin{aligned} U_g^{(\alpha)*} T_x^{(\alpha)} U_g^{(\alpha)} &= c_\alpha U_{g^{-1}}^{(\alpha)} \left\langle \cdot, k_x^{(\alpha)} \right\rangle k_x^{(\alpha)} U_{g^{-1}}^{(\alpha)*} = c_\alpha \left\langle \cdot, U_{g^{-1}}^{(\alpha)} k_x^{(\alpha)} \right\rangle U_{g^{-1}}^{(\alpha)} k_x^{(\alpha)} \\ &= c_\alpha \left\langle \cdot, k_{gx}^{(\alpha)} \right\rangle k_{gx}^{(\alpha)} = T_{gx}^{(\alpha)} , \end{aligned}$$

proving the invariance of the field $T_x^{(\alpha)}$.

Finally, from (2.4), (2.7), and (2.3) it follows that the measure

$$d\mu(z) := K(z,z)\, dm(z) = c_\alpha^{-1} K^{(\alpha)}(z,z)\, d\mu_\alpha(z)$$

on Ω is invariant under G. Thus the resulting operator calculus, defined for functions on Ω by

$$T_f^{(\alpha)} := \int_\Omega f(x)\; T_x^{(\alpha)}\; d\mu(x) ,$$

again transforms nicely under G:

$$U_g^{(\alpha)} T_f^{(\alpha)} U_g^{(\alpha)*} = T_{f \circ g}^{(\alpha)} ,$$

and, as in the case of the disc, in fact coincides with the Toeplitz operators:

$$\begin{aligned} \left\langle T_f^{(\alpha)} u, v \right\rangle &= c_\alpha \int_\Omega f(x) \; \left\langle u, k_x^{(\alpha)} \right\rangle \left\langle k_x^{(\alpha)}, v \right\rangle \; d\mu(x) \\ &= \int_\Omega f(x) \; \frac{\left\langle u, K_x^{(\alpha)} \right\rangle \left\langle K_x^{(\alpha)}, v \right\rangle}{K^{(\alpha)}(x,x)} \; K^{(\alpha)}(x,x)\, d\mu_\alpha(x) \\ &= \int_\Omega f(x)\; u(x)\; \overline{v(x)}\; d\mu_\alpha(x) \\ &= \langle fu, v \rangle = \left\langle T_f^{(\alpha)} u, v \right\rangle . \end{aligned}$$

3. Proof of Theorem 1

Part (a). Assume that $A \in \mathcal{B}$ and $f \in L^1(G)$. For any $u, v \in H$, we have

$$\begin{aligned} \left| \int_G f(g) \; \langle A_g u, v \rangle \; dg \right| &\le \|f\|_1 \cdot \sup_{g \in G} \left| \langle A U_g u, U_g v \rangle \right| \\ &\le \|f\|_1 \; \sup_g \|A\| \, \|U_g u\| \, \|U_g v\| \\ &= \|f\|_1 \; \|A\| \; \|u\| \; \|v\| . \end{aligned}$$

Thus the integral (1.10) exists in the weak operator topology, and

$$\|A_f\| \le \|f\|_1 \; \|A\|$$

as claimed.

Part (b). Assume now that $A \in \mathcal{S}^1$ and $f \in L^1(G)$. By part (a), we already know that A_f exists and is a bounded linear operator; we need to show that it is even trace-class. Recall that for any bounded linear operator X,

$$\|X\|_{\text{tr}} = \sup_{\{u_j\},\{v_j\}} \sum_j \left| \langle X u_j, v_j \rangle \right| , \tag{3.1}$$

the supremum being taken over all orthonormal bases $\{u_j\}$ and $\{v_j\}$ of H; here the equality also means that if X is not trace-class then the right-hand side is infinite. Apply this to A_f:

$$\begin{aligned}
\|A_f\|_{\text{tr}} &= \sup_{\{u_j\},\{v_j\}} \sum_j \left| \int_G f(g) \, \langle A U_g u_j, U_g v_j \rangle \, dg \right| \\
&\le \sup_{\{u_j\},\{v_j\}} \int_G |f(g)| \, \sum_j \left| \langle A U_g u_j, U_g v_j \rangle \right| dg \\
&\le \sup_{\{u_j\},\{v_j\}} \left[\|f\|_1 \cdot \sup_g \sum_j \left| \langle A U_g u_j, U_g v_j \rangle \right| \right] .
\end{aligned}$$

However, since $\{U_g u_j\}$, $\{U_g v_j\}$ are also orthonormal bases (as U_g is unitary), the last sum does not exceed $\|A\|_{\text{tr}}$ by (3.1); thus

$$\|A_f\|_{\text{tr}} \le \|f\|_1 \, \|A\|_{\text{tr}}$$

as asserted.

Part (c). Assume now that $A \in \mathcal{S}^1$ and $f \in L^\infty(G)$. Since A is trace-class, it has a spectral decomposition

$$A = \sum_j \lambda_j \langle \cdot, u_j \rangle v_j ,$$

with some orthonormal bases $\{u_j\}$, $\{v_j\}$ and numbers $\lambda_j \ge 0$ satisfying $\sum_j \lambda_j = \|A\|_{\text{tr}} < \infty$. For any $u, v \in H$, consider the function $A_{u,v}$ on G given by

$$A_{u,v}(g) := \langle A_g u, v \rangle . \tag{3.2}$$

Note that

$$\left| \langle A_f u, v \rangle \right| = \left| \int_G f(g) \, A_{u,v}(g) \, dg \right| \le \|f\|_\infty \, \|A_{u,v}\|_1 . \tag{3.3}$$

Thus it suffices to show that $A_{u,v} \in L^1$ with

$$\|A_{u,v}\|_1 \le \|A\|_{\text{tr}} \, \|u\| \, \|v\| \tag{3.4}$$

and the desired conclusion $\|A_f\| \le \|f\|_\infty \|A\|_{\text{tr}}$ will follow.

