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The Sturm-Hurwitz Theorem and its Extensions

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ABSTRACT. *The following principle is well-known in Harmonic Analysis: If a real function has a spectral gap at the origin then it must have many sign changes. We obtain some sharp estimates showing that the set of positivity of such functions cannot be too small. We also extend the principle above to complex functions: If a complex function has a spectral gap at the origin then the variation of argument of this function must be large.*

1. Sturm-Hurwitz Theorem and its Extensions

The reader is referred to [5] which gives an excellent account of the history of results on sign changes of real functions having a spectral gap at the origin. We shall state the classical Sturm-Hurwitz theorem and some of its extensions dealing with the lower density of sign changes.

(1) Functions on the unit circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

We shall identify $\mathbb{T}$ with the interval $[-\pi, \pi)$. For a function $f \in L^2(\mathbb{T})$ we denote by $\{f_j\} \in l^2(\mathbb{Z})$ the sequence of its Fourier coefficients:

$$f(t) \sim \sum_{j \in \mathbb{Z}} f_j e^{ijt} . \tag{1.1}$$

A function $f \in L^2(\mathbb{T})$ has a spectral gap (m, l) if $f_j = 0$ for $m < j < l$.

Theorem 1 (Sturm-Hurwitz). *Suppose a nontrivial function f is real and continuous on $\mathbb{T}$, and has a spectral gap $(-n, n)$ for some integer $n > 0$. Then f has at least $2n$ sign changes on $\mathbb{T}$.*

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Sturm stated this result for the case of trigonometric polynomials while Hurwitz generalized it to Fourier series. The interest and importance of the Sturm-Hurwitz theorem has been highlighted in several recent works of V. I. Arnold [2, 3] (see also [4] for some multidimensional results). In [2] Arnold outlines three very different proofs of this result.

(2) Functions on the set of integers $\mathbb{Z}$.

Let $\{f_j\}_{j=1}^N$ be a finite sequence of real numbers. We denote by $S(\{f_j\}, 1 \leq j \leq N)$ the number of sign changes of the sequence defined as the number of integers m, $1 \leq m < N$, such that $f_m f_{m+l} < 0$ where $m + l \leq N$, and l is the smallest positive integer such that $f_{m+l} \neq 0$.

A sequence $\{f_j\} \in l^2(\mathbb{Z})$ has a spectral gap on an interval $I \subseteq \mathbb{T}$ if the function f in (1.1) vanishes a.e. on I.

Theorem 2 (Logan [9]). *Let $\{f_j\} \in l^1(\mathbb{Z})$ be a nontrivial sequence of real numbers with a spectral gap $(-a\pi, a\pi)$, for some number $0 < a < 1$. Then*

$$\liminf_{j\to\infty} \frac{S(\{f_j\}, 1 \leq j \leq n)}{n} \geq a \ .$$

(3) Functions on the real line $\mathbb{R}$.

Let $f \in L^2(\mathbb{R})$ be a function on the real line $\mathbb{R}$. We denote by $\hat{f}$ its Fourier transform:

$$\hat{f}(x) \sim \int_{\mathbb{R}} e^{itx} f(t)\, dt \ .$$

A function $f \in L^2(\mathbb{R})$ has a spectral gap on an interval $I \subset \mathbb{R}$, if $\hat{f}(x) = 0$ a.e. for $x \in I$.

In fact, Logan [9] formulates a result about the sign changes of real functions, while Theorem 2 follows from the proof of that result. Namely, he proves that if a real continuous function $f \in L^1(\mathbb{R})$ has a spectral gap $(-\infty, -b\pi) \cup (-a\pi, a\pi) \cup (b\pi, \infty)$ for some $0 < a < b$ (i.e., the spectrum of f is compact), then f has a large number of sign changes:

$$\liminf_{r\to\infty} \frac{S(f, (0, r))}{r} \geq a \ . \tag{1.2}$$

Here $S(f, (0, r))$ denotes the number of sign changes of f on $(0, r)$. Since the right-hand side of (1.2) does not depend on the size of the support of $\hat{f}$, Logan conjectured that the assumption of compactness of support of $\hat{f}$ is redundant. This was confirmed in a recent result [5]. We state a corollary of the result from [5].

Theorem 3 (Eremenko-Novikov [5]). *Let $f \in L^1(\mathbb{R})$ be a nontrivial continuous real function with a spectral gap $(-a\pi, a\pi)$, for some number $a > 0$. Then* (1.2) *holds.*

It is shown in [5] that (1.2) still holds for some wider classes of functions and measures characterized by their rates of growth at infinity, but if the growth is faster than a certain threshold, the above statement is no longer true (see [5] for details).

The aim of this article is twofold: We show that the positivity set of functions having a spectral gap at the origin cannot be too small. We also show that the theorems above hold for complex functions, provided we replace the number of sign changes by the variation of argument. To emphasize the simple nature of the results, we shall avoid complications which arise in a general approach. Therefore, all functions we consider will be continuous and bounded, though the results can be extended to more general classes of functions and distributions.

2. Sturm Theorem on Cyclic Subgroups and Positivity Sets of Trigonometric Polynomials

The following known result is due to Arnold (see [10]): If a real trigonometric polynomial of degree m with vanishing constant term changes sign on a circle exactly twice, then the ratio of the lengths of the arcs they determine is at least $1/m$. Equality, up to a multiplicative constant and rotation, is attained by a polynomial whose double roots form a regular polygon with $m+1$ sides, with the exception of two consecutive simple roots.

We shall obtain a variant of the Sturm theorem on cyclic subgroups of $\mathbb{T}$, and show that the estimate in Arnold's result and some similar results can be deduced from it.

Let us denote by $\#(I)$ the number of elements of a set I, and by $\operatorname{meas}(I)$ the measure of I. Given a real function f on $\mathbb{T}$, we shall denote by $\{f > 0\} := \{t \in \mathbb{T} : f(t) > 0\}$ the positivity set of f, $\{f = 0\} := \{t \in \mathbb{T} : f(t) = 0\}$ the zero set of f, and $\{f \geq 0\} := \{f > 0\} \cup \{f = 0\}$.

