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Continuous Wavelets and Frames on Stratified Lie Groups I

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ABSTRACT. *Let G be a stratified Lie group and L be the sub-Laplacian on G. Let $0 \neq f \in \mathcal{S}(\mathbb{R}^+)$. We show that $Lf(L)\delta$, the distribution kernel of the operator $Lf(L)$, is an admissible function on G. It is always in the Schwartz space; one can choose f so that it has all moments vanishing, or has compact support with arbitrarily many moments vanishing. We also show that, if $\xi f(\xi)$ satisfies Daubechies' criterion, then $Lf(L)\delta$ generates a frame for any sufficiently fine lattice subgroup of G. Moreover, we show that the ratio of the frame bounds approaches 1 nearly quadratically as the dilation parameter approaches 1, so that the frame quickly becomes nearly tight (again assuming that the lattice subgroup is sufficiently fine). In particular, if the dilation parameter is $2^{1/3}$, and the lattice subgroup is sufficiently fine, then the "Mexican hat" wavelet, $Le^{-L/2}\delta$, generates a wavelet frame, for which the ratio of the optimal frame bounds is 1.0000 to four significant digits.*

1. Introduction

Let L denote the sub-Laplacian on a stratified group G [12] (for instance, the Heisenberg group $\mathbb{H}^n$). If $\phi \in \mathcal{S}(G)$ and $\int \phi = 0$, we say ϕ is *admissible* if for some $c \neq 0$, Calderón's reproducing formula:

$$\int_0^\infty \tilde{\phi}_a * \phi_a \, a^{-1} \, da = c\delta \, ,$$

holds in the sense of tempered distributions, where $\phi_a(x) = a^{-Q}\phi(a^{-1}x)$, Q is the homogeneous dimension of G, $\widetilde{\phi}(x) = \overline{\phi}(x^{-1})$ and δ denotes the point mass at $0 \in G$. (In Section 5, we shall show that this definition of "admissible" is equivalent to the one generally used in wavelet theory.) In Section 4, we shall show the following.

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Theorem 1. *Let f be a nonzero element of $\mathcal{S}(\mathbb{R}^+)$. Then $Lf(L)\delta \in \mathcal{S}(G)$ is admissible.*

For example, $Le^{-\frac{L}{2}}\delta$ is admissible. (Here $f(L)\delta$ is the distribution kernel of $f(L)$, so that if F is a Schwartz function, $f(L)F = F * [f(L)\delta]$.) Up to a constant, $Le^{-\frac{L}{2}}\delta$ is a very natural generalization of the Mexican Hat Wavelet to G. In case $G = \mathbb{H}^n$, Theorem 1 was shown for this function in Mayeli [29].

As a corollary of Theorem 1, we shall show in Sections 4 and 5.

Corollary 1.

(a) *There exist admissible $\phi \in \mathcal{S}(G)$ with all moments vanishing.*

(b) *There exist admissible $\phi \in C_c^\infty(G)$ with arbitrarily many moments vanishing.*

In Corollary 1 (a) and (b), we will in fact show that ϕ can be chosen to have the form $\phi = Lf(L)\delta$ for some $f \in \mathcal{S}(\mathbb{R}^+)$. As we will explain at the end of Section 4, Corollary 1 improves on Lemmas 1.61 and 1.62 of Folland-Stein [12] for stratified groups.

Moreover, we shall show in Section 7.

Theorem 2. *Let Γ be a lattice subgroup of G, and let f again be a nonzero element of $\mathcal{S}(\mathbb{R}^+)$.*

(a) *If $\xi f(\xi)$ satisfies "Daubechies' criterion" then for sufficiently small $b > 0$, the admissible function $Lf(L)\delta$ generates a wavelet frame for the lattice $b\Gamma$.*
(*Note: Daubechies' criterion holds here in particular if $f(\xi)$ does not vanish for any $\xi > 0$, or alternatively if the dilation parameter a is sufficiently close to* 1.)

(b) *As $a \to 1$, the ratio of the optimal frame bounds in (a) is*

$$1 + O\left(|a-1|^2 \log|a-1|\right) ,$$

for sufficiently small $b > 0$. (*Here a is again the dilation parameter*.)

Theorem 2 (b) says, in essence, that if a is close to 1, then the frame is "nearly tight," and that the convergence of the ratio of the optimal frame bounds to 1 is *nearly quadratic* in $|a-1|$. (Again, b must be sufficiently small, and is chosen after a is chosen.)

In particular, we shall show that, if one uses the dilation parameter $a = 2^{\frac{1}{3}}$, then for all sufficiently small $b > 0$, the admissible function $Le^{-\frac{L}{2}}\delta$ generates a wavelet frame for $b\Gamma$ which is "nearly tight:" There are frame bounds B_b, A_b with $\frac{B_b}{A_b} = 1.0000$ to four significant digits. This example shows that a need not be all that close to 1 for a nearly tight frame to be obtained in Theorem 2 (b); $a = 2^{\frac{1}{3}}$ is already very good.

Instead of using $Le^{-\frac{L}{2}}\delta$ as the admissible function, one could choose $f \in \mathcal{S}(\mathbb{R}^+)$ so that $\phi = Lf(L)\delta$ is as in Corollary 1 (a) or (b). One then obtains nearly tight frames of Schwartz functions with all moments vanishing, or nearly tight frames of C_c^∞ functions with arbitrarily many moments vanishing (for suitable a and b).

To clarify our terminology in Theorem 2:

- $b\Gamma = \{b\gamma : \gamma \in \Gamma\}$; here $b\gamma$, a dilate of γ, is defined in (3.1) below.
- For a fixed dilation parameter $a > 0$, if ϕ is a function on G, $j \in \mathbb{Z}$, and $\gamma \in \Gamma$, we set

$$\phi_{j,b\gamma}(x) = a^{-\frac{jQ}{2}}\phi\left([b\gamma]^{-1}\left[a^{-j}x\right]\right) .$$

- To say that an L^2 function ψ generates a wavelet frame for the lattice $b\Gamma$ is to say that $\{\phi_{j,b\gamma}(x) : j \in \mathbb{Z}, \gamma \in \Gamma\}$ is a frame.
- To say that a function $g \in \mathcal{S}(\mathbb{R}^+)$ satisfies Daubechies' criterion is to say that

$$A = \inf_{\lambda > 0} \sum_{j=-\infty}^{\infty} \left| g\left(a^{2j}\lambda\right) \right|^2 > 0 \, . \tag{1.1}$$

  In [6], p. 68, Daubechies observes that if $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$, then this is a necessary condition in Theorem 2 (a). Here we have put $g(\lambda) = \lambda f(\lambda)$, for $f \in \mathcal{S}(\mathbb{R}^+)$. Then it is easily seen that the series in (1.1) converges uniformly on compact subsets of $(0, \infty)$. Let $u(\lambda)$ denote the sum of that series; then clearly $u(a^2\lambda) = u(\lambda)$ for all $\lambda > 0$. Consequently, A is the just the minimum of the series for $\lambda \in [1, a^2]$. Thus, Daubechies' criterion is equivalent to the nonexistence of a $\lambda_0 > 0$ such that $g(a^{2j}\lambda_0) = 0$ for all integers j.

In fact, in Theorem 2, one does not even need the full force of the assumption that Γ is a lattice subgroup; all that one needs is that Γ is a discrete subset of G, and that there is a bounded measurable set $\mathcal{R}$, of positive measure, such that every $g \in G$ may be written uniquely in the form $g = x\gamma$ with $x \in \mathcal{R}$ and $\gamma \in \Gamma$.

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2. Earlier Work on Wavelets on Stratified Groups

Our results for stratified groups should be contrasted with those of Lemarié [25, 26]. He restricted himself to the case where Γ was the set of points all of whose coordinates are integers (to be sure, this is not always a lattice subgroup). He constructed an orthonormal basis of spline wavelets which were C^N (where N is arbitrary, but finite); which had arbitrarily (but finitely) many derivatives decaying exponentially; and which had arbitrarily (but finitely) many moments vanishing. His wavelets were definitely not smooth; they were built out of splines, that is, functions ψ with $L^M\psi$ a linear combination of Dirac measures for some M.

In this article, we are not seeking orthonormality. This however enables us to build in other features which may in certain circumstances be desirable. Specifically:

- As is well known, the redundancy of a frame is sometimes sought after;
- our continuous wavelets and frames are in the Schwartz space;
- in Corollary 1 (a), ϕ has all moments vanishing and is in the Schwartz space;
- in Corollary 1 (b), $\phi \in C_c^\infty(G)$;
- our prime example, the "stratified Mexican Hat wavelet" $Le^{-\frac{L}{2}}\delta$, has the property that it and all of its derivatives have "Gaussian" decay (by the work of Jersion/Sanchez-Calle [23] and of Varopoulos [31]). (Here we say a function F on G has "Gaussian" decay if for some $C, c > 0$,

$$|F(x)| \leq Ce^{-c|x|^2} \, .$$

  Here $|x|$ is the homogeneous norm of x; see Section 3 below for homogeneous norms.)

There are other previous results in wavelet theory on stratified Lie groups, but—except in the aforementioned results of Lemarié—high degrees of smoothness and decay, for continuous wavelets or nearly tight frames, were not previously obtained. The existence of admissible functions in L^2 was proved by Liu-Peng [27] for the Heisenberg group, and by Führ [14], (Corollary 5.28) for general homogeneous groups. (In contrast to those works, this article uses no representation theory whatsoever.) Frames consisting of L^2 functions were produced for the Heisenberg group in Maggioni [28].

In the latter article, Maggioni works on a space of homogeneous type which possesses an involution, and appropriate "dilations" and "translations;" examples are the stratified groups considered here (and hypergroups as well). He assumes that there is an admissible function and creates a wavelet frame from it. In the Heisenberg group situation, in order to get an admissible function, he cites the aforementioned result of Liu-Peng. If one instead uses our Theorem 1 and Corollary 1, together with Maggioni's results, one immediately obtains wavelet frames, in the Schwartz space, on general stratified groups. One even obtains frames with the properties stated in our Corollary 1 (a) or (b).

In this article, we prefer not to invoke the results of Maggioni, for the following reason. Maggioni requires that both the translation parameter (b in our Theorem 2) be sufficiently close to 0 and that the dilation parameter a be sufficiently close to 1. In Theorem 2 (a) we do not need to require that a be close to 1; for frames, all that is needed is that Daubechies' criterion be satisfied. This will then enable us to also demonstrate the nearly quadratic convergence as $a \to 1$ in Theorem 2 (b).

Let us clarify the similarities and differences between our methods and those of Maggioni, as well as those of earlier authors. Our method of constructing frames will be through discretizing a continuous problem. This idea goes back to the beginnings of wavelet theory, for instance, [7] and [13]. In these and other early works, one obtained various exact discretizations, where there was no error to be estimated in replacing an integral by a sum. More recently, such errors have been estimated, specifically in the work of Feichtinger and Gröchenig [10, 11, 19], Gilbert-Han-Hogan-Lakey-Weiland-Weiss [18], and Maggioni [28]. In the latter two references, the error is proved to have small norm on L^2, by use of the $T(1)$ theorem. In all of these references, the authors require that both the translation parameter (b in our Theorem 2) be sufficiently close to 0 and that the dilation parameter a be sufficiently close to 1.

We also will use the $T(1)$ theorem. The reason that we do not have to demand that a be close to 1, in order to get a frame, is because we shall discretize, not a continuous wavelet transform (as in the earlier works just cited), but rather the operator R_ψ which is the operator of convolution with $\sum_{j\in\mathbb{Z}} \tilde{\psi}_{a^j} * \psi_{a^j}$ (here $\psi = Lf(L)\delta$). We use the spectral theorem to show that R_ψ is bounded below if $\xi f(\xi)$ satisfies Daubechies' criterion.

In Section 8 we shall examine wavelet frame expansions in other Banach spaces (besides L^2). Again such questions have been discussed in the earlier works we have cited [10, 11, 19, 18], and [28] where again one requires a to be close to 1. (In particular, in [28], Maggioni addresses such questions on stratified groups.) Here however we shall again require only that the Daubechies criterion be satisfied (so that a need not be close to 1). The novel feature here will be the use of spectral multiplier theory (as in [12]) to invert R_ψ on appropriate Banach spaces (such as L^p $(1 < p < \infty)$ and the Hardy space H^1).

We also call attention to the important work of Han [20], on general spaces of homogeneous type. In Theorem 3.35 of that article, Han obtains frames by discretizing a discrete version of the Calderón reproducing formula in this general setting. He also uses a version of the $T(1)$ theorem to estimate errors. He also studies expansions in L^p $(1 < p < \infty)$. However, one cannot expect to obtain nearly tight frames by the methods in that article.

Since we hope this article will be of interest to both the "wavelet community" and the "stratified group community," we have supplied more details and introductory material than would be customary had we been writing for only one of these communities.

In future articles, we will study decay and regularity of dual frames, characterizations of various Banach spaces through wavelet frame expansion, and analogues of time-frequency localization for frames.

3. Notation

Following [12] (which we refer to for further details), we call a Lie group G stratified if it is nilpotent, connected and simply connected, and its Lie algebra $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g} = V_1 \oplus \cdots \oplus V_m$ such that $[V_1, V_k] = V_{k+1}$ for $1 \le k < m$ and $[V_1, V_m] = \{0\}$.

If G is stratified, its Lie algebra admits a canonical family of dilations, namely

$$\delta_r(X_1 + X_2 + \cdots + X_m) = rX_1 + r^2 X_2 + \cdots + r^m X_m \quad (X_j \in V_j) .$$

We identify G with $\mathfrak{g}$ through the exponential map. G is a Lie group with underlying manifold $\mathbb{R}^n$, for some n. G inherits dilations from g: If $x \in G$ and $r > 0$ we write

$$rx = \left(r^{d_1} x_1, \cdots, r^{d_n} x_n\right) . \tag{3.1}$$

(Here $d_1 \le \cdots \le d_n$ are those numbers for with $1 \le k \le m$ for which $V_k \ne 0$). The map $x \to rx$ is an automorphism of G.

The (element of) left (or right) Haar measure on G is simply $dx_1 \ldots dx_n$. The inverse of any $x \in G$ is simply $-x$. The group law must have the form

$$xy = (p_1(x, y), \ldots, p_n(x, y)) \tag{3.2}$$

for certain polynomials $p_1, \ldots, p_n$ in $x_1, \ldots, x_n, y_1, \ldots, y_n$.

We let $\mathcal{S}(G)$ denote the space of Schwartz functions on G. By definition $\mathcal{S}(G) = \mathcal{S}(\mathbb{R}^n)$.

The number $Q = \sum_1^m j(\dim V_j)$ will be called the *homogeneous dimension* of G. If ϕ is a function on G and $r > 0$, we define ϕ_r by

$$\phi_r(x) = r^{-Q}\phi\left(r^{-1}x\right) .$$

We fix a homogeneous norm function $|\ |$ on G which is smooth away from 0. Thus, [12] $|rx| = r|x|$ for all $x \in G$, $r \ge 0$, $|x^{-1}| = |x|$ for all $x \in G$, and $|x| > 0$ if $x \ne 0$. Moreover, for any $a > 0$, there is a finite $C_a > 0$ such that $\int_{|x|>R} |x|^{-Q-a} = C_a R^{-a}$ for all $R > 0$.

Let $X_1, \cdots, X_k$ be a basis for V_1 (viewed as left-invariant vector fields on G), let $L = -\sum_1^k X_i^2$ be the sub-Laplacian. This operator (which is hypoelliptic by Hörmander's theorem [21]) is well known to play on G much the same fundamental role on G as (minus) the ordinary Laplacian $\sum_1^N (\partial_{X_j})^2$ does on $\mathbb{R}^N$.

