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Pairs of Explicitly Given Dual Gabor Frames in $L^2(\mathbb{R}^d)$

Ole Christensen and Rae Young Kim

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ABSTRACT. *Given certain compactly supported functions $g \in L^2(\mathbb{R}^d)$ whose $\mathbb{Z}^d$-translates form a partition of unity, and real invertible $d \times d$ matrices B, C for which $\|C^T B\|$ is sufficiently small, we prove that the Gabor system $\{E_{Bm}T_{Cn}g\}_{m,n\in\mathbb{Z}^d}$ forms a frame, with a (noncanonical) dual Gabor frame generated by an explicitly given finite linear combination of shifts of g. For functions g of the above type and arbitrary real invertible $d \times d$ matrices B, C this result leads to a construction of a multi-Gabor frame $\{E_{Bm}T_{Cn}g_k\}_{m,n\in\mathbb{Z}^d,k\in\mathcal{F}}$, where all the generators g_k are dilated and translated versions of g. Again, the dual generators have a similar form, and are given explicitly. Our concrete examples concern box splines.*

1. Introduction

For $y \in \mathbb{R}^d$, the translation operator T_y and the modulation operator E_y are defined by

$$\begin{aligned} (T_y f)(x) &= f(x-y), \quad x \in \mathbb{R}^d , \\ (E_y f)(x) &= e^{2\pi i y\cdot x} f(x), \quad x \in \mathbb{R}^d , \end{aligned}$$

where $y \cdot x$ denotes the inner product between y and x in $\mathbb{R}^d$. Given two real and invertible $d \times d$ matrices B and C we consider Gabor systems of the form

$$\{E_{Bm}T_{Cn}g\}_{m,n\in\mathbb{Z}^d} = \left\{e^{2\pi i Bm\cdot x} g(x-Cn)\right\}_{m,n\in\mathbb{Z}^d} .$$

Our purpose is to construct a class of Gabor frames with generators that are easy to use in practice, and having the additional property that we can find a dual generator of the form

$$h = \sum_{k\in\mathcal{F}} c_k T_k g$$

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for some finite set $\mathcal{F} \subset \mathbb{Z}^d$ and explicitly given scalar coefficients c_k. One advantage of this is that the decay of the dual generator h in the frequency domain is controlled by the decay of $\hat{g}$. Our results extend the one-dimensional results in [2]. As we will see, the extension is nontrivial: It is not clear from the one-dimensional version how one has to define the dual generators in higher dimensions.

Our approach is strongly connected with the results by Janssen [5, 6], Labate [7], Hernandez, Labate, and Weiss [4], and Ron and Shen [8, 9]. However, in contrast to these articles, the focus is on explicit constructions rather than general characterizations. For more information about Gabor systems and their role in time-frequency analysis we refer to the book [3] by Gröchenig; for general frame theory we refer to [1].

In the rest of the introduction we collect a few conventions about notation and a basic result for obtaining a pair of dual frames. The dilation operator associated with a real $d \times d$ matrix C is

$$(D_C f)(x) = |\det C|^{1/2} f(Cx), \quad x \in \mathbb{R}^d .$$

Let C^T denote the transpose of a matrix C; then

$$D_C E_y = E_{C^T y} D_C, \quad D_C T_y = T_{C^{-1} y} D_C .$$

If C is invertible, we use the notation

$$C^\sharp = \left(C^T\right)^{-1} .$$

For $f \in (L^1 \cap L^2)(\mathbb{R}^d)$ we denote the Fourier transform by

$$\mathcal{F} f(\gamma) = \hat{f}(\gamma) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \gamma}\, dx .$$

As usual, the Fourier transform is extended to a unitary operator on $L^2(\mathbb{R}^d)$. The reader can check that

$$\mathcal{F} T_{Ck} = E_{-Ck} \mathcal{F} .$$

We conclude the introduction by stating a special case of a result from [4]; it will form the basis for all the results presented in the article. Let

$$\mathcal{D} := \left\{ f \in L^2\left(\mathbb{R}^d\right) : \hat{f} \in L^\infty\left(\mathbb{R}^d\right) \text{ and } \operatorname{supp} \hat{f} \text{ is compact} \right\} .$$

