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On the Mattila Integral Associated with Sign Indefinite Measures

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ABSTRACT. *In order to quantitatively illustrate the rôle of positivity in the Falconer distance problem, we construct a family of sign indefinite, compactly supported measures in* $\mathbb{R}^d$, *such that their Fourier transform and Fourier energy of dimension* $s \in (0, d)$ *are uniformly bounded. However, the Mattila integral, associated with the Falconer distance problem for these measures is unbounded in the range* $0 < s < \frac{d^2}{2d-1}$.

1. Introduction

Let μ be a compactly supported Borel measure in $\mathbb{R}^d$, $d \geq 2$. Suppose that μ is s-dimensional in the sense that its energy integral

$$I_s(\mu) = \int\int |x-y|^{-s}\, d\mu(x)\, d\mu(y) < \infty\,. \tag{1.1}$$

The question that arises often in geometric measure theory and related areas is to determine the rate of decay of the spherical average

$$\sigma_\mu(t) = \int_{S^{d-1}} \left|\widehat{\mu}(t\omega)\right|^2 d\omega, \quad t \geq 1\,. \tag{1.2}$$

The quantity $\sigma_\mu(t)$ plays the central rôle in restriction theory as well as the study of distance sets. See, for example, [9] and the references contained therein for background. Let us point out one important geometric context where the quantity $\sigma_\mu(t)$ arises.

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The Falconer distance conjecture [2] says that if the Hausdorff dimension of a compact set $E \subset \mathbb{R}^d$ is greater than $\frac{d}{2}$, then the Lebesgue measure of the distance set $\Delta(E)$ is positive, with

$$\Delta(E) = \{|x - y| : x, y \in E\} ,$$

where $|\cdot|$ is the standard Euclidean distance. Wolff [8] in dimension $d = 2$ and Erdoğan [1] in all $d \geq 2$ proved that the Lebesgue measure of $\Delta(E)$ is indeed positive if the Hausdorff dimension of E is greater than $\frac{d}{2}+\frac{1}{3}$. The proofs invoke highly nontrivial harmonic analysis, whereas a weaker exponent $\frac{d}{2}+\frac{1}{2}$ can be justified in a considerably more straightforward way. See e.g., [2, 4], and [5]. In fact, for a restricted class of measures, the "threshold dimension" $\frac{d}{2}+\frac{1}{3}$ has been improved by the authors [3] to $\frac{d^2}{2d-1}$, which we also expect to be true in the general case. The latter exponent appears in the present note as well, supposedly as an artifact of the construction we use. This suggests strongly that to exceed this exponent, positivity of the measure must be used in a very substantial way.

Mattila [4] initiated the L^2 approach to the Falconer distance problem, aiming to prove that there exists a positive probability Borel measure μ supported on E, such that

$$M(\mu) = \int_1^\infty \sigma_\mu^2(t) t^{d-1}\, dt \;<\; \infty\,. \tag{1.3}$$

We refer to the quantity $M(\mu)$ as the *Mattila integral.* If $M(\mu)$ is uniformly bounded, the Lebesgue measure of $\Delta(E)$ is positive. This represents the second moment argument, based on the fact that if ν is the (properly weighted) pull-forward of the measure $\mu \times \mu$ under the distance map $E \times E \to \mathbb{R}_+$, then the one-dimensional Fourier transform $\widehat{\nu}$ on $\mathbb{R}_+$ (coming from using polar coordinates) is such that

$$\widehat{\nu}(t) \;=\; C_d\, t^{\frac{d-1}{2}} \sigma_\mu(t)\,. \tag{1.4}$$

In the sequel, constants will be often buried in the notations $\lesssim$, $\gtrsim$ and $\approx$ in the usual way. These constants may depend on the dimension, the diameter of the support of μ, etc, and sometimes are written out explicitly as C, C_d, C_μ, c, with their values changing without warning.

It follows from (1.4) that if (1.3) is true, the distance measure ν, generated by μ, has an L^2 density, and its support has positive Lebesgue measure by the Cauchy-Schwartz inequality.

In order to prove (1.3), all of the aforementioned references followed Mattila [4] who sought to establish a point-wise estimate in the form

$$\sigma_\mu(t) \;\leq\; C_\mu t^{-\beta}, \quad t \geq 1\,, \tag{1.5}$$

for some $\beta \leq s$, cf. (1.1). If this is the case, polar coordinates and Plancherel imply that

$$\begin{aligned} M(\mu) &\lesssim \int \left|\widehat{\mu}(\xi)\right|^2 |\xi|^{-d+(d-\beta)}\, d\xi \\ &= C_d \int\int |x-y|^{-(d-\beta)}\, d\mu(x)\, d\mu(y) \\ &\approx I_{d-\beta}(\mu)\,, \end{aligned}$$

the energy integral of μ of order $(d-\beta)$. This integral is bounded if $d-\beta$ is smaller than the Hausdorff dimension of E, denoted by s.