The operator L, restricted to C_c^∞, is formally self-adjoint (see Proposition 3 below). Its closure has domain $\mathcal{D} = \{f \in L^2(G) : Lf \in L^2(G)\}$, where here we take Lf in the sense of distributions. (This is easily seen through use of subelliptic estimates.) From this fact it quickly follows that this closure is self-adjoint and is in fact the unique self-adjoint extension of $L|_{C_c^\infty}$. We now let L denote this self-adjoint operator. Suppose that L has spectral resolution

$$L = \int_0^\infty \lambda \, dP_\lambda \, .$$

One then has that $P_{\{0\}}\mathcal{H} = 0$. To see this, say $f \in L^2(G)$ and $Lf = 0$; we need to show that $f = 0$. Since L is the self-adjoint extension of $L|_{C_c^\infty}$, and $Lf = 0$, clearly $Lf = 0$ in the sense of distributions. But by [16], if $f \in \mathcal{S}'$ and $Lf = 0$, then f is a polynomial. If $f \in L^2(G)$, then surely $f = 0$, as claimed.

As usual, if f is a bounded Borel function on $[0, \infty)$, we define the operator $f(L)$ by

$$f(L) = \int_0^\infty f(\lambda) \, dP_\lambda \, ;$$

this is well defined and bounded on $L^2(G)$ by the spectral theorem. We denote by $f(L)\delta$ the corresponding distribution kernel of the bounded operator $f(L)$. Thus,

$$f(L)\eta = \eta * f(L)\delta \quad \forall \, \eta \in \mathcal{S}(G) \, .$$

Notation. We adopt the $f(L)\delta$ notation, because formally

$$f(L)\eta = f(L)\,[\eta * \delta] = \eta * f(L)\delta$$

since L is left-invariant.

Let $\mathbb{R}^+ = [0, \infty)$ and set

$$\mathcal{S}(\mathbb{R}^+) = \left\{ f \in C^\infty(\mathbb{R}^+) \ : \forall l, \, f^{(l)} \text{ decays rapidly at infinity and } \lim_{\lambda \to 0^+} f^{(l)}(\lambda) \text{ exists} \right\} .$$

Then by Borel's theorem on the existence of smooth functions with arbitrary Maclaurin series we have $\mathcal{S}(\mathbb{R}^+) = \mathcal{S}(\mathbb{R})|_{\mathbb{R}^+}$.

By [22] (or [15] if G is the Heisenberg group), one has the following.

Theorem 3. *Let $f \in \mathcal{S}(\mathbb{R}^+)$. Then the distribution kernel of the operator $f(L) = \int_0^\infty f(\lambda) \, dP_\lambda$ which we shall denote by $f(L)\delta$, is a Schwartz function on G.*

We have the following elementary lemma on distribution kernels.

Lemma 1. *Say $f, g \in \mathcal{S}(\mathbb{R}^+)$. The*

(1) $\bar{f}(L)\delta = \widetilde{f(L)\delta}$

(2) $[fg]\,(L)\delta = f(L)\delta * g(L)\delta$.

(3) *For $t > 0$ if the function f^t is given by $f^t(\lambda) = f(t\lambda) \ \forall \, \lambda \in [0, \infty)$, then*

$$\left[f^t(L)\delta \right] = [f(L)\delta]_{\sqrt{t}} \, .$$

Proof. For (1), using the spectral theorem we have $\bar{f}(L) = f(L)^*$, hence for any $\phi, \psi \in \mathcal{S}(G)$ we obtain

$$\begin{aligned} \left\langle \phi * \bar{f}(L)\delta, \psi \right\rangle &= \left\langle \bar{f}(L)\phi, \psi \right\rangle = \langle \phi, f(L)\psi \rangle \\ &= \langle \phi, \psi * f(L)\delta \rangle = \left\langle \phi * \widetilde{f(L)\delta}, \psi \right\rangle \end{aligned} \tag{3.3}$$

which implies the assertion.

For (2), say $\phi \in \mathcal{S}(G)$. By the spectral theorem,

$$[(fg)(L)]\phi = g(L)f(L)\phi = [\phi * f(L)\delta] * g(L)\delta \ ,$$

yielding (2). For the proof of (3) see Lemma 6.29 of [12]. □

C will always denote a constant, which may change from one occurrence to the next.

4. Proof of Theorem 1 and Corollary 1

To prove Theorem 1, we need the following lemma.

Lemma 2. *For any $f \in \mathcal{S}(\mathbb{R}^+)$ with $\int_0^\infty f(s)\,ds \neq 0$ we have*

$$K = \int_0^\infty (Lf(L)\delta)_t \frac{dt}{t} = \frac{1}{2} c\delta \ ,$$

where $c = \int_0^\infty f(s)\,ds$ is a nonzero constant.

Note that Theorem 1 follows immediately from this lemma, since

$$\left(\widetilde{Lf(L)\delta}\right)_t * (Lf(L)\delta)_t = \left[\widetilde{Lf(L)\delta} * (Lf(L)\delta)\right]_t \ ,$$

and by Lemma 1 $\widetilde{Lf(L)\delta} * (Lf(L)\delta) = Lg(L)\delta$, where $g(\lambda) = \lambda \mid f(\lambda) \mid^2$.

Proof. Let $h(\lambda) = \lambda f(\lambda)$. Write $\psi = h(L)\delta = Lf(L)\delta$; by Lemma 1, $\psi_t = h^{t^2}(L)\delta$ for any $t > 0$. Define $K_{\epsilon,A} = \int_\epsilon^A \psi_t \frac{dt}{t}$. Since $\int_G \psi = \int_G Lf(L)\delta = 0$, by Theorem 1.65 [12], $\int_\epsilon^A \psi_t \frac{dt}{t}$ converges in $\mathcal{S}'$ as $\epsilon \to 0$ and $A \to \infty$ to the tempered distribution $K = \int_0^\infty \psi_t \frac{dt}{t}$, which is C^∞ away from 0. Suppose $\phi_1 \in \mathcal{S}(G)$. Then $\phi_1 * K_{\epsilon,A} \in \mathcal{S}$ and for any $\phi_2 \in \mathcal{S}(G)$ we have

$$\begin{aligned}
\langle \phi_1 * K_{\epsilon,A}, \phi_2 \rangle = \left\langle K_{\epsilon,A}, \widetilde{\phi}_1 * \phi_2 \right\rangle &= \int_\epsilon^A \left\langle \psi_t, \widetilde{\phi}_1 * \phi_2 \right\rangle \frac{dt}{t} \\
&= \int_\epsilon^A \langle \phi_1 * \psi_t, \phi_2 \rangle \frac{dt}{t} \\
&= \int_\epsilon^A \left\langle \left[h^{t^2}(L)\right]\phi_1, \phi_2 \right\rangle \frac{dt}{t} \\
&= \int_\epsilon^A \int_0^\infty t^2 \lambda f\left(t^2\lambda\right) d\langle P_\lambda \phi_1, \phi_2 \rangle \frac{dt}{t} \\
&= \int_0^\infty \int_\epsilon^A t^2 \lambda f\left(t^2\lambda\right) \frac{dt}{t}\, d\langle P_\lambda \phi_1, \phi_2 \rangle \\
&= \frac{1}{2} \int_0^\infty \int_{\lambda\epsilon^2}^{\lambda A^2} f(t)\,dt\, d\langle P_\lambda \phi_1, \phi_2 \rangle \ .
\end{aligned}$$

Letting $F(x) = -\int_x^\infty f(s)\,ds$ (so that $F' = f$) we see that this double integral equals

$$\int_0^\infty \int_{\lambda\epsilon^2}^{\lambda A^2} f(t)\,dt\, d\langle P_\lambda \phi_1, \phi_2 \rangle = \int_0^\infty \left(F\left(\lambda A^2\right) - F\left(\lambda\epsilon^2\right)\right) d\langle P_\lambda \phi_1, \phi_2 \rangle \ .$$

Since the function F is bounded, and the measure $\langle P_\lambda\phi_1, \phi_2\rangle$ is supported on $(0,\infty)$ (in that $P_{\{0\}} = 0$), we see that

$$\lim_{\epsilon\to 0}\lim_{A\to\infty}\int_0^\infty \Big(F\big(\lambda A^2\big) - F\big(\lambda\epsilon^2\big)\Big)d\langle P_\lambda\phi_1, \phi_2\rangle = \int_0^\infty\int_0^\infty f(s)\,ds\,d\langle P_\lambda\phi_1, \phi_2\rangle$$
$$= \int_0^\infty f(s)ds\langle\phi_1, \phi_2\rangle\ .$$

This proves the lemma. Thus, Theorem 1 is established as well. □

To begin the proof of Corollary 1, if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, we let $|\alpha| = \sum_k d_k\alpha_k$. Note that $|\alpha|$ is the homogeneous degree of the monomial x^α, since $(rx)^\alpha = r^{|\alpha|}x^\alpha$ for $r > 0$. For any positive integer k, $L^k x^\alpha$ is a polynomial which is homogeneous of degree $|\alpha| - 2k$; it must therefore be identically zero if $|\alpha| - 2k < 0$. Integration by parts now at once shows the following proposition.

Proposition 1. *If $F \in \mathcal{S}(G)$, and if $|\alpha| < 2k$, then $\int_G x^\alpha L^k F = 0$.*

Proof of Corollary 1. For (a), select any nonzero $g \in \mathcal{S}(\mathbb{R}^+)$ which vanishes identically in a neighborhood of 0. For any positive integer k, define $g_k(x) = \frac{g(x)}{x^k}$; then $g_k \in \mathcal{S}(\mathbb{R}^+)$, and $g(L)\delta = L^k g_k(L)\delta$. By Theorem 1 and Proposition 1, $g(L)\delta$ is admissible and has all moments vanishing.

For (b), we note that if $g \in C_c^\infty(\mathbb{R})$ is real-valued and even, and if $m(\lambda) = \hat{g}(\sqrt{\lambda})$, then $m(L)\delta \in C_c^\infty(G)$. (This is proved in the Appendix to [17]; the argument is there attributed to J. Dziubanski, but he says the result was well-known; it appears to be based on ideas of Michael Taylor.) Thus, if $g \neq 0$, then for any positive integer k, $\phi_k = L^k m(L)\delta = L(L^{k-1}m(L)\delta)$ is admissible and in $C_c^\infty(G)$, and $\int x^\alpha\phi_k = 0$ whenever $|\alpha| < 2k$. (Note that ϕ_k cannot be identically zero, for then $\lambda^k m(\lambda)$ would be identically zero, so g would be zero.) This completes the proof. □

Remark. Corollary 1 (b) improves on Theorems 1.61 and 1.62 of Folland-Stein [12], at least for stratified G. There it was shown that there exist $\phi^1, \ldots, \phi^M, \psi^1, \ldots, \psi^M \in \mathcal{S}(G)$ with arbitrarily many moments vanishing, with the ψ^j having compact support, and with $\sum_1^M \int_0^\infty \phi_t^j * \psi_t^j \frac{dt}{t} = \delta$; here M depended on the number of moments one wanted to vanish. Now we see that we can always take $M = 1$ and $\psi^1 = \widetilde{\phi^1}$, so that both have compact support.

5. Continuous Wavelet Transform

In this section we study the continuous wavelet transform with respect to the quasiregular representation of the group $M := G \ltimes (0,\infty)$, where G is a stratified group with homogeneous degree Q and with Haar measure db. M is a locally compact group with left Haar measure $d\mu(M) = a^{-(Q+1)}\,da\,db$.

The positive number a defines an automorphism of the group G, which acts by dilation. The quasi-regular representation π of M acts on $L^2(G)$ as follows.

Let $\phi \in L^2(G)$, then

$$(\pi(x,a)\phi)(y) = (T_x D_a\phi)(y) = a^{-\frac{Q}{2}}\phi\big(a^{-1}\big(x^{-1}y\big)\big)\ \ \forall x, y \in G\ ,\ \forall a > 0\ . \tag{5.1}$$

Thus, T_x acts by left translation by x^{-1}, while D_a denotes a unitary dilation operator with respect to a.

The following definition and more details can be found for example in [14].

Definition 1. Let ϕ and ψ be any fixed functions in $L^2(G)$. Define the coefficient function $V_{\phi,\psi}$ on G by

$$V_{\phi,\psi} : (x, a) \mapsto \langle \psi, T_x D_a \phi \rangle \ . \tag{5.2}$$

The coefficient function $V_{\phi,\psi}$ is not necessarily square integrable on M. The function ϕ is called admissible when for any ψ the associated coefficient function $V_{\phi,\psi}$ is square integrable, and the operator

$$V_\phi : L^2(G) \longrightarrow L^2(M) \ ,$$

given by $[V_\phi(\psi)](x, a) = V_{\phi,\psi}(x, a)$, is an isometry. Then, for the admissible vector ϕ, the bounded operator V_ϕ is called a continuous wavelet transform of $L^2(G)$.

We shall soon show (in Proposition 2 below) that this (accepted) definition of admissible is consistent with our usage of the word *admissible* in Theorem 1.

The existence of admissible vectors in $L^2(G)$ for π was proved by Führ [14], (Corollary 5.28) for homogeneous groups. We recall this in the next theorem.

Theorem 4. *Let $M = G \ltimes H$, where G is a homogeneous Lie group and H is a one-parameter group of dilations. Then the quasi-regular representation π is contained in the left regular representation λ_M. Hence, there exists a continuous wavelet transform on G arising from the action of G by left translations and the action of the dilations.*

We now show (without use of Theorem 4) that there exist admissible $\phi \in \mathcal{S}(G)$. We claim the following.

Proposition 2. *Say $\phi \in \mathcal{S}(G)$ and $\int \phi = 0$, so that by Theorem 1.65 of [12], if*

$$K_{\epsilon,A} = \int_\epsilon^A \widetilde{\phi}_t * \phi_t \ \frac{dt}{t}$$

then $K = \lim_{\epsilon\to 0, A\to\infty} K_{\epsilon,A}$ exists in $\mathcal{S}'(G)$, C^∞ away from 0 and is homogeneous of degree $-Q$. Then ϕ is admissible (in the sense of Definition 1) if and only if $K = \delta$ up to a constant multiple. In particular if $0 \neq f \in \mathcal{S}(\mathbb{R}^+)$, then $\phi = Lf(L)\delta$ is admissible.

Proof. For $\psi \in L^2(G)$ we have:

$$\begin{aligned}\int_M \left|V_\phi \psi\right|^2 &= \int_G \int_0^\infty \left|\left\langle \psi, T_b D_a \widetilde{\phi}\right\rangle\right|^2 d\mu(M) \\ &= \int_G \int_0^\infty \left|\left(\psi * D_a \widetilde{\phi}\right)(b)\right|^2 a^{-(Q+1)}\, da\, db \quad \text{so} \ ,\end{aligned}$$

$$\int_M \left|V_\phi \psi\right|^2 = \int_0^\infty \left\|\psi * \left(D_a \widetilde{\phi}\right)\right\|^2 a^{-(Q+1)}\, da \ . \tag{5.3}$$

But for any $a > 0$,

$$\left\|\psi * D_a \widetilde{\phi}\right\|^2 a^{-Q} = \left\langle \psi, \psi * D_a \widetilde{\phi} * D_a \phi \right\rangle a^{-Q} = \left\langle \psi, \psi * \widetilde{\phi}_a * \phi_a \right\rangle . \tag{5.4}$$

Since $K_{\epsilon,A} \to K$ in $\mathcal{S}'$, if $g \in \mathcal{S}$, then $g * K_{\epsilon,A} \to g * K$ pointwise and for some N, C

$$\left|(g * K_{\epsilon,A})(x)\right| \leq C(1+ \mid x \mid)^N \qquad \text{for all} \quad x, \epsilon, A\ .$$

Using the dominated convergence theorem in (5.3) and (5.4), if $\psi \in \mathcal{S}(G)$, then

$$\| V_\phi \psi \|_{L^2}^2 = \langle \psi, \psi * K \rangle \leq C \| \psi \|_{L^2}^2 \tag{5.5}$$

since the map $\psi \to \psi * K$ is bounded on $L^2(G)$. V_ϕ thus maps $\mathcal{S}(G)$ to $L^2(M)$ and has a unique bounded extension to a map from $L^2(G)$ to $L^2(M)$. But if $\psi_k \to \psi$ in $L^2(G)$, surely $V_\phi \psi_k \to V_\phi \psi$ pointwise , so this extension can be none other than V_ϕ. Accordingly (5.5) holds for all $\psi \in L^2(G)$. We thus have

$$\begin{aligned} \| V_\phi \psi \|_{L^2(M)} = \| \psi \|_{L^2} \quad \forall \psi \in L^2 &\iff \langle \psi, \psi * K \rangle = \langle \psi, \psi \rangle \quad \forall \psi \in L^2 \\ &\iff \psi * K = \psi \quad \forall \psi \in L^2 \\ &\iff K = \delta \qquad \text{up to a constant}\ , \end{aligned}$$

as desired. (In the second implication, we have used polarization.) This completes the proof. □

6. Lemmas on Vector Fields

In this section we gather a number of facts which will be needed in our discussion of frames. These facts are analogues for G of very standard facts on $\mathbb{R}^n$ (such as the fundamental theorem of calculus—see Lemma 3 below).