Now

$$A_{u,v}(g) = \langle A U_g u, U_g v \rangle = \sum_j \lambda_j \, \langle U_g u, u_j \rangle \langle v_j, U_g v \rangle .$$

Hence,

$$\begin{aligned}\|A_{u,v}\|_1 &\le \sum_j \lambda_j \int_G \left|\langle U_g u, u_j\rangle\langle v_j, U_g v\rangle\right| dg \\ &\le \|A\|_{\rm tr} \sup_j \int_G \left|\langle U_g u, u_j\rangle\langle v_j, U_g v\rangle\right| dg \\ &\le \|A\|_{\rm tr} \sup_j \sqrt{\int_G \left|\langle U_g u_j, u\rangle\right|^2 dg}\,\sqrt{\int_G \left|\langle U_g v_j, v\rangle\right|^2 dg}\,.\end{aligned}$$

However, by the Schur orthogonality relations (2.2), the last two integrals are equal to $\|u_j\|^2\|u\|^2 = \|u\|^2$ and $\|v_j\|^2\|v\|^2 = \|v\|^2$, respectively. Consequently,

$$\|A_{u,v}\|_1 \le \|A\|_{\rm tr}\|u\|\ \|v\|\ ,$$

proving (3.4). This completes the proof of part (c).

Part (d). For fixed $u, v \in H$, consider again the function $A_{u,v}$ from (3.2). By the Cauchy-Schwarz inequality, $\|A_{u,v}\|_\infty \le \|A\|\ \|u\|\ \|v\|$, while in the proof of part (c) we have seen that $\|A_{u,v}\|_1 \le \|A\|_{\rm tr}\ \|u\|\ \|v\|$. Thus the map $A \mapsto A_{u,v}$ is a contraction from $\mathcal{B}$ into L^∞ as well as from $\mathcal{S}^1$ into L^1. By interpolation, it follows that for any $1 \le q' \le \infty$, $A \in \mathcal{S}^{q'}$ implies that $A_{u,v} \in L^{q'}$ and

$$\|A_{u,v}\|_{q'} \le \|A\|_{q'}\ \|u\|\ \|v\|\ .$$

(Here and below $\mathcal{S}^\infty := \mathcal{B}$.) As in (3.3), it therefore follows that if $A \in \mathcal{S}^{q'}$ and $f \in L^q$, where $\frac{1}{q} + \frac{1}{q'} = 1$, then A_f exists and

$$\|A_f\| \le \|f\|_q\ \|A\|_{q'}\ . \tag{3.5}$$

On the other hand, from parts (b) and (c), respectively, we know that if A is trace-class, then the map $f \mapsto A_f$ is a contraction from L^1 into $\mathcal{S}^1$ as well as from L^∞ into $\mathcal{B}$; by interpolation, it therefore follows that if A is trace-class and $f \in L^q$, then $A_f \in \mathcal{S}^q$ and

$$\|A_f\|_q \le \|f\|_q\ \|A\|_1\ . \tag{3.6}$$

Combining (3.5) and (3.6) and appealing to interpolation for one more time, we conclude that if $f \in L^q$ and $A \in \mathcal{S}^p$, where $\frac{1}{p} = \frac{\theta}{q'} + \frac{1-\theta}{1}$, $0 \le \theta \le 1$, then A_f is in $\mathcal{S}^r$ where $\frac{1}{r} = \frac{\theta}{\infty} + \frac{1-\theta}{q}$, and

$$\|A_f\|_r \le \|f\|_q\ \|A\|_p\ .$$

Eliminating θ gives $\theta = (1 - \frac{1}{p})q$, so that $0 \le \theta \le 1$ means $p \ge 1$ and $\frac{1}{p} + \frac{1}{q} \ge 1$, and $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} - 1$, thus completing the proof of part (d).

Corollary 2. *Let G be a unimodular locally compact group of transformations acting transitively on a manifold Ω and U_g a strongly continuous square-integrable irreducible unitary anti-representation of G in a separable Hilbert space H. Denote by $d\mu$ the image of the Haar measure of G under the quotient map $g \mapsto gx_0$ of G onto Ω (where x_0 is some fixed basepoint in Ω); consequently, $d\mu$ is a G-invariant measure on Ω.*

(a) *If A is bounded and $f \in L^1(\Omega, d\mu)$, then A_f exists and is bounded:*

$$\|A_f\| \le \|A\|\ \|f\|_1\ .$$

(b) *If A is trace-class and $f \in L^1(\Omega, d\mu)$, then A_f exists and is trace-class:*

$$\|A_f\|_{\mathrm{tr}} \le \|A\|_{\mathrm{tr}} \, \|f\|_1 \, .$$

(c) *If A is trace-class and $f \in L^\infty(\Omega, d\mu)$, then A_f exists and is bounded:*

$$\|A_f\| \le \|A\|_{\mathrm{tr}} \, \|f\|_\infty \, .$$

(d) *If $A \in \mathcal{S}^p$ and $f \in L^q(\Omega, d\mu)$, where $1 \le p \le \infty$, $1 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} \ge 1$, then A_f exists and belongs to the Schatten class $\mathcal{S}^r$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, and*

$$\|A_f\|_r \le \|A\|_p \, \|f\|_q \, .$$

Proof. Denote, for a moment, by πf the function on G defined, for a given function f on Ω, by $\pi f(g) := f(gx_0)$. In view of (1.9), the mapping $f \mapsto \pi f$ is an isometry from $L^p(\Omega, d\mu)$ into $L^p(G)$, for any p. Further, as we have already observed in the introduction, the operator A_f defined by (1.6) coincides with the operator $A_{\pi f}$ defined by (1.10). The corollary is therefore immediate from Theorem 1. □

Proof of Corollary 1. As we have seen at the end of Section 2, Toeplitz operators correspond to the calculus $f \mapsto A_f$ for the choice $A = c_\alpha \langle \cdot, k_{x_0}^{(\alpha)} \rangle k_{x_0}^{(\alpha)}$ (with some fixed basepoint $x_0 \in \Omega$). Since this is manifestly a trace-class operator (even rank-one), the desired assertion is immediate from part (d) of Corollary 2 (with $p = 1$) or from parts (b), (c) and interpolation. □

4. Berezin Transforms

Throughout this section, we assume that the operator A is trace-class. By Theorem 1, we then know that the map

$$\Gamma : f \mapsto A_f$$

maps L^p continuously into $\mathcal{S}^p$, for any $1 \le p \le \infty$. In particular, it makes sense to consider the adjoint $\Gamma^* : \mathcal{S}^p \to L^p$, $1 < p \le \infty$.