Suppose that $\Gamma \subset \mathbb{T}$ is a finite set, $\#(I) = n$. The number $S(f, \Gamma)$ of sign changes of f on Γ is defined as follows: We set $S(f, \Gamma) = 0$ if $\Gamma \subseteq \{f = 0\}$. Suppose f does not vanish on Γ. Pick $\gamma_1 < \ldots < \gamma_n < \gamma_{n+1} = \gamma_1 + 2\pi$ such that $\Gamma = \{e^{i\gamma_j}\}_{j=1}^{n}$. Then $S(f, \Gamma)$ is the number of j, $1 \leq j \leq n$, such that $f(\gamma_j) f(\gamma_{j+1}) < 0$. In general, we set $S(f, \Gamma) = S(f, \Gamma \setminus \{f = 0\})$. Clearly, $S(f, \Gamma)$ is either zero or an even number. Denote by $\mathbb{T}_N = \{\exp(2\pi i j/N)\}_{j=0}^{N-1}$ the cyclic group of N-th roots of unity, and $\mathbb{T}_N + \theta := \{\exp(i\theta + 2\pi i j/N)\}_{j=0}^{N-1}$ the rotation of $\mathbb{T}_N$ by θ.

We say that f in (1.1) is a trigonometric polynomial of degree (at most) m if $f_j = 0$ for $|j| > m$. If such a polynomial has a spectral gap $(-n, n)$, $n \leq m$, then, by the Sturm theorem, it must have $\geq 2n$ sign changes on the unit circle $\mathbb{T}$. We show that unless $f(t) = a \sin mt$ (i.e., $n = m$), f must have $\geq 2n$ sign changes on the cyclic subgroup $\mathbb{T}_{m+n}$.

Theorem 4. *Suppose that $\theta \in [-\pi, \pi)$ and that a real trigonometric polynomial f of degree m has a spectral gap $(-n, n)$, for some $0 < n \leq m$. Then either $f(t) = a \sin m(t - \theta)$, for some $a \in \mathbb{R}$ (i.e., $n = m$ and f is trivial on $\mathbb{T}_{2m} + \theta$), or f must have at least $2n$ sign changes on $\mathbb{T}_{m+n} + \theta$.*

Since the number of positive values of f on $\mathbb{T}_{m+n} + \theta$ is at least a half of the number of its sign changes on this set, we obtain the following.

Corollary 1. *Suppose that $\theta \in [-\pi, \pi)$ and that a real trigonometric polynomial f of degree m has a spectral gap $(-n, n)$ for some $0 < n \leq m$. Then either $f(t) = a \sin m(t-\theta)$, for some $a \in \mathbb{R}$, or $\#\left(\{f > 0\} \cap (\mathbb{T}_{m+n} + \theta)\right) \geq n$.*

One can easily reformulate Theorem 4 and Corollary 1 in terms of the discrete Fourier transform.

Proof of Theorem 4. Observe that it is easy to prove Theorem 4 by contradiction using the orthogonality of f and any polynomial of degree $< n$ on the group $\mathbb{T}_{m+n}$. We choose another method, which also works in a more general situation of Fourier transforms, see Section 5 below.

Clearly, it suffices to prove the theorem for $\theta = 0$. Assume first that the number $m + n$ is even. Set $l = (m+n)/2$ and $k = (m-n)/2$. Then f can be written as follows:

$$f(t) = \Re\left(\sum_{j=l-k}^{l+k} c_j e^{ijt}\right) = \Re\left(e^{ilt} \sum_{j=-k}^{k} c_{j+l} e^{ijt}\right), t \in \mathbb{T} .$$

Hence,

$$f\left(\frac{\pi j}{l}\right) = (-1)^j \varphi\left(\frac{\pi j}{l}\right),\ j = 0, 1, \ldots, 2l-1,\ \varphi(t) := \Re\left(\sum_{j=-k}^{k} c_{j+l} e^{ijt}\right).$$

We have to show that $S(f, \mathbb{T}_{2l}) \geq 2n$.

Observe that $\varphi \equiv 0$ if and only if $k = 0$ (so that $n = m$ and $l = m$) and $\Re(c_l) = 0$. This implies $f(t) = a \sin mt$, where $a = -\Im(c_l)$.

Assume that $\varphi \neq 0$. Since the degree of φ is at most k, it has at most $2k$ zeros (and so at most $2k$ sign changes) on $\mathbb{T}$. Let us show that $S(f, \mathbb{T}_{2l}) \geq 2l - \#\{\varphi(t) = 0\} \geq 2l - 2k = 2n$. If $\varphi(t)$ does not vanish on $\mathbb{T}_{2l}$, then $S(f, \mathbb{T}_{2l}) = 2l - S(\varphi, \mathbb{T}_{2l}) \geq 2l - \#\{\varphi(t) = 0\} \geq 2n$. Otherwise, by a small rotation of the zeros of φ which lie on $\mathbb{T}_{2l}$, one can find a real trigonometric polynomial $\varphi^*(t)$ of degree $\leq k$ which does not vanish on $\mathbb{T}_{2l}$, and such that the number of sign changes in the sequence $f^*(\pi j/l) := (-1)^j \varphi^*(\pi j/l), 0 \leq j \leq l$, is the same as in the sequence $f(\pi j/l), 0 \leq j \leq l$.

If the number $m + n$ is odd, we apply the argument above to the function $f(2t)$ to show that it must have at least $4n$ sign changes on $\mathbb{T}_{2(m+n)}$, so that $f(t)$ has at least $2n$ sign changes on $\mathbb{T}_{m+n}$. □

We apply Theorem 4 to obtain two known results (proven in [11] and [7] by a different approach) on the size of the positivity set and the size of the longest positivity arc.

Corollary 2. *Suppose $f(t)$ in (1.1) is a nontrivial real trigonometric polynomial of degree m with a spectral gap $(-n, n)$, $0 < n \leq m$. Then*

(i) ([11]) $\operatorname{meas}(\{f > 0\}) \geq 2\pi n/(n+m)$,

(ii) ([7]) *if f is nonnegative on an arc $I \subset \mathbb{T}$ then* $\operatorname{meas}(I) \leq 2\pi(m - n + 1)/(m + n)$.

