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Analytic Features of Reproducing Groups for the Metaplectic Representation

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ABSTRACT. *We introduce the notion of admissible subgroup H of $G = \mathbb{H}^d \rtimes Sp(d, \mathbb{R})$ relative to the (extended) metaplectic representation μ_e via the Wigner distribution. Under mild additional assumptions, it is shown to be equivalent to the fact that the identity $f = \int_H \langle f, \mu_e(h)\phi\rangle \mu_e(h)\phi \, dh$ holds (weakly) for all $f \in L^2(\mathbb{R}^d)$. We use this equivalence to exhibit classes of admissible subgroups of $Sp(2, \mathbb{R})$. We also establish some connections with wavelet theory, i.e., with curvelet and contourlet frames.*

1. Introduction

The study of reproducing formulae for functions in $L^2(\mathbb{R}^d)$ has attracted the interest of many authors, in physics [1], group representations [9] and applied mathematics, both in Gabor analysis [14] and in wavelet theory [4, 10, 18]. In a very general and abstract sense, they can all be recast in a formula of the type

$$f = \int_H \langle f, \phi_h \rangle \, \phi_h \, dh, \qquad f \in \mathcal{H}\,, \tag{1.1}$$

where $\mathcal{H}$ is a Hilbert space and $h \mapsto \phi_h$ is an $\mathcal{H}$-valued measurable function on some measure space (H, dh). Of course, the cases of greatest interest concern Hilbert spaces of functions, while the measure space H serves as parameter space. Thus, H takes into account the particular kind of analysis and synthesis processes that a formula like (1.1), known as

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reproducing formula, is meant to describe. We are mostly interested in the case in which H is a Lie group with left Haar measure dh, $\phi \in L^2(\mathbb{R}^d)$ is fixed and $h \mapsto \phi_h$ is an $L^2(\mathbb{R}^d)$-valued unitary representation of H. This rich structure often provides both a very efficient tool for computations and a means for finding new reproducing formulae, specially when H is chosen among the subgroups of some classical group of linear symmetries. A class of groups that has been widely studied is the class of semidirect products $H = \mathbb{R}^d \rtimes D$, where D is a closed matrix group (the so-called dilation group). They admit a natural unitary representation on $L^2(\mathbb{R}^d)$, the main ingredient for the construction of a wavelet transform. Initially, only irreducible square-integrable representations were considered [2, 12], but it soon became clear that nonirreducible representations [15, 19, 13] are of relevance as well.

Recently, the authors of [18] have proved a characterization of those dilation groups D which give rise to a reproducing formula (1.1). They introduce a notion of *admissibility,* a sufficient condition for a subgroup D of $GL(\mathbb{R}, d)$ to admit a window $\phi \in L^2(\mathbb{R}^d)$ such that (1.1) works for all $f \in L^2(\mathbb{R}^d)$. A dilation group D is admissible if there exists a Borel measurable $F \in L^1(\mathbb{R}^d)$ such that $F \geq 0$ and

$$\int_D F\big(x\,{}^t a\big)\,da = 1, \qquad \text{for a.e. } x \in \mathbb{R}^d \;, \tag{1.2}$$

where ${}^t a$ is the transpose of the matrix a, $x \mapsto x\,{}^t a$ is the right action of $a \in D$, and da is the left Haar measure on D. The above definition is motivated by the analysis of the "$ax + b$" group. In that case, any admissible wavelet ψ (in the usual Calderón sense) gives a function $F = |\hat{\psi}|^2$ for which formula (1.2) holds.

We work in a somewhat different setting. First, the Lie group H in (1.1) is a subgroup of the semidirect product $G = \mathbb{H}^d \rtimes Sp(d, \mathbb{R})$ of the Heisenberg group and the symplectic group. Secondly, the representation $h \mapsto \phi_h$ arises from the restriction to H of the reducible (extended) metaplectic representation μ_e of G as applied to a fixed and suitable window function $\phi \in L^2(\mathbb{R}^d)$. A group H for which there exists a window ϕ such that (1.1) holds is said to be *reproducing.* A complete classification of reproducing subgroups in the case $d = 1$ is given in [8], but for the case $d \geq 2$, the groups we treat here are the only known examples.