Lemma 1. *Let B be an invertible $d \times d$ matrix, and let $\{g_n\}_{n \in \mathbb{Z}^d}$ and $\{h_n\}_{n \in \mathbb{Z}^d}$ be collections of functions in $L^2(\mathbb{R}^d)$. Assume that $\{T_{Bm} g_n\}_{m,n \in \mathbb{Z}^d}$ and $\{T_{Bm} h_n\}_{m,n \in \mathbb{Z}^d}$ are Bessel sequences and that for all $f \in \mathcal{D}$,*

$$\sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} \int_{\operatorname{supp} \hat{f}} \left|\hat{f}\left(\gamma + B^\sharp m\right)\right|^2 \left|\hat{g}_n(\gamma)\right|^2 d\gamma \quad < \quad \infty , \tag{1.1}$$

$$\sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} \int_{\operatorname{supp} \hat{f}} \left|\hat{f}\left(\gamma + B^\sharp m\right)\right|^2 \left|\hat{h}_n(\gamma)\right|^2 d\gamma \quad < \quad \infty . \tag{1.2}$$

Then $\{T_{Bm} g_n\}_{m,n \in \mathbb{Z}^d}$ and $\{T_{Bm} h_n\}_{m,n \in \mathbb{Z}^d}$ are dual frames for $L^2(\mathbb{R}^d)$ if and only if

$$\sum_{k \in \mathbb{Z}^d} \overline{\hat{g}_k\left(\gamma - B^\sharp n\right)} \hat{h}_k(\gamma) = |\det B| \delta_{n,0}, \quad a.e. \ \gamma ,$$

for all $n \in \mathbb{Z}^d$.

2. Dual Pairs of Gabor Frames

We first prove a time-domain version of Lemma 1 for Gabor systems. As we will see, we can remove the technical conditions (1.1) and (1.2) in the Gabor case. We begin with a lemma.

Lemma 2. *Let $g \in L^2(\mathbb{R}^d)$ and assume that B and C are invertible matrices. Then for all $f \in \mathcal{D}$,*

$$\sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d}\int_{\operatorname{supp}\hat{f}} \left|\hat{f}(\gamma + B^\sharp m)\right|^2 |g(\gamma - Cn)|^2\, d\gamma < \infty\,.$$

Proof. Let $f \in \mathcal{D}$. Then

$$\sum_{m\in\mathbb{Z}^d}\left|\hat{f}(\gamma + B^\sharp m)\right|^2 \le \sup_{\gamma\in B^\sharp[0,1]^d}\sum_{m\in\mathbb{Z}^d}\left|\hat{f}(\gamma + B^\sharp m)\right|^2\,. \tag{2.1}$$

Independently of the choice of $\gamma \in B^\sharp[0,1]^d$, only a fixed finite number of $m \in \mathbb{Z}^d$ will give nonzero contributions to the sum on the right-hand side of (2.1); since $\hat{f}$ is bounded, this implies that there exists a constant K such that

$$\sum_{m\in\mathbb{Z}^d}\left|\hat{f}(\gamma + B^\sharp m)\right|^2 \le K, \quad \text{a.e. } \gamma\,.$$

Hence,

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d}\int_{\operatorname{supp}\hat{f}} \left|\hat{f}(\gamma + B^\sharp m)\right|^2 |g(\gamma - Cn)|^2\, d\gamma \\
&\quad = \int_{\operatorname{supp}\hat{f}}\sum_{m\in\mathbb{Z}^d}\left|\hat{f}(\gamma + B^\sharp m)\right|^2 \sum_{n\in\mathbb{Z}^d}|g(\gamma - Cn)|^2\, d\gamma \\
&\quad \le K\int_{\operatorname{supp}\hat{f}}\sum_{n\in\mathbb{Z}^d}|g(\gamma - Cn)|^2\, d\gamma\,.
\end{aligned}$$

Choose an integer $a > 0$ such that

$$\operatorname{supp}\hat{f} \subseteq C[-a,a]^d\,.$$

Then

$$\begin{aligned}
\int_{\operatorname{supp}\hat{f}}\sum_{n\in\mathbb{Z}^d}|g(\gamma - Cn)|^2\, d\gamma &\le \int_{C[-a,a]^d}\sum_{n\in\mathbb{Z}^d}|g(\gamma - Cn)|^2\, d\gamma \\
&\le |\det C|\int_{[-a,a]^d}\sum_{n\in\mathbb{Z}^d}|g(C(\xi - n))|^2\, d\xi\,.
\end{aligned}$$

Now, using that (modulo null-sets)

$$[-a,a]^d = \bigcup_{k\in[-a,a-1]^d\cap\mathbb{Z}^d}\left(k + [0,1]^d\right)$$

and that the function $\xi \mapsto \sum_{n\in\mathbb{Z}^d}|g(C(\xi - n))|^2$ is $\mathbb{Z}^d$-periodic,

$$\begin{aligned}\int_{[-a,a]^d} \sum_{n\in\mathbb{Z}^d} |g(C(\xi-n))|^2\,d\xi \\ &= (2a)^d \int_{[0,1]^d} \sum_{n\in\mathbb{Z}^d} |g(C(\xi-n))|^2\,d\xi \\ &= (2a)^d \int_{\mathbb{R}^d} |g(C\xi)|^2\,d\xi \\ &= |\det C|^{-1}(2a)^d \int_{\mathbb{R}^d} |g(\eta)|^2\,d\eta < \infty\,. \end{aligned}$$

□

The following is the frame-pair version of Corollary 3.3 in [7]. It can also be considered as the time-domain version of Lemma 1. Results of that type already appeared in [8] by Ron and Shen, and (in the one-dimensional case) in [5] by Janssen. We provide the short proof for the sake of completeness.