The inequalities in (i) *and* (ii) *are sharp.*

Proof.

(i) Since $\operatorname{meas}(\{f = 0\}) = 0$, it suffices to establish the inequality in (i) for the set $\{f \geq 0\}$.

Corollary 1 implies

$$\#(\{f \geq 0\} \cap (\mathbb{T}_{m+n} + \theta)) \geq n,\ \ \theta \in [-\pi, \pi).$$

Hence, the set $\{f \geq 0\}$ and its $m + n - 1$ translations $\{f \geq 0\} + 2\pi j/(n + m)$, $j = 1, \ldots, n + m - 1$, cover the circle with multiplicity n, i.e., each point in the circle belongs to at least n of these sets. This proves (i).

The sharpness of (i) for $n = 1$ follows from the example:

$$f(t) = \sum_{j=1}^{m} \sin\left(\frac{j\pi}{m+1}\right) \cos jt = \sin\left(\frac{\pi}{m+1}\right) \frac{\cos^2 \frac{(m+1)t}{2}}{\cos t - \cos \frac{\pi}{m+1}}.$$

To show the sharpness of (i) for $m + n$ even, we set

$$f(t) := -\frac{\sin^2 \frac{(n+m)t}{2}}{\prod_{j=0}^{n-1} \sin\left(t - \frac{2\pi j}{n+m}\right)}. \tag{2.1}$$

One can check that f is a trigonometric polynomial of degree m with a spectral gap $(-n, n)$ (see [11] for similar examples), and the zero set of f is $\mathbb{T}_{m+n}$. Moreover, f has $2n$ sign changes which occur at the zeros of the denominator, i.e., the points $t = 2\pi j/(m+n)$ and $t = \pi + 2\pi j/(m+n)$, $j = 0, \dots, n-1$. Since $f(t)$ is negative in small left neighborhoods of the origin and π, there are exactly n intervals of the length $2\pi/(n+m)$ each where f is positive, i.e., the measure of its positivity set is exactly $2\pi n/(n+m)$. Examples for odd $m+n$ can be found in [11].

To verify (ii) we observe that f has at least $2n-2$ sign changes on $\mathbb{T} \setminus I$, so that the intersection of $\mathbb{T} \setminus I$ with $\mathbb{T}_{m+n} + \theta$ contains at least $2n-1$ points, for each $\theta \in [-\pi, \pi)$. Hence, $\text{meas}(\mathbb{T} \setminus I) \geq 2\pi(2n-1)/(m+n)$, which proves (ii). The sharpness of (ii) follows from the example considered by Yudin (see [11]):

$$f(t) := \frac{(-1)^n \cos^2 \frac{(n+m)t}{2}}{\prod_{j=0}^{n-1} \left(\cos\left(t - \frac{4\pi j}{n+m}\right) - \cos \frac{\pi}{n+m} \right)} .$$

One can check that f is a trigonometric polynomial of degree m with a spectral gap $(-n, n)$ (see [11]). Since the denominator vanishes at $t = 4\pi j/(n+m) \pm \pi/(n+m) + 2\pi k$, $j = 0, \dots, n-1, k \in \mathbb{Z}$, and $f(0) < 0$, we see that f is nonnegative on the interval $[4\pi(n-1)/(n+m) + \pi/(n+m), 2\pi - \pi/(n+m)]$, the length of this interval being $2\pi(m-n+1)/(n+m)$. □

Observe also that Theorem 4 and Corollary 1 are sharp in the sense that for every $0 < n < m$ and $N < m+n$ there exists a nontrivial trigonometric polynomial f of degree m with a spectral gap $(-n, n)$ which has $\leq n-1$ positive values (and so $\leq 2n-2$ sign changes) on $\mathbb{T}_N$.

In the case $n = 1$ (only constant term is missing), Corollary 2 (i) follows from an elegant result on measure-preserving transformations due to Tabachnikov [10].

3. Positive Fourier Coefficients

Recall that a sequence $\{f_j\} \in l^2(\mathbb{Z})$ has a spectral gap on an interval $I \subset \mathbb{T}$ if the function f in (1.1) vanishes a.e. on I. We shall denote by $\{f_j > 0\} := \{j : f_j > 0\}$ the positivity set of the sequence. We are interested in the size of this set for the sequences with a spectral gap at the origin.

Theorem 5.

(i) *Let* $\{f_j\} \in l^1(\mathbb{Z})$ *be a nontrivial real sequence with a spectral gap* $(-a\pi, a\pi)$ *for some* $0 < a < 1$. *Then*

$$\liminf_{j\to\infty} \frac{\#(\{f_j > 0\} \cap \{1, \dots, n\})}{n} \geq \frac{a}{2} .$$

(ii) *For every* $0 < a < 1$ *there is a nontrivial real sequence* $\{f_j\} \in l^1(\mathbb{Z})$ *with a spectral gap* $(-a\pi, a\pi)$ *such that*

$$\lim_{j\to\infty} \frac{\#(\{f_j > 0\} \cap \{1, \dots, n\})}{n} = \frac{a}{2} . \tag{3.1}$$

Proof. Statement (i) follows from Theorem 2 and the observation that the number of positive elements in any finite real sequence $f_1, \dots, f_n$ cannot be smaller then a half of the number of sign changes: $\#(\{f_j > 0\} \cap \{1, \dots, n\}) \geq S(\{f_j\}, 1 \leq j \leq n)/2$.

Let us establish (ii) for a rational. Write $a = (q-p)/q$ where $0 < p < q$ are some even numbers, and set

$$\varphi(x) := \frac{1}{(x+1/2)(x+3/2)} \prod_{j=0}^{p-1} \sin\left(\pi \frac{2x-2j-1}{2q}\right) .$$

Set $f_j := (-1)^{j+1}\varphi(j)$, $j \in \mathbb{Z}$. Statement (ii) of the theorem now follows from the following lemma:

Lemma 1. *The sequence $\{f_j\}$ belongs to $l^1(\mathbb{Z})$, has a spectral gap $[-a\pi, a\pi]$ and satisfies* (3.1).