The right-invariant vector fields Y_l ($1 \leq l \leq n$) may be defined by

$$Y_l g = -\widetilde{X_l \tilde{g}}$$

for $g \in C^1(G)$.

We note the following.

Proposition 3. *Suppose $\phi \in \mathcal{S}(G)$. Then, for all l, $\int_G X_l \phi = 0$ and $\int_G Y_l \phi = 0$.*

Proof. Note that each X_l is homogeneous of degree a_l. This forces X_l to have the form

$$X_l = \frac{\partial}{\partial x_l} + \sum_{k>l} p_k(x) \frac{\partial}{\partial x_k}\ ,$$

where p_k is a homogeneous polynomial of degree $a_k - a_l < a_k$. (See [12] for a detailed proof of this.) Accordingly $p_k(x)$ must actually be a polynomial in $x_1, \ldots, x_l$, so multiplication by it commutes with $\partial/\partial x_k$ for $k > l$.

Accordingly $\int_G X_l \phi = 0$. By using $\tilde{}$ we see that $\int_G Y_l \phi = 0$ as well.

If $x = (x_1, \ldots, x_n) \in G$, and $t > 0$, for want of a better notation, let us define

$$[t]x = (tx_1, \ldots, tx_n)$$

[recall that tx means something else, see (3.1)].

Recall that we are identifying G with $\mathfrak{g}$ through the exponential map. Then, if $x \in G$, we say that the point $\exp(x \cdot X)(0)$ has coordinates x. □

Lemma 3.

(a) *Suppose that $x \in G$ and that U is an open neighborhood of the line segment $\{[t]x : 0 \le t \le 1\}$. If $g \in C^1(U)$, then*

$$g(x) - g(0) = \int_0^1 [(x \cdot X)g]([t]x)\, dt \tag{6.1}$$

and

$$g(x) - g(0) = \int_0^1 [(x \cdot Y)g]([t]x)\, dt\ . \tag{6.2}$$

(b) *Suppose that $x, u \in G$ and that U is an open neighborhood of the set $\{u([t]x) : 0 \le t \le 1\}$. If $h \in C^1(U)$, then*

$$h(ux) - h(u) = \int_0^1 [(x \cdot X)h](u([t]x))\, dt\ .$$

(c) *Suppose that $x, u \in G$ and that U is an open neighborhood of the set $\{([t]x)u : 0 \le t \le 1\}$. If $h \in C^1(U)$, then*

$$h(xu) - h(u) = \int_0^1 [(x \cdot Y)h](([t]x)u)\, dt\ .$$

Proof. For (a), we note that

$$\begin{aligned} g(x) - g(0) &= g(\exp(x \cdot X)(0)) - g(0) \\ &= \int_0^1 \frac{d}{dt} g(\exp(t[x \cdot X])(0))\, dt \\ &= \int_0^1 [(x \cdot X)g]([t]x)\, dt\ , \end{aligned}$$

proving (6.1). Applying $\tilde{}$ to (6.1), we find (6.2) as well. For (b), we apply (6.1) to the function $g = h_u$ where $h_u(x) = h(ux)$. For (c) we apply (6.2) to the function $g = h_u$ where $h_u(x) = h(xu)$. This completes the proof. □

We will be needing two applications of Lemma 3, Propositions 4 and 7 below. First, however, some remarks on homogeneous norms.

A homogeneous norm function satisfies a type of triangle inequality [12], Equation (1.8): For some $C > 0$, $|xy| \le C(|x| + |y|)$ for all $x, y \in G$. We shall need three other facts about homogeneous norms.

Proposition 4. *There exists $c > 0$ such that for all $R > 0$, if $|u^{-1}x| \ge 2R$, then*

$$\min_{|u^{-1}y| \le R} \left|x^{-1}y\right| \ge c\left|u^{-1}x\right|\ .$$

Proof. Since $x^{-1}y = (x^{-1}u)(u^{-1}y) = (u^{-1}x)^{-1}(u^{-1}y)$, we may after a translation assume $u = 0$. It is enough to show that, for some $c > 0$, if $|y| \le \frac{|x|}{2}$, then $|x^{-1}y| > c|x|$. After a dilation we may assume $|x| = 2$ and $|y| \le 1$. By the triangle inequality, for some $C > 0$, $|x^{-1}y| \ge \frac{|x|}{C} - |y|$, so we may assume also that $|x| \le 2C$. But $|x^{-1}y|$ does not vanish for (x, y) in the compact set

$$\{x : 2 \le |x| \le 2C\} \times \{y : |y| \le 1\}\ ,$$

so it has a positive minimum there, as desired. □

Sometimes we use the "standard homogeneous norm function" on G, defined by

$$|x| = \left(\sum_{k=1}^{n} |x_k|^{2b_k}\right)^{\frac{1}{2A}} ,$$

where $A = a_1 \ldots a_n$, and each $b_k = \frac{A}{a_k}$. We shall clearly indicate when we do this the following.

Proposition 5. *There is a constant $C > 0$ such that for all $x = (x_1, \ldots, x_n) \in G$, $|x_m| \le C|x|^{a_m}$ for $1 \le m \le n$.*

Proposition 6. *There is a constant $C > 0$ such that for all $x \in G$ and all t with $0 \le t \le 1$, we have $|[t]x| \le C|x|$. If $|\,|$ is the standard homogeneous norm function, we can take $C = 1$.*

Proof of Propositions 5 and 6. Since any two homogeneous norms are equivalent, we may assume that $|\,|$ is the standard homogeneous norm function. But in that case the propositions are evident (and we can take $C = 1$ in both). □

We now turn to the applications of Lemma 3. We define a *normalized bump function* to be a C^1 function with support in the unit ball $B(0, 1) = \{x : |x| < 1\}$ with C^1 norm less than or equal to 1. For any function $f : G \to \mathbb{C}$, if $R > 0$ and $u \in G$, we let $f^{R,u}(x) = f(R^{-1}(u^{-1}x))$. We claim the following.

Lemma 4. *There exists a constant $C > 0$ such that for all normalized bump functions f, all $R > 0$, and all $u, x, y \in G$ we have*

$$\left| f^{R,u}(xy) - f^{R,u}(x) \right| \le C \sum_{k=1}^{n} \frac{|y_k|}{R^{a_k}} .$$

Proof. We have

$$f^{R,u}(xy) - f^{R,u}(x) = f\left(R^{-1}\big(u^{-1}xy\big)\right) - f\left(R^{-1}\big(u^{-1}x\big)\right) = f\big(x'y'\big) - f\big(x'\big) ,$$

where $x' = R^{-1}(u^{-1}x)$ and

$$y' = R^{-1}y = \left(\frac{y_1}{R^{a_1}}, \ldots, \frac{y_n}{R^{a_n}}\right) .$$

In proving the lemma we may therefore assume that $R = 1$ and $u = 0$, so that $f^{R,u} = f$. In that case we use Lemma 3 (b) to find that

$$|f(xy) - f(x)| = \int_0^1 [(y \cdot X)f]\,(x\,([t]y))\; dt \le C \sum_{k=1}^{n} |y_k|$$

as claimed, since the functions $X_k f$ are bounded (uniformly for all normalized bump functions f). □

We now turn to our second application of Lemma 3. First we define a *Calderon-Zygmund kernel* to be a complex-valued function $K(x, y)$, defined for all $x, y \in G$ with

$x \neq y$, which is continuous (off the diagonal), and which, for some $C, c > 0$, satisfies the following three estimates (for all $x, y \in G$ with $x \neq y$):

$$|K(x, y)| \leq \frac{C}{\left|y^{-1}x\right|^Q} \, ; \tag{6.3}$$

$$\text{If} \quad \left|x^{-1}x'\right| \leq c\left|y^{-1}x\right|, \text{ then} \quad \left|K\left(x', y\right) - K(x, y)\right| \leq C\frac{\left|x^{-1}x'\right|}{\left|y^{-1}x\right|^{Q+1}} \, ; \tag{6.4}$$

$$\text{If} \quad \left|y^{-1}y'\right| \leq c\left|y^{-1}x\right|, \text{ then} \quad \left|K\left(x, y'\right) - K(x, y)\right| \leq C\frac{\left|y^{-1}y'\right|}{\left|y^{-1}x\right|^{Q+1}} \, . \tag{6.5}$$

We then claim the following.

Proposition 7. *Suppose $K(x, y)$ is defined and C^1 away from the diagonal in $G \times G$, and that for some $A > 0$,*

$$\left|X_x^\alpha X_y^\beta K(x, y)\right| \leq A\left|y^{-1}x\right|^{-(Q+|\alpha|+|\beta|)} , \tag{6.6}$$

whenever $0 \leq \alpha_1 + \ldots + \alpha_n + \beta_1 + \ldots + \beta_n \leq 1$, and whenever $x, y \in G$ with $x \neq y$. Then K is a Calderon-Zygmund kernel. (*Here $X_x^\alpha = X_1^{\alpha_1} \ldots X_n^{\alpha_n}$, where the X_k are taken in the x variable.*)

Proof. By taking $\alpha = \beta = 0$ in (6.6), we have (6.3). To prove (6.4), we may assume we are using the standard homogeneous norm function; we will then show that (6.4) holds with $c = \frac{1}{2}$.

In this proof, it will be convenient let $X_{k,1}K(x, y)$ denote the result of applying X_k to K in the x variables.

Suppose $x \neq y$ and

$$\left|x^{-1}x'\right| \leq \frac{\left|y^{-1}x\right|}{2} = \frac{\left|x^{-1}y\right|}{2} \, .$$

If $0 \leq t \leq 1$, then by Proposition 6,

$$\left|[t]\left(x^{-1}x'\right)\right| \leq \left|x^{-1}x'\right| \leq \frac{\left|x^{-1}y\right|}{2}$$

as well. In particular, $[t]\left(x^{-1}x'\right) \neq x^{-1}y$, so $x\left([t]\left(x^{-1}x'\right)\right) \neq y$. Moreover, by Proposition 4, there exists a $c_1 > 0$ (independent of the specific values of x, y, x', t) such that

$$\left|y^{-1}x\left([t]\left(x^{-1}x'\right)\right)\right| \geq c_1\left|y^{-1}x\right| \, .$$

We write $x' = x(x^{-1}x')$. Using Lemma 3 and Proposition 5, we find that for some

$C_1, C_2, C_3 > 0$,

$$\begin{aligned}\left|K(x', y) - K(x, y)\right| &= \left| \int_0^1 \sum_{k=1}^n \left(x^{-1}x'\right)_k \left(X_{k,1} K\left(x\left([t]\left(x^{-1}x'\right)\right), y\right) dt \right| \\ &\le C_1 A \sum_{k=1}^n \frac{\left|\left(x^{-1}x'\right)_k\right|}{\left|y^{-1}x\right|^{Q+a_k}} \\ &\le C_2 A \sum_{k=1}^n \frac{\left|x^{-1}x'\right|^{a_k}}{\left|y^{-1}x\right|^{Q+a_k}} \\ &\le C_3 A \frac{\left|x^{-1}x'\right|}{\left|y^{-1}x\right|^{Q+1}}\end{aligned}$$

so that (6.4) holds. (Note for later purposes that C_1, C_2, C_3 depend only on the group G and not in any way on K.) The proof of (6.5) is exactly analogous. This proves the proposition. □

We will be using Lemma 4 and Proposition 7 in conjunction with the $T(1)$ theorem for stratified groups. We review this theorem in a moment.

First, however, some definitions. Suppose that a linear operator $T : C_c^1(G) \to L^2(G)$. One says that T is *restrictedly bounded* if there is a $C > 0$ such that $\|T(f^{R,u})\|_2 \le C R^{Q/2}$ for all normalized bump functions f, all R and all u.

If $T : C_c^1 \to L^2(G)$ is linear, we say that a linear operator $T^* : C_c^1 \to L^2(G)$ is its formal adjoint if for all $f, g \in C_c^1$ we have

$$\langle Tf, g\rangle = \left\langle f, T^* g\right\rangle .$$

T^* is evidently unique if it exists.

We will be using the "easier case" of the David-Journé $T(1)$ theorem [8] for stratified groups ([24] or [30], pp. 293–300). (The latter reference is only for $G = \mathbb{R}^n$, but the proof for general G requires only minor changes—see the Appendix to this article.) We may formulate this theorem as follows.

Theorem 5. *Suppose that $T : C_c^1(G) \to L^2(G)$ has a formal adjoint $T^* : C_c^1(G) \to L^2(G)$. Suppose further:*

(i) *T and T^* are restrictedly bounded;*

(ii) *there is a Calderon-Zygmund kernel K such that if $f \in C_c^1$, then for x outside the support of f, $(Tf)(x) = \int K(x, y) f(y)\, dy$; and*

(iii) *$T(1) = T^*(1) = 0$.*

Then T extends to a bounded operator on L^2.

Condition (iii) means precisely ([30], pp. 300–301) that whenever $f \in C_c^\infty(G)$ and $\int_G f = 0$, we have that $\int_G Tf = \int_G T^* f = 0$. In fact we shall show that this is true for all $f \in C_c^1(G)$ with $\int_G f = 0$. (Even without condition (iii), conditions (i) and (ii) imply that for all such f, Tf, and $T^* f$ are in $L^1(G)$.)

In fact we shall need a quantitative version of Theorem 5.

Theorem 6. *There exist $C_0, N > 0$, such that for any $A > 0$, we have the following. Whenever $T : C_c^1(G) \to L^2(G)$ has a formal adjoint $T^* : C_c^1(G) \to L^2(G)$, and whenever T, T^* satisfy:*

(i) $\|Tf^{R,u}\|_2 \leq AR^{\frac{Q}{2}}$ *and* $\|T^* f^{R,u}\|_2 \leq AR^{\frac{Q}{2}}$ *for all normalized bump functions* f;

(ii) *there is a kernel* $K(x, y)$, C^1 *off the diagonal, such that if* $f \in C_c^1$, *then for* x *outside the support of* f, $(Tf)(x) = \int K(x, y) f(y)\, dy$; *and whenever at most one of* $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ *is not zero, and whenever* $x, y \in G$ *with* $x \neq y$, *we have*

$$\left| X_x^\alpha X_y^\beta K(x, y) \right| \leq A \left| y^{-1} x \right|^{-(Q+|\alpha|+|\beta|)} ;$$

(iii) *and* $T(1) = T^*(1) = 0$,

then T *extends to a bounded operator on* L^2, *and* $\|T\| \leq C_0 A$.

Proof. This follows at once from an examination of the proofs of Theorem 5 (in [24] or [30]) and of Proposition 7 above. □

7. Frames

Suppose now that one has a discrete subset Γ of G, and a bounded measurable set $\mathcal{R} \subseteq G$ of positive measure, such that every $g \in G$ may be written uniquely in the form $g = x\gamma$ with $x \in \mathcal{R}$ and $\gamma \in \Gamma$.

For example, one could choose Γ to be any lattice subgroup of G, if one is available. Thus, Γ is a discrete subgroup of G, such that G/Γ is compact. (Note: by [5], p. 197, Equation (2), it is equivalent to assume that Γ is a discrete subgroup of G, such that G/Γ has finite volume with respect to the induced invariant measure. If the coefficients of all the polynomials appearing in (3.2) are integers, as is the case for the Heisenberg group, one could take Γ to be the integer lattice, namely the set of points all of whose coordinates are integers.) We then let $\mathcal{R}$ be a fundamental region for G/Γ. (By this we mean a bounded measurable subset of G, of positive measure, consisting of precisely one representative of each right coset of Γ.) Thus, every $g \in G$ may be written uniquely in the form $g = \gamma x$ with $x \in \mathcal{R}$, $\gamma \in \Gamma$.

