The Journal of Fourier Analysis and Applications

Volume 13, Issue 3, 2007

A Look at $BMO_\varphi(\omega)$ through Carleson Measures

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Communicated by Hans Triebel

ABSTRACT. As Fefferman and Stein showed, there is a tight connection between Carleson measures and BMO functions. In this work we extend this type of results to the more general scope of the $BMO_\varphi(\omega)$ spaces. As a byproduct a weighted version of the Triebel-Lizorkin space $\dot{F}^0_{\infty,2}$ is introduced, which turns out to be isomorphic to $BMO(\omega)$ as in the unweighted case.

1. Introduction

Given a growth function φ and a weight ω, we shall consider the $BMO_\varphi(\omega)$ space, that is the set of functions whose oscillation, when averaged over balls, is controlled by means of φ and ω, measuring their degree of smoothness. More precisely, we shall say that a locally integrable function f belongs to $BMO_\varphi(\omega)$ if there exists a constant C such that the inequality

$$\frac{1}{\omega(B)}\int_B |f(y) - m_B f|\, dy \leq C\varphi\left(|B|^{\frac{1}{n}}\right) \tag{1.1}$$

holds for every ball B in $\mathbb{R}^n$, where, as usual, $m_B f$ denotes the average of f over B with respect to the Lebesgue measure. The first appearance of this kind of weighted spaces goes back to [8] and [14]. In the latter article, the authors introduced $BMO(\omega)$ ($\varphi \equiv 1$ in our context) as the natural space where weighted L^∞ functions are mapped by $\mathcal{H}$, the Hilbert transform on the line, generalizing the well known *BMO* space of John and Nirenberg. In the more general context $\varphi(t) = t^\beta$, $0 < \beta < 1$, it is shown in [10] that the fractional integral operator I_α maps $L^p(w)$ with $p > n/\alpha$ into these spaces, under suitable conditions on the weight. Later, this result was extended to weighted Orlicz spaces [11] giving rise to

Math Subject Classifications. Primary: 42B25.
Keywords and Phrases. Weigths, bounded mean oscillation, characterizations, Carleson measure.
Acknowledgements and Notes. The authors were supported by the Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina and by the Universidad Nacional del Litoral.

ISSN 1069-5869 DOI: 10.1007/s00041-005-5044-3

the spaces under consideration in their full generality. Finally, in [13], it is shown that they are preserved by the Hilbert transform on the line.

In their celebrated article [7], Fefferman and Stein shed light on the tight connection between *BMO*-functions and Carleson measures. Let us remind that a measure μ on $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ is said to be a Carleson measure when a constant C exists such that for any ball $B(x_0, r) \subset \mathbb{R}^n$

$$\mu(B(x_0, r) \times (0, r)) \leq Cr^n .$$

With this notation the Fefferman-Stein result can be stated as:

$$f \in BMO \Leftrightarrow \int \frac{f(x)}{1+|x|^{n+1}}\,dx < \infty \text{ and}$$
$$t|\nabla(P_t * f)|^2(x)\,dx\,dt \text{ is a Carleson measure, where } P_t \text{ denotes the Poisson kernel .}$$

Later on, W. Smith, in [17], proved an extension of this result to the spaces BMO_φ (i.e., $BMO_\varphi(\omega)$ with $\omega = 1$) giving a suitable definition of φ-Carleson measures.

A more recent version of this kind of characterization of functions in *BMO* appears in Stein's book [15] (see Theorem 3, p. 159). Basically, Stein's statement is the following.

Theorem 1. *Let $\psi \in \mathcal{S}$ with $\int \psi = 0$.*

(a) *If $f \in BMO$ then $d\mu = |f * \psi_t|^2 \frac{dx\,dt}{t}$ is a Carleson measure.*

(b) *Conversely, suppose ψ satisfies a Tauberian condition, $\int \frac{|f(x)|}{1+|x|^{n+1}}\,dx < \infty$ and $d\mu = |f * \psi_t|^2 \frac{dx\,dt}{t}$ is a Carleson measure; then f is in BMO.*

Here, by a Tauberian condition we mean that $\widehat{\psi}$ does not vanish identically in any ray emanating from the origin (i.e., for every $\xi \neq 0$ there exists a $t > 0$ with $\hat{\psi}(t\xi) \neq 0$) and, as usual, $\psi_t = t^{-n}\psi(x/t)$.

Also, it is well known that *BMO* coincides with the Triebel-Lizorkin space $\dot{F}^0_{\infty,2}$ (see [4] or [5]). The above result, even if very close, does not allow to conclude such characterization: One should prove part (b) of the theorem under the more general situation of a distribution in $\mathcal{S}'/\mathcal{P}$ ($\mathcal{P}$ the set of polynomials) instead of the integrability condition on the function f.

In this work we give an extension of the theorem above to the more general spaces $BMO_\varphi(\omega)$ under appropriate assumptions on φ and ω, which, at the same time, allows us to obtain, as a corollary, the identification of $BMO(\omega)$ with a weighted version of $\dot{F}^0_{\infty,2}$.

It is worth mentioning that Bui and Taibleson defined in [1] weighted $\dot{F}^\alpha_{\infty,q}$ spaces. However, as we will show, for $\alpha = 0$ and $q = 2$, their definition does not give the weighted space $BMO(\omega)$. In fact we prove that, at least for weights in the Muckenhoupt class A_1, it coincides rather with the unweighted *BMO* space.

In proving our main theorem we establish a duality inequality involving generalized Carleson measures and tent spaces. This is achieved by means of an adequate atomic decomposition of the latter spaces.

The structure of the article is as follows: Section 2 contains some basic facts and the statement of our main theorem; Sections 3 and 4, respectively, contain some needed results, interesting by themselves, about generalizations of Hardy and tent spaces; the proof of the main theorem is given in Section 5, and, finally, Section 6 is devoted to the above remark on weighted Triebel-Lizorkin spaces.

2. Preliminaries and the Main Result

We start by reminding some basic notions about growing functions and weights.

For a nonnegative and nondecreasing function φ defined in $[0, \infty)$, we shall say that it is of upper type β, if there exists a constant C such that

$$\varphi(\theta t) \le C\theta^\beta \varphi(t) \tag{2.1}$$

for all $\theta \ge 1$ and $t \ge 0$. If there exists such number β, we shall denote by $I(\varphi) = \inf\{\beta : \varphi \text{ is of upper type } \beta\}$. Let us notice that our assumptions on φ guarantee that $I(\varphi) \ge 0$. Similarly, whenever (2.1) holds for $0 \le \theta \le 1$, φ is said to be of lower type β.

Next we remind that a weight ω belongs to the Muckenhoupt class A_r, $r > 1$, if there exists a constant C such that for any ball $B \subset \mathbb{R}^n$

$$\frac{1}{|B|}\int_B \omega \left(\frac{1}{|B|}\int_B \omega^{-\frac{1}{r-1}}\right)^{r-1} \le C ,$$

and a weight ω belongs to A_1 if there exists a constant C such that for any ball $B \subset \mathbb{R}^n$

$$\frac{1}{|B|}\int_B \omega \le C \inf_B \omega .$$

Any constant C satisfying the above inequalities will be called A_p-constant or A_1-constant, respectively. Finally, a weight is in the class A_∞ when it belongs to some A_r, $r \ge 1$.

The following two lemmas contain some technical results about weights which will be important to get our main theorem.