Proof. The function φ is an entire function of exponential type $p\pi/q$ and is integrable on the real line. It follows that its Fourier transform $\hat{\varphi}$ is continuous on the real line and vanishes outside $[-\pi p/q, \pi p/q]$. Since

$$\hat{\varphi}(t) = \sum_{j\in\mathbb{Z}} \varphi(j)e^{ijt},\ t \in \mathbb{T} ,$$

we have

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}(t+2\pi j) = \sum_{j\in\mathbb{Z}} \varphi(j)e^{ijt},\ t \in \mathbb{R} .$$

We see that

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}(t+\pi+2\pi j) = \sum_{j\in\mathbb{Z}} (-1)^j\varphi(j)e^{ijt} = 0,\ t \in \left[-\pi+\frac{\pi p}{q}, \pi-\frac{\pi p}{q}\right] = [-a\pi, a\pi] .$$

This gives

$$\sum_{j\in\mathbb{Z}} f_j e^{ijt} = -\sum_{j\in\mathbb{Z}} (-1)^j\varphi(j)e^{ijt} = 0,\ t \in [-a\pi, a\pi] .$$

Hence, the sequence $\{f_j\}$ has a spectral gap $[-a\pi, a\pi]$.

Observe that $\varphi(x)(x+1/2)(x+3/2)$ is a $2q$-periodic function, so that the sign of f_j is also $2q$-periodic for $j \geq 0$. Moreover, $\varphi(0) > 0$, $\varphi(q) > 0$, and on the interval $[0, 2q)$, the function φ changes its sign at the points $t = j + 1/2$ and $t = j + 1/2 + q$, $j = 0, \ldots, p-1$. We see that $f_j < 0$ for $j \in \{0, 1, \ldots, p\} \cup \{q, q+1, \ldots, q+p\}$. Hence, for $0 \leq j < 2q$, the f_j are positive if and only if $j = p+1, p+3, \ldots, q-1$, or $j = q+p+1, q+p+3, \ldots, 2q-1$. This gives

$$(2q)^{-1}\#\left(\{f_j > 0\} \cap \{0, 1, \ldots, 2q-1\}\right) = \frac{q-p}{2q} = \frac{a}{2} ,$$

which proves the lemma. □

We shall now sketch the proof for a irrational. Choose two large even integers $0 < p < q$ such that $p/q < 1-a$, and the number $\delta := (1-a-p/q)/2$ is sufficiently small (it suffices to assume that $\delta < 1/(8q)$).

For every $k \in \mathbb{N}$ fix an integer $l(k) \in \mathbb{N}$ such that $|ql(k) - k/\delta| < q$. Denote $\Lambda := \{l = l(k), k \in \mathbb{N}\}$, and set

$$\psi(x) := \varphi(x) \prod_{l\in\Lambda} \left(1 - \frac{x^2}{(p+1/2+ql)^2}\right)\left(1 - \frac{x^2}{(p+3/2+ql)^2}\right) ,$$

where φ is the function defined above. The l^∞-distance between the sequence $\{p+1/2+ql, l \in \Lambda\}$ and the sequence $(1/\delta)\mathbb{N} := \{k/\delta, k \in \mathbb{N}\}$ does not exceed $p+1/2+q < 2q$, which is a small number compared to $1/\delta$. The same is true for the distance between $\{p+3/2+ql, l \in \Lambda\}$ and $(1/\delta)\mathbb{N}$. Hence, the infinite product in the right-hand side can be regarded as a 'small' perturbation of the product

$$\left(\prod_{k\in\mathbb{N}}\left(1-\frac{\delta^2x^2}{k^2}\right)\right)^2 = \left(\frac{\sin(\pi\delta x)}{\pi\delta x}\right)^2 .$$

Therefore, one may check that ψ is an entire function of exponential type $\pi(p/q+2\delta) = \pi(1-a)$, $\psi \in L^1(\mathbb{R})$ and $\{\psi(j), j \in \mathbb{Z}\} \in l^1(\mathbb{Z})$.

Set $f_j := (-1)^{j+1}\psi(j)$, $j \in \mathbb{Z}$. The same argument as in Lemma 1 shows that the sequence f_j has a spectral gap $[-a\pi, a\pi]$. Observe that the set of all positive roots of ψ is as follows:

$$\left(\left\{\frac{1}{2},\dots,p-\frac{1}{2}\right\}+q(\mathbb{N}\setminus\Lambda)\right)\bigcup\left(\left\{\frac{1}{2},\dots,p-\frac{1}{2},p+\frac{1}{2},p+\frac{3}{2}\right\}+q\Lambda\right) .$$

One may check that $\psi(ql) > 0, l = 0, 1, \cdots$, which gives $f_j < 0$ for $j = ql, \dots, p+ql, l \in \mathbb{N}\setminus\Lambda$, and $j = ql, \dots, p+2+ql, l \in \Lambda$. Outside this values of j, the sequence f_j shifts its sign. Hence,

$$\#\left(\{f_j > 0, ql \le j \le q(l+1)\}\right) = \begin{cases} \frac{q-p}{2} & l \in \mathbb{N}\setminus\Lambda , \\ \frac{q-p}{2}-1 & l \in \Lambda . \end{cases}$$

Since the l^∞-distance between Λ and $(1/q\delta)\mathbb{N}$ is finite, the density of the positivity set of f_j for $j > 0$ is as follows:

$$\frac{q-p}{2q}(1-q\delta)+\left(\frac{q-p}{2q}-\frac{1}{q}\right)q\delta = \frac{a}{2} .$$

□

4. Positivity Sets of Fourier Transforms

To include the trigonometric polynomials into consideration, we shall consider real functions f which can be represented as the Fourier-Stieltjes transform of a finite complex-valued Borel measure. Such functions f admit a representation

$$f(x) = \Re\left(\int_0^\infty e^{isx}\,d\sigma(s)\right) , \tag{4.1}$$

where σ is a complex finite measure. The function f has a spectral gap on a symmetric interval $I, -I = I$, if $\sigma = 0$ on $I \cap [0,\infty)$.