In order to motivate the main result of this note we need a few bits of notation. First, for a not necessarily positive Schwartz class function f, define

$$\sigma_f(t) = \int_{S^{d-1}} \left|\widehat{f}(t\omega)\right|^2 d\omega, \quad t \geq 1\,. \tag{1.6}$$

The spherical averages of Fourier transforms for general sign-indefinite measures have recently been addressed in a number of articles. See, for example, [6, 7] and the references contained therein.

Define

$$FI_s(f) \;=\; \int \left|\widehat{f}(\xi)\right|^2 |\xi|^{-d+s}\, d\xi \;=\; C_d \int_0^\infty \sigma_f(t) t^{s-1}\, dt\ . \tag{1.7}$$

Observe that in the case when f defines a positive measure in $\mathbb{R}^d$, the quantities $FI_s(f)$ and $I_s(f)$ are the same up to a constant. See, for example, [9] for a detailed proof. Similarly, the fact that the Fourier transform of f is bounded clearly implies that f has bounded variation, which is not the case in the sign-indefinite case.

The following construction-based theorem indicates that without the positivity assumption, even though the "Fourier energy" integral $FI_s(f)$ is bounded, the Mattila integral $M(f)$ corresponding to the measure $d\mu = f\,dx$ may behave quite badly in the sense that it will diverge not only for small s, but also in some range of $s > \frac{d}{2}$ where the Falconer conjecture is supposed to hold. This emphasizes the rôle of positivity in the context of the latter conjecture, which turns out to be crucial even though the Fourier transform of f in the construction below is positive and uniformly bounded, as in the lattice based examples explored previously [2, 3]. Our main result is the following.

Theorem 1. *For any $0 < s < \frac{d^2}{2d-1}$, there exists a one-parameter family $\mathcal{F}_q$ of compactly supported C^∞ functions, with support contained in a ball of radius $\lesssim 1$, such that for $f_{q,s} \in \mathcal{F}$, $|\widehat{f}_{q,s}|$ is uniformly bounded, $FI_s(f_{q,s}) \approx 1$, while $M(f_{q,s}) \to \infty$ as $q \to \infty$.*

We note that the assumptions of Theorem 1 are meant to mimic the positive case. More precisely, the condition that $FI_s(f) \approx 1$ says in the positive measure context that f is supported on a set of Hausdorff dimension of at least s. In particular, if we start out with a positive probability Borel measure μ supported on a set of positive Hausdorff dimension, then many reasonable localizations of μ on the Fourier transform side lead to functions satisfying the assumptions of Theorem 1. Our desire to understand the behavior of such functions in the context of the Falconer distance problems led us to the construction behind Theorem 1.

Remark 1 (C. Thiele, private communication). If we are willing to relax the conditions in Theorem 1, a simpler construction can be used to yield a similar conclusion. Let $\widehat{f_T}(\xi) = |\xi|^{\frac{d-s}{2}}\phi(\xi - T\omega_0)$, where ϕ further denotes a fixed nonnegative radial Schwartz-class bump function supported in the unit ball, T is a large positive real parameter, and ω_0 is a fixed vector of modulus 1. Then clearly

$$FI_s(f_T) \;=\; \int \left|\widehat{f_T}(\xi)\right|^2 |\xi|^{-d+s}\, d\xi \approx 1\ ,$$

while

$$\int_{S^{d-1}} \left|\widehat{f_T}(t\omega)\right|^2 d\omega \;\approx\; T^{1-s}\chi_{[T-1,T+1]}(t)\ .$$

This leads immediately to the lower bound

$$M(f_T) \;\gtrsim\; T^{-2s+2} \int_{T-1}^{T+1} t^{d-1}\, dt \;\approx\; T^{-2s+d+1} \to \infty, \quad \text{as } \; T \to \infty\ ,$$

provided that $s < \frac{d+1}{2}$, a stronger conclusion than the one offered by Theorem 1. However, this example clearly does not satisfy the assumption that $\widehat{f_T}$ is uniformly bounded.