Lemma 1. *Let ρ be a nonnegative increasing function of finite upper type and ω a weight in A_∞. Then*

(a) *there exists a constant C_0 such that for any $C > 0$*

$$\rho\left(\frac{1}{|Q|}\int_Q C\omega\right) \le C_0 \frac{1}{|Q|}\int_Q \rho(C\omega)$$

for any cube $Q \subset \mathbb{R}^n$.

(b) *If in addition ρ is concave, then there exists $p > 1$ such that for any $C > 0$, the weights $\rho(C\omega)$ belong to A_p with an A_p-constant independent of C.*

The proof of (a) is straightforward using the following characterization of A_∞ (see, for example, [2]).

There exist $0 < \alpha, \beta < 1$ such that for any cube $Q \subset \mathbb{R}^n$

$$|\{x \in Q : \omega(x) > \beta m_Q \omega\}| \ge \alpha |Q| .$$

Moreover, let us remark that, following the proof that this condition implies A_∞, it is easy to check that the index p such that $w \in A_p$, as well as the A_p-constant, only depend on the constants α and β.

For (b), since $\omega \in A_\infty$, the above inequality holds for some values α and β. Using the concavity of η, it is easy to see that the same type of inequality holds for $\rho(C\omega)$ with exactly the same constants α and β. From the above remark (b) follows.

Lemma 2. *Let ρ be a weight in A_p, $1 \le p < \infty$. Then there exists a constant C such that*

$$\left(\frac{|E|}{|B|}\right)^p \le C\frac{\omega(E)}{\omega(B)}$$

holds for every measurable set $E \subset B$ and every ball $B \subset \mathbb{R}^n$.

The proof follows easily by using Hölder's inequality and the A_p condition.

As we said in the introduction, we shall consider the spaces $BMO_\varphi(\omega)$ for φ a concave function as above, and ω a weight in A_∞. These spaces consist of locally integrable functions on $\mathbb{R}^n$ such that (1.1) holds. Moreover if we set $\|f\|_{BMO_\varphi(\omega)}$ as the infimum of the constants for which (1.1) holds, $BMO_\varphi(\omega)$ turns out to be a Banach space modulo constants. In particular, when $\omega \in A_p$, $1 \le p < \infty$, it can be proved (see [13], Theorem 2.2, p. 7) that a function f belongs to $BMO_\varphi(\omega)$ if and only if for any $r < \infty$ and $1 < r \le p'$, there exists a constant C_r such that

$$\left(\frac{1}{\omega(B)}\int_B |f(y) - m_B f|^r \omega(y)^{1-r}\,dy\right)^{1/r} \le C_r\varphi\left(|B|^{1/n}\right)$$

holds for every ball B in $\mathbb{R}^n$. Moreover, for every fixed r satisfying these conditions, the infimum constant C_r defines an equivalent norm in $BMO_\varphi(\omega)$.

Next we introduce a generalization of the notion of Carleson measures. For φ and ω as above, we shall say that a measure $d\mu$ on R^{n+1}_+ is a (φ, ω)-Carleson measure when there exists a constant such that

$$\int_{\hat{B}} |d\mu| \le C\omega(B)\varphi^2\left(|B|^{1/n}\right) \tag{2.2}$$

for any ball $B \subset \mathbb{R}^n$. Here $\hat{B}$ denotes the tent corresponding to the ball $B = B(x_0, r)$, that is, $\hat{B} = \{(x,t) \in \mathbb{R}^{n+1}_+ : |x - x_0| + t < r\}$. As usual, we denote by $[d\mu]_{\varphi,\omega}$ the infimum of the constants appearing in (2.2). This definition is a weighted extension of the notion given in [17]. Now we are in position to state our main result.

Theorem 2. *Let φ be a nonnegative, nondecreasing concave function defined on $[0, \infty)$ with $I(\varphi) < 1$. Let $q_0 = 1 + \frac{(1-I(\varphi))}{n}$ and ω be an A_q weight. Further, let ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ with a null integral. The following statements hold:*

(a) *If $f \in BMO_\varphi(\omega)$, then $d\mu = |\psi_t * f|^2(x)\frac{t^n}{\omega(B(x,t))}\,dx\frac{dt}{t}$ is a (φ,ω)-Carleson measure with*

$$[d\mu]_{\varphi,\omega} \le C\|f\|^2_{BMO_\varphi(\omega)}\ .$$

(b) *Assume further that ψ satisfies a Tauberian condition. Then any distribution $f \in \mathcal{S}'/\mathcal{P}$ such that $d\mu = |\psi_t * f|^2(x)\frac{t^n}{\omega(B(x,t))}\,dx\frac{dt}{t}$ is a (φ,ω)-Carleson measure, can be seen as a $BMO_\varphi(\omega)$ function with*

$$\|f\|^2_{BMO_\varphi}(\omega) \le C[d\mu]_{\varphi,\omega}\ .$$

We notice that part (a) is a generalization of (a) in Theorem 1 while part (b) looks slightly different. However, we may obtain as a corollary of our theorem such an extension.

Corollary 1. *Let φ and ω be as above and ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying the Tauberian condition and with a null integral. Then, if f is such that $\int_{\mathbb{R}^n}\frac{|f(x)|}{1+|x|^{n+1}}\,dx < \infty$ and $d\mu = |f * \psi_t|^2(x)\frac{t^n}{\omega(B(x,t))}\,dx\frac{dt}{t}$ is a (φ,ω)-Carleson measure, f is also in $BMO_\varphi(\omega)$ with $\|f\|^2_{BMO_\varphi(\omega)} \le C[d\mu]_{\varphi,\omega}$.*

This corollary follows from the theorem by noting that a function f satisfying $\int_{\mathbb{R}^n} \frac{|f(x)|}{1+|x|^{n+1}}\, dx < \infty$ defines an element of $\mathcal{S}'/\mathcal{P}$.

3. Some Basic Facts About $H^q_\eta(\omega)$

In this section we present some results concerning weighted Hardy-Orlicz atomic spaces that will be useful to our purposes. Mostly, they are spread in the literature, perhaps not with the degree of generality we need here. We state them and outline their proofs for the sake of completeness.

In the sequel we shall work with a nonnegative, increasing and concave function η with $\eta(0) = 0$ and of lower type $\ell > \frac{n}{n+1}$. Notice that η, being concave, is also of upper type one. Given such η and a weight $\omega \in A_q$, we shall say that a function a is an (η, q, ω)-atom if a is supported in a ball B, has zero average and

$$\|a\|_{L^q(\omega)} \leq \frac{|B|}{(\omega(B))^{1/q'}}\ \eta^{-1}\left(\frac{1}{|B|}\right) , \tag{3.1}$$

where η^{-1} denotes the inverse function of η. With this notion, we define the atomic space $H^q_\eta(\omega)$ as the set of distributions $f \in \mathcal{S}'$ that can be written as $f = \Sigma_{i=1}^\infty b_i$ (in the sense of distributions), where $\{b_i\}$ is a sequence of multiples of (η, q, ω)-atoms such that

$$\sum_{i=1}^\infty |B_i| \eta\left(\frac{\omega(B_i)^{1/q'}}{|B_i|}\|b_i\|_{L^q(\omega)}\right) < \infty ,$$

where B_i is a ball containing the support of b_i. For any such decomposition we introduce the quantity

$$\Lambda_q(\{b_i\}) = \inf\left\{\lambda : \sum |B_i|\eta\left(\frac{\omega(B_i)^{1/q'}}{\lambda^{1/\ell}|B_i|}\|b_i\|_{L^q(\omega)}\right) \leq 1\right\} ,$$

and we denote by $[f]_{H^q_\eta(\omega)}$ the infimum of $\Lambda_q(\{b_i\})$ taken over all decompositions of f. It is easy to check that $[.]_{H^q_\eta(\omega)}$ defines a quasi-metric invariant under translations, which is positively homogeneous when raised to the $(1/\ell)$th-power (i.e., : $[\alpha f]^{1/\ell}_{H^q_\eta}(\omega) = |\alpha|[f]^{1/\ell}_{H^q_\eta(\omega)}$ for every $\alpha \in \mathbb{R}$).