We are interested in the size of the positivity set $\{f > 0\} = \{x \in \mathbb{R} : f(x) > 0\}$ when f has a spectral gap at the origin.

Let us first assume that the support of σ is compact, i.e., there exist $0 < a < b$ such that

$$d\sigma = 0 \text{ on } [0, a\pi) \cup (b\pi, \infty) . \tag{4.2}$$

The following result extends Corollary 2 (i) to Fourier-Stieltjes transforms.

Theorem 6. *Let f be a nontrivial real function in* (4.1) *satisfying* (4.2)*. Then*

$$\liminf_{r\to\infty} \frac{\operatorname{meas}(\{f>0\}\cap(0,r))}{r} \geq \frac{a}{a+b} . \tag{4.3}$$

Theorem 6 is sharp: It is possible to show that for every $0<a<b$ there exists a function f in (4.1) which satisfies (4.2), and

$$\lim_{r\to\infty} \frac{\operatorname{meas}(\{f>0\}\cap(0,r))}{r} = \frac{a}{a+b} .$$

For $a, b \in 2\mathbb{N}$, this follows from example (2.1).

In general, if the support of σ is unbounded, the set $\{f>0\}$ can be arbitrarily small. This is already true for functions $f\in L^1(\mathbb{R})$. However, it turns out that the set $\{f>0\}$ cannot be too small when $f\in L^p(\mathbb{R})$, $p>1$.

Theorem 7.

(i) *For any positive increasing function $\psi(x)\nearrow\infty$ defined on $[0,\infty)$ there is a real function $f\in L^1(\mathbb{R})$ which has a spectral gap $(-\pi,\pi)$, and such that*

$$\int_{\{f>0\}} \psi(|x|)\,dx < \infty . \tag{4.4}$$

(ii) *Suppose a real function $f\in L^1(\mathbb{R})\cap L^p(\mathbb{R})$, $p>1$, has a spectral gap $(-\alpha,\alpha)$ for some $\alpha>0$. Set*

$$M_n := \int_{\{f>0\}} |x|^n\,dx,\ n=0,1,\dots . \tag{4.5}$$

Then either $M_n=\infty$ for some $n\geq 0$ or

$$\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} < \infty .$$

Corollary 3. *Suppose a real function $f\in L^1(\mathbb{R})\cap L^p(\mathbb{R})$, $p>1$, has a spectral gap $(-\alpha,\alpha)$, $\alpha>0$. Then*

$$\int_{\{f>0\}} e^{\epsilon|x|}\,dx = \infty ,$$

for every positive ϵ.

Proof of Theorem 6. Assume a real $f\in L^1(\mathbb{R})$ satisfies (4.2), where $d\sigma(x)=\hat{f}(x)\,dx$. Then the function $\hat{f}(bt)$ is concentrated on $a\pi/b\leq|t|\leq\pi$. Theorem 5 now gives a (sharp) estimate of the positivity set $\{j: f(j/b)>0\}$:

$$\liminf_{n\to\infty} \frac{\#(\{j: f(j/b)>0\}\cap\{1,\dots,n\})}{n} \geq \frac{a}{2b} .$$

The proof of Theorem 6 is based on the fact that the positivity set of the sequence $\{f(2j/(a+b))\}$ is 'larger' than the one of the sequence $\{f(j/b)\}$.

Lemma 2. *Let f be a nontrivial real function in* (4.1) *satisfying* (4.2). *Then for every $x \in [0, 2/(a+b))$ we have*

$$\liminf_{n\to\infty} \frac{\#\left(\{j : f(2j/(a+b)+x) > 0\} \cap \{1, \dots, n\}\right)}{n} \geq \frac{a}{a+b} .$$

Proof. It suffices to establish Lemma 2 for $x = 0$. The proof is similar to the proof of Theorem 4 above.

By using a suitable change of variable, in what follows we may assume that $a = 1-\alpha$ and $b = 1+\alpha$ for some $0 < \alpha < 1$, so that the spectrum of f belongs to $\pi(1-\alpha) \leq |x| \leq \pi(1+\alpha)$:

$$f(t) = \Re\left(\int_{\pi(1-\alpha)}^{\pi(1+\alpha)} e^{its}\, d\sigma(s)\right) = \Re\left(e^{i\pi t}\int_{-\pi\alpha}^{\pi\alpha} e^{its}\, d\sigma(s+\pi)\right) .$$

This gives

$$f(k) = (-1)^k \varphi(k),\ k \in \mathbb{Z},\ \varphi(t) := \Re\left(\int_{-\pi\alpha}^{\pi\alpha} e^{its}\, d\sigma(s+\pi)\right) .$$

To prove the lemma, we have to show that

$$\liminf_{n\to\infty} \frac{\#\left(\{j : f(j) > 0\} \cap \{1, \dots, n\}\right)}{n} \geq \frac{1-\alpha}{2} .$$

However, since the number of positive elements in any finite sequence is not smaller then a half of the number of sign changes, it suffices to show that

$$\liminf_{n\to\infty} \frac{S\left(\{f(j)\}, 1 \leq j \leq n\right)}{n} \geq 1-\alpha . \tag{4.6}$$

Since φ has at least one zero between any two integers j, l such that $\varphi(j)\varphi(l) < 0$, one can verify that the number of sign changes in a sequence $f(1), \dots, f(j)$ is not smaller that $j - Z(\varphi, (0, j))$, where $Z(\varphi, (0, j))$ is the number of zeros (including multiplicities) of $\varphi(t)$ on $(0, j)$ (see [1] for details). Hence, to establish (4.6) it suffices to show that

$$\limsup_{r\to\infty} \frac{Z(\varphi, (0, r))}{r} \leq \alpha .$$

This follows from the theorem of Cartwright and Levinson ([8], Chapter V, Theorem 7). The reason for this inequality to hold is that φ is an entire function of exponential type $\pi\alpha$ bounded on the real line. This implies that φ is of completely regular growth in the sense of Levin and Pfluger. The latter means that the sequence of complex zeros of φ in any open angle containing the positive ray has density α, so the upper density of its positive zeros is at most α. □

We can now finish the proof of Theorem 6. Since $\text{meas}(\{f = 0\}) = 0$ it suffices to establish (4.3) for the set $\{f \geq 0\}$ instead of $\{f > 0\}$. This easily follows.