2. A Lower Bound for the Spherical Average

Let $q \gg 1$ and the family $\mathcal{F}$ contain functions $f_{q,s}$, defined by the following relation:

$$\widehat{f}_{q,s}(\xi) = \widehat{\phi}\big(q^{-\frac{d}{s}}\xi\big) \sum_{a\in A} \widehat{\phi}(aq-\xi) ,$$

where ϕ is the radial cut-off function as above; let us also assume that the Fourier transform $\widehat{\phi}$ is nonnegative. Furthermore, A is a Delone (alias Delaunay, or well-distributed) set in the sense that elements of A are separated by at least some C^{-1}, while any cube of side length C in $\mathbb{R}^d$ contains some element of A. The function $f_{q,s}$, as well as the constants that are not written out explicitly in the estimates below, depend on the specific Delone set A to be described, as well as the parameters q, s. We shall need the following estimates.

Lemma 1. *For every Delone set A, all $s \in (0,d)$ and $q \gg 1$, the function $f_{q,s}$ is supported in the ball of radius 2, and $|\widehat{f}_{q,s}|$ is uniformly bounded.*

Lemma 2. *For every Delone set A,*

$$FI_s(f_{q,s}) \approx 1 , \tag{2.1}$$

with constants independent of q.

Let us take Lemma 1 and 2 for granted for the moment and obtain a lower bound for the spherical average $\sigma_{f_{q,s}}(t)$ (which will also serve as an upper bound). We have, for some $0 < c = O(1)$ that

$$\big|\widehat{f}_{q,s}(\xi)\big|^2 \;=\; c\big|\widehat{\phi}\big(q^{-\frac{d}{s}}\xi\big)\big|^2 \sum_{a\in A} B_{a,\frac{c}{q}}(\xi/q) + R_q , \tag{2.2}$$

where $B_{x_0,r}(x)$ denotes the characteristic function of the ball of radius r centered at x_0. Since the Fourier transform of ϕ is rapidly decaying and, in fact, nonnegative, the term $R_q \geq 0$ is easily seen to be $O(1)$, being actually handled by the same argument that we use to estimate the main term below.

Since ϕ is radial, let ϕ_0 be the function of one variable such that $\phi(x) = \phi_0(|x|)$, with $\widehat{\phi_0}$ (with a slight abuse of notation) denoting a Schwartz class nonnegative function of one variable, such that $\widehat{\phi}(\xi) = \widehat{\phi_0}(|\xi|)$. Clearly, $\phi_0(|\xi|)$ vanishes rapidly as $|\xi| \to \infty$. It follows that

$$\begin{aligned} \int_{S^{d-1}} \big|\widehat{f}_{q,s}(t\omega)\big|^2 d\omega \;&\gtrsim\; \big|\widehat{\phi_0}\big(q^{-\frac{d}{s}}t\big)\big|^2 \textstyle\sum_{a\in A} \int_{S^{d-1}} B_{a,\frac{1}{q}}(t\omega/q)\, d\omega \\ &\approx\; \big|\widehat{\phi_0}\big(q^{-\frac{d}{s}}t\big)\big|^2 t^{1-d} \textstyle\sum_{a\in A:\, \frac{t}{q}\leq|a|\leq\frac{t}{q}+\frac{1}{q}} 1 . \end{aligned} \tag{2.3}$$

At this point, in order to get the desired family $\mathcal{F}$, we consider a special Delone set A with the properties we need. Let A contain the origin, plus on each sphere of integer radius m we place $\approx m^{d-1}$ 1-separated points of A. It follows that whenever t is a positive integer multiple of q, we have

$$\sum_{\frac{t}{q}\leq|a|\leq\frac{t}{q}+\frac{1}{q}} 1 \approx \left(\frac{t}{q}\right)^{d-1} , \tag{2.4}$$

so for $n = 1, 2, \ldots$, all $\tau \in (-c, c)$, and all $t = nq + \tau$,

$$\int_{S^{d-1}} \left|\widehat{f}_{q,s}(t\omega)\right|^2 d\omega \;\gtrsim\; \left|\phi_0\big(q^{-\frac{d}{s}}t\big)\right|^2 \frac{1}{q^{d-1}} \,, \tag{2.5}$$

in particular whenever

$$nq \lesssim q^{\frac{d}{s}}, \quad \sigma_{f_{q,s}}(t) \gtrsim q^{1-d} \,. \tag{2.6}$$

Observe that choosing $q \approx t^{\frac{s}{d}}$, as we may, we obtain

$$\int_{S^{d-1}} \left|\widehat{f}_q(t\omega)\right|^2 d\omega \;\gtrsim\; t^{-s+\frac{s}{d}} \,, \tag{2.7}$$

for a sequence of ts going to infinity, tuned to a sequence $f_{q,s}$ of members of the family $\mathcal{F}$.