Since we know a priori that Γ^* is continuous, it is enough to find its values on the rank-one operators $\langle \cdot, u \rangle v$, since the span of these is dense in any $\mathcal{S}^p$, $1 \le p < \infty$, as well as w^*-dense in $\mathcal{S}^\infty = (\mathcal{S}^1)^*$. But for any $u, v \in H$, and $T = \langle \cdot, u \rangle v$,

$$\begin{aligned} \int_G \Gamma^* T \cdot \overline{f} \, dg &= \langle \Gamma^* T, f \rangle = \langle T, \Gamma f \rangle = \mathrm{tr}\big(T A_f^*\big) = \langle v, A_f u \rangle \\ &= \int_G \overline{f(g)} \, \langle v, A_g u \rangle \, dg \\ &= \int_G \overline{f(g)} \, \mathrm{tr}\big(T A_g^*\big) \, dg \, . \end{aligned}$$

Thus

$$\Gamma^* T(g) = \mathrm{tr}\big(T A_g^*\big) \, .$$

Theorem 2. *Under the hypothesis of Theorem 1, the map Γ^* satisfies: If $A \in \mathcal{S}^p$, $1 \le p \le \infty$, $1 \le q \le \infty$, $\frac{1}{p} + \frac{1}{q} \ge 1$, then $\Gamma^* : \mathcal{S}^{r'} \to L^{q'}$ contractively, where $\frac{1}{q'} = 1 - \frac{1}{q}$ and $\frac{1}{r'} = 1 - \frac{1}{r}$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ (i.e., $\frac{1}{r'} = \frac{1}{q'} + \frac{1}{p'}$ where $\frac{1}{p'} = 1 - \frac{1}{p}$).*

Proof. For $1 < q' \leq \infty$ and $1 < r' \leq \infty$ the claim follows from part (d) of Theorem 1 and the dualities $(\mathcal{S}^r)^* = \mathcal{S}^{r'}$, $(L^q)^* = L^{q'}$. Thus we only need to deal with the cases $r' = 1$, i.e., $\Gamma^* : \mathcal{S}^1 \to L^p$ if $A \in \mathcal{S}^p$; and $q' = 1$, i.e., $\Gamma^* : \mathcal{S}^1 \to L^1$ if $A \in \mathcal{S}^1$.

We start with the latter, so assume that A and T are trace-class. Let $A = \sum_j \lambda_j \langle \cdot, u_j \rangle v_j$ and $T = \sum_k t_k \langle \cdot, \phi_k \rangle \psi_k$ be the spectral decompositions of A and T, respectively, with $\{u_j\}$, $\{v_j\}$, $\{\phi_k\}$, $\{\psi_k\}$ orthonormal bases in H and $\lambda_j, t_k \geq 0$, $\sum_j \lambda_j = \|A\|_{\text{tr}}$, $\sum_k t_k = \|T\|_{\text{tr}}$. Then

$$\Gamma^* T(g) = \operatorname{tr}\big(T U_g^* A^* U_g\big) = \sum_{j,k} \lambda_j t_k \big\langle u_j, U_g \phi_k \big\rangle \big\langle U_g \psi_k, v_j \big\rangle \ .$$

Thus

$$\begin{aligned}
\big\|\Gamma^* T\big\|_1 &\leq \sum_{j,k} \lambda_j t_k \int_G \big| \big\langle u_j, U_g \phi_k \big\rangle \big\langle U_g \psi_k, v_j \big\rangle \big| \, dg \\
&\leq \Big(\sum_{j,k} \lambda_j t_k \Big) \sup_{j,k} \int_G \big| \big\langle u_j, U_g \phi_k \big\rangle \big\langle U_g \psi_k, v_j \big\rangle \big| \, dg \\
&\leq \|A\|_{\text{tr}} \ \|T\|_{\text{tr}} \ \sup_{j,k} \Big(\int_G \big| \big\langle u_j, U_g \phi_k \big\rangle \big|^2 \, dg \Big)^{1/2} \Big(\int_G \big| \big\langle U_g \psi_k, v_j \big\rangle \big|^2 \, dg \Big)^{1/2} .
\end{aligned}$$

But by the Schur orthogonality relations (2.2), the last two integrals are equal to $\|u_j\|^2 \|\phi_k\|^2$ and $\|\psi_k\|^2 \|v_j\|^2$, respectively, i.e., to 1. Consequently,

$$\big\|\Gamma^* T\big\|_1 \leq \|A\|_{\text{tr}} \ \|T\|_{\text{tr}} \ ,$$

which is the desired claim.