Let us observe that any function g in $L^q(\omega)$, supported in a ball and with zero average, belongs to $H^q_\eta(\omega)$ and, moreover, if it satisfies (3.1) then, $[g]_{H^q_\eta(\omega)} \leq 1$.

The first result we need is quite standard.

Proposition 1. *Let L be a functional in the dual of $H^q_\eta(\omega)$. Then there exists $h \in BMO_\varphi(\omega)$ with $\varphi(t^{1/n}) = 1/t\eta^{-1}(1/t)$ such that*

$$L(g) = \int h(x)g(x)\, dx ,$$

for any $g \in L^q(\omega)$ with compact support and zero average.

Moreover,

$$\|h\|_{BMO_\varphi(\omega)} \leq [L] = \inf\left\{C : |L(f)| \leq C[f]^{1/\ell}_{H^q_\eta(\omega)}\right\} . \tag{3.2}$$

Proof. It is easy to see that for any ball B, L defines a bounded linear functional on $L^q_0(B,\omega)$, the subspace of functions in $L^q(\omega)$, supported in B with zero average, since for such f we have

$$|L(f)| \leq C[f]^{1/\ell}_{H^q_\eta(\omega)} \leq C\frac{\omega(B)^{1/q'}}{|B|\eta^{-1}\left(\frac{1}{|B|}\right)}\|f\|_{L^q(\omega)}\,.$$

Extending L by the Hahn-Banach Theorem we know that there exists a function $h_B \in L^{q'}(\omega^{1-q'})$ supported in B, such that

$$L(f) = \int_B h_B f = \int_B (h_B - m_B h_B) f, \quad f \in L^q_0(B,\omega)$$

and, moreover, we have

$$\left(\frac{1}{\omega(B)}\int_B |h_B - m_B h_B|^{q'}\omega^{1-q'}\right)^{1/q'} \leq C\varphi\big(|B|^{1/n}\big)\,, \tag{3.3}$$

with C independent of B. Taking now an increasing sequence of balls, by a standard argument, a function h may be defined, modulo constants, satisfying (3.3) for any ball. Since $\omega \in A_q$ it is known that such inequality implies $h \in BMO_\varphi(\omega)$, producing an equivalent norm, (see comments after Lemma 2). Therefore (3.2) also follows. □

The next result shows that functions in $\mathcal{S}$ with zero moments of any order are dense in our spaces. Related results appear in [18], however their spaces are not quite the same as ours.

Proposition 2. *$\mathcal{S}_\infty = \{f \in \mathcal{S} : \operatorname{supp}\hat{f} \subset \{x : \epsilon < |x| < \frac{1}{\epsilon}\}$, for some $\epsilon > 0\}$ is a dense subspace of $H^q_\eta(\omega)$ as long as $q < 2 + \frac{1}{n} - \frac{1}{\ell}$.*

Proof. As in [16], given a ball $B_0 = B(x_0, r)$, a function $g \in L^q(\omega) \cap L^1$ with zero integral can be split, pointwisely and in the sense of $\mathcal{S}'$, as

$$g = \sum_{k\geq 0}(g - m_k)\chi_{E_k} + \sum_{k\geq 0}\beta_k R_k\,, \tag{3.4}$$

where $E_0 = B_0$, $E_k = B(x_0, r2^k) - B(x_0, r2^{k-1}) = B_k - B_{k-1}$, $m_k = \frac{1}{|E_k|}\int_{E_k} g$, $\beta_k = \sum_{i\geq k+1} m_i|E_i| = \int_{B_k^c} g$ and $R_k = |E_{k+1}|^{-1}\chi_{E_{k+1}} - |E_k|^{-1}\chi_{E_k}$.

Clearly each term in the above sums is a multiple of an atom. Moreover, if $g \in \mathcal{S}$, it is easy to check that this decomposition implies that $g \in H^q_\eta(\omega)$. Thus, $\mathcal{S}_\infty$ is a subspace of $H^q_\eta(\omega)$.

To obtain the density, we observe that it is enough to approximate functions in $L^q(\omega)\cap L^1$ with compact support and zero average. Let b be one such function and σ be a radial function in $\mathcal{S}$ such that $\hat{\sigma}(\xi) = 1$ for $|\xi| \leq 1$ and $\hat{\sigma}(\xi) = 0$ for $|\xi| \geq 2$. For any t, $0 < t \leq 1$, the function $\sigma_t * b - \sigma_{1/t} * b$ belongs to $\mathcal{S}_\infty$ and moreover we will show that

$$\|\sigma_t * b - b\|_{H^q_\eta(\omega)} \longrightarrow 0 \tag{3.5}$$

and

$$\|\sigma_{1/t} * b\|_{H^q_\eta(\omega)} \longrightarrow 0 \tag{3.6}$$

when t goes to zero.

To this end we use the above decomposition for $g = \sigma_t * b - b$, $B_0 = 2B^*$, with B^* a ball containing the support of b. We denote m_k^t and β_k^t the corresponding coefficients.

For $x \in E_k$, $k \geq 1$, using the decay of σ, we get the estimate

$$|\sigma_t * b|(x) \leq C(N, \sigma, b, B_0, \omega) \frac{t^{N-n}}{2^{kN}}$$

for any positive integer N. Then

$$\left\| \left(\sigma_t * b - b - m_k^t\right) \mathcal{X}_{E_k} \right\|_{L^q(\omega)} \leq C t^{N-n} 2^{k(n-N)}, \quad k \geq 1 .$$

Besides, for $k = 0$, using that σ_t is an approximation to the identity and that $\omega \in A_q$, we get

$$\|\mathcal{X}_{E_0}(\sigma_t * b - b)\|_{L^q(\omega)} \to 0 \quad \text{for} \quad t \to 0 .$$

Therefore, setting $h_k^t = (\sigma_t * b - b - m_k^t)\chi_{E_k}$, choosing N large enough and using again that $\omega \in A_q$, we easily obtain that $\Lambda_q(\{h_k^t\}) \longrightarrow 0$, as desired.

Also, for any $k \geq 0$, due to the decay of σ we get

$$\left|\beta_k^t\right| = \left| \int_{B_k^c} \sigma_t * b \right| \leq C t^{N-n} 2^{k(n-N)} \|b\|_1 ,$$

and hence

$$\left\| \beta_k^t R_k \right\|_{L^q(\omega)} \leq C t^{N-n} 2^{-kN} (\omega(B_k))^{1/q} .$$

Arguing as above, we also get $\Lambda_q(\{\beta^{t_k} R_k\}) \longrightarrow 0$. Then (3.5) is proved.

To show (3.6) we use (3.4) again, now for $g = \sigma_{1/t} * b$, and we denote by $\tilde{m}_k^t$ and $\tilde{\beta}_k^t$ the corresponding coefficients.