Lemma 3. *For any positive $\epsilon < (1-\alpha)/2$ there exists an integer $N(\epsilon)$ such that*

$$n^{-1}\,\text{meas}\left(\{f \geq 0\} \cap (0, n)\right) \geq \frac{1-\alpha}{2} - \epsilon,\ n \geq N(\epsilon) .$$

Proof. By Lemma 2, for every $0 < \epsilon < (1-\alpha)/2$ and $x \in [0, 1)$ there exists an integer $N(x, \epsilon)$ such that

$$\#\left(\{f \geq 0\} \cap \{1+x, 2+x, \dots, n+x\}\right) \geq \Big(\frac{1-\alpha}{2} - \epsilon\Big)n, \; n \geq N(x, \epsilon) .$$

Since f is continuous, it is clear that for every $X \in (0, 1)$ we have

$$\begin{aligned} &\#\left(\{f \geq 0\} \cap \{1+X, 2+X, \dots, n+X\}\right) \\ &\quad \leq \limsup_{x \to X} \#\left(\{f \geq 0\} \cap \{1+x, 2+x, \dots, n+x\}\right) , \end{aligned}$$

the strict inequality may occur when f has an even order zero at some point $X + j$ and f is negative in a neighborhood of this point. This gives

$$N(X, \epsilon) \geq \limsup_{x \to X} N(x, \epsilon) .$$

Hence, $N(\epsilon) := \sup_{x \in [0,1)} N(x, \epsilon) < \infty$ for every $\epsilon > 0$. The functions

$$k_n(x) := \#\left(\{f \geq 0\} \cap \{1+x, 2+x, \dots, n+x\}\right), n \geq N(\epsilon) ,$$

are piece-wise constant, and satisfy $k_n(x) \geq ((1-\alpha)/2 - \epsilon)n$. We conclude that

$$n^{-1} \operatorname{meas}\left(\{f \geq 0\} \cap (0, n)\right) = n^{-1} \int_0^1 k_n(x)\, dx \geq \frac{1-\alpha}{2} - \epsilon, \; n \geq N(\epsilon) ,$$

which proves Lemma 3. □

Proof of Theorem 7.

(i) Let $\psi(x) \nearrow \infty$ be a positive increasing function. We shall construct a function $f \in L^1(\mathbb{R})$ which satisfies (4.4) and such that $\hat{f}(t) = 0$ on $(-\pi, \pi)$. Our approach is somewhat similar to that in [6].

Set

$$f_1(x) := \frac{1}{\pi} \frac{1}{1+x^2} .$$

Then $\hat{f}_1(t) = e^{-|t|}$. Write

$$e^{-|t|} = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos nt, \; |t| < \pi , \tag{4.7}$$

where

$$c_n := \frac{2}{\pi} \int_0^{\pi} e^{-t} \cos nt\, dt = \frac{1-(-1)^n e^{-\pi}}{\pi\left(1+n^2\right)}, \; n = 0, 1, \dots .$$

We see that

$$\sum_{n=0}^{\infty} |c_n| < \infty . \tag{4.8}$$

We now choose any sequence of positive numbers $\epsilon_n < 1/2, n \geq 0$, which satisfy:

$$n^k \epsilon_n \to 0, n \to \infty, \quad \text{for every } \ k > 0\ , \tag{4.9}$$

and

$$\sum_{n=0}^{\infty} \int_{(n-\epsilon_{|n|}, n+\epsilon_{|n|})} \psi(x)\, dx < \infty\ . \tag{4.10}$$

Set

$$\hat{f}_2(t) := \frac{c_0}{2} \frac{\sin \epsilon_0 t}{\epsilon_0 t} + \sum_{n=1}^{\infty} c_n \cos nt \frac{\sin \epsilon_n t}{\epsilon_n t}\ .$$

Denote by $\chi_{[a,b]}(x)$ the characteristic function of the interval $[a, b]$. Since

$$\frac{1}{2a} \hat{\chi}_{[-a,a]}(t) = \frac{\sin at}{at}\ ,$$

it is clear that $\hat{f}_2$ is the Fourier transform of the function

$$f_2(x) := \sum_{n \in \mathbb{Z}} \frac{c_{|n|}}{4\epsilon_{|n|}} \chi_{[n-\epsilon_{|n|}, n+\epsilon_{|n|}]}(x)\ .$$

This shows that

$$\operatorname{supp}(f_2) = \bigcup_{n \in \mathbb{Z}} [n - \epsilon_{|n|}, n + \epsilon_{|n|}]\ , \tag{4.11}$$

where $\operatorname{supp}(f_2)$ denotes the support of f_2. Observe that, by (4.8)

$$\int_{\mathbb{R}} |f_2(x)|\, dx \leq \sum_{n \in \mathbb{Z}} \frac{|c_{|n|}|}{4\epsilon_{|n|}} \int_{n-\epsilon_{|n|}}^{n+\epsilon_{|n|}} 1\, dx \leq \sum_{n=0}^{\infty} |c_n| < \infty\ .$$

Hence, $f_2 \in L^1(\mathbb{R})$.

Since

$$\frac{\sin \epsilon_n t}{\epsilon_n t} = 1 - \frac{1}{3!}(\epsilon_n t)^2 + \frac{1}{5!}(\epsilon_n t)^4 - \ldots$$

we see by (4.7) that

$$\hat{f}_2(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos nt + \varphi(t) = e^{-|t|} + \varphi(t), t \in (-\pi, \pi)\ , \tag{4.12}$$

where

$$\varphi(t) := -\frac{t^2}{3!} \left(\frac{c_0 \epsilon_0^2}{2} + \sum_{n=1}^{\infty} c_n \epsilon_n^2 \cos nt \right) + \frac{t^4}{5!} \left(\frac{c_0 \epsilon_0^4}{2} + \sum_{n=0}^{\infty} c_n \epsilon_n^4 \cos nt \right) - \ldots\ .$$

It follows from (4.8) and (4.9) that $\varphi \in C^\infty(\mathbb{R})$. Clearly there exists a function $\hat{f}_3$ which belongs to $C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $\hat{f}_3(t) = \varphi(t), t \in (-\pi, \pi)$. Its inverse Fourier transform f_3 satisfies:

$$\left| x^k f_3(x) \right| \to 0, |x| \to \infty, \quad \text{for every } k > 0\ . \tag{4.13}$$

In particular, $f_3 \in L^1(\mathbb{R})$.