Definition 2. A countable subset $\{e_n\}_{n \in I}$ of a Hilbert space $\mathcal{H}$ is said to be a **frame** if there exist two positive numbers $A \leq B$ such that, for any $f \in \mathcal{H}$,

$$A \parallel f \parallel^2 \leq \sum_{n \in I} | \langle f, e_n \rangle |^2 \leq B \parallel f \parallel^2 \;,$$

the positive numbers A and B are called *frame bounds*.

Note that the frame bounds are not unique. The *a lower frame bound* is the supremum over all lower frame bounds, and the *optimal upper frame bound* is the infimum over all upper frame bounds. The optimal frame bounds are of course frame bounds. The frame is called a *tight frame* when we can take $A = B$. (Informally, we also say the frame is *"nearly tight"* if $\frac{B}{A}$ is close to 1.) Frames were introduced in [9]. (For more information on frames, see [6] or [2].)

We consider $\phi \in \mathcal{S}(G)$ with $\int \phi = 0$. For $a, b > 0$, we define

$$\phi_{j,b\gamma}(x) = [D_{a^j} T_{b\gamma} \phi](x) = a^{-\frac{jQ}{2}} \phi\big([b\gamma]^{-1}\big[a^{-j} x\big]\big) \;,$$

(a will usually be fixed.) The set $\{\phi_{j,b\gamma}\}$ is called the wavelet system generated by ϕ. We seek conditions on ϕ and the numbers $a, b > 0$ which guarantee that this wavelet system is a frame (in which case it is called a wavelet frame).

In order to do this we study the operator

$$S_{\phi,b} : f \to \sum_{\gamma\in\Gamma, j\in\mathbb{Z}} \langle f, \phi_{j,b\gamma}\rangle \phi_{j,b\gamma} .$$

It is not hard to see [6] that $\{\phi_{j,b\gamma}\}$ is a frame if and only if: For any $f \in L^2(G)$, the series defining $S_{\phi,b} f$ converges unconditionally to f in $L^2(G)$; and $S_{\phi,b}$ is bounded on $L^2(G)$; and $S_{\phi,b} \geq AI$ for some *strictly positive* number A. (If the frame is "nearly tight," that is if, for certain frame bounds A, B one knows that $\frac{B}{A} - 1 = \epsilon$ is small, then $(\frac{1}{A})S_{\phi,b} f$ is a good approximation to f. Indeed, for any $f \in L^2$,

$$A\|f\|^2 \leq \langle S_{\phi,b} f, f\rangle \leq B\|f\|^2$$

implies that, as operators, $0 \leq (\frac{1}{A})S_{\phi,b} - I \leq \epsilon I$, whence $\|(\frac{1}{A})S_{\phi,b} - I\| \leq \epsilon$. For this reason, one generally prefers "nearly tight" frames.)

More generally we shall need to consider $\phi, \psi \in \mathcal{S}(G)$ with $\int \phi = \int \psi = 0$ and look at operators of the form

$$S_{\phi,\psi,b} : f \to \sum_{\gamma\in\Gamma, j\in\mathbb{Z}} \langle f, \phi_{j,b\gamma}\rangle \psi_{j,b\gamma} .$$

Theorem 7. *Fix $a > 0$. In parts (a), (b), (c), and (d) we also fix $\phi, \psi \in \mathcal{S}(G)$ with $\int \phi = \int \psi = 0$.*

(a) *For any $0 < b < 1$ and $f \in C_c^1(G)$, the series defining $S_{\phi,\psi,b} f$ converges absolutely, uniformly on G.*

(b) *For any $0 < b < 1$ and $f \in C_c^1(G)$, $S_{\phi,\psi,b} f \in L^2(G)$.*

(c) *For some $C > 0$, $\|S_{\phi,\psi,b} f\|_2 \leq Cb^{-Q}\|f\|_2$ for all $0 < b < 1$ and $f \in C_c^1(G)$. Consequently, $S_{\phi,\psi,b}$ extends to be a bounded operator on $L^2(G)$. (In fact, if we put $T = S_{\phi,\psi,b}$, then T satisfies the hypotheses of Theorem 5.)*

(d) *If $f, g \in L^2(G)$, then*

$$\langle S_{\phi,\psi,b} f, g\rangle = \sum_{\gamma\in\Gamma, j\in\mathbb{Z}} \langle f, \phi_{j,b\gamma}\rangle \langle \psi_{j,b\gamma}, g\rangle ; \tag{7.1}$$

here the series converges absolutely.

(e) *Say $\mathcal{B}_0$ is a bounded subset of $\mathcal{S}(G)$, $f \in C_c^1(G)$ and $b > 0$. Then the series defining $[S_{\phi,\psi,b} f](x)$ converges absolutely, uniformly for $x \in G$ and $\phi, \psi \in \mathcal{B}_0$ with $\int \psi = \int \phi = 0$.*

(f) *If $\mathcal{B}_0$ is a bounded subset of $\mathcal{S}(G)$, then there exists a constant C such that $\|S_{\phi,\psi,b}\| \leq Cb^{-Q}$ for all $0 < b < 1$ and all $\psi, \phi \in \mathcal{B}_0$ with $\int \psi = \int \phi = 0$.*

Remark. For the boundedness of $S_{\phi,\psi,b}$ on L^2, one may also consult Section 6 of Maggioni [28]. If $G = \mathbb{R}^n$, the fact that $T = S_{\phi,\psi,b}$ satisfies the conclusions of Theorem 5 has a long history. For instance, in Lemma 9.1.5 of [6], condition (ii) of Theorem 5 (b) is verified for this T, if $G = \mathbb{R}$. If $G = \mathbb{R}^n$, Theorem 5 is verified for this T, and more general operators T, in [18], Sections 2.1–2.3.

To prove the theorem, we shall need the following technical lemma.

Lemma 5. *For $N > 0$ define the function g_N on G by*

$$g_N(x) = (1 + |x|)^{-N} .$$

Then:

(a) *Let B_0 be a bounded subset of G. Then for some $C > 0$,*

$$g_N(x) \leq C g_N\big(y^{-1}x\big)$$

for all $x \in G$ and $y \in B_0$.

(b) *Say $M, N > \frac{Q}{2}$, and suppose $0 < L < \min(M - \frac{Q}{2}, N - \frac{Q}{2})$. Then for some $C > 0$,*

$$(g_M * g_N)(x) \leq C g_L(x)$$

for all $x \in G$.

Proof. For (a), we use the triangle inequality for G: For some $C > 0$,

$$\left|y^{-1}x\right| \leq C(|y| + |x|)$$

for all $x, y \in G$. From this we find at once that

$$\big(1 + \left|y^{-1}x\right|\big)^N \leq C^N (1 + |y|)^N (1 + |x|)^N ,$$

and (a) now follows.

For (b), we similarly observe that

$$(1 + |x|)^L \leq C^L (1 + |y|)^L \big(1 + \left|y^{-1}x\right|\big)^L .$$

Accordingly

$$\begin{aligned}(1 + |x|)^L (g_M * g_N)(x) &= \int (1 + |x|)^L g_M(y) g_N\big(y^{-1}x\big)\, dy \\ &\leq C^L \int g_{M-L}(y) g_{N-L}\big(y^{-1}x\big)\, dy \\ &\leq C^L \|g_{M-L}\|_2 \|g_{N-L}\|_2\end{aligned}$$

which is finite, since $M - L, N - L > \frac{Q}{2}$. This completes the proof. □

Note that Lemma 5 (a) implies that for any measurable subset $E \subseteq B_0$ of positive measure, we have that

$$g_N(x) \leq \frac{C}{m(E)} \int_E g_N\big(y^{-1}x\big)\, dy ,$$

for all $x \in G$. Such facts will be used without further comment in the proof which follows.

Proof of Theorem 7. We first prove (a). Since we shall be using Theorem 6 in our proof of (c), we shall actually need a stronger conclusion than (a).

We shall in fact show that:

(*) For all normalized bump functions f and all $R > 0$ and $u \in G$, there exists $C >$

0 such that the series defining $S_{\phi,\psi,b} f^{R,u}$ converges absolutely, uniformly on G, and $\|S_{\phi,\psi,b} f^{R,u}\|_\infty \le Cb^{-Q}$.

We begin by noting that there exists $C > 0$ such that for any f, R, u as above we have:

$$\left|\langle f^{R,u}, \phi_{j,b\gamma}\rangle\right| \le \left\| f^{R,u}\right\|_1 \|\phi_{j,b\gamma}\|_\infty \le CR^Q a^{-\frac{jQ}{2}} .$$

We let

$$C_{j,R} = \sup \left|\langle f^{R,u}, \phi_{j,b\gamma}\rangle\right| ,$$

the sup being taken over all normalized bump functions f, all $u \in G$, and all $\gamma \in \Gamma$. Thus,

$$C_{j,R} \le CR^Q a^{-\frac{jQ}{2}} . \tag{7.2}$$

Note also that if $R \ge a^j$, then

$$C_{j,R} \le CR^{-1} a^{j(\frac{Q}{2}+1)} . \tag{7.3}$$

Indeed, say f, R, u, γ are as above. Since $\int \phi = 0$, putting $v = b\gamma$ we have by Lemma 4 that

$$\begin{aligned}
\left|\langle f^{R,u}, \phi_{j,b\gamma}\rangle\right| &= a^{-\frac{jQ}{2}} \left| \int_G f^{R,u}(y) \phi\left(a^{-j}\left[\left(a^j v^{-1}\right) y\right]\right) dy \right| \\
&= a^{-\frac{jQ}{2}} \left| \int_G f^{R,u}\left(\left(a^j v\right) y\right) \phi\left(a^{-j} y\right) dy \right| \\
&= a^{-\frac{jQ}{2}} \left| \int_G \left[f^{R,u}\left(\left(a^j v\right) y\right) - f^{R,u}\left(a^j v\right)\right] \phi\left(a^{-j} y\right) dy \right| \\
&\le Ca^{-\frac{jQ}{2}} \int_G \left(\sum_{k=1}^n \frac{|y_k|}{R^{a_k}} \right) \left|\phi\left(a^{-j} y\right)\right| dy \\
&= Ca^{\frac{jQ}{2}} \sum_{k=1}^n \left[\frac{a^j}{R}\right]^{a_k} \left(\int_G |y_k \phi(y)| \, dy \right) \\
&\le C \frac{a^{j(\frac{Q}{2}+1)}}{R} ,
\end{aligned}$$

since we are here assuming that $\frac{a^j}{R} \le 1$.

Now select any $N > Q+1$, and note that $|\psi| \le Cg_N$ for some C. Fixing $j \in \mathbb{Z}$, we now see that

$$\begin{aligned}
\sum_{\gamma\in\Gamma} \left|\langle f^{R,u}, \phi_{j,b\gamma}\rangle \psi_{j,b\gamma}(x)\right| &\le C_{j,R} C \sum_{\gamma\in\Gamma} D_{a^j} T_{b\gamma} g_N(x) \\
&= Ca^{-\frac{jQ}{2}} C_{j,R} \sum_{\gamma\in\Gamma} g_N\left([b\gamma]^{-1}\left[a^{-j} x\right]\right) \\
&\le C \frac{a^{-\frac{jQ}{2}} C_{j,R}}{b^Q} \sum_{\gamma\in\Gamma} \int_{b\mathcal{R}} g_N\left(y^{-1}[b\gamma]^{-1}\left[a^{-j} x\right]\right) dy \\
&= C \frac{a^{-\frac{jQ}{2}} C_{j,R}}{b^Q} \int_G g_N(z) \, dz \\
&= C \frac{a^{-\frac{jQ}{2}} C_{j,R}}{b^Q} .
\end{aligned}$$

Given $R > 0$, we now select $j_0 \in \mathbb{Z}$ with $a^{j_0} \leq R \leq a^{j_0+1}$. Recalling (7.2) and (7.3), we now obtain

$$\begin{aligned}
\sum_{\gamma\in\Gamma, j\in\mathbb{Z}} \left|\langle f^{R,u}, \phi_{j,b\gamma}\rangle \psi_{j,b\gamma}(x)\right| &\leq \frac{C}{b^Q}\left[\sum_{j\leq j_0} C_{j,R}a^{-\frac{jQ}{2}} + \sum_{j>j_0} C_{j,R}a^{-\frac{jQ}{2}}\right] \\
&\leq \frac{C}{b^Q}\left[\sum_{j\leq j_0} \frac{a^j}{R} + \sum_{j>j_0} R^Q a^{-jQ}\right] \\
&\leq \frac{C}{b^Q}\left[\sum_{j\leq j_0} a^{j-j_0} + \sum_{j>j_0} a^{(j_0-j+1)Q}\right] \\
&\leq \frac{C}{b^Q}\,,
\end{aligned}$$

proving (*) and hence (a).

We next prove (b). Again, we shall prove a stronger conclusion, which will be needed in our proof of (c).

For $x, y \in G$, $x \neq y$, we wish to define

$$K_{\phi,\psi,b}(x,y) = \sum_{\gamma\in\Gamma, j\in\mathbb{Z}} \psi_{j,b\gamma}(x)\overline{\phi_{j,b\gamma}}(y)\,; \tag{7.4}$$

we will soon show that the sum converges absolutely. The reason for this definition is that formally

$$\left[S_{\phi,\psi,b}f\right](x) = \int_G K_{\phi,\psi,b}(x,y)f(y)\,dy\,;$$

we will soon show that this is true if $f \in C_c^1$, for x outside the support of f. These facts are immediate consequences of the following assertion (with $J = 0$, $\psi = \Phi$ and $\phi = \overline{\Psi}$:)

(**) Say $\Phi, \Psi \in \mathcal{S}(G)$, and $J > 0$. Then for some $C > 0$,

$$\sum_{\gamma\in\Gamma, j\in\mathbb{Z}} a^{-jJ}\left|\Phi_{j,b\gamma}(x)\Psi_{j,b\gamma}(y)\right| \leq \frac{C}{b^Q}\left|y^{-1}x\right|^{-Q-J} \tag{7.5}$$

for all $x, y \in G$, $x \neq y$. Moreover, the series converges uniformly on compact subsets of $(G \setminus \{0\}) \times (G \setminus \{0\})$.

To prove (7.5), define

$$\begin{aligned}
K_I(x,y) &= \sum_{\gamma\in\Gamma, j\in\mathbb{Z}} a^{-jJ}\left|y^{-1}x\right|^{Q+J}\left|\Phi_{j,b\gamma}(x)\Psi_{j,b\gamma}(y)\right| \\
&= \sum_{\gamma\in\Gamma, j\in\mathbb{Z}} a^{-j(Q+J)}\left|y^{-1}x\right|^{Q+J}\left|\Phi\left((b\gamma)^{-1}\left[a^{-j}x\right]\right)\Psi\left((b\gamma)^{-1}\left[a^{-j}y\right]\right)\right|.
\end{aligned}$$

We need to show that K_I is bounded for $x \neq y$. Observe that for any $x, y \in G$ we have that $K_I(ax, ay) = K_I(x, y)$. Therefore we need only consider those x, y with $|y^{-1}x| \in [1, a]$. Choose $L > Q + J$ and $N > \frac{Q}{2} + L$. For some C_0, $\Phi, \Psi \leq C_0 g_N$. Thus, for fixed j,

$$\begin{aligned}
&\sum_{\gamma\in\Gamma}\left|\Phi\left((b\gamma)^{-1}\left[a^{-j}x\right]\right)\Psi\left((b\gamma)^{-1}\left[a^{-j}y\right]\right)\right| \\
&\quad\leq C\sum_{\gamma\in\Gamma} g_N\left((b\gamma)^{-1}\left[a^{-j}x\right]\right) g_N\left((b\gamma)^{-1}\left[a^{-j}y\right]\right) \\
&\quad\leq \frac{C}{b^Q}\sum_{\gamma\in\Gamma}\left[\int_{b\mathcal{R}} g_N\left(z^{-1}(b\gamma)^{-1}\left[a^{-j}x\right]\right) g_N\left(z^{-1}(b\gamma)^{-1}\left[a^{-j}y\right]\right)\,dw\right] \\
&\quad= \frac{C}{b^Q}\int_G g_N\left(w^{-1}\left[a^{-j}x\right]\right) g_N\left(w^{-1}\left[a^{-j}y\right]\right)\,dw \\
&\quad= \frac{C}{b^Q}\left(g_N * g_N\right)\left(a^{-j}\left[y^{-1}x\right]\right) \\
&\quad\leq \frac{C}{b^Q} g_L\left(a^{-j}\left[y^{-1}x\right]\right)\ .
\end{aligned}$$

Consequently, for $|y^{-1}x| \in [1, a]$, we have that

$$\begin{aligned}
|K_I(x,y)| &\leq \frac{Ca^{Q+J}}{b^Q}\sum_{j\in\mathbb{Z}} a^{-j(Q+J)} g_L\left(a^{-j}\left[y^{-1}x\right]\right) \\
&\leq \frac{C}{b^Q}\left[\sum_{j\geq 0} a^{-j(Q+J)} + \sum_{j<0} a^{-j(Q+J)}\left|a^{-j}\left[y^{-1}x\right]\right|^{-L}\right] \\
&= \frac{C}{b^Q}\left[\sum_{j\geq 0} a^{-j(Q+J)} + \sum_{j<0} a^{j(L-Q-J)}\right] \\
&\leq \frac{C}{b^Q}\ ,
\end{aligned}$$

proving (7.5). The uniform convergence asserted in (**) follows from an examination of the proof of (7.5). This proves (**).