Although, the setups $\mathbb{R}^d \rtimes D$ and $\mathbb{H}^d \rtimes Sp(d, \mathbb{R})$ are quite different in spirit, there is a crucial conceptual link between them. The point is that both are intimately related to the geometry of affine actions on Euclidean space. Indeed, one of the most important features of μ_e is that it may be realized by affine actions on $\mathbb{R}^{2d}$ by means of the Wigner distribution. The reader is referred to [5, 11, 14] for a thorough discussion of this basic construct in time-frequency analysis, some of whose properties will be recalled in Section 3. The cross-Wigner distribution $W_{f,g}$ of $f, g \in L^2(\mathbb{R}^d)$ is

$$W_{f,g}(x,\xi) = \int e^{-2\pi i \langle \xi, y \rangle} f\left(x + \frac{y}{2}\right) \overline{g\left(x - \frac{y}{2}\right)}\,dy \;. \tag{1.3}$$

The quadratic expression $W_f := W_{f,f}$ is usually called the Wigner distribution of f. The crucial property of W alluded to above is that it intertwines μ_e and the affine action on $\mathbb{R}^{2d}$. In other words:

$$W_{\mu_e(g)\phi}(x,\xi) = W_\phi\left(g^{-1} \cdot (x,\xi)\right), \qquad g \in G \;,$$

where $g \cdot (x, \xi)$ is the natural affine action of G on phase space. Actually, since the reproducing formula is insensitive to phase factors, i.e., to the action of the center of $\mathbb{H}^d$, the group G is truly $\mathbb{R}^{2d} \rtimes Sp(d, \mathbb{R})$, whose affine action on $\mathbb{R}^{2d}$ is rather obvious [see the next section and in particular (2.2)]. This is why in our Definition 2 the Wigner distribution plays the same role as F plays in (1.2). Thus, we call *admissible* a connected Lie subgroup $H \subset G$ if there exists $\phi \in L^2(\mathbb{R}^d)$ such that

$$\int_H W_\phi \left(h^{-1} \cdot (x, \xi)\right) dh = 1, \quad \text{for a.e. } (x, \xi) \in \mathbb{R}^{2d} .$$

The outline of the article is as follows. In Section 2 we establish some background and notation. In Section 3 we prove Theorem 1, which states that under mild additional assumptions on the mapping $h \mapsto W_\phi(h^{-1} \cdot (x, \xi))$, a subgroup H is reproducing if and only if it is admissible. Theorem 1 has consequences on the geometric properties related to admissibility. Some of these appear in [7]; a deeper study of the geometry of admissible groups is developed in a forthcoming article [6]. In Section 4, Theorem 1 is applied to prove the admissibility of a subgroup H of $Sp(2, \mathbb{R})$ that we denote $TDS(2)$. In Section 5 we establish some connections with wavelet theory. We exhibit another reproducing subgroup of $Sp(2, \mathbb{R})$, which is a covering of the similitude group of the plane $SIM(2)$. We then show that our theory, for both $TDS(2)$ and $SIM(2)$, parallels the theory developed in the context of two-dimensional wavelets. The groups $TDS(2)$ and $SIM(2)$ are the forerunners of the *curvelet* and *contourlet* frames, nowadays heavily employed in the context of signal processing [4, 10]. In particular, curvelets are actively investigated from the point of view of statistical estimation, sparsity of the representation and rate of approximation. Our approach starts from the whole time-frequency plane $\mathbb{R}^{2d}$, instead of looking at either time or frequency, as is typical in the philosophy of the setting $\mathbb{R}^d \rtimes D$. This justifies the use of the Wigner distribution and its time-frequency properties. In Section 6 we prove that a class of groups, parametrized by $\beta \in \mathbb{R}$ and including $SIM(2)$ when $\beta = 0$, is reproducing. This time, however, our proof is direct, namely we show (1.1) without using Theorem 1.