Set $f(x) := -f_1(x) + f_2(x) - f_3(x)$. Then $f \in L^1(\mathbb{R})$. By construction, $\hat{f}_3(x) = \varphi(x)$ on $(-\pi, \pi)$. Hence, by (4.12), $\hat{f}(x) = 0$ on $(-\pi, \pi)$. By (4.13), there is a constant $x_0 > 0$ such that

$$-f_1(x) - f_3(x) = -\frac{1}{\pi\left(1+x^2\right)} - f_3(x) < 0, \; |x| \geq x_0 .$$

Hence,

$$\{f > 0\} \subseteq [-x_0, x_0] \bigcup \{f_2 > 0\} \subseteq [-x_0, x_0] \bigcup \operatorname{supp}(f_2) .$$

By (4.11) and (4.10), we see (4.4) holds.

(ii) Let $N_n, n \in \mathbb{N}$, be a logarithmically convex sequence (i.e., $N_n^2 \leq N_{n-1}N_{n+1}, n > 1$). For an interval I, denote by $C(N_n; I)$ the class of functions $F \in C^\infty(I)$ with the property that $|F^{(n)}(x)| \leq N_n$ for all $n \in \mathbb{N}$ and all $x \in I$. We call the class $C(N_n, I)$ quasi-analytic (in the sense of Denjoy-Carleman), if every function $F \in C(N_n, I)$ is uniquely determined by the sequence of all its derivatives at every fixed point on I. We shall use the following

Lemma 4 (Denjoy-Carleman). *The class $C(N_n, I)$ is quasi-analytic, if and only if $\sum_{n=2}^{\infty} N_{n-1}/N_n = \infty$.*

Set $f_+(t) := f(t)\chi_{\{f>0\}}(t)$, where $\chi_{\{f>0\}}(t)$ is the characteristic function of the set $\{f > 0\}$.

Let us assume that Theorem 7 (ii) is not true, i.e., it is possible that

$$\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} = \infty . \tag{4.14}$$

To show that assumption (4.14) leads to a contradiction, we use the following

Lemma 5. *Suppose* (4.14) *is true, where M_n are defined in* (4.5). *Then the function $\hat{f}_+(x)$ belongs to some quasi-analytic class $C(N_n, \mathbb{R})$.*

A convenient way to generate a logarithmically convex sequence N_n is to fix a non-decreasing function $\eta : [1, \infty) \to (0, \infty)$ and set

$$N_n = N_{n-1}\eta(n), n \geq 2 .$$

Then the Denjoy-Carleman result is equivalent to the statement that $C(N_n, I)$ is quasi-analytic if and only if

$$\int_1^\infty \frac{1}{\eta(x)}\, dx = \infty . \tag{4.15}$$

Proof of Lemma 5. Since $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, we have $f \in L^s(\mathbb{R})$ for every $s, 1 < s < p$. Take a number s so that $q := s/(s-1) \in \mathbb{N}$. Clearly, $1/q + 1/s = 1$.

Let M_n be the (logarithmically convex!) sequence defined in (4.5), and let $\eta_0(x)$ be a function satisfying (4.15) and such that

$$M_n = M_{n-1}\eta_0(n), n \geq 2 .$$

Observe that

$$\max_{x\in\mathbb{R}} \left|\hat{f}_+^{(n)}(x)\right| \le \int_{\{f>0\}} |t|^n |f(t)|\,dt$$
$$\le \left(\int_{\{f>0\}} |t|^{nq}\,dt\right)^{1/q} \left(\int_{\{f>0\}} |f(t)|^s\,dt\right)^{1/s} \le c M_{nq}^{1/q} ,$$

where $c := \|f\|_s < \infty$ is a constant. Set

$$\eta(x) := \eta_0(qx),\, N_n := \prod_{k=1}^{n} \eta(k) .$$

Then $\eta(x)$ satisfies (4.15), so that N_n satisfy (4.14), N_n is logarithmically convex, and we have

$$N_n = \prod_{k=1}^{n} \eta_0(qk) \ge \left(\prod_{k=1}^{qn} \eta_0(k)\right)^{1/q} = M_{nq}^{1/q} .$$

We conclude that the function $\hat{f}_+(x)/c$ belongs to $C(N_n, \mathbb{R})$, and so $\hat{f}_+(x)$ is quasi-analytic on the real line. □

To prove Theorem 7 (ii) write $f(t) = f_+(t) - f_-(t)$, where

$$f_-(t) = f_+(t) - f(t) = f(t)\left(\chi_{\{f>0\}}(t) - 1\right) \ge 0,\ t \in \mathbb{R} .$$

Since $\hat{f}(x) = 0$ on some interval $(-\alpha, \alpha)$, we see that $\hat{f}_-(x) = \hat{f}_+(x)$, $x \in (-\alpha, \alpha)$, so that $\hat{f}_-/c \in C(N_n, (-\alpha, \alpha))$. However, since $f_-(t)$ is a nonnegative function, we get for every $x \in \mathbb{R}$ and $n = 0, 1, 2, \ldots$ that

$$\left|\hat{f}_-^{(2n)}(x)\right| \le \left|\int_{\mathbb{R}} (it)^{2n} e^{itx} f_-(t)\,dt\right| \le \int_{\mathbb{R}} t^{2n} f_-(t)\,dt = \left|\hat{f}_-^{(2n)}(0)\right| \le c N_{2n} .$$

Moreover, by the Cauchy-Schwartz inequality,

$$\left|\hat{f}_-^{(2n+1)}(x)\right|^2 \le \left|\hat{f}_-^{(2n)}(0)\hat{f}_-^{(2n+2)}(0)\right| \le c^2 N_{2n} N_{2n+2},\ n = 0, 1, 2, \ldots .$$