(**) now implies at once that the series in (7.4) converges absolutely for $x \neq y$, and we define its sum to be $K_{\phi,\psi,b}(x, y)$.

We can now easily prove (b). Actually, in order to also later prove (c), we will note the following stronger statement:

(***) $K_{\phi,\psi,b}$ is smooth away from the diagonal; moreover for all multiindices α, β there exists $C_{\alpha,\beta} > 0$ such that for all $x, y \in G$ with $x \neq y$ and for all $0 < b < 1$ we have

$$\left|X_x^\alpha X_y^\beta K_{\phi,\psi,b}(x,y)\right| \leq \frac{C_{\alpha,\beta}}{b^Q}\left|y^{-1}x\right|^{-(Q+|\alpha|+|\beta|)}\ . \tag{7.6}$$

(***) follows at once from (**). Indeed, we claim that, if $x \neq y$, then

$$X_x^\alpha X_y^\beta K_I(x,y) = \sum_{\gamma\in\Gamma, j\in\mathbb{Z}} a^{-j(|\alpha|+|\beta|)}\left(X^\alpha\psi\right)_{j,b\gamma}(x)\left(\overline{X^\beta\phi}\right)_{j,b\gamma}(y)\ . \tag{7.7}$$

To see this, note that by (**), the series in (7.7) converges uniformly on compact subsets of $(G \setminus \{0\}) \times (G \setminus \{0\})$. On such a compact set, any (usual) differential monomial

$\partial_x^\rho \partial_y^\tau$ is a linear combination, with polynomial coefficients, of the $X_x^\alpha X_y^\beta$. Thus, the series in (7.4) converges in the topology of $C^\infty((G \setminus \{0\}) \times (G \setminus \{0\}))$. This implies that K is smooth off the diagonal, and also that, when we differentiate K, we can bring derivatives past the summation sign. This proves (***).

We now show that (***) implies (b). In fact it implies the following stronger statement, which we shall also need in the proof of (c):

(****) For all normalized bump functions f and all $R > 0$ and $u \in G$, there exists $C > 0$ such that

$$\left\| S_{\phi,\psi,b} f^{R,u} \right\|_2 \leq \frac{C}{b^Q} R^{\frac{Q}{2}} .$$

To see this, we observe that if $|x^{-1}u| \geq 2R$, then

$$\begin{aligned}
\left| \left[S_{\phi,\psi,b} f^{R,u} \right](x) \right| &\leq \int_G \left| K_{\phi,\psi,b}(x,y) f^{R,u}(y) \right| dy \\
&\leq C b^{-Q} \int_{|u^{-1}y| \leq R} \left| x^{-1} y \right|^{-Q} dy \\
&\leq C R^Q b^{-Q} \max_{|u^{-1}y| \leq R} \left| x^{-1} y \right|^{-Q} \\
&\leq C R^Q b^{-Q} \left| x^{-1} u \right|^{-Q} . && (7.8)
\end{aligned}$$

(The last inequality follows from Proposition 4.) Finally, if $g = S_{\phi,\psi,b} f^{R,u}$, recalling (*), we have that

$$\begin{aligned}
\|g\|_2^2 &= \int_{|x^{-1}u| < 2R} |g(x)|^2 \, dx + \int_{|x^{-1}u| > 2R} |g(x)|^2 \, dx \\
&\leq C b^{-2Q} \left[(2R)^Q + R^{2Q} \int_{|x^{-1}u| > 2R} \left| x^{-1} u \right|^{-2Q} dx \right] \\
&\leq C b^{-2Q} \left[(2R)^Q + R^{2Q} \int_{|y| > 2R} |y|^{-2Q} \, dy \right] \\
&\leq C b^{-2Q} \left[(2R)^Q + R^{2Q} (2R)^{-Q} \right] \\
&= C R^Q b^{-2Q} .
\end{aligned}$$

This proves (****), and hence (b).

We now claim that (c) follows directly from (***), (****), and Theorem 6. In order to apply Theorem 6, we make the following two additional observations.

(1) *The formal adjoint of $S_{\phi,\psi,b}$ is $S_{\psi,\phi,b}$.* What we are claiming is that for all $f, g \in C_c^1$, we have

$$\langle S_{\phi,\psi,b} f, g \rangle = \langle f, S_{\psi,\phi,b} g \rangle . \tag{7.9}$$

Indeed, by (*), the left side of (7.9) clearly equals

$$\sum_{\gamma \in \Gamma, j \in \mathbb{Z}} \langle f, \phi_{j,b\gamma} \rangle \langle \psi_{j,b\gamma}, g \rangle .$$

Evidently this equals the right side of (7.9), as claimed. Note, for later purposes, that this observation also proves (d) if $f, g \in C_c^1$.

(2) $S_{\phi,\psi,b}(1) = S_{\psi,\phi,b}(1) = 0$. We need to show that whenever $f \in C_c^1(G)$ and $\int f = 0$, we have that

$$\int S_{\phi,\psi,b} f = 0$$

(and similarly $\int S_{\psi,\phi,b} f = 0$.) To see this, for any finite subset $\mathcal{F}$ of $\mathbb{Z} \times \Gamma$, define the operator

$$S^{\mathcal{F}}_{\phi,\psi,b} : \ f \to \sum_{(j,\gamma)\in\mathcal{F}} \langle f, \phi_{j,b\gamma}\rangle \psi_{j,b\gamma} \ .$$

We regard this as an operator on $C_c^1(G)$, and it maps this space into $C^\infty(G)$, since it has smooth kernel

$$K^{\mathcal{F}}_{\phi,\psi,b}(x,y) = \sum_{(j,\gamma)\in\mathcal{F}} \psi_{j,b\gamma}(x)\overline{\phi}_{j,b\gamma}(y) \ .$$

For any integer $N > 0$ we also let $S^N_{\phi,\psi,b} = S^{\mathcal{F}_N}_{\phi,\psi,b}$ and $K^N_{\phi,\psi,b} = K^{\mathcal{F}_N}_{\phi,\psi,b}$, where

$$\mathcal{F}_N = \{(j,\gamma) : \ |j| \le N, \ |\gamma| \le N\} \ .$$

Since $\psi \in \mathcal{S}$ has integral zero, it is evident that for all $f \in C_c^1(G)$, we have

$$\int S^{\mathcal{F}}_{\phi,\psi,b} f = 0$$

for all $\mathcal{F}$. Fix f with $\int f = 0$, and fix $b > 0$; we need to deduce that $\int S_{\phi,\psi,b} = 0$. This will follow at once from the dominated convergence theorem if we can show:

(i) $S^N_{\phi,\psi,b} f \to S_{\phi,\psi,b} f$ pointwise as $N \to \infty$; and

(ii) for some $C > 0$, $|S^{\mathcal{F}}_{\phi,\psi,b} f| \le C g_{Q+1}$ for all $\mathcal{F}$.

(Here g_{Q+1} is as in Lemma 5.) Since (i) follows at once from the absolute convergence proved in (*), we need only establish (ii). (*) similarly shows that, for some $C > 0$, $|[S^{\mathcal{F}}_{\phi,\psi,b} f](x)| \le C$ for all $\mathcal{F}$ and all $x \in G$. Suppose then that the support of f is contained in $\{x : |x| < R\}$; we need only show that for some C, A (independent of $\mathcal{F}$),

$$\left| \left[S^{\mathcal{F}}_{\phi,\psi,b} f \right](x) \right| \le C|x|^{-Q-1} \tag{7.10}$$

whenever $|x| > AR$. But by (**), for any multiindices α, β, there is a $C_{\alpha,\beta} > 0$ (independent of $\mathcal{F}$) such that for all $x, y \in G$ with $x \ne y$,

$$\left| X_x^\alpha X_y^\beta K^{\mathcal{F}}_{\phi,\psi,b}(x,y) \right| \le C_{\alpha,\beta} \left| y^{-1}x \right|^{-(Q+|\alpha|+|\beta|)} \ .$$

By Proposition 7 (and its proof), the $K^{\mathcal{F}}_{\phi,\psi,b}$ satisfy the Calderon-Zygmund inequalities (6.3), (6.4), and (6.5) with constants c, C independent of $\mathcal{F}$. By the triangle inequality, there is a number $A > 0$ such that whenever $|x| > AR$ and $|y| < R$, we have $|y^{-1}x| > cR$. Thus, if $|x| > AR$, we have that

$$\left|\left[S^{\mathcal{F}}_{\phi,\psi,b} f\right](x)\right| = \left|\int_{|y|<R} \left[K^{\mathcal{F}}_{\phi,\psi,b}(x,y) - K^{\mathcal{F}}_{\phi,\psi,b}(x,0)\right] f(y)\, dy\right|$$
$$\leq C|x|^{-(Q+1)} \int_{|y|<R} |y||f(y)|\, dy$$

as claimed.

These observations now prove (c) at once.

We next prove (d). We fix $b > 0$. In observation (1) above, we have already seen that (d) holds for $f, g \in C^1_c$. To prove it in general, we let $S^{\mathcal{F}}_{\phi,\psi,b}$, $K^{\mathcal{F}}_{\phi,\psi,b}$, $S^N_{\phi,\psi,b}$, and $K^N_{\phi,\psi,b}$ be as in observation (2). We observe that, for any $f, g \in L^2$, we have

$$\left\langle S^N_{\phi,\psi,b} f, g\right\rangle = \sum_{\gamma\in\Gamma,\, j\in\mathbb{Z},\, |\gamma|\leq N,\, |j|\leq N} \langle f, \phi_{j,b\gamma}\rangle\langle\psi_{j,b\gamma}, g\rangle\ , \tag{7.11}$$

and we claim that

$$\left\langle S^N_{\phi,\psi,b} f, g\right\rangle \to \langle S_{\phi,\psi,b} f, g\rangle\ . \tag{7.12}$$

Since (d) holds for $f, g \in C^1_c$, which is dense in L^2, it is enough to show that the norms $\|S^{\mathcal{F}}_{\phi,\psi,b}\|$ are uniformly bounded in $\mathcal{F}$. But this follows from Theorem 6, together with a repetition of the proofs of (*), (***), and (****) with $S^{\mathcal{F}}_{\phi,\psi,b}$, $K^{\mathcal{F}}_{\phi,\psi,b}$ in place of $S_{\phi,\psi,b}$, $K_{\phi,\psi,b}$; here one must note that all bounds are independent of $\mathcal{F}$.

Now in (7.12), take the special case $\psi = \phi$ and $g = f$. All the terms in the series in (7.11) are then nonnegative, so the series in (7.1) converges absolutely to the left side of that equation—in that special case. But in the general case, Cauchy-Schwarz as applied to the series in (7.12) now shows that this series always converges absolutely. Moreover, by (7.12), this series converges to the left side of (7.1). This proves (d).

(e) follows from an examination of the proofs of (a). (f) follows from an examination of the proofs of (a), (b), and (c). (In particular, note for later purposes that the constants $C_{\alpha,\beta}$ in (7.6) may be taken independent of $\phi, \psi \in \mathcal{B}_0$ with $\int\psi = \int\phi = 0$.) This completes the proof of Theorem 7. □

The main idea in our proof of Theorem 2 (a) is to show that, for b sufficiently small, $Vb^Q S_{\psi,\psi,b}$ is "well approximated" by the operator $R_\psi = \sum_{j\in\mathbb{Z}} R_j$, where if $f \in L^2(G)$ we put

$$R_j f = f * \widetilde{\psi}_{a^j} * \psi_{a^j}\ ,$$

then to use the spectral theorem to show that R_ψ is bounded below if $\psi = g(L)\delta$ and g satisfies Daubechies' criterion. We begin to make these ideas rigorous, by noting the following proposition. (In this proposition, $\mathcal{R}$ is, once again, a bounded measurable subset of G, of positive measure, such that every $g \in G$ may be written uniquely in the form $g = x\gamma$ with $x \in \mathcal{R}$ and $\gamma \in \Gamma$.)

Proposition 8. *Suppose $\psi \in \mathcal{S}(G)$ and $\int\psi = 0$. Suppose $f \in C^1_c(G)$. Then*

$$\sum_{j\in\mathbb{Z}} (R_j f)(x) = \int_{b\mathcal{R}} \left[S_{T_z\psi,T_z\psi,b} f\right](x)\, dz \tag{7.13}$$

where the series on the left side converges absolutely, uniformly for x on G. Consequently, $\sum R_j f$ converges to an L^2 function, and the map $R_\psi : C_c^1 \to L^2$ given by $R_\psi f = \sum R_j f$ extends to a bounded positive operator on L^2.

Proof. Fix j for now and put $\eta = \psi_{a^j}$. Then

$$(R_j f)(x) = \int_G f(y) \left[\tilde{\eta} * \eta\right] \left(y^{-1}x\right) dy .$$

But

$$\begin{aligned}
\left[\tilde{\eta} * \eta\right] \left(y^{-1}x\right) dy &= \int_G \tilde{\eta}\big(y^{-1}z\big)\eta\big(z^{-1}x\big)\, dz \\
&= \int_G \overline{\eta}\big(z^{-1}y\big)\eta\big(z^{-1}x\big)\, dz \\
&= a^{-2jQ} \int_G \overline{\psi}\left(a^{-j}\big[z^{-1}y\big]\right) \psi\left(a^{-j}\big[z^{-1}x\big]\right) dz \\
&= a^{-jQ} \int_G \overline{\psi}\left(z^{-1}\big[a^{-j}y\big]\right) \psi\left(z^{-1}\big[a^{-j}x\big]\right) dz \\
&= a^{-jQ} \sum_{\gamma \in \Gamma} \int_{b\mathcal{R}} \overline{\psi}\left(z^{-1}[b\gamma]^{-1}\big[a^{-j}y\big]\right) \psi\left(z^{-1}[b\gamma]^{-1}\big[a^{-j}x\big]\right) dz \\
&= \sum_{\gamma \in \Gamma} \int_{b\mathcal{R}} \overline{(T_z\psi)}_{j,b\gamma}(y)\, (T_z\psi)_{j,b\gamma}(x)\, dz .
\end{aligned}$$

(In the fourth line, we have made the change of variables $z \to a^j z$.) By Theorem 7 (e), the series

$$\sum_{j \in \mathbb{Z}, \gamma \in \Gamma} \langle f, (T_z\psi)_{j,b\gamma}\rangle (T_z\psi)_{j,b\gamma}(x)$$

converges absolutely, uniformly for $x \in G$ and $z \in b\mathcal{R}$. This would therefore also surely be true if we fixed a j and summed only over γ. Thus,

$$(R_j f)(x) = \int_{b\mathcal{R}} \sum_{\gamma \in \Gamma} \langle f, (T_z\psi)_{j,b\gamma}\rangle\, (T_z\psi)_{j,b\gamma}(x)\, dz$$

and finally, summing over j, we find (7.13) as well, the sum on the left side converging absolutely, uniformly for $x \in G$. The remaining conclusions of the proposition now follow at once from Theorem 7 (f) and Minkowski's inequality. (Note: To show that R_ψ is positive, it is enough to show that $\langle R_\psi f, f\rangle \geq 0$ for all $f \in C_c^1$, and for this it is enough to show that $\langle R_j f, f\rangle \geq 0$ for all $f \in C_c^1$ and all j. But this is clear, since for such f, $\langle R_j f, f\rangle = \|f * \psi_{a^j}\|_2^2$.) This completes the proof. □

We can now reach an understanding of why $Vb^Q S_{\psi,\psi,b}$ is well approximated by R_ψ for b small.