2. Preliminaries and Notation

The symplectic group is defined by

$$Sp(d, \mathbb{R}) = \left\{ g \in GL(2d, \mathbb{R}) : \ {}^t g J g = J \right\} ,$$

where

$$J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}$$

is the standard symplectic form

$$\omega(x, y) = {}^t x J y, \qquad x, y \in \mathbb{R}^{2d} . \tag{2.1}$$

The metaplectic representation μ of (the two-sheeted cover of) the symplectic group arises as intertwining operator between the standard Schrödinger representation ρ of the Heisenberg group $\mathbb{H}^d$ and the representation that is obtained from it by composing ρ with the action of $Sp(d, \mathbb{R})$ by automorphisms on $\mathbb{H}^d$ (see, e.g., [11]). We briefly review its construction.

The Heisenberg group $\mathbb{H}^d$ is the group obtained by defining on $\mathbb{R}^{2d+1}$ the product

$$(z, t) \cdot \left(z', t'\right) = \left(z + z', t + t' - \frac{1}{2}\omega\left(z, z'\right)\right) ,$$

where ω stands for the standard symplectic form in $\mathbb{R}^{2d}$ given in (2.1). We denote the translation and modulation operators on $L^2(\mathbb{R}^d)$ by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\xi f(t) = e^{2\pi i\langle \xi, t\rangle} f(t) \ .$$

The Schrödinger representation of the group $\mathbb{H}^d$ on $L^2(\mathbb{R}^d)$ is then defined by

$$\rho(x,\xi,t) f(y) = e^{2\pi i t} e^{\pi i\langle x,\xi\rangle} e^{2\pi i\langle \xi, y-x\rangle} f(y-x) = e^{2\pi i t} e^{\pi i\langle x,\xi\rangle} T_x M_\xi f(y) \ ,$$

where we write $z = (x, \xi)$ when we separate space components (that is x) from frequency components (that is ξ) in a point z in phase space $\mathbb{R}^{2d}$. The symplectic group acts on $\mathbb{H}^d$ via automorphisms that leave the center $\{(0,t) : t \in \mathbb{R}\} \simeq \mathbb{R}$ of $\mathbb{H}^d$ pointwise fixed:

$$A \cdot (z,t) = (Az, t) \ .$$

Therefore, for any fixed $A \in Sp(d,\mathbb{R})$ there is a representation

$$\rho_A : \mathbb{H}^d \to \mathcal{U}\big(L^2\big(\mathbb{R}^d\big)\big), \qquad (z,t) \mapsto \rho\,(A \cdot (z,t))$$

whose restriction to the center is a multiple of the identity. By the Stone-von Neumann theorem, ρ_A is equivalent to ρ. That is, there exists an intertwining unitary operator $\mu(A) \in \mathcal{U}(L^2(\mathbb{R}^d))$ such that $\rho_A(z,t) = \mu(A) \circ \rho(z,t) \circ \mu(A)^{-1}$, for all $(z,t) \in \mathbb{H}^d$. By Schur's lemma, μ is determined up to a phase factor e^{is}, $s \in \mathbb{R}$. It turns out that the phase ambiguity is really a sign, so that μ lifts to a representation of the (double cover of the) symplectic group. It is the famous metaplectic or Shale-Weil representation.