Lemma 3. *Two Bessel sequences $\{E_{Bm}T_{Cn}g\}_{m,n\in\mathbb{Z}^d}$ and $\{E_{Bm}T_{Cn}h\}_{m,n\in\mathbb{Z}^d}$ form dual frames for $L^2(\mathbb{R}^d)$ if and only if*

$$\sum_{k\in\mathbb{Z}^d} \overline{g\big(x - B^\sharp n - Ck\big)}\,h(x-Ck) = |\det B|\delta_{n,0}\,. \tag{2.2}$$

Proof. We note that $\{E_{Bm}T_{Cn}g\}_{m,n\in\mathbb{Z}^d}$ and $\{E_{Bm}T_{Cn}h\}_{m,n\in\mathbb{Z}^d}$ form dual frames if and only if $\{\mathcal{F}^{-1}E_{Bm}T_{Cn}g\}_{m,n\in\mathbb{Z}^d}$ and $\{\mathcal{F}^{-1}E_{Bm}T_{Cn}h\}_{m,n\in\mathbb{Z}^d}$ are dual frames. Now, $\mathcal{F}^{-1}E_{Bm}T_{Cn}g = T_{-Bm}\mathcal{F}^{-1}T_{Cn}g$; thus, the result follows from Lemma 1 and Lemma 2 with $g_n = \mathcal{F}^{-1}T_{Cn}g$, $h_n = \mathcal{F}^{-1}T_{Cn}h$. □

We now present the first version of our results. For simplicity we consider the case $C = I$. For any $d\times d$ matrix we define the norm $\|B\|$ by

$$\|B\| = \sup_{\|x\|=1} \|Bx\|\,.$$

Theorem 1. *Let $N\in\mathbb{N}$. Let $g\in L^2(\mathbb{R}^d)$ be a real-valued bounded function with* supp $g\subseteq[0,N]^d$, *for which*

$$\sum_{n\in\mathbb{Z}^d} g(x-n) = 1\,.$$

Assume that the $d\times d$ matrix B is invertible and $\|B\|\le\frac{1}{\sqrt{d}(2N-1)}$. For $i=1,\dots,d$, let F_i be the set of lattice points $\{k_j\}_{j=1}^d\in\mathbb{Z}^d$ for which the coordinates k_j, $j=1,\dots,d$, satisfy the requirements

$$\begin{cases} \textit{if} \quad j=1,\dots,i-1, & \textit{then} \quad |k_j|\le N-1\,; \\ \textit{if} \quad j=i, & \textit{then} \quad 1\le k_j\le N-1\,; \\ \textit{if} \quad j=i+1,\dots,d, & \textit{then} \quad k_j=0\,. \end{cases} \tag{2.3}$$

Define $h\in L^2(\mathbb{R}^d)$ by

$$h(x) := |\det B|\left[g(x) + 2\sum_{i=1}^{d}\sum_{k\in F_i} g(x+k)\right]. \tag{2.4}$$

Then the function g and the function h generate dual frames $\{E_{Bm}T_n g\}_{m,n\in\mathbb{Z}^d}$ and $\{E_{Bm}T_n h\}_{m,n\in\mathbb{Z}^d}$ for $L^2(\mathbb{R}^d)$.

Proof. We apply Lemma 3. Since B is invertible, for any $n \in \mathbb{Z}^d$ we have

$$|n| = \left\| B^T B^\sharp n \right\| \leq \|B\| \left\| B^\sharp n \right\| ;$$

thus, for $n \neq 0$, $\|B^\sharp n\| \geq 1/\|B\|$. Note that with the definition (2.4), we have $\operatorname{supp} h \subseteq [-N+1, 2N-1]^d$; thus, (2.2) is satisfied for $n \neq 0$ if $1/\|B\| \geq \sqrt{d}(2N-1)$, i.e., if

$$\|B\| \leq \frac{1}{\sqrt{d}(2N-1)} .$$

Thus, we only need to check that

$$\sum_{k\in\mathbb{Z}^d} g(x-k)h(x-k) = |\det B|, \ x \in [0,1]^d ;$$

due to the compact support of g, this is equivalent to

$$\sum_{n\in[0,N-1]^d\cap\mathbb{Z}^d} g(x+n)h(x+n) = |\det B|, \ x \in [0,1]^d . \tag{2.5}$$