Theorem 8. *Suppose $\psi \in \mathcal{S}(G)$, and $\int_G \psi = 0$. Let V be the measure of $\mathcal{R}$, and let R_ψ be as in Proposition 8. For $1 \leq l \leq n$, let $\psi_l = Y_l\psi$ (so that, by Proposition 3, $\int_G \psi_l = 0$ for all l). Then:*

(a) *If* $f \in C_c^1(G)$,

$$\left[\frac{1}{Vb^Q} R_\psi f - S_{\psi,\psi,b} f\right](x) = \frac{1}{Vb^Q} \sum_{l=1}^{n} \int_{b\mathcal{R}} \int_0^1 z_l \left(\left[S_{T_{[t]z} Y_l \psi, T_{[t]z}\psi, b} f\right] + \left[S_{T_{[t]z}\psi, T_{[t]z} Y_l \psi, b} f\right]\right)(x)\, dt\, dz\ .$$

(b) *There exists* $C > 0$ *such that for all* $0 < b < 1$*, the norm on* $L^2(G)$

$$\left\|\frac{1}{Vb^Q} R_\psi - S_{\psi,\psi,b}\right\| \le \frac{C}{b^{Q-1}}\ .$$

(c) *If* $R_\psi \ge AI$ *for some* $A > 0$*, then there exists* $b_0 > 0$ *such that* $\{\psi_{j,b\gamma}\}$ *is a frame whenever* $0 < b < b_0$*. More precisely, choose* $B > 0$ *such that* $R_\psi \le BI$ (*of course we can choose* $B = \|R_\psi\|$)*. Then, for* $0 < b < b_0$*, we can choose* $A_b, B_b > 0$ *such that*

$$A_b \|f\|_2^2 \le \sum_{j\in\mathbb{Z},\gamma\in\Gamma} |\langle f, \psi_{j,b\gamma}\rangle|^2 \le B_b \|f\|_2^2 \tag{7.14}$$

for all $f \in L^2$*, and such that*

$$\lim_{b\to 0^+} \frac{B_b}{A_b} = \frac{B}{A}\ . \tag{7.15}$$

Proof. Of course, the measure of $b\mathcal{R}$ is Vb^Q. Using Theorem 7 (e), together with Proposition 8, we see that

$$\left[\frac{1}{Vb^Q} R_\psi f - S_{\psi,\psi,b} f\right](x) = \frac{1}{Vb^Q} \int_{b\mathcal{R}} \left(\left[S_{T_z\psi,T_z\psi,b} f\right](x) - \left[S_{\psi,\psi,b} f\right](x)\right) dz$$
$$= \frac{1}{Vb^Q} \sum_{j\in\mathbb{Z},\gamma\in\Gamma} \int_{b\mathcal{R}} \left[\langle f, (T_z\psi)_{j,b\gamma}\rangle (T_z\psi)_{j,b\gamma}(x) - \langle f, \psi_{j,b\gamma}\rangle \psi_{j,b\gamma}(x)\right] dx. \tag{7.16}$$

However, fixing j, γ, we have that

$$\langle f, (T_z\psi)_{j,b\gamma}\rangle (T_z\psi)_{j,b\gamma}(x) - \langle f, \psi_{j,b\gamma}\rangle \psi_{j,b\gamma}(x) = \int_G f(y) K(x,y)\, dy \tag{7.17}$$

where

$$K(x,y) = \overline{(T_z\psi)}_{j,b\gamma}(y)\, (T_z\psi)_{j,b\gamma}(x) - \overline{\psi}_{j,b\gamma}(y) \psi_{j,b\gamma}(x)\ .$$

Fix x, y as well, and, for $w \in G$, let

$$F(w) = \overline{\left(T_{w^{-1}}\psi\right)}_{j,b\gamma}(y) \left(T_{w^{-1}}\psi\right)_{j,b\gamma}(x)\ .$$

Explicitly

$$F(w) = a^{-jQ} \overline{\psi}\left(w\,(b\gamma)^{-1}\left(a^{-j}y\right)\right) \psi\left(w\,(b\gamma)^{-1}\left(a^{-j}x\right)\right)\ .$$

Since each Y_l is right-invariant, note that

$$\begin{aligned}(Y_1 F)(w) &= a^{-jQ}\Big[\overline{(Y_l\psi)}\left(w\,(b\gamma)^{-1}\left(a^{-j}y\right)\right)\psi\left(w\,(b\gamma)^{-1}\left(a^{-j}x\right)\right) \\ &\quad + \overline{\psi}\left(w\,(b\gamma)^{-1}\left(a^{-j}y\right)\right)(Y_l\psi)\left(w\,(b\gamma)^{-1}\left(a^{-j}x\right)\right)\Big] \\ &= \overline{\left(T_{w^{-1}}\left[Y_l\psi\right]\right)}_{j,b\gamma}(y)\left(T_{w^{-1}}\psi\right)_{j,b\gamma}(x) \\ &\quad + \overline{\left(T_{w^{-1}}\psi\right)}_{j,b\gamma}(y)\left(T_{w^{-1}}\left[Y_l\psi\right]\right)_{j,b\gamma}(x)\ .\end{aligned}$$

Then, by Lemma 3, and the facts that $z^{-1} = -z$ and that the Y_l are right-invariant, we have

$$\begin{aligned}K(x,y) &= F\left(z^{-1}\right) - F(0) \\ &= -\sum_{l=1}^{n} z_l \int_0^1 (Y_l F)\left(([t]z)^{-1}\right) dt \\ &= -\sum_{l=1}^{n} z_l \int_0^1 \Big[\overline{\left(T_{[t]z}\left[Y_l\psi\right]\right)}_{j,b\gamma}(y)\left(T_{[t]z}\psi\right)_{j,b\gamma}(x) \\ &\quad + \overline{\left(T_{[t]z}\psi\right)}_{j,b\gamma}(y)\left(T_{[t]z}\left[Y_l\psi\right]\right)_{j,b\gamma}(x)\Big]\, dt\ .\end{aligned}$$

Since $f \in C_c^1$,

$$\begin{aligned}\int_G f(y)K(x,y)\,dy = -\sum_{l=1}^{n} z_l \int_0^1 \Big[&\langle f, (T_{[t]z}[Y_l\psi])_{j,b\gamma}\rangle (T_{[t]z}\psi)_{j,b\gamma}(x) \\ &+ \langle f, (T_{[t]z}\psi)_{j,b\gamma}\rangle (T_{[t]z}[Y_l\psi])_{j,b\gamma}(x)\Big]\, dt\ .\end{aligned}$$

Part (a) of the theorem now follows at once from this, (7.16), (7.17), and Theorem 7 (e).

For (b), choose a number $M > 0$ such that $|z_l| \le M$ whenever $z \in \mathcal{R}$ and $1 \le l \le n$. Then, surely, $|z_l| \le b^{a_l}M \le bM$ whenever $z \in b\mathcal{R}$ and $1 \le l \le n$. Accordingly, by Theorem 7 (f) and Minkowski's inequality, there exist $C_1, C > 0$ such that for all $f \in C_c^1$ and all $0 < b < 1$, we have

$$\begin{aligned}\left\|\frac{1}{Vb^Q}R_\psi f - S_{\psi,\psi,b}f]\right\|_2 &\le \frac{bM}{Vb^Q}\sum_{l=1}^{n}\int_{b\mathcal{R}}\int_0^1 \left(\left\|S_{T_{[t]z}Y_l\psi,T_{[t]z}\psi,b}f\right\|_2\right. \\ &\quad \left. + \left\|S_{T_{[t]z}\psi,T_{[t]z}Y_l\psi,b}f\right\|_2\right)\, dt\, dz \\ &\le \frac{2nbM}{Vb^Q}m(b\mathcal{R})\frac{C_1}{b^Q}\|f\|_2 \\ &= \frac{C}{b^{Q-1}}\|f\|_2\ .\end{aligned}$$

Since C_c^1 is dense in L^2, (b) now follows.

Finally, for (c), take C as in (b). Then for any $f \in L^2$,

$$\langle S_{\psi,\psi,b}f, f\rangle = \frac{1}{Vb^Q}\langle R_\psi f, f\rangle + \left\langle\left[S_{\psi,\psi,b} - \frac{1}{Vb^Q}R_\psi\right]f, f\right\rangle$$

so that, for all $f \in L^2$,

$$A_b\|f\|_2^2 \ \le\ \langle S_{\psi,\psi,b}f, f\rangle \ \le\ B_b\|f\|_2^2\ ,$$

where

$$A_b = \frac{A - CVb}{Vb^Q} \tag{7.18}$$

and

$$B_b = \frac{B + CVb}{Vb^Q} \ . \tag{7.19}$$

Put $b_0 = \min(\frac{A}{CV}, 1)$. If $0 < b < b_0$, then $A_b > 0$, and, moreover, by Theorem 7 (d), (7.14) holds for all $f \in L^2$. Finally, (7.15) is immediate from (7.18) and (7.19). This proves (c) and completes the proof of the theorem. □

Accordingly, the search for frames reduces to the question of finding ψ with $R_\psi \geq AI$ for some $A > 0$.

If $G = \mathbb{R}^n$ with the usual addition, then if $f \in L^2$,

$$\widehat{(R_j f)}(\xi) = \left|\hat{\psi}\left(a^j \xi\right)\right|^2 \hat{f}(\xi) \ .$$

Define

$$m_{\hat{\psi},a}(\xi) = \sum_{j \in \mathbb{Z}} \left|\hat{\psi}\left(a^j \xi\right)\right|^2 \ .$$

(Usually a is fixed and understood, and we will just write $m_{\hat{\psi}} = m_{\hat{\psi},a}$.) Since we are assuming that $\int \psi = 0$, surely $\hat{\psi}(0) = 0$, so that $|\hat{\psi}(\xi)| \leq C|\xi|$ for $|\xi| < 1$. Also, since $\hat{\psi} \in \mathcal{S}$, $|\hat{\psi}(\xi)| \leq \frac{C}{|\xi|}$ for $|\xi| \geq 1$. From these facts it is easy to see that the series defining $m_{\hat{\psi}}(\xi)$ converges uniformly on any compact subset of $\mathbb{R}^n$ which excludes the origin. We claim that

$$\widehat{(R_\psi f)}(\xi) = m_{\hat{\psi}}(\xi) \hat{f}(\xi)$$

for all $f \in L^2$. This is not hard to see, but since we shall need an analogue for general G, let us present the argument in detail.

First note that $m_{\hat{\psi}}(a\xi) = m_{\hat{\psi}}(\xi)$ for all ξ, so $m_{\hat{\psi}}$ is uniformly bounded on $\mathbb{R}^n$. Define an operator $Q : L^2 \to L^2$ by $\widehat{(Qf)}(\xi) = m_{\hat{\psi}}(\xi) \hat{f}(\xi)$; we want to show that $R_\psi = Q$. For $N > 0$, set $Q_N = \sum_{j=-N}^{N} R_j$, an operator on L^2; then $\|Q_N\| \leq \|m_{\hat{\psi}}\|_\infty$ for all N. If

$$V = \left\{ f \in L^2 : \hat{f} = 0 \text{ a.e. outside some compact subset of } \mathbb{R}^n \setminus \{0\} \right\} ,$$

then $Q_N f \to Qf$ in L^2 for all $f \in V$. Since V is dense in L^2 and the $\|Q_N\|$ are uniformly bounded, we see that $Q_N f \to Qf$ for all $f \in L^2$. However, $Q_N f \to R_\psi f$ pointwise on $\mathbb{R}^n$ if $f \in C_c^1$. Consequently, $Qf = R_\psi f$ for all $f \in C_c^1$, and hence for all $f \in L^2$, as claimed.

If we now let

$$B = \sup_{\xi \neq 0} m_{\hat{\psi}}(\xi), \quad A = \inf_{\xi \neq 0} m_{\hat{\psi}}(\xi) \ ,$$

we now see that

$$A\|f\|_2^2 \leq \langle R_\psi f, f \rangle \leq B\|f\|_2^2$$

for all $f \in L^2$. Theorem 8 then tells us in particular that $\{\psi_{j,b\gamma}\}$ is a frame for all sufficiently small b, *provided that* $A > 0$. The condition $A > 0$ is called *Daubechies' criterion.* (In [6], Daubechies shows that $\{\psi_{j,b\gamma}\}$ is a frame if $n = 1$ and Γ is the integer lattice, if Daubechies' criterion holds. Her methods are very different from those of this article—she uses Plancherel and Parseval.) Since, for all ξ, $m_{\hat{\psi}}(a\xi) = m_{\hat{\psi}}(\xi)$, A is the minimum value of $m_{\hat{\psi}}$ on the compact annulus $\{\xi : 1 \leq |\xi| \leq a\}$. Thus, Daubechies' criterion is equivalent to the hypothesis that there does not exist a nonzero $\xi_0 \in \mathbb{R}^n$ such that $\hat{\psi}(a^j\xi_0) = 0$ for all integers j.

Now we turn to general stratified Lie groups G.

Proof of Theorem 2. We change notation from the statement of Theorem 2, writing H in place of f. Thus, we restrict attention to ψ of the form $\psi = F(L)\delta = LH(L)\delta$, where $F(\lambda) = \lambda H(\lambda)$ and $H, F \in \mathcal{S}(\mathbb{R}^+)$. In that case, if $f \in L^2$,

$$R_j f = F_j(L) f$$

where

$$F_j(\lambda) = \left|F\left(a^{2j}\lambda\right)\right|^2 .$$

Define

$$m_{F,a^2}(\lambda) = \sum_{j\in\mathbb{Z}} \left|F\left(a^{2j}\lambda\right)\right|^2 .$$

(Usually a is fixed and understood, and we will just write $m_F = m_{F,a^2}$.) As before, the series defining m_F converges uniformly on any compact subset of $\mathbb{R}^+$ which excludes the origin. We claim that

$$R_\psi f = m_F(L) f \tag{7.20}$$

for all $f \in L^2$.

As before, $m_F(a^2\lambda) = m_F(\lambda)$ for all λ, so m_F is uniformly bounded on $\mathbb{R}^+$. Set $Q = m_F(L)$, and put $\mathcal{H} = L^2(G)$. For $N > 0$, set $Q_N = \sum_{j=-N}^{N} R_j$, an operator on $\mathcal{H}$; then, by the spectral theorem, $\|Q_N\| \leq \|m_F\|_\infty$ for all N. Let

$$V = \bigcup_{0<\epsilon<N<\infty} P_{[\epsilon,N]}\mathcal{H} .$$

(Recall that the $P_{[a,b]}$ are spectral projectors of L.) Then $Q_N f \to Qf$ in $\mathcal{H}$ for all $f \in V$. But, since $P_{\{0\}} = 0$, V is dense in $\mathcal{H}$. Since the $\|Q_N\|$ are uniformly bounded, it follows that $Q_N f \to Qf$ for all $f \in \mathcal{H}$. However, $Q_N f \to R_\psi f$ pointwise on G if $f \in C_c^1$. Consequently, $Qf = R_\psi f$ for all $f \in C_c^1$, and hence for all $f \in L^2$, as claimed.

If we now let

$$B = \sup_{\lambda>0} m_F(\lambda), \quad A = \inf_{\lambda>0} m_F(\lambda) ,$$

and again note that $P_{\{0\}} = 0$, we see that

$$A\|f\|_2^2 \leq \langle R_\psi f, f\rangle \leq B\|f\|_2^2$$

for all $f \in L^2$. Theorem 8 then tells us in particular that $\{\psi_{j,b\gamma}\}$ is a frame for all sufficiently small b, *provided that* $A > 0$—in other words, if F satisfies Daubechies' criterion (where of course we use a^2 in place of the a we used on $\mathbb{R}^n$).