The representations ρ and μ can be combined and give rise to the extended metaplectic representation of the group $G = \mathbb{H}^d \rtimes Sp(d,\mathbb{R})$, the semidirect product of $\mathbb{H}^d$ and $Sp(d,\mathbb{R})$. The group law on G is

$$((z,t), A) \cdot \big((z',t'), A'\big) = \big((z,t) \cdot \big(Az', t'\big), AA'\big)$$

and the extended metaplectic representation μ_e of G is

$$\mu_e\,((z,t), A) = \rho(z,t) \circ \mu(A) \ .$$

A slight simplification in our formalism comes from the observation that the reproducing formula (1.1) is insensitive to phase factors: If we replace $\mu_e(h)\phi := \phi_h$ with $e^{is}\mu_e(h)\phi$ the formula is unchanged, for any $s \in \mathbb{R}$. The role of the center of the Heisenberg group is thus irrelevant, so that the "true" group under consideration is $\mathbb{R}^{2d} \rtimes Sp(d,\mathbb{R})$, which we denote again by G. Thus, G acts naturally by affine transformations on phase space, namely

$$g \cdot (x,\xi) = ((q,p), A) \cdot (x,\xi) = A\,{}^t(x,\xi) + {}^t(q,p) \ . \tag{2.2}$$

For elements of $Sp(d,\mathbb{R})$ in special form, the metaplectic representation can be computed explicitly in a simple way. For $f \in L^2(\mathbb{R}^d)$, we have

$$\mu\left(\begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix}\right) f(x) = (\det A)^{-1/2} f\big(A^{-1}x\big) \tag{2.3}$$

$$\mu\left(\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}\right) f(x) = \pm e^{i\pi\langle Cx,x\rangle} f(x) \tag{2.4}$$

$$\mu\,(J) = (-i)^{d/2}\mathcal{F} \ , \tag{2.5}$$

where $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i\langle x,\xi\rangle}\,dx, \qquad f \in L^1\big(\mathbb{R}^d\big) \cap L^2\big(\mathbb{R}^d\big)\,.$$

In the above formula and elsewhere, $\langle x, \xi\rangle$ denotes the inner product of $x, \xi \in \mathbb{R}^d$. Similarly, for $f, g \in L^2(\mathbb{R}^d)$, $\langle f, g\rangle$ will denote their inner product in $L^2(\mathbb{R}^d)$. Other notation is as follows. We put $\dot{\mathbb{R}} = \mathbb{R} \setminus \{0\}$, $\mathbb{R}_\pm = (0, \pm\infty)$. For $1 \le p \le \infty$, $\|\cdot\|_p$ stands for the L^p-norm of measurable functions on $\mathbb{R}^d$ with respect to Lebesgue measure. The left Haar measure of a group H will be written dh and we always assume that the Haar measure of a compact group is normalized so that the total mass is one.

3. The Reproducing Condition

Definition 1. We say that a connected Lie subgroup H of $G = \mathbb{R}^{2d} \rtimes Sp(d, \mathbb{R})$ is a *reproducing group* for μ_e if there exists a function $\phi \in L^2(\mathbb{R}^d)$ such that

$$f = \int_H \langle f, \mu_e(h)\phi\rangle \mu_e(h)\phi\,dh, \quad \text{for all } f \in L^2\big(\mathbb{R}^d\big)\,. \tag{3.1}$$

Any $\phi \in L^2(\mathbb{R}^d)$ for which (3.1) holds is called a reproducing function.

Notice that we do require formula (3.1) to hold for all functions in $L^2(\mathbb{R}^d)$ for the same window ϕ, but we do not require the restriction of μ_e to H to be irreducible. Equivalently, formula (3.1) can be written in term of the L^2-norm of f

$$\|f\|_2^2 = \int_H |\langle f, \mu_e(h)\phi\rangle|^2\,dh, \quad \text{for all } f \in L^2\big(\mathbb{R}^d\big)\,. \tag{3.2}$$

3.1 The Wigner Distribution and Some Useful Properties

We collect some well-known properties of the Wigner distribution and then we establish Lemmata 2 and 3, which will be used in Section 4. For the proof of Proposition 1 see [11, 14], whereas Lemma 1 is from [14]. Recall that the cross-Wigner distribution is defined, for $f, g \in L^2(\mathbb{R}^d)$, by (1.3).