First, for $k \geq 1$, using the smoothness and decay of σ, we have for $x \in E_k$

$$|\sigma_{1/t} * b|(x) \leq C \frac{t^{n+1-M}}{2^{kM}} \|b\|_{L^q(\omega)} ,$$

for M as large as we want. Choosing $M = n + 1 - \delta$, with $0 < \delta < 1$, we get

$$\|(\sigma_{1/t} * b)\mathcal{X}_{E_k}\|_{L^q(\omega)} \leq C t^\delta 2^{-k(1-\delta)} \|b\|_{L^q(\omega)} .$$

As for $k = 0$, we clearly have

$$\|(\sigma_{1/t} * b)\mathcal{X}_{E_0}\|_{L^q(\omega)} \leq C t^n \|b\|_{L^q(\omega)} .$$

Therefore, setting $\tilde{h}_k^t = (\sigma_{1/t} * b - \tilde{m}_k^t)\chi_{E_k}$ and using that η is of lower type ℓ, we obtain

$$\sum_{k \geq 0} |B_k| \eta \left(\frac{\omega(B_k)^{1/q'}}{|B_k|} \left\| \tilde{h}_k^t \right\|_{L^q(\omega)} \right) \leq C \eta\left(t^\delta\right) \sum_{k \geq 0} 2^{k(n + \ell(n(q-2)+\delta-1))} .$$

Since $q < 2 + \frac{1}{n} - \frac{1}{\ell}$, we may choose δ small enough to make the last series convergent. This shows that $\Lambda_q(\{\tilde{h}_k^t\}) \longrightarrow 0$.

A similar argument proves the convergence to zero of $\Lambda_q(\{\tilde{\beta}_k^t R_k\})$, finishing the proof of the proposition. □

4. Some Basic Facts on the Tent Spaces $T_\eta(\omega)$

In what follows, for a measurable function G defined on $\mathbb{R}^{n+1}_+$, we set

$$\mathcal{V}(G)(x) = \left(\int_{\Gamma(x)} |G(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2} \tag{4.1}$$

where $\Gamma(x)$ denotes the cone $\{(y,t) : |x-y| < t\}$.

For a nonnegative increasing and concave function η, with $\eta(0) = 0$ and lower type $\ell > n/(n+1)$, and a weight ω in $L^1_{\text{loc}}(\mathbb{R}^n)$, we introduce the tent space $T_\eta(\omega)$ as those functions G such that

$$[\mathcal{V}(G)\omega]_{L^\eta} \equiv [G]_{T_\eta(\omega)} < \infty \,,$$

where by $[g]_{L^\eta}$ we mean $\inf\{\lambda : \int \eta(g/\lambda^{1/\ell}) \leq 1\}$.

The main goal of this section is to get an atomic decomposition of $T_\eta(\omega)$, extending the result contained in [3] for $\eta(t) = t$ and $\omega \equiv 1$. We start with the notion of atoms.

Given a ball $B = B(x,r) \subset \mathbb{R}^n$ we denote by $\hat{B}$ the tent over B, i.e., $\hat{B} = \{(y,t) : |x-y| + t < r\}$. Now, a function $a(y,t)$ is said to be an atom whenever it is supported in some $\hat{B}$ and

$$\left(\int_{\mathbb{R}^{n+1}_+} |a(y,t)|^2 \frac{\omega(B(y,t))}{t^n} \frac{dy\,dt}{t} \right)^{1/2} \leq \frac{|B|}{(\omega(B))^{1/2}} \eta^{-1}\left(\frac{1}{|B|} \right) . \tag{4.2}$$

Observe that if we set $W(y,t) = \omega(B(y,t))/t^{n+1}$ the left-hand side of (4.2) is just $\|a\|_{L^2(W)}$. Also, due to the concavity of η, it is easy to check that atoms do belong to $T_\eta(\omega)$ and, moreover, $[a]_{T_\eta(\omega)} \leq 1$. With this notation we obtain the following result.

Theorem 3 (Atomic decomposition of $T_\eta(\omega)$). *Let η be a function as above and ω be a weight in $A_{2-(1/\ell-1/n)}$. Given $F \in T_\eta(\omega)$, there exists a sequence of multiple of atoms, $\{b_j\}$, such that*

$$F = \sum b_j \qquad a.e.$$

Moreover, if we denote by B_j the ball associated to b_j such that supp $b_j \subset \hat{B}_j$ *and define*

$$\Lambda(\{b_j\}) = \inf \left\{ \lambda > 0 : \sum |B_j| \eta \left(\frac{(\omega(B_j))^{1/2}}{\lambda^{1/\ell}|B_j|} \|b_j\|_{L^2(W)} \right) \leq 1 \right\} , \tag{4.3}$$

we have $\Lambda(\{b_j\}) < \infty$ and, moreover,

$$\Lambda(\{b_j\}) \leq C[F]_{T_\eta(\omega)} \,. \tag{4.4}$$

Before proving the theorem we need the following proposition which gives the weighted version of a key estimate given in [3]. Before stating it, we introduce the definition of the tent over a general measurable set $\Omega \subset \mathbb{R}^n$ as the union of the tents $\hat{B}$ for all the balls $B \subset \Omega$.

Proposition 3. *Let ω be a weight in A_∞ and B_0 a ball in $\mathbb{R}^n$. Then there exists a constant C such that, for every measurable function F defined on $\mathbb{R}^n \times (0,\infty)$ and every measurable set $E \subset B_0$, we have*

$$\int_{\hat{B}_0 - \hat{\Omega}} |F(x,t)|^2 \frac{\omega(B(x,t))}{t^n} \frac{dx\,dt}{t} \leq C \int_{B_0 - E} \mathcal{V}^2(F)(x)\omega(x)\,dx \,,$$

where $\Omega = \{x \in B_0 : M(\mathcal{X}_E)(x) > \frac{1}{2}\}$, with M being the Hardy-Littlewood maximal operator.

Proof. Set $Z = \{(x, y, t) \in (B_0 - E) \times (\widehat{B}_0 - \widehat{\Omega}) : |x - y| < t\}$ and $Z_{(y,t)} = (B_0 - E) \cap B(y, t)$. We claim that there exists $\alpha > 0$, such that for any $(y, t) \in \widehat{B}_0 - \widehat{\Omega}$.

$$\omega(Z_{(y,t)}) \geq \alpha\omega(B(y, t)) \,. \tag{4.5}$$

In fact, if $(y, t) \in \widehat{B}_0 - \widehat{\Omega}$, there exists $x_0 \notin \Omega$ with $x_0 \in B(y, t) \subset B_0$. So $M(\mathcal{X}_E)(x_0) \leq \frac{1}{2}$, and, in particular

$$|E \cap B(y, t)| \leq \frac{1}{2}|B(y, t)| \,.$$

Therefore $|(B_0 - E) \cap B(y, t)| \geq \frac{1}{2}|B(y, t)|$ and (4.5) follows from the A_∞ condition.