For the remaining part, i.e., $\Gamma^* : \mathcal{S}^1 \to L^p$ if $A \in \mathcal{S}^p$, observe first of all that it is enough to settle the cases $p = 1$ and $p = \infty$: Indeed, for a fixed $T \in \mathcal{S}^1$, we will then have that $A \mapsto \Gamma_A^* T$ is a contraction from $\mathcal{S}^1$ into L^1 and from $\mathcal{S}^\infty$ into L^∞; by interpolation, it will therefore follow that it is a contraction from $\mathcal{S}^p$ into L^p for all $1 \leq p \leq \infty$, and we will be done. However, the claim for $p = 1$ has been just settled in the previous paragraph; while the claim for $p = \infty$ is straightforward from the inequality

$$\big|\Gamma^* T(g)\big| = \big|\operatorname{tr}\big(T A_g^*\big)\big| \leq \|T\|_{\text{tr}} \, \|A_g\| = \|T\|_{\text{tr}} \|A\|$$

(since $\|A_g\| = \|U_g^* A U_g\| = \|A\|$). This completes the proof. □

Remark 1. Note that the case $q = 1$ of the last theorem follows trivially from the "Hölder inequality"

$$\big|\operatorname{tr}\big(T A_g^*\big)\big| \leq \|T\|_{p'} \, \|A\|_p, \qquad 1 \leq p \leq \infty \, .$$

By duality, this can likewise be used to get an alternative proof of the case $q = 1$, $1 < p \leq \infty$ of Theorem 1 (otherwise obtainable by interpolation from parts (a) and (b) thereof). However, it does not seem to simplify the proofs of the remaining cases of Theorems 1 and 2 in any way.

Of course, as with Theorem 1 and Corollary 2, we again have also a completely analogous statement concerning operator fields and calculi on Ω [i.e., based on (1.6) not (1.10)], whose statement and proof can safely be left to the reader.

Combining the last theorem with Theorem 1 yields the following.

Corollary 3. *If A is trace-class, then $\Gamma^*\Gamma$ is a contraction $L^p \to L^p$, and $\Gamma\Gamma^*$ is a contraction $\mathcal{S}^p \to \mathcal{S}^p$, for any $1 \le p \le \infty$.*

The mapping $\Gamma^*\Gamma$,

$$\Gamma^*\Gamma f(g) = \int_G f\big(g'\big)\ \mathrm{tr}\big(A_{g'}A_g^*\big)\ dg'\ ,$$

is known as the (generalized) *Berezin transform* on G; similarly, the mapping $\Gamma\Gamma^*$,

$$\Gamma\Gamma^* T = \int_G \mathrm{tr}\big(TA_g^*\big)\ A_g\ dg\ ,$$

is called the (generalized) *operator Berezin transform*. One also has, of course, the analogous objects for operator fields on Ω instead of G. In the traditional case of Toeplitz operators on the Bergman space of the disc, these formulas become

$$\Gamma^*\Gamma f(x) = \int_{\mathbf{D}} f(y)\ \frac{|K(x,y)|^2}{K(x,x)}\ dy\ ,$$
$$\Gamma\Gamma^* T = T_{\widetilde{T}}, \qquad \widetilde{T}(x) := \langle Tk_x, k_x\rangle\ .$$

A nontrivial example of the operator Berezin transform on the disc is the m-Berezin transform on $\mathbf{D}$ of Suarez [44], corresponding to

$$A = (m+1)\sum_{j=0}^{m}\binom{m}{j}(-1)^j\big\langle\cdot, z^j\big\rangle z^j \qquad (m = 0, 1, 2, \dots)\ .$$

For $m = 0$, this reduces to the ordinary Berezin transform above; while as $m \to \infty$, the transforms $\Gamma_m\Gamma_m^*$ turn out to be a sort of "approximate identity," in that $\Gamma_m\Gamma_m^* T \to T$ in the weak or strong operator topology for many $T \in \mathcal{B}$; this fact is utilized for studying the structure of Toeplitz algebras on the Bergman space of the disc. All this has subsequently been generalized also to the unit ball, see Nam, Zheng, and Zhong [34].

Observe that in view of the G-invariance properties (1.11) and (1.12) [or (1.7) and (1.8)], both transforms $\Gamma^*\Gamma$ and $\Gamma\Gamma^*$ are invariant:

$$\Gamma^*\Gamma(f\circ g) = \big(\Gamma^*\Gamma f\big)\circ g, \qquad \Gamma\Gamma^*\big(U_gTU_g^*\big) = U_g\big(\Gamma\Gamma^*T\big)U_g^*, \qquad \forall g \in G\ .$$

In particular, in the context of bounded symmetric domains (cf. the end of Section 2), this means that $\Gamma^*\Gamma$ is a Fourier multiplier with respect to the *Helgason-Fourier* transform on symmetric spaces. Namely, there is a family of *conical functions* $e_{\lambda,b}$, indexed by $\lambda \in \mathbf{R}^r$ and $b \in K/M$, where r is the rank of the symmetric domain Ω and K/M is the quotient of the stabilizer K in G of a chosen basepoint $x_0 \in \Omega$ by the normalizer M in K of the maximal Abelian subgroup A of G; and for $f \in L^2(\Omega, d\mu)$, one defines the Helgason-Fourier transform $\widetilde{f}$ of f as

$$\widetilde{f}(\lambda, b) = \int_\Omega f(x)\ e_{\lambda,b}(x)\ d\mu(x)\ .$$

It turns out that there is an inversion formula

$$f(x) = \int_{\mathbf{R}^r}\int_{K/M} \widetilde{f}(\lambda, b)\ e_{-\lambda,b}(x)\ |c(\lambda)|^2\ db\ d\lambda$$

with some function c on $\mathbf{R}^r$ (the Harish-Chandra c-function) and db the Haar measure on K/M; and a Plancherel isometry

$$\int_\Omega |f(x)|^2 \, d\mu(x) = \int_{\mathbf{R}^r} \int_{K/M} |f(\lambda, b)|^2 \, |c(\lambda)|^2 \, db \, d\lambda \, ,$$

which establishes a unitary isomorphism between $L^2(\Omega, d\mu)$ and a subspace $\mathcal{M}$ of all functions in $L^2(\mathbf{R}^r \times K/M, |c(\lambda)|^2 \, db \, d\lambda)$ satisfying a certain symmetry condition. Under this isomorphism, an operator on $L^2(\Omega, d\mu)$ commuting with the action of G corresponds to the operator on $\mathcal{M}$ of multiplication by a certain function depending only on λ. See e.g., Helgason [30] for the details. All this also makes sense, of course, for the complex n-space $\mathbf{C}^n$ in the place of the bounded symmetric domains — then the Helgason-Fourier transform becomes the ordinary Fourier transform and $\Gamma^*\Gamma$, an operator on $L^2(\mathbf{C}^n)$ commuting with all translations and rotations, will be an ordinary Fourier multiplier (or, in other words, a convolution operator).