To check that (2.5) holds, we use that for $x \in [0,1]^d$,

$$\sum_{n\in[0,N-1]^d\cap\mathbb{Z}^d} g(x+n) = 1 . \tag{2.6}$$

For $n := \{n_j\}_{j=1}^d \in [0, N-1]^d \cap \mathbb{Z}^d$, and $i = 1, \ldots, d$, let E_i^n denote the set of lattice points $\{k_j\}_{j=1}^d \in \mathbb{Z}^d$ whose coordinates k_j satisfy the requirements

$$\begin{cases} \text{if} \quad j = 1, \ldots, i-1, & \text{then} \quad 0 \leq k_j \leq N-1 ; \\ \text{if} \quad j = i, & \text{then} \quad n_j + 1 \leq k_j \leq N-1 ; \\ \text{if} \quad j = i+1, \ldots, d, & \text{then} \quad k_j = n_j . \end{cases}$$

Define $\tilde{h}_n \in L^2(\mathbb{R}^d)$ by

$$\tilde{h}_n(x) := |\det B| \left[g(x+n) + 2\sum_{i=1}^d \sum_{k\in E_i^n} g(x+k) \right] .$$

We now consider the finite set $[0, N-1]^d \cap \mathbb{Z}^d$. Using lexicographic ordering, i.e.,

$$\begin{aligned} &(i_1, \ldots, i_d) > (j_1, \ldots, j_d) \\ &\Leftrightarrow (i_d > j_d) \vee ((i_d = j_d) \wedge (i_{d-1} > j_{d-1})) \vee \cdots \\ &\vee((i_d = j_d) \wedge \cdots \wedge (i_2 = j_2) \wedge i_1 > j_1) , \end{aligned}$$

we write

$$[0, N-1]^d \cap \mathbb{Z}^d = \{n_1, n_2, \cdots, n_{N^d}\} ,$$

with $n_j < n_k$ for $j < k$. Then for $x \in [0,1]^d$, (2.6) implies that

$$
\begin{aligned}
1 &= \left(\sum_{j=1}^{N^d} g(x+n_j)\right)^2 \\
&= (g(x+n_1)+g(x+n_2)+\cdots+g(x+n_{N^d})) \\
&\quad \times (g(x+n_1)+g(x+n_2)+\cdots+g(x+n_{N^d})) \\
&= g(x+n_1)[g(x+n_1)+2g(x+n_2)+2g(x+n_3)+\cdots+2g(x+n_{N^d})] \\
&\quad + g(x+n_2)[g(x+n_2)+2g(x+n_3)+2g(x+n_4)+\cdots+2g(x+n_{N^d})] \\
&\quad +\cdots \\
&\quad +\cdots \\
&\quad + g(x+n_{N^d-1})[g(x+n_{N^d-1})+2g(x+n_{N^d})] \\
&\quad + g(x+n_{N^d})[g(x+n_{N^d})] \\
&= \frac{1}{|\det B|}\sum_{j=1}^{N^d} g(x+n_j)\tilde{h}_{n_j}(x)\ .
\end{aligned}
$$

It remains to show that for $x \in [0,1]^d$ and $n = \{n_j\}_{j=1}^d \in [0, N-1]^d \cap \mathbb{Z}^d$,

$$h(x+n) = \tilde{h}_n(x)\ .$$

In order to do so, it is sufficient to show that for any $i = 1, \dots, d$,

$$\sum_{k\in F_i} g(x+n+k) = \sum_{k\in E_i^n} g(x+k),\ \ x \in [0,1]^d\ . \tag{2.7}$$

Fix $i \in \{1,\dots,d\}$. If $1 \le j < i$, then

$$
\begin{aligned}
\left\{n_j+k_j : \{k_j\}_{j=1}^d \in F_i\right\} &= [n_j-N+1, n_j+N-1]\cap\mathbb{Z} \\
&\supseteq [0, N-1]\cap\mathbb{Z}\ .
\end{aligned} \tag{2.8}
$$

If $j = i$, then

$$
\begin{aligned}
\left\{n_j+k_j : \{k_j\}_{j=1}^d \in F_i\right\} &= [n_j+1, n_j+N-1]\cap\mathbb{Z} \\
&\supseteq [1+n_j, N-1]\cap\mathbb{Z}\ .
\end{aligned} \tag{2.9}
$$