Then, we have

$$\begin{aligned}
\int_{\widehat{B}_0-\widehat{\Omega}} |F(y,t)|^2 \frac{\omega(B(y,t))}{t^n} \frac{dy\,dt}{t} &\leq \frac{1}{\alpha}\int_{\widehat{B}_0-\widehat{\Omega}} |F(y,t)|^2 \int_{Z_{(y,t)}} \omega(x)\,dx \frac{dy\,dt}{t^{n+1}} \\
&= \frac{1}{\alpha}\int_Z |F(y,t)|^2 \omega(x)\,dx\, \frac{dy\,dt}{t^{n+1}} \\
&\leq \frac{1}{\alpha}\int_{B_0-E} \omega(x) \left(\int_{\Gamma(x)} |F(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\right) dx \\
&= \frac{1}{\alpha}\int_{B_0-E} \mathcal{V}^2(F)(x)\omega(x)\,dx \,.
\end{aligned}$$

□

Now, we are in a position to proceed with the decomposition into atoms.

Proof of Theorem 3. For $k \in \mathbb{Z}$ let $E_k = \{x : \mathcal{V}(F)(x) > 2^k\}$ and $\Omega_k = \{x : M(\mathcal{X}_{E_k})(x) > \frac{1}{2}\}$. It is not hard to check that, except for a zero measure set, supp $F \subset \cup\widehat{\Omega}_k$. In fact, for any Lebesgue point (x, t) not belonging to any $\widehat{\Omega}_k$, there exists a sequence $\{y_k\} \subset B(x, t)$ with $M(\mathcal{X}_{E_k})(y_k) \leq \frac{1}{2}$. Therefore for any k, $|B(x, t) \cap \{z : \mathcal{V}(F)(z) \leq 2^k\}| \geq \frac{1}{2}|B(x, t)|$, and taking the limit for k tending to $-\infty$, we get

$$|B(x, t) \cap \{z : \mathcal{V}(F)(z) = 0\}| \geq \frac{1}{2}|B(x, t)| \,.$$

From here we easily conclude that for some $y \in B(x, t)$, $F = 0$ a.e. in $\Gamma(y)$ and hence $F(x, t) = 0$.

Now, for each k we make a Whitney decomposition of Ω_k into cubes Q_k^j. Next we choose a family of corresponding concentric balls B_k^j, containing Q_k^j and with radii C-times the diameter of Q_k^j, in such a way that for the sets

$$A_k^j = \widehat{B_k^j} \cap \left(Q_k^j \times (0, \infty)\right) \cap \left(\widehat{\Omega}_k - \widehat{\Omega}_{k+1}\right),$$

it holds that

$$\widehat{\Omega}_k - \widehat{\Omega}_{k+1} \subset \cup_j A_k^j \,.$$

In fact, it is not difficult to see that it is enough to take C greater than $C_0 + 1$, where C_0 is the constant of the Whitney covering.

Now we define $b_k^j = F\chi_{A_k^j}$. It is clear that they are multiples of atoms and that $F = \sum b_k^j$. It remains to show that $\Lambda(\{b_k^j\}) \le C[F]_{T_\eta(\omega)}$. First observe that, by Proposition 3, we have

$$\begin{aligned}\|b_k^j\|^2_{L^2(W)} &\le \int_{\hat{B}_k^j - \hat{\Omega}_{k+1}} |F(y,t)|^2 \frac{\omega(B(y,t))}{t^n}\frac{dy\,dt}{t} \\ &\le C\int_{B_k^j - E_{k+1}} |\mathcal{V}(F)(x)|^2\omega(x)\,dx \\ &\le C2^{2(k+1)}\omega\big(B_k^j\big)\,.\end{aligned}$$

We set $\gamma = [F]^{1/\ell}_{T_\eta(\omega)}$. Since η is assumed concave, and hence of upper type one, we may apply Lemma 1 to get

$$\begin{aligned}\sum_{k,j}\big|B_k^j\big|\eta\left(\frac{\omega\big(B_k^j\big)^{1/2}}{\big|B_k^j\big|\gamma}\|b_k^j\|_{L^2(W)}\right) &\le C\sum_{k,j}\big|Q_k^j\big|\eta\left(2^{k+1}\frac{\omega\big(Q_k^j\big)}{\gamma\big|Q_k^j\big|}\right) \\ &\le C\sum_{k,j}\int_{Q_k^j}\eta\left(\omega(z)\frac{2^{k+1}}{\gamma}\right)dz \\ &\le C\sum_k\int_{\Omega_k}\eta\left(\omega(z)\frac{2^{k+1}}{\gamma}\right)dz\,.\end{aligned}$$

But, by part (b) of the same Lemma 1, there exists $p > 1$ such that $\eta(C\omega) \in A_p$ with a uniform constant. Therefore the Hardy-Littlewood maximal operator is of weak type (p, p) with respect to $\eta(C\omega)$ with a uniform constant. Thus, we have

$$\int_{\{M(\chi_{E_k}) > \frac{1}{2}\}}\eta\left(\frac{2^{k+1}}{\gamma}\omega(z)\right)dz \le C\int_{E_k}\eta\left(\frac{2^{k+1}}{\gamma}\omega(z)\right)dz\,.$$

With this estimate, the above sum over k is bounded by

$$\begin{aligned}C\sum_k\int_{E_k}\eta\left(\frac{2^{k+1}}{\gamma}\omega(z)\right)dz &\le C\int_{\mathbb{R}^n}\sum_{k<\log_2\mathcal{V}(F)(z)}\eta\left(\frac{2^{k+1}}{\gamma}\omega(z)\right)dz \\ &\le C\int_{\mathbb{R}^n}\sum_{k<\log_2\mathcal{V}(F)(z)}\int_{2^{k+1}}^{2^{k+2}}\eta\left(\frac{s}{\gamma}\omega(z)\right)\frac{ds}{s}\,dz \\ &\le C\int_{\mathbb{R}^n}\left(\int_0^{4\omega(z)\mathcal{V}(F)(z)/\gamma}\eta(s)\frac{ds}{s}\right)dz \\ &\le C\int_{\mathbb{R}^n}\eta\left(\frac{\mathcal{V}(F)(z)}{\gamma}\omega(z)\right)dz\,,\end{aligned}$$

where we have used that the positive lower type of η implies $\int_0^t \eta(s)\frac{ds}{s} \le C\eta(t)$. Since by definition of γ, the last quantity is less than or equal to one, the theorem is completely proved. □

5. Proof of the Main Result

Proof of Theorem 2. Part (a). Let $B = B(x_0, r)$ be a ball in $\mathbb{R}^n$. We split f as

$$f = (f - m_B f)\chi_{\tilde{B}} + (f - m_B f)\chi_{\tilde{B}^c} + m_B f = f_1 + f_2 + f_3 ,$$

where $\tilde{B} = B(x_0, 2r)$. Since ψ_t has zero average, $\psi_t * f_3 \equiv 0$. For f_1, using Hölder's inequality, we have

$$\begin{aligned}
I &= \int_{\hat{B}} |\psi_t * f_1|^2(y) \frac{t^n}{\omega(B(y,t))} \frac{dy\,dt}{t} \\
&\le C \int_{\hat{B}} |\psi_t * f_1|^2(y) \frac{\omega^{-1}(B(y,t))}{t^n} \frac{dy\,dt}{t} \\
&= C \int_{\hat{B}} |\psi_t * f_1|^2(y) \left(\int_{B(y,t)} \omega^{-1}(z)\,dz \right) \frac{dy\,dt}{t^{n+1}} \\
&\le C \int_{B(x_0,r)} \omega^{-1}(z) \left(\int_{\Gamma(z)} |\psi_t * f_1|^2(y) \frac{dy\,dt}{t^{n+1}} \right) dz .
\end{aligned}$$