In the case of the disc, for instance, the conical functions are given by

$$e_{\lambda,b}(x) = \left(\frac{1 - |x|^2}{|b - x|^2} \right)^{\frac{1}{2} + i\lambda}, \qquad x \in \mathbf{D}, \ \lambda \in \mathbf{R}, \ b \in \mathbf{T} \, ;$$

and the ordinary Berezin transform $B = \Gamma^*\Gamma$ on $\mathbf{D}$ (corresponding to the Toeplitz calculus $\Gamma f = T_f$ on the Bergman space $L^2_{\text{hol}}(\mathbf{D})$) satisfies $\widetilde{(Bf)}(\lambda, b) = \beta(\lambda) \widetilde{f}(\lambda, b)$, where

$$\beta(\lambda) = \frac{\left(\frac{1}{4} + \lambda^2 \right) \pi}{\cosh \pi \lambda} \, . \tag{4.1}$$

(See [9, 50].)

The reason why we have stated the last formula is that it shows that the condition in our Corollary 1 for the membership of the Toeplitz operators in $\mathcal{S}^p$ is only sufficient, but not necessary — i.e., there exist $f \notin L^p(d\mu)$ for which $T_f \in \mathcal{S}^p$. To see this, note that e.g., for $p = 2$,

$$\|T_f\|^2_{\mathcal{S}^2} = \langle T_f, T_f \rangle_{\mathcal{S}^2} = \langle \Gamma^*\Gamma f, f \rangle_{L^2} = \left\| \left(\Gamma^*\Gamma \right)^{1/2} f \right\|^2_{L^2} \, .$$

Thus if we had $f \in L^2 \iff T_f \in \mathcal{S}^2$, it would follow that $f \in L^2 \iff (\Gamma^*\Gamma)^{1/2} f \in L^2$, or upon using the Plancherel isomorphism above,

$$\widetilde{f} \in L^2 \iff \beta^{1/2} \widetilde{f} \in L^2 \, .$$

(Here the two occurrences of L^2 stand for $L^2(\mathbf{R} \times \mathbf{T}, |c(\lambda)|^2 \, db \, d\lambda)$.) The last condition is clearly equivalent to β and β^{-1} being bounded; however, in view of (4.1), for β^{-1} this is not the case. Up to the author's knowledge, a necessary and sufficient condition for the membership of T_f in $\mathcal{S}^p$ is unknown.

Similarly, although there are examples easy to construct showing that the various assertions in Theorem 1 cannot be improved in general, in some instances one gets better results due to various "cancellations" in the integral defining A_f (in the proof of Theorem 1, we have always been looking just at the "size" of the integrand, with respect to various norms). A prime instance of this phenomenon is the classical Weyl calculus of operators on $L^2(\mathbf{R}^{2n})$ or, upon employing the Bargmann isometry, on the Fock space

$\mathcal{F} := L^2_{\text{hol}}(\mathbf{C}^n, e^{-|z|^2}\pi^{-n}\, dm(z))$: It corresponds to taking for A the elementary symmetry operator

$$Wu(z) := u(-z), \qquad f \in \mathcal{F}\,.$$

Since W does not belong to $\mathcal{S}^p$ for any finite p, our Theorem 1 yields only the implication

$$f \in L^1\big(\mathbf{C}^n, dm\big) \implies W_f \in \mathcal{B}, \qquad \|W_f\| \le \|f\|_1\,.$$

However, in reality things are much better, since by the old result of Pool [41],

$$f \in L^2\big(\mathbf{C}^n, dm\big) \implies W_f \in \mathcal{S}^2, \qquad \|W_f\|_{\mathcal{S}^2} = \|f\|_{L^2}\,.$$

By interpolation, $f \mapsto W_f$ is bounded from $L^p \to \mathcal{S}^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$, for any $1 \le p \le 2$. It was shown by Simon and Wong that one cannot do better: For any $2 < p \le \infty$ there exists $f \in L^p$ for which $W_f \notin \mathcal{B}$. On the other hand, there are results by Cordes, Folland, Howe, Meyer, Hwang, Boulkhemair, and others [13], showing that $W_f \in \mathcal{B}$ if $f \in BC^{n+2}(\mathbf{C}^n)$ (the space of function with bounded derivatives of orders up to $n+2$). It would be nice to know whether our Theorem 1 can be improved in some way in this respect, i.e., whether any better estimates can possibly be established for A_f even if A is only bounded but subject to some additional hypothesis.

5. General Operator Fields

The notion of the operator field makes perfect sense, of course, also in situations when there is no group action present: That is, we can consider a mapping

$$x \mapsto A_x \tag{5.1}$$

from a completely arbitrary domain $\Omega \subset \mathbf{C}^n$ into operators on some Hilbert space H, and define the A-Toeplitz operators by

$$A_f := \int_\Omega f(x)\; A_x\; dx$$

for some measure dx on Ω, provided the integral exists in the weak operator topology. Naturally, there is little hope to prove anything without some additional hypothesis on the map (5.1). For operator fields of the form

$$A_x = U(x)^* A U(x)$$

with A fixed and $x \mapsto U(x)$ a weakly-measurable function from Ω into $\mathcal{B}(H)$, it was proved by Arsu ([7, Lemma 4.3]) that if A is trace-class, then

$$\sup_y \|U(y)\|^2 \le C < \infty,\;\; f \in L^1 \quad\implies\quad A_f \in \mathcal{S}^1 \;\text{ and }\; \|A_f\|_{\text{tr}} \le C\;\|f\|_1\;\|A\|_{\text{tr}}$$

and

$$\sup_y \sup_{\|u\|,\|v\|\le 1} \int_\Omega |\,\langle U(x)u, v\rangle\,|^2\, dx \le C < \infty,\;\; f \in L^\infty$$
$$\implies \quad A_f \in \mathcal{B} \;\text{ and }\; \|A_f\| \le C\|f\|_\infty \|A\|_{\text{tr}}\,.$$

The proofs proceed along very similar lines as for parts (b) and (c) of our Theorem 1.