If $j > i$, then

$$\left\{n_j+k_j : \{k_j\}_{j=1}^d \in F_i\right\} = \{n_j\}\ .$$

Via the definition of the set E_i^n this shows that

$$E_i^n \subseteq \left\{n+k : k = \{k_j\}_{j=1}^d \in F_i\right\}\ . \tag{2.10}$$

In order to show that we have equality in (2.7), we again fix $i \in \{1,\dots,d\}$. Suppose that $m := \{m_j\}_{j=1}^d \in \{n+k : k = \{k_j\}_{j=1}^d \in F_i\} \setminus E_i^n$. Then either, by (2.8), there exists $j \in \{1,\dots,i-1\}$ such that

$$m_j := n_j + k_j \notin [0, N-1]\cap\mathbb{Z}\ ;$$

or, by (2.9),

$$m_i := n_i + k_i \in \big([1+n_j, n_j+N-1] \setminus [1+n_j, N-1]\big) \cap \mathbb{Z} = [N, n_j+N-1] \cap \mathbb{Z}\,.$$

In both cases, since $\operatorname{supp} g \subseteq [0, N]^d$, this implies that $g(x+m) = 0$ for $x \in [0,1]^d$. Hence,

$$\sum_{k \in F_i} g(x+n+k) = \sum_{k \in E_i^n} g(x+k)\ ,$$

as desired. □

Example 1. For $d = 1$, the Gabor system considered in Theorem 1 is $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ for some $b > 0$. The reader can check that

$$F_1 = \{1, \ldots, N-1\}\ ;$$

thus, the expression for the dual generator h in (2.4) is

$$h(x) = bg(x) + 2b \sum_{k=1}^{N-1} g(x+k)\ .$$

This result corresponds to the one-dimensional case treated in [2].

For $d = 2$, (2.3) leads to the sets

$$\begin{aligned} F_1 &= \left\{(k_1,k_2) \in \mathbb{Z}^2 \,\middle|\, 1 \le k_1 \le N-1, k_2 = 0\right\}, \\ F_2 &= \left\{(k_1,k_2) \in \mathbb{Z}^2 \,\middle|\, |k_1| \le N-1, 1 \le k_2 \le N-1\right\}. \end{aligned}$$

For $N = 3$, the sets F_1 and F_2 are marked on Figure 1.

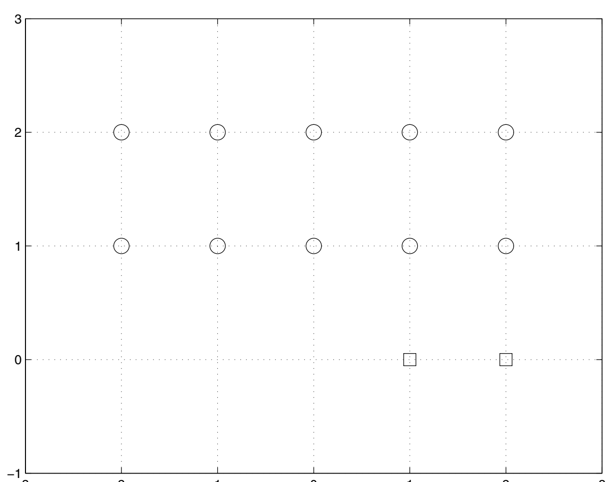


FIGURE 1 The sets F_1 (marked by □) and F_2 (marked by ◯) corresponding to $N = 3$ and $d = 2$.

Via a change of variable Theorem 1 leads to a construction of frames of the type $\{E_{Bm}T_{Cn}g\}_{m,n\in\mathbb{Z}^d}$ and convenient duals.

Theorem 2. *Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R}^d)$ be a real-valued bounded function with* $\operatorname{supp} g \subseteq [0, N]^d$*, for which*

$$\sum_{n \in \mathbb{Z}^d} g(x-n) = 1 \,.$$

Let B and C be invertible $d \times d$ matrices such that $\|C^T B\| \le \frac{1}{\sqrt{d}(2N-1)}$, and let (with the sets F_i defined as in Theorem 1)

$$h(x) = \left|\det\left(C^T B\right)\right| \left[g(x) + 2 \sum_{i=1}^{d} \sum_{k \in F_i} g(x+k) \right]. \tag{2.11}$$

Then the function $D_{C^{-1}} g$ and the function $D_{C^{-1}} h$ generate dual Gabor frames $\{E_{Bm} T_{Cn} D_{C^{-1}} g\}_{m,n \in \mathbb{Z}^d}$ and $\{E_{Bm} T_{Cn} D_{C^{-1}} h\}_{m,n \in \mathbb{Z}^d}$ for $L^2(\mathbb{R}^d)$.