This shows that $\hat{f}_-/c \in C(N_n^*, \mathbb{R})$, where we set $N_n^* = N_n, n = 0, 2, 4, \ldots$, and $N_n^* = \sqrt{N_{n-1}N_{n+1}}, n = 1, 3, \ldots$. Clearly, the sequence N_n^* is logarithmically convex, and since N_n satisfy (4.14) then so do N_n^*. We conclude that both $\hat{f}_+$ and $\hat{f}_-$ are quasi-analytic on $\mathbb{R}$. Hence, $\hat{f}$ is quasi-analytic on $\mathbb{R}$. This implies $\hat{f}(x) \equiv 0$, which contradicts the assumption $f \neq 0$. □

5. Variation of Argument of Functions with a Spectral Gap at the Origin

First let us define the variation of argument Var (arg f, Γ) of a complex function f on a finite set $\Gamma \subset \mathbb{T}$, $\#(\Gamma) = n$. We set Var(arg $f, \Gamma) = 0$ if $\Gamma = \emptyset$. If f does not vanish on Γ, we set

$$\mathrm{Var}(\arg f, \Gamma) = \sum_{j=1}^{n} \left|\arg \frac{f(\gamma_{j+1})}{f(\gamma_j)}\right| ,$$

where $\Gamma = \{e^{i\gamma_j}\}_{j=1}^n$, $\gamma_1 < \ldots < \gamma_{n+1} = \gamma_1 + 2\pi$, and we choose $|\arg w| \le \pi$ for each complex number $w \neq 0$. In general, we set

$$\mathrm{Var}(\arg f, \Gamma) := \mathrm{Var}(\arg f, \Gamma \setminus \{f = 0\}) .$$

The variation of argument of f on the circle is defined as follows:

$$\mathrm{Var}(\arg f, \mathbb{T}) := \sup_{\Gamma} \mathrm{Var}(\arg f, \Gamma) ,$$

where the supremum is taken over all finite subsets $\Gamma \subset \mathbb{T}$. Clearly, if f is a real continuous function, then $\mathrm{Var}(\arg f, \Gamma) = \pi S(f, \Gamma)$, and $\mathrm{Var}(\arg f, \mathbb{T}) = \pi S(f, \mathbb{T})$, where $S(f, \Gamma)$ and $S(f, \mathbb{T})$ are the number of sign changes of f on $\Gamma \subset \mathbb{T}$ and $\mathbb{T}$, respectively. Observe also that if $f \in C^1(\mathbb{T})$ then

$$\mathrm{Var}(\arg f, \mathbb{T}) = \int_{-\pi}^{\pi} |d \arg f(t)| . \tag{5.1}$$

There is a simple connection between the variation of argument and the number of sign changes based on the formula:

$$\left| \arg \frac{w_1}{w_2} \right| = \int_0^{\pi} S\left(\Re\big(w_1 e^{-i\theta}\big), \Re\big(w_2 e^{-i\theta}\big) \right) d\theta ,$$

where $w_1, w_2 \neq 0$ are arbitrary complex numbers, $|\arg w| \le \pi$ for every complex $w \neq 0$, and $S(a, b)$ for a, b real means the number of sign changes: $S(a, b) = 1$ if $ab < 0$ and $S(a, b) = 0$ otherwise. Let $\Gamma \subset \mathbb{T}$ be a finite set. Using the previous formula, one can easily deduce the following formula:

$$\mathrm{Var}(\arg f, \Gamma) = \int_0^{\pi} S\big(\Re\big(f(t)e^{-i\theta}\big), \Gamma\big) \, d\theta , \tag{5.2}$$

where $S(\Re(f(t)e^{-i\theta}), \Gamma)$ is the number of sign changes of $\Re(f(t)e^{-i\theta})$ on $\Gamma \subset \mathbb{T}$. The same formula is true for $\Gamma = \mathbb{T}$.

Equality (5.2) allows to estimate the variation of argument of a complex function f via the number of the sign changes of its projections to the lines $\arg z = \theta$. Let us for example extend the Sturm-Hurwitz theorem and Theorem 4 to complex functions.

Theorem 8.

(i) *Suppose a nontrivial complex continuous function f on the circle $\mathbb{T}$ has a spectral gap $(-n, n)$. Then* Var $(\arg f, \mathbb{T}) \ge 2\pi n$.

(ii) *If we additionally assume that f is a trigonometric polynomial of degree $m \ge n$ then either $f(t) = a \sin mt$ for some $a \in \mathbb{C}$ (i.e., $n = m$ and f is trivial on $\mathbb{T}_{2m}$), or Var $(\arg f, \mathbb{T}_{m+n}) \ge 2\pi n$.*

Observe that Theorem 8 ceases to be true if we replace the variation of argument by the change of argument [i.e., if we set the modulus sign outside the integral in (5.1)].

Let us prove part (i). If a nontrivial continuous function f has a spectral gap $(-n, n)$, then every function $\Re(f(t)\exp(-i\theta))$, $\theta \in [-\pi, \pi)$, is real, continuous and has a spectral gap $(-n, n)$. By the Sturm-Hurwitz theorem, the number of sign changes of $\Re(f(t)\exp(-i\theta))$ is at least $2n$, for each θ. Hence, by (5.2), we obtain:

$$\mathrm{Var}(\arg f, \mathbb{T}) = \int_0^{\pi} S\big(\Re\big(f(t)e^{-i\theta}\big), \mathbb{T}\big) \, d\theta \ge 2\pi n .$$

The proof of (ii) is similar. □

One can extend the definition of variation of argument to sequences on $\mathbb{Z}$ and functions on $\mathbb{R}$. Then one can formulate extensions of Theorems 2 and 3 to complex functions in which the number of sign changes is replaced by the variation of argument. The proofs are similar to the proof of Theorem 8 plus an extra argument: One needs to obtain a uniform estimate of the number of sign changes of the projections $\Re(fe^{-i\theta})$. This can be done in a similar way as in Lemma 3.

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