3. Proof of Lemma 1 and Lemma 2

To prove that $f_{q,s}$ in Lemma 1 is compactly supported, we write

$$\begin{aligned} f_{q,s}(x) &= \textstyle\sum_{a\in A} \int e^{2\pi i x\cdot\xi} \widehat{\phi}\big(q^{-\frac{d}{s}}\xi\big)\widehat{\phi}(aq-\xi)\, d\xi \\ &= \textstyle\sum_{a\in A} \int\int e^{2\pi i(x-y)\cdot\xi} \widehat{\phi}\big(q^{-\frac{d}{s}}\xi\big)\, d\xi\, e^{2\pi i q a\cdot y}\phi(y)\, dy \\ &= q^{\frac{d^2}{s}} \textstyle\sum_{a\in A} \int e^{2\pi i q a\cdot y} \phi\big(q^{\frac{d}{s}}(x-y)\big)\phi(y)\, dy \end{aligned} \tag{3.1}$$

from which it is apparent that f vanishes identically outside the ball of radius 2.

To prove Lemma 2, let us use (2.2) once again, and by the standard cut-off argument the term R_q representing the rapidly vanishing "tails" of the nonnegative $\widehat{\phi}(aq-\xi)$ for $|aq-\xi| \gtrsim 1$ will satisfy the same bound. The principal contribution into the quantity $FI_s(f_{q,s})$ comes from the term

$$\begin{aligned} &\sum_{a\in A} \int \left|\widehat{\phi}\big(q^{-\frac{d}{s}}\xi\big)\right|^2 B_{a,\frac{c}{q}}(\xi/q)|\xi|^{-d+s}\, d\xi \\ &\quad = C + \sum_{a\in A,\, a\neq 0} \int \left|\widehat{\phi}\big(q^{-\frac{d}{s}}\xi\big)\right|^2 B_{a,\frac{c}{q}}(\xi/q)|\xi|^{-d+s}\, d\xi \,, \end{aligned}$$

where the uniformly bounded quantity $C > 0$ comes from the term $a = (0, \ldots, 0)$ in the summation. On the other hand, the second term is nonnegative and bounded from above by

$$\begin{aligned} &C_N q^{-d+s} \textstyle\sum_{a\in A,\, a\neq 0} |a|^{-d+s} \int \big(1 + q^{-\frac{d}{s}}|\xi|\big)^{-N} B_{a,\frac{1}{q}}(\xi/q)\, d\xi \\ &\quad = C_N q^s \textstyle\sum_{a\in A,\, a\neq 0} |a|^{-d+s} \int \big(1 + q^{-\frac{d}{s}+1}|\xi|\big)^{-N} B_{a,\frac{1}{q}}(\xi)\, d\xi \\ &\quad \lesssim q^s q^{-d} \textstyle\sum_{1\leq |a|\leq q^{\frac{d}{s}-1}} |a|^{-d+s} \;\approx\; q^s q^{-d}\big(q^{\frac{d}{s}-1}\big)^s \;\approx 1 \,. \end{aligned}$$

As all the quantities involved are nonnegative, the quantity $FI_s(f_{q,s})$ is also bounded from below by some c, and the proof of Lemma 2 is complete.

4. Conclusion of the Proof of Theorem 1

We must finally estimate

$$\int_1^\infty \left(\int_{S^{d-1}} \left| \widehat{f_q}(t\omega) \right|^2 d\omega \right)^2 t^{d-1}\, dt\ .$$

From (2.5) and (2.7) we know that

$$\int_{S^{d-1}} \left| \widehat{f_q}(t\omega) \right|^2 d\omega \gtrsim \frac{1}{q^{d-1}}\ ,$$

whenever $\frac{t}{q}$ is an integer, and we can without loss of generality reckon that the estimate persists as t varies from mq to $mq+1$. It follows that

$$\begin{aligned} M(f) &\gtrsim \sum_{m=1}^{q^{\frac{d}{s}-1}} \int_{mq}^{(m+1)q} \left(\int_{S^{d-1}} \left| \widehat{f_q}(t\omega) \right|^2 d\omega \right)^2 t^{d-1}\, dt \\ &\gtrsim \sum_{m=1}^{q^{\frac{d}{s}-1}} \frac{1}{q^{2(d-1)}} (mq)^{d-1} \\ &\gtrsim q^{\frac{d^2}{s}-2d+1}\ . \end{aligned}$$

The power of q on the right-hand side is positive unless

$$s \geq \frac{d^2}{2d-1}\ ,$$

as claimed. This completes the proof of Theorem 1.

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