Proof. By assumptions and Theorem 1, the Gabor systems $\{E_{C^T Bm} T_n g\}_{m,n \in \mathbb{Z}^d}$ and $\{E_{C^T Bm} T_n h\}_{m,n \in \mathbb{Z}^d}$ form dual frames; since

$$D_{C^{-1}} E_{C^T Bm} T_n = E_{Bm} T_{Cn} D_{C^{-1}} \,,$$

the result follows from $D_{C^{-1}}$ being unitary. □

For functions g of the above type and arbitrary real invertible $d \times d$ matrices B and C, Theorem 2 leads to a construction of a (finitely generated) multi-Gabor frame $\{E_{Bm} T_{Cn} g_k\}_{m,n \in \mathbb{Z}^d, k \in \mathcal{F}}$, where all the generators g_k are dilated and translated versions of g. Again, the dual generators have a similar form, and are given explicitly.

Theorem 3. *Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R}^d)$ be a real-valued bounded function with* $\operatorname{supp} g \subseteq [0, N]^d$*, for which*

$$\sum_{n \in \mathbb{Z}^d} g(x-n) = 1 \,.$$

Let B and C be invertible $d \times d$ matrices and choose $J \in \mathbb{N}$ such that $J \ge \|C^T B\| \sqrt{d}(2N-1)$. Define the function h by (2.11)*. Then the functions*

$$g_k = T_{\frac{1}{J}Ck} D_{JC^{-1}} g, \;\; h_k = T_{\frac{1}{J}Ck} D_{JC^{-1}} h, \;\; k \in \mathbb{Z}^d \cap [0, J-1]^d$$

generate dual multi-Gabor frames $\{E_{Bm} T_{Cn} g_k\}_{m,n \in \mathbb{Z}^d, k \in \mathbb{Z}^d \cap [0,J-1]^d}$ and $\{E_{Bm} T_{Cn} h_k\}_{m,n \in \mathbb{Z}^d, k \in \mathbb{Z}^d \cap [0,J-1]^d}$ for $L^2(\mathbb{R}^d)$.

Proof. The choice of J implies that the matrices B and $\frac{1}{J}C$ satisfy the conditions in Theorem 2; thus

$$\left\{ e^{2\pi i Bm \cdot x} (D_{JC^{-1}} g) \left(x - \frac{1}{J} Cn \right) \right\}_{m,n \in \mathbb{Z}^d} \quad \text{and} \quad \left\{ e^{2\pi i Bm \cdot x} (D_{JC^{-1}} h) \left(x - \frac{1}{J} Cn \right) \right\}_{m,n \in \mathbb{Z}^d}$$

form a pair of dual Gabor frames for $L^2(\mathbb{R}^d)$. Now,

$$\left\{ \frac{1}{J} Cn \right\}_{n \in \mathbb{Z}^d} = \bigcup_{k \in \mathbb{Z}^d \cap [0,J-1]^d} \left\{ \frac{1}{J} Ck + Cn \right\}_{n \in \mathbb{Z}^d} \,.$$