We present another result in this direction, which seems neither to include nor to be included in the results of Arsu.

Theorem 3. *Assume that* (5.1) *is measurable in the weak operator topology.*

(a) *If* $\sup_x \|A_x\| \le C < \infty$ *and* $f \in L^1(dx)$*, then* A_f *is bounded and*

$$\|A_f\| \le C\ \|f\|_1\ .$$

(b) *If* $\sup_x \|A_x\|_{\rm tr} \le C < \infty$ *and* $f \in L^1(dx)$*, then* A_f *is trace-class and*

$$\|A_f\|_{\rm tr} \le C\ \|f\|_1\ .$$

(c) *If* $f \in L^\infty(dx)$ *and*

$$\begin{aligned} \int_\Omega \Big\langle \big(A_x^* A_x\big)^{1/2} u, u\Big\rangle\, dx \le C\ \|u\|^2\ , \\ \int_\Omega \Big\langle \big(A_x A_x^*\big)^{1/2} u, u\Big\rangle\, dx \le C\ \|u\|^2\ , \end{aligned} \tag{5.2}$$

for all $u \in H$*, then* A_f *is bounded and*

$$\|A_f\| \le C\ \|f\|_\infty\ .$$

(d) *If* $1 < p < \infty$*,* $f \in L^{p'}(dx)$*,* $\frac{1}{p} + \frac{1}{p'} = 1$*, and*

$$\begin{aligned} \int_\Omega \Big\langle \big(A_x^* A_x\big)^{1/2} u, u\Big\rangle^p\ dx \le C^p\ \|u\|^{2p}\ , \\ \int_\Omega \Big\langle \big(A_x A_x^*\big)^{1/2} u, u\Big\rangle^p\ dx \le C^p\ \|u\|^{2p}\ , \end{aligned} \tag{5.3}$$

for all $u \in H$*, then* A_f *is bounded and*

$$\|A_f\| \le C\ \|f\|_{p'}\ .$$

(e) *Part* (d) *also holds if* (5.3) *is replaced by*

$$\begin{aligned} \int_\Omega \Big\langle \big(A_x^* A_x\big)^{p/2} u, u\Big\rangle\, dx \le C^p\ \|u\|^2\ , \\ \int_\Omega \Big\langle \big(A_x A_x^*\big)^{p/2} u, u\Big\rangle\, dx \le C^p\ \|u\|^2\ . \end{aligned} \tag{5.4}$$

Note that the conditions in (c)-(e) cannot be weakened to $\sup_x \|A_x\|_{\rm tr} < \infty$: For instance, if $A_x = \langle \cdot, a\rangle\, a\ \forall x$ for some fixed $a \in H$, then $A_f = (\int f) \cdot \langle \cdot, a\rangle\, a$ which makes sense only for $f \in L^1$.

Proof. Part (a) is just a simple consequence of the Schwarz inequality:

$$\big|\big\langle A_f u, v\big\rangle\big| \le \|f\|_1 \cdot \sup_x |\langle A_x u, v\rangle| \le \|f\|_1\ C\ \|u\|\ \|v\|\ .$$

For part (b), we again use the criterion (3.1):

$$\begin{aligned}\|A_f\|_{\mathrm{tr}} &= \sup_{\{u_j\},\{v_j\}} \sum_j \left| \int_\Omega f(x)\ \langle A_x u_j, v_j\rangle\ dx \right| \\ &\le \sup_{\{u_j\},\{v_j\}} \int_\Omega |f(x)|\ \sum_j \left|\langle A_x u_j, v_j\rangle\right| dx \\ &\le \sup_{\{u_j\},\{v_j\}} \int_\Omega |f(x)|\ \|A_x\|_{\mathrm{tr}}\, dx \\ &\le \|f\|_1\ \sup_x \|A_x\|_{\mathrm{tr}}\ .\end{aligned}$$

For part (c), we recall first of all that for any bounded operator X,

$$|\langle Xu, v\rangle|^2 \le \langle |X|u, u\rangle\ \langle |X^*|v, v\rangle\ , \tag{5.5}$$

where $|X| := (X^*X)^{1/2}$. Indeed, let $X = W|X|$ be the polar decomposition of X, so that W is a partial isometry with initial space $\overline{\operatorname{Ran} X^*} = \overline{\operatorname{Ran}|X|}$ and final space $\overline{\operatorname{Ran} X}$; in particular, W^*W is identity of $\operatorname{Ran}|X|$. It follows that $|X^*| = (W|X|^2W^*)^{1/2} = W|X|W^*$. Since $|X|$ and $|X^*|$ are nonnegative operators, by the Schwarz inequality

$$\begin{aligned}|\langle Xu, v\rangle|^2 &= |\langle W|X|u, v\rangle|^2 = \left|\langle |X|u, W^*v\rangle\right|^2 \\ &\le \langle |X|u, u\rangle \langle |X|W^*v, W^*v\rangle = \langle |X|u, u\rangle \langle |X^*|v, v\rangle\ ,\end{aligned}$$

as claimed. Applying now (5.5) to the operators A_x, it follows that

$$\begin{aligned}\left|\langle A_f u, v\rangle\right| &\le \int_\Omega |f(x)|\ |\langle A_x u, v\rangle|\ dx \\ &\le \int_\Omega |f(x)|\ \langle |A_x|u, u\rangle^{1/2}\ \langle |A_x^*|v, v\rangle^{1/2}\ dx \\ &\le \|f\|_\infty \left(\int_\Omega \langle |A_x|u, u\rangle\ dx\right)^{1/2} \left(\int_\Omega \langle |A_x^*|v, v\rangle\ dx\right)^{1/2} \\ &\le \|f\|_\infty\ C\ \|u\|\ \|v\|\end{aligned}$$

by (5.2). This establishes the claim.