Proposition 1. *The Wigner distribution of $f, g \in L^2(\mathbb{R}^d)$ satisfies:*

(i) *$W_{f,g}$ is uniformly continuous on $\mathbb{R}^{2d}$, and $\|W_{f,g}\|_\infty \le 2^d\|f\|_2\|g\|_2$.*

(ii) *$W_{f,g} = \overline{W_{g,f}}$; in particular, W_f is real-valued.*

(iii) *Moyal's identity: $\langle W_f, W_g\rangle_{L^2(\mathbb{R}^{2d})} = \langle f, g\rangle_{L^2(\mathbb{R}^d)}\overline{\langle f, g\rangle}_{L^2(\mathbb{R}^d)}$.*

(iv) *If $f, g \in \mathcal{S}(\mathbb{R}^d)$, then $W_{f,g} \in \mathcal{S}(\mathbb{R}^{2d})$.*

(v) *If $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then $\|f\|_2^2 = \int_{\mathbb{R}^{2d}} W_f(x, \xi)\,dx\,d\xi$.*

(vi) *Marginal property:*

$$\int_{\mathbb{R}^d} W_f(x, \xi)\,d\xi = |f(x)|^2, \quad \forall f \in L^2\big(\mathbb{R}^d\big) \text{ with } \hat{f} \in L^1\big(\mathbb{R}^d\big)\,. \tag{3.3}$$

An alternative description of $W_{f,g}$ is provided by the lemma below.

Lemma 1. *Let* $\mathcal{T}_s F(x,t) = F(x+\frac{t}{2}, x-\frac{t}{2})$ *be the symmetric coordinate transform and* $\mathcal{F}_2 F(x,\xi) = \int_{\mathbb{R}^d} F(x,t)e^{-2\pi i\langle \xi,t\rangle}\,dt$ *be the Fourier transform in the second variable. Then*

$$W_{f,g} = \mathcal{F}_2\mathcal{T}_s\big(f\otimes\bar{g}\big)\,. \tag{3.4}$$

We use Lemma 1 to prove the following density result.

Lemma 2. *If* $R(x,\xi)$ *is a real, slowly increasing measurable function on* $\mathbb{R}^{2d}$ *such that*

$$\int_{\mathbb{R}^{2d}} R(x,\xi)W_f(x,\xi)\,dx\,d\xi = 0, \qquad \text{for all} \quad f\in\mathcal{S}\big(\mathbb{R}^d\big)\,,$$

then $R(x,\xi)=0$ *for a.e.* $(x,\xi)\in\mathbb{R}^{2d}$.

Proof. By Lemma 1 it follows that $V := \operatorname{span}\{W_{f,g}\ f,g\in\mathcal{S}(\mathbb{R}^d)\}$ is dense in $\mathcal{S}(\mathbb{R}^{2d})$. For $f,g\in\mathcal{S}(\mathbb{R}^d)$, a straightforward computation gives

$$W_{f+g} = W_f + W_g + 2\mathrm{Re}\,W_{f,g}, \qquad W_{f+ig} = W_f + W_g + 2\mathrm{Im}\,W_{f,g}$$

and the assumption implies $\langle \mathrm{Re}\,W_{f,g}, R\rangle = 0$ and $\langle \mathrm{Im}\,W_{f,g}, R\rangle = 0$. Since R is real, these two identities are equivalent to $\langle W_{f,g}, R\rangle = 0$. The conclusion follows from the density of V, because for every $F\in\mathcal{S}(\mathbb{R}^{2d})$ the functional $F\mapsto \int_{\mathbb{R}^{2d}} R(x,\xi)F(x,\xi)\,dx\,d\xi$ is a tempered distribution and we have