Since $\omega \in A_2$, from the theory of vector valued singular integrals we have that the square function $S_\psi f(z) = (\int_{\Gamma(z)} |\psi_t * f|^2(y) \frac{dy\,dt}{t^{n+1}})^{1/2}$ is bounded from $L^2(\omega^{-1})$ into $L^2(\omega^{-1})$, then

$$\begin{aligned}
I \le C \int_{\mathbb{R}^n} |f_1(z)|^2 \omega^{-1}(z)\,dz &= C \int_{\tilde{B}} |f(z) - m_B f|^2 \omega^{-1}(z)\,dz \\
&\le C \left(\int_{\tilde{B}} |f(z) - m_{\tilde{B}} f|^2 \omega^{-1}(z)\,dz \right. \\
&\qquad \left. + |m_B f - m_{\tilde{B}} f|^2 w^{-1}(\tilde{B}) \right) \\
&\le C \omega(B) \varphi^2(|B|^{1/n}) \|f\|^2_{BMO_\varphi(\omega)} .
\end{aligned}$$

The last inequality is due to the equivalence of norms in $BMO_\varphi(\omega)$ (see comments after Lemma 2) and the fact that $\omega \in A_2$. Now, for f_2, denoting by $B_k = B(x_0, 2^k r)$, we have

$$\begin{aligned}
|\psi_t * f_2|(y) &\le \sum_{k=2}^{\infty} \int_{B_k - B_{k-1}} |f(x) - m_B f| |\psi_t(y-x)|\,dx \\
&\le \sum_{k=2}^{\infty} \int_{B_k - B_{k-1}} |f(x) - m_{B_k} f| |\psi_t(y-x)|\,dx \\
&\quad + \sum_{k=2}^{\infty} \sum_{j=1}^{k} \frac{1}{|B_j|} \left(\int_{B_j} |f(z) - m_{B_j} f|\,dz \right) \left(\int_{B_k - B_{k-1}} |\psi_t(y-x)|\,dx \right) = D_1 + D_2 ,
\end{aligned} \tag{5.1}$$

where we have used the inequality $|f(x) - m_B f| \le |f(x) - m_{B_k} f| + \sum_{j=1}^{k} |m_{B_j} f - m_{B_{j-1}} f|$.

Using that $\psi \in \mathcal{S}$, we get for $(y,t) \in \widehat{B}$

$$\begin{aligned} D_1 &\le C \sum_{k=2}^{\infty} \int_{B_k - B_{k-1}} |f(x) - m_{B_k} f| \frac{t^\alpha}{(t+|y-x|)^{n+\alpha}}\, dx \\ &\le C\left(\frac{t}{r}\right)^\alpha \sum_{k=2}^{\infty} \frac{1}{2^{k\alpha}} \frac{1}{|B_k|} \int_{B_k} |f(x) - m_{B_k} f|\, dx \\ &\le C\left(\frac{t}{r}\right)^\alpha \|f\|_{BMO_\varphi(\omega)} \sum_{k=2}^{\infty} \frac{1}{2^{k\alpha}} \frac{\omega(B_k)}{|B_k|} \varphi\left(2^k r\right) . \end{aligned}$$

If φ is of upper type β, then taking $\alpha = n + \beta$ and recalling that an A_2 weight is in $A_{2-\epsilon}$ for some $\epsilon > 0$, Lemma 2 with $p = 2 - \epsilon$, $B = B_k$ and $E = B$ allows us to obtain

$$\begin{aligned} D_1 &\le C\left(\frac{t}{r}\right)^{n+\beta} \|f\|_{BMO_\varphi(\omega)} \frac{\omega(B)}{|B|} \varphi(r) \sum_{k=2}^{\infty} 2^{k(n(2-\epsilon)-2n-\beta+\beta)} \\ &= C\|f\|_{BMO_\varphi(\omega)} \left(\frac{t}{r}\right)^{n+\beta} \varphi(r) \frac{\omega(B)}{|B|} . \end{aligned}$$

To estimate D_2 we observe that

$$\int_{B_k - B_{k-1}} |\psi_t(y-x)|\, dx \le C \frac{t^\alpha}{\left(t + 2^k r\right)^{n+\alpha}} |B_k| .$$

Therefore, with similar arguments to those used for D_1, we get

$$\begin{aligned} D_2 &\le C\|f\|_{BMO_\varphi(\omega)} \sum_{k=2}^{\infty} \frac{t^\alpha}{\left(2^k r\right)^\alpha} \sum_{j=1}^{k} \frac{\omega(B_j)}{|B_j|} \varphi\left(2^j r\right) \\ &\le C\|f\|_{BMO_\varphi(\omega)} \left(\frac{t}{r}\right)^\alpha \frac{\omega(B)}{|B|} \varphi(r) \sum_{k=2}^{\infty} \frac{1}{2^{k\alpha}} \sum_{j=1}^{k} 2^{jn(2-\epsilon)} 2^{j(\beta-n)} \\ &\le C\|f\|_{BMO_\varphi(\omega)} \left(\frac{t}{r}\right)^{n+\beta} \varphi(r) \frac{\omega(B)}{|B|} . \end{aligned}$$

So we obtain the same estimate for D_1 and D_2. Then, integrating over $\hat{B}$ and applying the A_2 condition, we get

$$\begin{aligned} &\int_{\hat{B}} |\psi_t * f_2|^2(y) \frac{\omega^{-1}(B(y,t))}{t^n} \frac{dy\, dt}{t} \\ &\quad \le C\|f\|^2_{BMO_\varphi(\omega)} \frac{\varphi^2(r)}{r^{2(n+\beta)}} \left(\frac{\omega(B)}{|B|}\right)^2 \int_{\hat{B}} t^{n+2\beta} \omega^{-1}(B(y,t)) \frac{dy\, dt}{t} \\ &\quad \le C\|f\|^2_{BMO_\varphi(\omega)} \frac{\varphi^2(r)}{r^{2(n+\beta)}} \left(\frac{\omega(B)}{|B|}\right)^2 \omega^{-1}(B)|B| \int_0^r t^{n+2\beta} \frac{dt}{t} \\ &\quad \le C \,\|\, f \,\|^2_{BMO_\varphi(\omega)}\, \varphi^2(r)\omega(B) . \end{aligned}$$

Finally, from this estimate and that obtained for I we finish the proof of (a).

Now we turn into the proof of (b). Under our assumptions on ψ, there exists $\bar{\psi} \in \mathcal{S}$ with $\int \bar{\psi} = 0$ such that, for any $g \in \mathcal{S}$,

$$g_\epsilon = \int_\epsilon^{1/\epsilon} \bar{\psi}_t * \psi_t * g \frac{dt}{t} \longrightarrow g$$

pointwise, when ε goes to zero (see, for example, [15], p. 159). The above formula first appeared in [12] and is often referred to as the "Calderón reproducing formula" or "Calderón representation theorem." Furthermore, for $g \in \mathcal{S}_\infty$ (see Proposition 2 for the definition), we may follow the same steps as in [5], p. 122, to conclude that the above convergence occurs also in the topology of $\mathcal{S}$. Therefore, for f as in the hypothesis, $g \in \mathcal{S}_\infty$, and denoting $\tilde{g}(x) = g(-x)$, we have