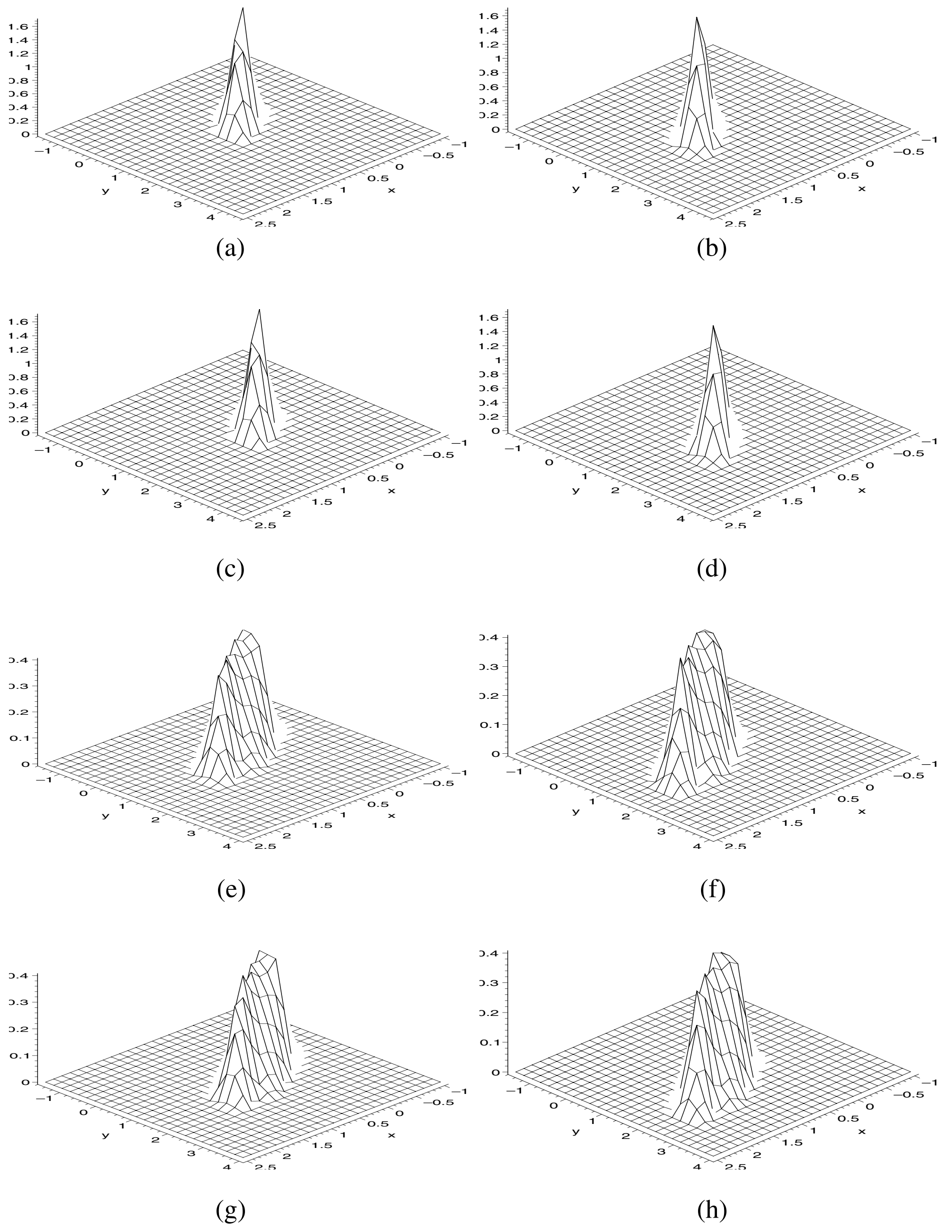


FIGURE 2 Plots of the generators in Example 2: (a) $g_{(0,0)}$; (b) $g_{(1,0)}$; (c) $g_{(0,1)}$; (d) $g_{(1,1)}$; (e) $h_{(0,0)}$; (f) $h_{(1,0)}$; (g) $h_{(0,1)}$; (h) $h_{(1,1)}$.

Thus,

$$\begin{aligned}\left\{(D_{JC^{-1}}g)\left(\cdot - \frac{1}{J}Cn\right)\right\}_{n\in\mathbb{Z}^d} &= \bigcup_{k\in\mathbb{Z}^d\cap[0,J-1]^d} \left\{(D_{JC^{-1}}g)\left(\cdot - \frac{1}{J}Ck - Cn\right)\right\}_{n\in\mathbb{Z}^d} \\ &= \bigcup_{k\in\mathbb{Z}^d\cap[0,J-1]^d} \left\{T_{Cn}T_{\frac{1}{J}Ck}D_{JC^{-1}}g(\cdot)\right\}_{n\in\mathbb{Z}^d} .\end{aligned}$$

Inserting this into the expression for the pair of dual frames leads to the result. □

Note that multi-generated Gabor system have appeared in various applications for a long time, see, e.g., [10].

Via our results we now construct Gabor frames for $L^2(\mathbb{R}^d)$ with box spline generators and dual generators having a similar form.

Example 2. Let B_2 be the one-dimensional B-spline of order 2 defined by

$$B_2(x) = \begin{cases} x, & x \in [0,1[\ ; \\ 2-x, & x \in [1,2[\ ; \\ 0, & x \notin [0,2[\ . \end{cases}$$

Define $g \in L^2(\mathbb{R}^2)$ by

$$g(x,y) = B_2(x)\ B_2(y)\ ; \tag{2.12}$$

then $\text{supp}\, g \subseteq [0,2]^2$, and

$$\sum_{n\in\mathbb{Z}^2} g(x-n) = 1,\ x \in \mathbb{R}^2 ,$$

since the integer-translates of B_2 form a partition of unity. Let the 2×2 matrices B and C be defined by

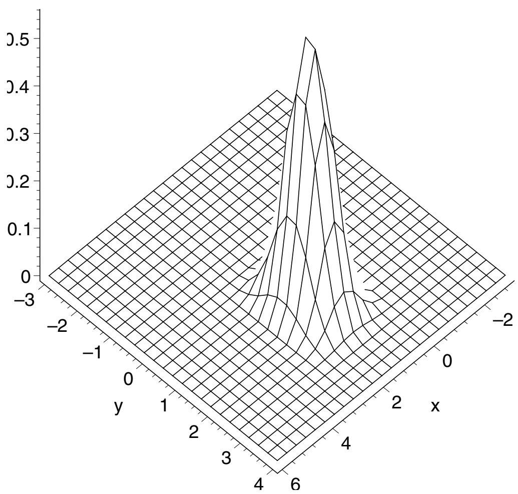
$$B = \frac{1}{10}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\ C = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} .$$