For part (d), consider again the functions

$$A_{u,v}(x) := \langle A_x u, v\rangle\ .$$

By (5.5),

$$\begin{aligned}\|A_{u,v}\|_p^p &= \int_\Omega |\langle A_x u, v\rangle|^p\ dx \\ &\le \int_\Omega \langle |A_x|u, u\rangle^{p/2}\ \langle |A_x^*|v, v\rangle^{p/2}\ dx \\ &\le \left(\int_\Omega \langle |A_x|u, u\rangle^p\ dx\right)^{1/2} \left(\int_\Omega \langle |A_x^*|v, v\rangle^p\ dx\right)^{1/2} .\end{aligned}$$

By (5.3), this does not exceed $C^p\|u\|^p\|v\|^p$. Thus $A_{u,v} \in L^p$ and $\|A_{u,v}\|_p \leq C\|u\|\,\|v\|$. Since

$$\langle A_f u, v\rangle = \int_\Omega f(x)\; A_{u,v}(x)\; dx\;,$$

the desired assertion follows.

Finally, for part (e), observe that for any nonnegative operator X and $p > 1$,

$$\langle Xu, u\rangle \leq \langle X^p u, u\rangle^{1/p}\;\; \|u\|^{2-2/p}\;. \tag{5.6}$$

Indeed, let E_t, $0 \leq t < \infty$, be the spectral measure of X; then by the Spectral Theorem and Hölder's inequality,

$$\begin{aligned}\langle Xu, u\rangle &= \int_0^\infty t\; d(E_t u, u)\\ &\leq \left(\int_0^\infty t^p\; d(E_t u, u)\right)^{1/p}\left(\int_0^\infty d(E_t u, u)\right)^{1/p'}\\ &= \langle X^p u, u\rangle^{1/p}\;\; \|u\|^{2/p'}\;,\end{aligned}$$

which is (5.6). Taking now $X = |A_x|$, we see that

$$\langle |A_x|u, u\rangle^p \leq \langle |A_x|^p u, u\rangle\;\; \|u\|^{2p-2}\;,$$

and similarly for $\langle |A_x^*|u, u\rangle$; thus (5.4) implies (5.3). This completes the proof of Theorem 3. □

Note that the inequality (5.2) is easily seen to be fulfilled in the situation as in Theorem 1, i.e., for $A_g = U_g^* A U_g$ on a group G with A trace-class; this follows by the Schur orthogonality relations much in the same way as in the proof of part (c) of Theorem 1. Similarly, (5.4) is easily seen to hold in that case if $A \in \mathcal{S}^p$; using interpolation, we thus see that the last theorem completely includes Theorem 1.

Likewise, both (5.2) and (5.4) are straightforward to verify for the operator field corresponding to Toeplitz operators,

$$T_x := \langle\cdot, k_x\rangle\, k_x, \qquad \text{where} \quad k_x := \frac{K_x}{\|K_x\|}\;,$$

with $K_x := K(\cdot, x)$ the reproducing kernel of the weighted Bergman space with respect to any weight (or even measure) $d\nu$ on Ω such that $\|K_x\| > 0\ \forall x$, and

$$dx = K(x, x)\, d\nu(x)\;.$$

Indeed, since T_x are rank-one selfadjoint projections, we have $|T_x| = |T_x^*| = |T_x|^p = |T_x^*|^p = T_x$ for any $p > 0$, and

$$\int_\Omega \langle T_x u, u\rangle\;\; dx = \int_\Omega |\,\langle u, k_x\rangle\,|^2\; K(x, x)\, d\nu(x) = \int_\Omega |u(x)|^2\; d\nu(x) = \|u\|^2$$

by the reproducing property of the kernel. Thus both (5.2) and (5.4) hold with $C = 1$. In particular, the following analogue of Corollary 1 remains in force. (We warn the reader, however, that this is hardly a new result — even though the author is not aware of any explicit reference for it in the literature — because the proof for the disc given e.g., in [54, Lemma 6.3.4] extends without changes also to the present situation.)

Corollary 4. *Let Ω be an arbitrary domain in $\mathbf{C}^n$ and consider the Bergman space $L^2_{\text{hol}}(\Omega, d\nu)$ with respect to any measure ν such that this space has a reproducing kernel $K(x, y)$ and $K(x, x) > 0\ \forall x \in \Omega$. Then the Toeplitz operator T_f on $L^2_{\text{hol}}(\Omega, d\nu)$ belongs to $\mathcal{S}^p$ whenever $f \in L^p(K(z, z)\, d\nu(z))$, and the map $f \mapsto T_f$ from this L^p into $\mathcal{S}^p$ is continuous* (*even a contraction*).

6. Other Directions

We cannot resist concluding this article by a brief (and, necessarily, selective and incomplete) survey of the various other contexts in which Toeplitz operators, or, more generally, operator calculi of the form (1.6) or (1.10), appear naturally and have been studied by other authors. As has already been remarked, (1.10) probably appeared first in the theory of representations of locally compact groups, where it is in fact nowadays a standard device for extending the action of a group G to its group algebra $L^1(G)$, see, for instance, the book [32]; and in mathematical physics, in the context of quantization of symplectic manifolds, where in addition to the article of Ali and Doebner [1] already cited before we should mention the "quantizers" (=our operator fields) of Gracia-Bondia [27] — see e.g., the recent survey [2] for further information on the various aspects. In the context of bounded symmetric domains, operator calculi were first systematically treated by Arazy and Upmeier [4], who in fact have in the meantime taken the whole program a good deal further by treating also the *real* symmetric domains [6], where fundamentally new phenomena arise (the "operator calculi" then act not from functions on the domain into operators on some Hilbert space of holomorphic functions on it, but from functions on the real bounded symmetric domain into holomorphic functions on its complexification).