$$\begin{aligned}
\left(f, \tilde{g}\right) &= \lim_{\epsilon \longrightarrow 0} \left(f, \tilde{g}_\epsilon\right) \\
&= \lim_{\epsilon \longrightarrow 0} \int_\epsilon^{1/\epsilon} \left(f, \bar{\psi}_t * \psi_t * \tilde{g}\right) \frac{dt}{t} \\
&= \lim_{\epsilon \longrightarrow 0} \int_\epsilon^{1/\epsilon} \left(\bar{\psi}_t * f, \psi_t * g\right) \frac{dt}{t} \\
&= \lim_{\epsilon \longrightarrow 0} \int_\epsilon^{1/\epsilon} \left(\int_{\mathbb{R}^n} (\psi_t * f)(x) \left(\bar{\psi}_t * g\right)(x)\, dx \right) \frac{dt}{t} \,,
\end{aligned} \tag{5.2}$$

where for the last equality we use that $\psi_t * f$ and $\bar{\psi}_t * g$ are C^∞ functions and that, as we will see below, the integral is absolutely convergent. We claim that for any pair of measurable functions on $\mathbb{R}^{n+1}_+$, say F and G, we have

$$\int_{\mathbb{R}^{n+1}_+} |F(x,t)||G(x,t)| \frac{dx\,dt}{t} \leq C[dF]^{1/2}_{\varphi,\omega} [G]^{1/\ell}_{T_\eta(\omega)} \,, \tag{5.3}$$

where $dF = |F(x,t)|^2 \frac{t^n}{\omega(B(x,t))} \frac{dx\,dt}{t}$, $\eta^{-1}(\frac{1}{t}) = \frac{1}{t\varphi(t^{1/n})}$ and $\ell = 1/(1+\beta/n)$, being β an upper type for φ, such that $\omega \in A_q$ for some $q < 1 + \frac{1-\beta}{n}$. Such choice of β is possible due to the property '$\omega \in A_p \Rightarrow \omega \in A_{p-\varepsilon}$, for some $\varepsilon > 0$'. Let us notice that if φ is of upper type β, then, from its definition, η^{-1} is of upper type $1 + \beta/n$ and, consequently, η is of lower type ℓ.

To show (5.3) note that if $G \in T_\eta(\omega)$, in view of the atomic decomposition (see Theorem 3), G can be written a.e. as

$$G(x,t) = \sum_j b_j(x,t)$$

in such a way that

$$\Lambda(\{b_j\}) \leq C[G]_{T_\eta(\omega)} \,.$$

Observe that Theorem 3 can be applied since, as we remarked above, $\omega \in A_q \subset A_{1+\frac{1-\beta}{n}} = A_{2-(1/\ell - 1/n)}$.

Therefore, if B_j is the ball associate to b_j such that $\operatorname{supp}(b_j) \subset \hat{B}_j$, we get

$$\begin{aligned}\left|\int_{\mathbb{R}^{n+1}_+} F(x,t)G(x,t)\frac{dx\,dt}{t}\right| &\leq \sum_j \left(\int_{\hat{B}_j} |F(x,t)|^2 \frac{t^n}{\omega(B(x,t))}\frac{dx\,dt}{t}\right)^{1/2} \\ &\quad \left(\int_{\hat{B}_j} |b_j(x,t)|^2 \frac{\omega(B(x,t))}{t^n}\frac{dx\,dt}{t}\right)^{1/2} \\ &\leq [dF]^{1/2}_{\varphi,\omega} \sum_j \omega(B_j)^{1/2}\varphi\big(|B_j|^{1/n}\big)\|b_j\|_{L^2(W)} .\end{aligned}$$

Now, if σ denotes the last sum, it is easy to check that

$$\sum_j |B_j|\eta\left(\frac{\omega(B_j)^{1/2}}{|B_j|\sigma}\|b_j\|_{L^2(W)}\right) \geq 1 .$$

In fact, replacing in σ, $\varphi(|B_j|^{1/n})$ by $1/(|B_j|\eta^{-1}(1/|B_j|))$, the above inequality follows using the fact that η is of upper type less than or equal to one. Therefore, in view of the definition of Λ, we get

$$\sigma^\ell \leq \Lambda(\{b_j\})$$

which together with (4.4) gives (5.3). Now, applying this inequality in (5.2), we have

$$\begin{aligned}\big|(f,\tilde{g})\big| &\leq C[d\mu]^{1/2}_{\varphi,\omega}\left[\left(\int_{\Gamma(.)}\big|\bar{\psi}_t * g\big|^2(y)\frac{dy\,dt}{t^{n+1}}\right)^{1/2}\omega(.)\right]^{1/\ell}_{L^\eta} \\ &= C[d\mu]^{1/2}_{\varphi,\omega}\Big[\big(S_{\bar{\psi}}g\big)\omega\Big]^{1/\ell}_{L^\eta} ,\end{aligned}$$

where $d\mu$ denotes the measure associate to $\psi_t * f$, as given in the statement of the theorem. Then, it is clear that part (b) of our theorem would follow from the above inequality by using Propositions 1 and 2, provided we can prove that

$$[(S_{\bar{\psi}}g)]_{L^\eta} \leq C[g]_{H^q_\eta(\omega)} . \tag{5.4}$$

Notice that we may apply Proposition 2 since $\omega \in A_q$ for $q < 2 + \frac{1}{n} - \frac{1}{\ell}$. In order to check that (5.4) holds, we recall, as in the proof of part (a), that $S_{\bar{\psi}}$ is nothing else but the square function.

Then, the theory of vector valued singular integrals allows us to assert that it is bounded on $L^q(\omega)$ for any $\omega \in A_q$. Now, for a function b in $L^q(\omega)$ with compact support on a ball $B_0 = B(x_0, r_0)$ and zero average, Jensen's inequality leads us to the following estimate

$$\begin{aligned}\int_{\tilde{B}_0} \eta(S_{\bar{\psi}}b(x)\omega(x))\,dx &\leq C|B_0|\eta\left(\frac{\omega(B_0)^{1/q'}}{|B_0|}\|S_{\bar{\psi}}b\|_{L^q(\omega)}\right) \\ &\leq C|B_0|\eta\left(\frac{\omega(B_0)^{1/q'}}{|B_0|}\|b\|_{L^q(\omega)}\right) ,\end{aligned} \tag{5.5}$$

where $\tilde{B}_0 = B(x_0, 2r_0)$. On the other hand, it is also known that

$$S_{\bar{\psi}}b(x) \leq C\left(\frac{r_0}{|x-x_0|}\right)^{n+1}\frac{\|b\|_{L^q(\omega)}}{\omega(B_0)^{1/q}} ,$$

for $x \notin \tilde{B}_0$. Then, a standard reasoning using this estimate, Jensen's inequality and the fact that $\omega \in A_q$ allows us to get

$$\begin{aligned}\int_{\tilde{B}_0^c} \eta(S_{\tilde{\psi}} b(x)\omega(x))\, dx &\le C|B_0| \sum_{j=2}^{\infty} 2^{jn} \\ &\quad \times \eta\left(\frac{\omega(B_0)^{1/q'}}{|B_0|} \frac{\|b\|_{L^q(\omega)}}{2^{(2n-nq+1)j}}\right) \\ &\le C|B_0|\eta\left(\frac{\omega(B_0)^{1/q'}}{|B_0|}\|b\|_{L^q(\omega)}\right)\end{aligned} \tag{5.6}$$

because of our assumptions on q and ℓ. Therefore, if b is an atom in $H^q_\eta(\omega)$, (5.5) and (5.6) imply (5.4) for $g = b$. Consequently, (5.4) holds for every g in $H^q_\eta(\omega)$, finishing our proof. □