This establishes Theorem 2 (a). (Here Daubechies' criterion is clearly equivalent to the nonexistence of a $\lambda_0 > 0$ such that $F(a^{2j}\lambda_0) = 0$ for all integers j.)

We now prove Theorem 2 (b). By (7.15) we need only show that if a is close enough to 1, then

$$A = \inf_{\lambda>0} m_{F,a^2}(\lambda) > 0$$

(so that the Daubechies condition holds), and that

$$\frac{B}{A} = 1 + O\left(|a-1|^2 \log|a-1|\right) , \tag{7.21}$$

as $a \to 1$. We may assume $a > 1$ (otherwise replace a by $\frac{1}{a}$, and note that $m_{F,a^2} = m_{F,(1/a)^2}$, and $O\big(|a-1|^2 \log|a-1|\big) = O\big(|\frac{1}{a}-1|^2 \log|\frac{1}{a}-1|\big)$. However, if $a > 1$, then (7.21) follows at once from the following elementary lemma.

Lemma 6. *Suppose H is a nonzero element of $\mathcal{S}(\mathbb{R}^+)$ and let $F(s) = sH(s)$. Let $I \in (0,\infty)$ be defined by*

$$I = \int_0^\infty |F(t)|^2 \frac{dt}{t} = \int_0^\infty |F(ts)|^2 \frac{dt}{t} \tag{7.22}$$

(*for any $s > 0$*)*, as in Calderón's reproducing formula. Suppose $a > 1$. Then for all $s > 0$,*

$$A(a) \le \sum_{n=-\infty}^{\infty} \left|F\big(a^{2n}s\big)\right|^2 \le B(a) < \infty ,$$

where, as $a \to 1$,

$$A(a) = \frac{I}{2\log a}\left(1 - O\left((a-1)^2 \Big|(\log|a-1|)\Big|\right)\right) ,$$
$$B(a) = \frac{I}{2\log a}\left(1 + O\left((a-1)^2 \Big|(\log|a-1|)\Big|\right)\right) .$$

Proof. Define a new function $G : \mathbb{R} \to \mathbb{R}$ by

$$G(u) = \left|F\big(e^u\big)\right|^2 = \left|e^u H\big(e^u\big)\right|^2 ;$$

then

$$G \in \mathcal{S}(\mathbb{R}), \text{ and } \quad |G(u)| \le Ke^{-2|u|}$$

for some constant K.

If we put $t = e^u$ in Calderón's identity (7.22), and also write $s = e^v$, that identity becomes the simpler identity

$$\int_{-\infty}^{\infty} G(u+v)\, du = I$$

(independent of v). If we again put $s = e^v$, and now write $a^2 = e^c$, we see that the sum we need to estimate has the simpler form

$$\sum_{n=-\infty}^{\infty} \left|F\big(a^{2n}s\big)\right|^2 = \sum_{n=-\infty}^{\infty} G(nc+v) . \tag{7.23}$$

Since the sum on the right side of (7.23) is periodic with period c, the sum need only be estimated for $0 \leq v \leq c$. Since we are letting $a \to 1^+$, we may assume $0 < c = 2\log a < \frac{1}{e}$.

We note that $c\sum_{n=-\infty}^{\infty} G(nc+v)$ is a Riemann sum for the integral $\int_{-\infty}^{\infty} G(u+v)\,du = I$. To estimate the difference, we recall the midpoint rule: Say f is C^2 in a neighborhood of $[a,b]$. Divide $[a,b]$ into n intervals of equal length $\Delta x = \frac{b-a}{n}$ and let x_k^* be the midpoint of the kth interval. Let

$$E = \left| \int_a^b f(x)\,dx - \sum_{k=1}^{n} f\left(x_k^*\right)\Delta x \right| .$$

Then

$$E \leq \frac{1}{24}\left\| f'' \right\|_\infty (b-a)(\Delta x)^2 .$$

Thus, there is a constant $P > 0$ such that whenever $0 \leq v \leq c < \frac{1}{e}$, and whenever $N > 0$ is an integer,

$$\begin{aligned}
&\left| c\sum_{n=-\infty}^{\infty} G(nc+v) - I \right| \\
&\quad = \left| c\sum_{n=-\infty}^{\infty} G(nc+v) - \int_{-\infty}^{\infty} G(u+v)\,du \right| \\
&\quad \leq \left| c\sum_{n=-N}^{N} G(nc+v) - \int_{-(Nc+\frac{c}{2})}^{Nc+\frac{c}{2}} G(u+v)\,du \right| \\
&\qquad + c\sum_{|n|>N} G(nc+v) + \int_{|u|>Nc+\frac{c}{2}} G(u+v)\,du \\
&\quad \leq \frac{1}{24}\left\| G'' \right\|_\infty [(2N+1)c]c^2 + 2Ke^{-2(N+1)c}e^{2v}\frac{c}{1-e^{-2c}} + Ke^{-(2N+1)c}e^{2v} \\
&\quad \leq P\left(Nc^3 + e^{-2Nc}\right) .
\end{aligned}$$

Note that for $x > e$, $x\log x > e\log e > 1$. Since we are assuming $\frac{1}{c} > e$, there is an integer N with

$$\frac{\log\left(\frac{1}{c}\right)}{c} < N < \frac{2\log\left(\frac{1}{c}\right)}{c} .$$

Using such an N we see that

$$\left| \sum_{n=-\infty}^{\infty} \left|F\left(a^{2n}s\right)\right|^2 - \frac{I}{c} \right| = \left| \sum_{n=-\infty}^{\infty} G(nc+v) - \frac{I}{c} \right| \leq P\left(2c\log\left(\frac{1}{c}\right) + c\right) \leq 3Pc\log\left(\frac{1}{c}\right) .$$

Accordingly $\sum_{n=-\infty}^{\infty} |F(a^{2n}s)|^2$ is between $(\frac{I}{c})(1 \pm Qc^2|\log c|)$, where $Q = \frac{3P}{I}$. Since $c = 2\log a$, and since $\frac{\log a}{a-1} \to 1$ as $a \to 1^+$, we have completed the proof of Lemma 6, and, with it, the proof of Theorem 2. □

Example. Daubechies ([6], especially p. 77 and pp. 71–72), calculated for instance, that if $a = 2^{\frac{1}{3}}$, $\psi(x) = c(1-x^2)e^{-\frac{x^2}{2}}$ (for $x \in \mathbb{R}$; here $c \neq 0$ could be chosen arbitrarily), $B = \sup_{\xi>0} m_{\hat{\psi},a}(\xi)$, $A = \inf_{\xi>0} m_{\hat{\psi},a}(\xi)$, then $\frac{B}{A} = 1.0000$ to four significant digits.[1]

In that case ψ is a multiple of the second derivative of $e^{-\frac{x^2}{2}}$, so $\hat{\psi}(\xi) = c'\xi^2 e^{-\frac{\xi^2}{2}}$. Again c' is nonzero and arbitrary; let us now take it to be 1. If we let $F(\lambda) = \hat{\psi}(\sqrt{\lambda}) = \lambda e^{-\frac{\lambda}{2}}$ (essentially[2] making the change of variables $\lambda = \xi^2$), then

$$F(L)\delta = Le^{-L/2}\delta = \Psi, \text{ say },$$

and

$$m_{F,a^2}(\lambda) = m_{\hat{\psi},a}\big(\sqrt{\lambda}\big),$$

so that $\sup_{\lambda>0} m_{F,a^2}(\lambda) = \sup_{\xi>0} m_{\hat{\psi},a}(\xi) = B$, say, and $\inf_{\lambda>0} m_{F,a^2}(\lambda) = \inf_{\xi>0} m_{\hat{\psi},a}(\xi) = A$, say. By the aforementioned calculation of Daubechies, $\frac{B}{A} = 1.0000$ to four significant digits. Thus, by Theorem 8, we can choose $b_0 > 0$ such that $\{\Psi_{j,b\gamma}\}$ is a frame whenever $0 < b < b_0$, with frame bounds A_b, B_b and such that, moreover, $\frac{B_b}{A_b} = 1.0000$ to four significant digits.

Ψ is, up to a constant multiple, a natural generalization of the Mexican Hat wavelet to G.

8. Frames in Other Banach Spaces

In this section we discuss the invertibility of $S_{\psi,\psi,b}$ on other Banach spaces, such as L^p or H^1. Let us clarify which Banach spaces we can allow.

Definition 3. We call a Banach space $\mathcal{B}$ of measurable functions on G *acceptable* if $L^2 \cap \mathcal{B}$ is dense in $\mathcal{B}$, $\mathcal{B} \subseteq \mathcal{S}'$ (continuous inclusion), and if the following condition holds:

There exist $C_0, N > 0$, such that for any $A_0 > 0$, we have the following. Whenever $T : L^2 \to L^2$ is linear and satisfies:

(i) The operator norm of T on L^2 is less than or equal to A_0;

(ii) There is a kernel $K(x, y)$, C^1 off the diagonal, such that if $f \in C_c^1$, then for x outside the support of f, $(Tf)(x) = \int K(x, y) f(y)\, dy$; and whenever $0 \le \alpha_1 + \ldots + \alpha_n + \beta_1 + \ldots + \beta_n \le 1$, and whenever $x, y \in G$ with $x \neq y$, we have

$$\left|X_x^\alpha X_y^\beta K(x, y)\right| \le A_0 \left|y^{-1}x\right|^{-(Q+|\alpha|+|\beta|)}; \text{ and} \tag{8.1}$$

(iii) $T^*(1) = 0$;

then $T|_{L^2\cap\mathcal{B}}$ extends to a bounded operator on $\mathcal{B}$, with norm $\|T\| \le C_0 A_0$.

Surely [3, 4] L^p $(1 < p < \infty)$ and H^1 are acceptable Banach spaces. In this section, we shall show the following.

[1] Actually, Daubechies took a specific value of c, but clearly that is irrelevant in computing B/A. Also, in her table on p. 77 of [6], her B/A is larger than $\sup_{\xi>0} m_{\hat{\psi},a}(\xi)/\inf_{\xi>0} m_{\hat{\psi},a}(\xi)$, [see her Equations (3.3.19) and (3.3.20)], but that is an even stronger assertion than the one we are making.

[2] If $G = \mathbb{R}$ we are of course passing from the spectral resolution of d/dx to that of d^2/dx^2.

Theorem 9. *Suppose $H \in \mathcal{S}(\mathbb{R}^+)$, $F(\lambda) = \lambda H(\lambda)$, and that F satisfies Daubechies' criterion* (*i.e., that* $\inf_{\lambda>0} m_{F,a^2}(\lambda) > 0$). *Let $\psi = F(L)\delta$. Suppose $\mathcal{B}$ is acceptable, and that $\psi_{j,b\gamma} \in \mathcal{B}$ for all $j \in \mathbb{Z}$ and $0 < b < 1$. Then:*

(a) *For some $b_0 > 0$, $S_{\psi,\psi,b}$ is invertible on $\mathcal{B}$ whenever $0 < b < b_0$. Suppose now that $0 < b < b_0$.*

(b) *Suppose that for some dense subspace $\mathcal{D}$ of $\mathcal{B}$, the series $\sum_{j,\gamma}\langle f, \psi_{j,b\gamma}\rangle \psi_{j,b\gamma}$ converges unconditionally to $S_{\psi,\psi,b}f$ in $\mathcal{B}$ for all $f \in \mathcal{D}$. Then this series converges unconditionally to $S_{\psi,\psi,b}f$ for all $f \in \mathcal{B}$. Moreover, if we let $\phi^{j,b\gamma} = S^{-1}_{\psi,\psi,b}\psi_{j,b\gamma}$, then for any $f \in \mathcal{B}$,*

$$f = \sum_{j,\gamma}\langle f, \psi_{j,b\gamma}\rangle \phi^{j,b\gamma} ,$$

where the series converges unconditionally to f in $\mathcal{B}$. In particular, the set $\{\phi^{j,b\gamma}\}$ is a complete system in $\mathcal{B}$ (*i.e., the closure of the linear span of this set is all of $\mathcal{B}$*).

(c) *The hypotheses, and hence the conclusion, of (b) hold if $\mathcal{B} = L^p$ $(1 < p < \infty)$ or H^1. Here we may take $\mathcal{D} = C_c^\infty \cap \mathcal{B}$.*

Proof. We retain all the notation of the proofs of Theorems 7 and 8.

For (a), by Theorem 7 and Definition 3, $S_{\psi,\psi,b}|_{L^2\cap\mathcal{B}}$ extends to a bounded operator on $\mathcal{B}$. Also, by Proposition 8, Theorem 7 (f), the second last sentence of the proof of Theorem 7 (f), and Definition 3, we have that $R_\psi|_{L^2\cap\mathcal{B}}$ extends to a bounded operator on $\mathcal{B}$. Further, by Theorem 8 (a) and Minkowski's inequality, there exists $C > 0$ such that for all $0 < b < 1$, the norm on $\mathcal{B}$

$$\left\| \frac{1}{Vb^Q} R_\psi - S_{\psi,\psi,b} \right\| \le \frac{C}{b^{Q-1}} .$$

To prove (a), it suffices to show that R_ψ is invertible on $\mathcal{B}$. Indeed, say this were known. For (a), it is clearly enough to show that the operator

$$L_b = Vb^Q R_\psi^{-1} S_{\psi,\psi,b}$$

is invertible on $\mathcal{B}$ for all sufficiently small b. But this is clear, since

$$\|I - L_b\| \le Cb \left\| R_\psi^{-1} \right\|$$

which is less than 1 if b is sufficiently small.

So it is enough to show that R_ψ is invertible on $\mathcal{B}$. By (7.20), $R_\psi \equiv m_F(L)$ on L^2. By Daubechies' criterion, $\frac{1}{m_F} = G$, say, is a bounded function on $\mathbb{R}^+$, so surely, by the spectral theorem, the inverse of R_ψ on L^2 is $G(L)$. It suffices then to show that $G(L)$, restricted to $L^2 \cap \mathcal{B}$, has an extension to a bounded operator on $\mathcal{B}$. (Indeed, we would then know that $m_F(L)G(L)f = G(L)m_F f = f$ for all $f \in L^2 \cap \mathcal{B}$, so this would hold for all $f \in \mathcal{B}$ and $m_F(L)$ would be invertible on $\mathcal{B}$.) It suffices then to show that $T = G(L)$ satisfies (i), (ii), and (iii) of Definition 3, for some $A_0 > 0$. Surely (i) is satisfied.

First note that $m_F(\lambda)$ is smooth for $\lambda > 0$. Indeed, if $V = |H|^2$, then $V \in \mathcal{S}(\mathbb{R}^+)$, and

$$m_F(\lambda) = \sum_{j\in\mathbb{Z}} a^{4j}\lambda^2 V\left(a^{2j}\lambda\right) .$$

Since V and all its derivatives are bounded and decay rapidly at ∞, the smoothness of m_F follows at once.

Thus, $G \in C^\infty((0,\infty))$. If $\lambda > 0$, choose l with $a^{2l} \leq \lambda \leq a^{2(l+1)}$. Surely $G(\lambda) = G(a^{-2l}\lambda)$, so for any k

$$\left|G^{(k)}(\lambda)\right| = a^{-2kl}\left|G^{(k)}\left(a^{-2l}\lambda\right)\right| \leq a^{2k}\lambda^{-k}M ,$$

where $M = \max_{1\leq\lambda\leq a^2} |G^{(k)}(\lambda)|$. This shows that $\|\lambda^k G^k(\lambda)\|_\infty < \infty$ for any k. (ii) and (iii) now follow by the spectral multiplier theorem of Hulanicki-Stein [12], Theorem 6.25; see also [1]. (Indeed, by that theorem, $G(L) : H^1 \to H^1$, so (iii) holds. Also, in the terminology of [12], the proof of their Theorem 6.25 shows that $G(L)$ is given by convolution with a kernel of type $(0, r)$ for any r, so (ii) holds as well.) (a) is therefore established.