However, the area where Toeplitz operators and related operator calculi have featured most prominently in various guises is probably the time-frequency analysis. Historically, the first appearance was in the context of the Segal-Bargmann (or Fock) spaces, where Toeplitz operators are traditionally known as "anti-Wick" operators, and the most time-honored symbol-operator correspondence is of course the classical Weyl calculus. (An excellent treatment can be found in Folland's book [25], which also provides ample historical and bibliographic references.) Here the group G is generated by the translations $f(x) \mapsto f(x-y)$ and modulations $f(x) \mapsto e^{\pi i \omega \cdot x} f(x)$, $y, \omega \in \mathbf{R}^n$, and, if one wants not only projective but honest representation, one also adds the multiplications $f(x) \mapsto e^{\pi i t} f(x)$, $t \in \mathbf{R}$, thus getting the Heisenberg group. The resulting calculi, most often known as various kinds of "localization operators," were extensively studied by many authors (see e.g., Boggiatto, Gröchenig, and Cordero [11], Boggiatto [10], Cordero, Pilipovic, Rodino, and Teofanov [15], Cordero and Rodino [16], Toft [47], to mention just a few recent ones, and the book by Gröchenig [28]), who also considered these calculi for symbols in various *modulation spaces* [23], a direction which we have not tackled at all in this article. Another kind of "Toeplitz operators" which falls under our Ansatzs (1.6) or (1.10) arises in multiresolution analysis, cf. Jiang and Peng [31] (who also obtain some necessary-and-sufficient criteria for $\mathcal{S}^p$-membership of the resulting operators (involving Wiener amalgam spaces), but again only for symbols which are either holomorphic or nonnegative; some sufficient (but not necessary) criteria are likewise given in [11] and [47]). Connections between localization operators and the ordinary Toeplitz operators on the

Segal-Bargmann space were studied by Coburn [14] and Lo [33]. (These can in fact be handled rather efficiently by the "Berezin transforms" $\Gamma^*\Gamma$ from this article, see [22]). Of a different flavor are the so-called Calderon-Toeplitz operators on $L^2(\mathbf{R}^n)$ studied by Rochberg and Nowak in [42] and [37].

Most of these developments make, in principle, perfect sense also when the Segal-Bargmann-Fock space is replaced by the standard weighted Bergman spaces on bounded symmetric domains. Toeplitz operators on these spaces have been studied extensively and are by now well understood. The situation is a bit more tricky with the Weyl calculus, of which there exist several versions. One is obtained upon taking for A_0 the reflection operator $f(z) \mapsto f(-z)$, as on the Fock space; this was studied by Unterberger and Unterberger [49], Upmeier [51], and Arazy and Upmeier [5]. From the point of view of quantization, this corresponds to the case when the bounded symmetric domain is the *phase* space of the quantized system. A different Weyl calculus, for which the domain is not the phase but the *configuration* space, was considered by Tate [45]. Still another point of view, leading to a third variant of the Weyl calculus, consists in identifying it with the partial-isometric part $\mathcal{W}$ of the polar decomposition of the Berezin transform $\Gamma = \mathcal{W}|\Gamma| : L^2(d\mu) \to \mathcal{S}^2$ associated to the Toeplitz calculus; this was studied by Orsted and Zhang [38]. (On the unit disc or, equivalently, the upper half-plane, yet another Weyl calculus, leading ultimately to the appearance of Bessel functions, can be found in Chapter III of the book of Terras [46].) However, many things which are well understood in the Fock space setting become much more complicated on bounded symmetric domains, so there is quite a lot of unexplored territory; for instance, though there is a very good counterpart of Fourier transform on symmetric domains [30], no one seems to have investigated modulation spaces in that context.

Another topic we have not even touched is the spectrum of the generalized Toeplitz/localization operators (1.6), (1.10), and its behavior. For instance, for f the characteristic function of some open set, one would naively expect A_f to be some kind of "projection operator," hence with spectrum concentrated at 0 and 1 and having some sort of "plunge region" in between. The extent to which this is true has been studied, again in the context of Toeplitz, or localization, operators (sometimes also called "concentration operators" in this setting), on the Fock space and bounded symmetric domains by several authors (De Mari, Feichtinger, and Nowak [18, 24, 19]), and is of interest from the point of view of practical applications.

One more topic not addressed in this article is the question of compactness of the operators A_f from Theorem 1. Typically, A_f is compact if f is compactly supported, but this condition is far from necessary. For Toeplitz operators on bounded symmetric domains, a known necessary and sufficient condition is that the Berezin transform $\Gamma^*\Gamma f$ vanish at the boundary [8, 21]. Once again, generalizations to other operator calculi are open to investigation.

Finally, we have excluded completely from our discussion the Toeplitz operators on Hardy and Dirichlet spaces, since they are not of the form (1.6) or (1.10). The reason is that for the Hardy space e.g., on the unit circle $\mathbf{T}$, there exists no measure on $\mathbf{T}$ invariant under G $(= SU(1, 1))$; while for the Dirichlet spaces, there is even no good action U_g of G on the space. As a result, the theory of Toeplitz and similar calculi has a very different flavor in these settings — for instance, there are no compact Toeplitz operators on the Hardy space except the zero operator (hence the analogue of our Theorem 1 concerning the membership of T_f in Schatten ideals becomes trivial). Similarly, our theory does not apply to *Hankel* operators on Bergman (and, even more so, on Hardy and Dirichlet) spaces: While in principle one can modify the proofs to work for operators from one space into another, the problem is that the action of U_g on the target space of Hankel operators — the orthogonal complement $L^2 \ominus L^2_{\text{hol}}$ — is no longer irreducible [40]. The reader is referred e.g., to Rochberg and Wu [43] and Duistermaat and Lee [20] for Toeplitz operators on the Dirichlet space, while excellent sources for the Hardy space are the recent book by Peller [39] for Hankel operators, and those by Nikolskii [35, 36] for Hankel and Toeplitz operators alike.

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