A direct calculation shows that

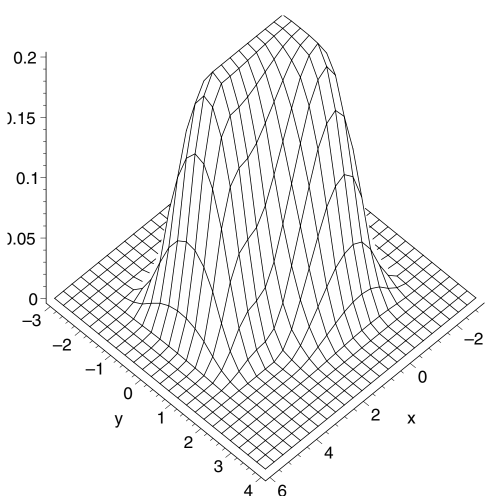
$$\begin{aligned}\left\|C^T B\right\|^2 &= \left\|\frac{1}{10}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\right\|^2 = \sup_\theta \left\|\frac{1}{10}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right\|^2 \\ &= \left(\frac{1}{10}\right)^2 (\sqrt{2}+1)^2 .\end{aligned}$$

Thus,

$$\left\|C^T B\right\|\sqrt{d}(2N-1) = \frac{3}{10}(2+\sqrt{2}) = 1.02\cdots .$$



(a)



(b)

FIGURE 3 The functions g [Figure (a)] and h [Figure (b)] in Example 3.

Thus, we can apply Theorem 3 with $J = 2$. Define the function $h \in L^2(\mathbb{R}^2)$ by (2.11), i.e.,

$$\begin{aligned} h(x,y) &= \left|\det\left(C^T B\right)\right| [g(x,y) + 2g((x,y)+(1,0)) \\ &\quad + 2g((x,y)+(-1,1)) + 2g((x,y)+(0,1)) + 2g((x,y)+(1,1))] \\ &= \frac{1}{10} \begin{cases} 2xy+2x+2y+2, & (x,y) \in [-1,0[\times[-1,0[; \\ 2x+2, & (x,y) \in [-1,0[\times[0,1[; \\ 4x-2xy+4-2y, & (x,y) \in [-1,0[\times[1,2[; \\ 2y+2, & (x,y) \in [0,1[\times[-1,0[; \\ -xy+2, & (x,y) \in [0,1[\times[0,1[; \\ -2x+xy+4-2y, & (x,y) \in [0,1[\times[1,2[; \\ 2y+2, & (x,y) \in [1,2[\times[-1,0[; \\ -xy+2, & (x,y) \in [1,2[\times[0,1[; \\ -2x+xy+4-2y, & (x,y) \in [1,2[\times[1,2[; \\ 6y+6-2xy-2x, & (x,y) \in [2,3[\times[-1,0[; \\ 6-6y-2x+2xy, & (x,y) \in [2,3[\times[0,1[; \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{2.13}$$

By Theorem 3, the four functions

$$g_k = T_{\frac{1}{2}Ck} D_{2C^{-1}} g, \ k \in \mathbb{Z}^2 \cap [0,1]^2 \tag{2.14}$$

generate a multi-Gabor frame $\{E_{Bm}T_{Cn}g_k\}_{m,n\in\mathbb{Z}^2, k\in\mathbb{Z}^2\cap[0,1]^2}$, with a dual frame $\{E_{Bm}T_{Cn}h_k\}_{m,n\in\mathbb{Z}^2, k\in\mathbb{Z}^2\cap[0,1]^2}$, where

$$h_k = T_{\frac{1}{2}Ck} D_{2C^{-1}} h, \ k \in \mathbb{Z}^2 \cap [0,1]^2 . \tag{2.15}$$

Example 3. Similar calculations can be performed for any tensor product of B-splines. On Figure 3 we plot the box spline $g(x,y) = B_3(x)B_3(y)$ and the function h in (2.11) for the choice

$$B = \frac{1}{10}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} .$$

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Department of Mathematics, Technical University of Denmark, Building 303 2800 Lyngby, Denmark
e-mail: Ole.Christensen@mat.dtu.dk

Department of Mathematics, Yeungnam University, 214-1, Dae-dong, Gyeongsan-si,
Gyeongsangbuk-do, 712-749, Republic of Korea
e-mail: rykim@ynu.ac.kr