6. A Weighted Triebel-Lizorkin Space

As we said in Section 1, Bui and Taibleson introduced in [1] a weighted version of the Triebel-Lizorkin spaces $\dot{F}^{\alpha}_{\infty,q}$. Specifically, they take $\psi \in \mathcal{S}$ with supp $\hat{\psi} \subset \{\xi : \frac{1}{2} \le |\xi| \le 2\}$ and $\sum_{j=-\infty}^{\infty} |\hat{\psi}(2^{-j}\xi)|^2 = 1$ for $|\xi| \ne 0$. Then, for $\omega \in A_\infty$, the space $\dot{F}^{\alpha,\omega}_{\infty,q}$, $\alpha \in \mathbb{R}$, $0 < q < \infty$ is defined as the set of f in $\mathcal{S}'/\mathcal{P}$ such that

$$\sup_Q \left(\frac{1}{\omega(Q)} \int_Q \sum_{j=-\log_2(\ell(Q))}^{\infty} \left(2^{j\alpha}|\psi_{2^{-j}} * f|(x)\right)^q \omega(x)\, dx\right)^{1/q} < \infty\,, \tag{6.1}$$

where Q denotes a dyadic cube in $\mathbb{R}^n$ with side length $\ell(Q)$. For the case $\alpha = 0$ and $q = 2$ we have the following result.

Proposition 4. *The space BMO is contained in* $\dot{F}^{0,\omega}_{\infty,2}$ *for any* ω *in* A_2. *Moreover, if in addition, we assume* $\omega \in A_1$, *then both spaces coincide.*

In view of the above proposition and Lemma 2, we consider that $\dot{F}^{\alpha,\omega}_{\infty,q}$ should be rather defined as the set of f in $\mathcal{S}'/\mathcal{P}$ such that

$$\sup_B \left(\frac{1}{\omega(B)} \int_{\hat{B}} \left(t^{-\alpha}|\psi_t * f|(x)\right)^q \frac{t^n}{\omega(B(x,t))} \frac{dx\, dt}{t}\right)^{1/q} < \infty\,.$$

When $\alpha = 0$ and $q = 2$, note that for $\omega \in A_{1+1/n}$ and ψ in $\mathcal{S}$ satisfying a Tauberian condition, Lemma 2 allows us to obtain two facts: First, a weighted version of the well known result $BMO \simeq \dot{F}^{0}_{\infty,2}$, and, second, as a consequence, that the definition of the space does not depend on the choice of ψ.

Proof of Proposition 4. Let f be in BMO. Given a cube Q, we split f as follows

$$f = (f - m_{Q,\omega} f)\mathcal{X}_{\tilde{Q}} + (f - m_{Q,\omega} f)\chi_{\tilde{Q}^c} + m_{Q,\omega} f = f_1 + f_2 + f_3\,,$$

where $m_{Q,\omega} f = \frac{1}{\omega(Q)} \int_Q f\omega$ and $\tilde{Q}$ denotes the concentric cube with Q and side length

$2\ell(Q)$. In order to prove that f satisfies (6.1) for $\alpha = 0$ and $q = 2$, we first estimate

$$\begin{aligned} I &= \int_Q \sum_{j=-\log_2 \ell(Q)}^{\infty} |\psi_{2^{-j}} * f_1|^2(x)\omega(x)\,dx \\ &\le \int_{\mathbb{R}^n} \sum_{j=-\infty}^{\infty} |\psi_{2^{-j}} * f_1|^2(x)\omega(x)\,dx\ . \end{aligned}$$

It is well known that this discrete version of the square function, that is $G(f)(x) = (\sum_{j=-\infty}^{\infty} |\psi_{2-j} * f|^2(x))^{1/2}$, is bounded on $L^p(\omega)$ as long as $\omega \in A_p$ (see, for example, [4], p. 125, or, for more details, [9]). So, we have

$$\begin{aligned} I \le C \int_{\mathbb{R}^n} |f_1(x)|^2\omega(x)\,dx = C \int_{\tilde{Q}} |f - m_{Q,\omega} f|^2 \omega(x)\,dx \\ \le C\|f\|_{BMO}^2\,\omega(Q)\ , \end{aligned}$$

where the last inequality follows from the fact that

$$\sup_Q \left(\frac{1}{\omega(Q)} \int_Q |f - m_{Q,\omega} f|^p \omega \right)^{1/p} \simeq \|f\|_{BMO}\ ,$$

(see, for example, [6], Corollary 2.4).

On the other hand, denoting by Q_k the concentric cube with Q and side length $2^k\ell(Q)$, a similar reasoning to that applied for getting estimate (5.1) allows us to get

$$\begin{aligned} |\psi_{2^{-j}} * f_2|(y) \le C\left(\frac{2^{-j}}{\ell(Q)}\right)^{\alpha} \Bigg(\sum_{k=2}^{\infty} \frac{1}{2^{k\alpha}} \frac{1}{|Q_k|} \int_{Q_k - Q_{k-1}} |f(x) - m_{Q_k,\omega} f|\,dx \\ + \sum_{k=2}^{\infty} \sum_{i=2}^{k} \frac{1}{\omega(Q_i)} \int_{Q_i} |f(x) - m_{Q_i,\omega} f|\omega(x)\,dx \\ \times \int_{Q_k - Q_{k-1}} |\psi_{2^{-j}}(y-x)|\,dx \Bigg)\ , \end{aligned}$$

for every $j \in Z$, for $\alpha > 0$ fixed. Then, since $\omega \in A_2$, Hölder's inequality and the fact that the norms are equivalent, yield to

$$|\psi_{2^{-j}} * f_2|(y) \le C\left(\frac{2^{-j}}{\ell(Q)}\right)^{\alpha} \|f\|_{BMO}\ .$$

Now, from this inequality, we get

$$\begin{aligned} \int_Q \sum_{j=-\log_2 \ell(Q)}^{\infty} |\psi_{2^{-j}} * f_2|^2(x)\omega(x)\,dx &\le C\frac{\omega(Q)}{(\ell(Q))^{\alpha}} \|f\|_{BMO} \sum_{j=-\log_2 \ell(Q)}^{\infty} 2^{-j\alpha} \\ &\le C\omega(Q)\|f\|_{BMO}\ . \end{aligned}$$

So, the above estimate, the one obtained for I, and the fact that $\psi_2 - j * f_3 = 0$ prove $f \in \dot{F}^{0,\omega}_{\infty,2}$, in other words, $BMO \subset \dot{F}^{0,\omega}_{\infty,2}$.

Now, assuming $\omega \in A_1$ and using the well known unweighted result $BMO \simeq \dot{F}^{0}_{\infty,2}$ (see [4], for instance), we can write

$$\begin{aligned}\|f\|^2_{BMO} &\leq \sup_Q \frac{1}{|Q|}\int_Q \sum_{j=-\log_2 \ell(Q)}^{\infty} |\psi_{2^{-j}} * f|^2(x)\,dx \\ &\leq C \sup_Q \frac{1}{\omega(Q)}\int_Q \sum_{j=-\log_2 \ell(Q)}^{\infty} |\psi_{2^{-j}} * f|^2(x)\omega(x)\,dx\,.\end{aligned}$$

Clearly, this implies $\dot{F}^{0,\omega}_{\infty,2} \subset BMO$. □

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Received August 01, 2005

Revision received March 20, 2007

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