For (b), note that, since $L^2 \cap \mathcal{B}$ is dense in $\mathcal{B}$, and since $\mathcal{B} \subseteq \mathcal{S}'$ (continuous inclusion), the operators $S^{\mathcal{F}}_{\psi,\psi,b}$, acting on $L^2 \cap \mathcal{B}$, may be extended to operators on $\mathcal{B}$, where they are given by

$$S^{\mathcal{F}}_{\psi,\psi,b} f = \sum_{(j,\gamma)\in\mathcal{F}} \langle f, \psi_{j,b\gamma}\rangle \psi_{j,b\gamma} .$$

It suffices then to show that for some $C > 0$, the operator norms on $\mathcal{B}$ of $S^{\mathcal{F}}_{\psi,\psi,b}$ are all less than C, for all $\mathcal{F}$. But, during the proof of Theorem 7 [see the discussion after (7.12)], we have observed that the operator norms of $S^{\mathcal{F}}_{\psi,\psi,b}$ on L^2 are uniformly bounded in $\mathcal{F}$, and that the kernels $K = K^{\mathcal{F}}_{\psi,\psi,b}$ satisfy the inequality (8.1) for some A_0 independent of $\mathcal{F}$. This proves our assertion, by definition of acceptable Banach space.

For (c), first take $\mathcal{B} = L^p$, and say $f \in C_c^\infty$. By Theorem 7 (a):

(*) Say $\epsilon_1 > 0$. There is a finite set $\mathcal{F}_1 \subseteq \mathbb{Z} \times \Gamma$, such that for any finite set $\mathcal{G}$ with $\mathcal{F}_1 \subseteq \mathcal{G} \subseteq \mathbb{Z} \times \Gamma$, we have

$$\left\| S_{\psi,\psi,b} f - S^{\mathcal{G}}_{\psi,\psi,b} f \right\|_\infty < \epsilon_1 .$$

Moreover, since the $K^{\mathcal{F}}_{\psi,\psi,b}$ satisfy (8.1) uniformly in $\mathcal{F}$, the argument leading to (7.8) shows that there is a C such that $\left|[S^{\mathcal{G}}_{\psi,\psi,b} f](x)\right| \leq C|x|^{-Q}$ for all x and all finite $\mathcal{G}$. These facts imply that for any $\epsilon > 0$, and any number $0 < q < Q$, there is a finite set $\mathcal{F} \subseteq \mathbb{Z} \times \Gamma$, such that for any finite set $\mathcal{G}$ with $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathbb{Z} \times \Gamma$, we have

$$\left| S_{\psi,\psi,b} f - S^{\mathcal{G}}_{\psi,\psi,b} f \right| < \epsilon g_q . \tag{8.2}$$

(Here g_q is as in Lemma 5.) If now also q is also required to satisfy $q > Q/p$, so that $g_q \in L^p$, then in (8.2), $\|S_{\psi,\psi,b} f - S^{\mathcal{G}}_{\psi,\psi,b} f\|_p < \epsilon\|g_q\|_p$. The unconditional convergence in L^p, for $f \in C_c^\infty$, follows at once.

Finally, in (c), take $\mathcal{B} = H^1$; then

$$\mathcal{D} = C_c^\infty \cap H^1 = \left\{ f \in C_c^\infty : \int f = 0 \right\} .$$

We define a standard molecule to be an L^2 function M with $\|M\|_2 \leq 1$, $\int |M(x)|^2 |x|^{Q+1} \leq 1$, and $\int M = 0$. In [4], it is shown that $M \in H^1$, and further that there is an $A_0 > 0$ such that $\|M\|_{H^1} \leq A_0$ for all standard molecules M.

Suppose now $f \in \mathcal{D}$. Combining (*) above and (7.10), we see that for any $\epsilon > 0$, and any number $0 < q < Q+1$, there is a finite set $\mathcal{F} \subseteq \mathbb{Z} \times \Gamma$, such that for any finite set $\mathcal{G}$ with $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathbb{Z} \times \Gamma$, we have

$$\left| S_{\psi,\psi,b} f - S^{\mathcal{G}}_{\psi,\psi,b} f \right| < \epsilon g_q \ . \tag{8.3}$$

If now also q is also required to satisfy $q > Q + \frac{1}{2}$, then $\max\left(\|g_q\|_2, \left[\int |g_q(x)|^2 |x|^{Q+1}\, dx\right]^{\frac{1}{2}}\right) = C_0$, say, is finite. Thus, in (8.3), $S_{\psi,\psi,b} f - S^{\mathcal{G}}_{\psi,\psi,b} f$ is ϵC_0 times a standard molecule, so its H^1 norm is less than $\epsilon C_0 A_0$. The unconditional convergence in H^1, for $f \in \mathcal{D}$, follows at once. □

9. Remarks

(1) When studying frames, one often takes several different ψs, say $\psi^1, \ldots, \psi^N$, all having integral zero, and asks when $\cup_{k=1}^N \{\psi^k_{j,b\gamma}\}$ is a frame. In our situation, by Theorem 8 (b),

$$\left\| \sum_{k=1}^N \left[\frac{1}{V b^Q} R_{\psi^k} - S_{\psi^k,\psi^k,b} \right] \right\| \le \frac{C}{b^{Q-1}} \ .$$

Thus, a simple modification of the proof of Theorem 8 (c) shows that, if we can find positive A, B with

$$AI \le \sum_{k=1}^N R_{\psi^k} \le BI \ ,$$

then for some $b_0 > 0$, if $0 < b < b_0$, we can choose $A_b, B_b > 0$ such that

$$A_b \|f\|_2^2 \le \sum_{k=1}^N \sum_{j\in\mathbb{Z},\gamma\in\Gamma} \left|\langle f, \psi^k_{j,b\gamma}\rangle\right|^2 \le B_b \|f\|_2^2$$

for all $f \in L^2$, and such that

$$\lim_{b\to 0^+} \frac{B_b}{A_b} = \frac{B}{A} \ .$$

We restrict attention to ψ^k of the form $\psi^k = F^k(L)\delta = LH^k(L)\delta$, where $F^k(\lambda) = \lambda H^k(\lambda)$ and $H^k, F^k \in \mathcal{S}(\mathbb{R}^+)$. Then $\sum_{k=1}^N R_{\psi^k} = \sum_{k=1}^N m_{F^k}(\delta)$, and we can take

$$B = \sup_{\lambda>0} \sum_{k=1}^N m_{F^k}(\lambda), \quad A = \inf_{\lambda>0} \sum_{k=1}^N m_{F^k}(\lambda) \ ,$$

provided this A is positive. (This will be so if there does not exist a $\lambda_0 > 0$ such that $F^k(a^{2j}\lambda_0) = 0$ for all k and all integers j.) With higher N, one has more flexibility in making $\sum_{k=1}^N m_{F^k}$ nearly constant, thereby getting a nearly tight frame.

(2) In this article, we have let L be the sub-Laplacian for simplicity, but the our main results (Theorem 1, Corollary 1, and Theorem 2) continue to hold if L is any positive Rockland operator. (In (1.1), one must change a^{2j} to a^{kj} where k is the homogeneous degree of L.) Indeed, the key fact that we have used about L is Theorem 3, and that theorem continues to hold if L is a positive Rockland operator. (See [22] for this fact and the definition of a Rockland operator.)

A. Appendix: T(1) Theorem Technicalities

As we have said, the "easier case" of the $T(1)$ theorem for stratified Lie groups, as used in this article, may be proved by making only minor changes in the proof for $\mathbb{R}^n$ in [30], pp. 293–300. However, one change requires a little thought.

Stein assumes that T is given to be a continuous linear mapping from $\mathcal{S}$ to $\mathcal{S}'$. He however only uses this assumption in the argument at the top of p. 296. Moreover, the argument at the top of p. 296 uses the Fourier transform. We need to present a replacement for that argument, for general G, in which only C_c^∞ functions are used.

For $R > 0$, let $B(0, R) = \{x \in G : |x| < R\}$. We begin by observing the following.

Proposition A.1. *Say $T : C_c^1(G) \to L^2(G)$ is linear and restrictedly bounded. Then T is continuous from C_c^∞ to L^2.*

Proof. By definition we need to show that for any compact set $K \subseteq G$, there exist C_0, N such that $\|Tf\|_2 \le C_0$ for all $f \in C_c^\infty$ with support contained in K and with $\|f\|_{C^N} \le 1$. We claim that we can always take $N = 1$. Indeed, fix K and choose $R > 0$ with $K \subseteq B(0, R)$. If f is as above, set $F(x) = cf(Rx)$, where $c = \min(1, R^{-a_1}, \ldots, R^{-a_n})$. Then F is a normalized bump function, and $f = (\frac{1}{c})F^{R,0}$. Since T is restrictedly bounded, for some $C > 0$, $\|Tf\|_2 \le \frac{C}{c} R^{\frac{Q}{2}}$, as desired. □

To replace the argument at the top of p. 296 in [30], we now proceed as follows. Say $\phi \in C_c^\infty(G)$ has support contained in the unit ball $B(0, 1)$. For $f \in L^2(G)$, let

$$S_j f = f * \phi_{2^{-j}} .$$

We claim the following.

Proposition A.2. *Suppose a linear operator $T : C_c^1(G) \to L^2(G)$ is restrictedly bounded. Then:*

(a) *For all $f \in C_c^\infty$, $S_j T S_j f \to Tf$ in L^2 as $j \to \infty$; and*
(b) *for all $f \in C_c^\infty$, $S_j T S_j f \to 0$ in L^2 as $j \to -\infty$.*

Proof. Of course S_j is bounded on L^2 for all j, and $\|S_j\| \le \|\phi_{2^{-j}}\|_1 = \|\phi\|_1 = A$, say. For (a), we observe

$$\begin{aligned}\|S_j T S_j f - Tf\|_2 &\le \|S_j T(S_j f - f)\|_2 + \|S_j Tf - Tf\|_2 \\ &\le A\|T(S_j f - f)\|_2 + \|S_j Tf - Tf\|_2 \longrightarrow 0\end{aligned}$$

as $j \to \infty$, since $S_j f \to f$ in C_c^∞, $T : C_c^\infty \to L^2$ is continuous, and $S_j Tf \to Tf$ in L^2.

For (b) we observe $\|S_j T S_j f\|_2 \le A\|T S_j f\|_2$, so we need only show $T S_j f \to 0$ in L^2. Write $J = -j$, and note

$$S_j f = f * \phi_{2^J} = \left(f_{2^{-(J+1)}} * \phi_{2^{-1}}\right)_{2^{J+1}} .$$

As $J \to \infty$, $f_{2^{-(J+1)}} * \phi_{2^{-1}} \to \phi_{2^{-1}}$ in C_c^∞, where $c = \int_G f$; moreover, for J sufficiently large the supports of all these functions are contained in the unit ball. Thus, we may choose C_1 such that for J sufficiently large, any one of these functions is C_1 times a normalized bump function. But for any function F,

$$F_{2^{J+1}} = 2^{-(J+1)} F^{2^{J+1},0} ,$$

so $\|TS_j f\|_2 \leq CC_1 2^{-\frac{J+1}{2}} \to 0$ as $J \to \infty$, as desired. □

Another very small point: We have defined a normalized bump function to be a C^1 function with support contained in the unit ball, whose C^1 norm is less than or equal to 1; Stein assumes in addition that the function is smooth. But our definition only makes the hypotheses of Theorems 5 and 6 stronger, so of course the theorems hold.

References

[1] Christ, M. (1991). L^p bounds for spectral multipliers on nilpotent groups, *Trans. Amer. Math. Soc.* **328**, 73–81.

[2] Christensen, O. (2002). *An Introduction to Frames and Riesz Bases,* Birkhäuser.

[3] Coifman, R. and Weiss, G. (1971). Analyse harmonique noncommutative sur certains espaces homogenes, *Lecture Notes in Math.* **242**, Springer-Verlag, Berlin and New York.

[4] Coifman, R. and Weiss, G. (1989). Extensions of Hardy space and their use in analysis, *Bull. Amer. Math. Soc. (N. S.)* **83**, 569–645.

[5] Corwin, L. and Greenleaf, F. P. (1977). *Representations of Nilpotent Lie Groups and Their Applications,* Cambridge University Press, Cambridge.

[6] Daubechies, I. (1992). *Ten Lectures on Wavelets,* SIAM, Philadelphia, Pennsylvania.

[7] Daubechies, I., Grossman, A., and Meyer, Y. (1986). Painless nonorthogonal expansions, *J. Math. Phys.* **27**, 1271–1283.

[8] David, G. and Journé, J. L. (1984). A boundedness criterion for generalized Calderón-Zygmund operators, *Ann. Math.* **120**, 371–397.

[9] Duffin, R. J. and Schaeffer, A. C. (1952). A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.* **72**, 341–366.

[10] Feichtinger, H. G. and Gröchenig, K. (1989). Banach spaces related to integrable group representations and their atomic decompositions, I, *J. Funct. Anal.* **86**, 307–340.

[11] Feichtinger, H. G. and Gröchenig, K. (1989). Banach spaces related to integrable group representations and their atomic decompositions, II, *Mh. Math.* **108**, 129–148.

[12] Folland, G. B. and Stein, E. M. (1982). *Hardy Spaces on Homogeneous Groups,* Mathematical Notes 28, Princeton University Press.

[13] Frazier, M., Jawerth, B., and Weiss, G. (1991). *Littlewood-Paley Theory and the Study of Function Spaces,* CBMS Reg. Conf. Series in Math., No. 79, Amer. Math. Soc., Providence, RI.

[14] Führ, H. (2005). Abstract harmonic analysis of continuous wavelet transforms, *Lecture Notes in Math.* **1863**, Springer Verlag.

[15] Geller, D. (1980). Fourier analysis on the Heisenberg group, I, Schwartz space, *J. Funct. Anal.* **36**, 205–254.

[16] Geller, D. (1983). Liouville's theorem for homogeneous groups, *Comm. Partial Differential Equations* **8**, 1665–1677.

[17] Grafakos, L. and Li, X. (2000). Bilinear operators on homogeneous groups, *J. Operator Theory* **44**, 63–90.

[18] Gilbert, J. E., Han, Y. S., Hogan, J. A., Lakey, J. D., Weiland, D., and Weiss, G. (2002). Smooth molecular decompositions of functions and singular integral operators, *Mem. Amer. Math. Soc.* **156**(742).

[19] Gröchenig, K. (1991). Describing functions: Atomic decompositions versus frames, *Mh. Math.* **112**, 1–41.

[20] Han, Y. S. (2000). Discrete Calderón-type reproducing formula, *Acta Math. Sin. (Engl. Ser.)* **16**, 277–294.

[21] Hörmander, L. (1967). Hypoelliptic second-order differential equations, *Acta Math.* **119**, 147–171.

[22] Hulanicki, A. (1984). A functional calculus for Rockland operators on nilpotent Lie groups, *Stud. Math.* **78**, 253–266.

[23] Jerison, D. and Sanchez-Calle, A. (1986). Estimates for the heat kernel for a sum of squares of vector fields, *Indiana Univ. Math. J.* **35**, 835–854.

[24] Lemarié, P. G. (1984). *Algèbres D'opérateurs et Semi-Groupes de Poisson sur un Espace de Nature Homogène,* Université de Paris-Sud, Département de Mathématiques, Orsay.

[25] Lemarié, P. G. (1989). Base d'ondelettes sur les groupes stratifiés, *Bull. Soc. Math. France* **117**, 211–232.

[26] Lemarié, P. G. (1991). Wavelets, spline interpolation and Lie groups, *Harmonic Analysis (Sendai, 1990)*, Springer, Tokyo, 154–164.

[27] Liu, H. and Peng, L. (1997). Admissible wavelets associated with the Heisenberg group, *Pacific J. Math.* **180**, 101–123.

[28] Maggioni, M. (2004). Wavelet frames on groups and hypergroups via discretization of Calderón formulas, *Monatshefte für Mathematik,* **143**, 299–331.

[29] Mayeli, A. (2005). Discrete and continuous wavelet transformation on the Heisenberg group, PhD thesis, Technische Universität München.

[30] Stein, E. M. (1993). *Harmonic Analysis,* Princeton Mathematical Series 45, Princeton University Press.

[31] Varopoulos, N. (1988). Analysis on Lie groups, *J. Funct. Anal.* **76**, 346–410.

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