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} R(x,\xi)F(x,\xi)\,dx\,d\xi &= \langle F, R\rangle \\
&= \left\langle \lim_{n\to\infty}\sum_{k=0}^{n} c_k W_{f_k,g_k}, R\right\rangle \\
&= \lim_{n\to\infty}\left\langle \sum_{k=0}^{n} c_k W_{f_k,g_k}, R\right\rangle \\
&= \lim_{n\to\infty}\sum_{k=0}^{n} c_k \langle W_{f_k,g_k}, R\rangle \\
&= 0\,,
\end{aligned}$$

so that $R(x,\xi)=0$, for a.e. $(x,\xi)\in\mathbb{R}^{2d}$ and the proof is complete. □

The next lemma will be used in Section 4.

Lemma 3. *Let* $\phi_0,\phi_1\in L^2(\mathbb{R}^d)$ *and define* $\phi := \phi_0\otimes\phi_1\in L^2(\mathbb{R}^{2d})$. *Then*

$$W_\phi((z_1,z_2),(\zeta_1,\zeta_2)) = W_{\phi_0}(z_1,\zeta_1)W_{\phi_1}(z_2,\zeta_2)\,, \tag{3.5}$$

where the variables $z_1, z_2, \zeta_1, \zeta_2$ *are in* $\mathbb{R}^d$.

Proof. Simply compute the Wigner distribution (1.3) of $\phi = \phi_0\otimes\phi_1$. □

3.2 The Admissibility Condition

In this section we find an admissibility condition that, together with some additional integrability and boundedness properties of $h\mapsto W_\phi(h^{-1}\cdot(x,\xi))$ implies that a subgroup H of $G=\mathbb{R}^{2d}\rtimes Sp(d,\mathbb{R})$ is reproducing.

Theorem 1. *Suppose that $\phi \in L^2(\mathbb{R}^d)$ is such that the mapping*

$$h \mapsto W_{\mu_e(h)\phi}(x,\xi) = W_\phi\big(h^{-1} \cdot (x,\xi)\big) \tag{3.6}$$

is in $L^1(H)$ for a.e. $(x,\xi) \in \mathbb{R}^{2d}$ and

$$\int_H \big|W_\phi\big(h^{-1} \cdot (x,\xi)\big)\big|\, dh \leq M, \qquad \textit{for a.e. } (x,\xi) \in \mathbb{R}^{2d}\ . \tag{3.7}$$

Then condition (3.1) *holds for all $f \in L^2(\mathbb{R}^d)$ if and only if the following admissibility condition is satisfied:*

$$\int_H W_\phi\big(h^{-1} \cdot (x,\xi)\big)\, dh = 1, \quad \textit{for a.e. } (x,\xi) \in \mathbb{R}^{2d}\ . \tag{3.8}$$

Proof. It is enough to test the reproducing formula (3.2) on the Schwartz class. Namely, if we show the mapping $f \mapsto \langle f, \mu_e(h)\phi\rangle$ is an isometry on $\mathcal{S}(\mathbb{R}^d)$ into $L^2(H)$, the pointwise convergence of the coefficients $\langle f, \mu_e(h)\phi\rangle$ guarantees that (3.2) holds for all $f \in L^2(\mathbb{R}^d)$ as well.

Sufficiency. Assume that (3.8) is true and take $f \in \mathcal{S}(\mathbb{R}^d)$. By (v) of Proposition 1, its L^2-norm can be computed via its Wigner distribution, that is:

$$\begin{aligned}\|f\|_2^2 &= \int_{\mathbb{R}^{2d}} W_f(x,\xi)\, dx\, d\xi = \int_{\mathbb{R}^{2d}} \left(\int_H W_\phi\big(h^{-1} \cdot (x,\xi)\big)\, dh\right) W_f(x,\xi)\, dx\, d\xi \\ &= \int_H \left(\int_{\mathbb{R}^{2d}} W_\phi\big(h^{-1} \cdot (x,\xi)\big)\ W_f(x,\xi)\, dx\, d\xi\right) dh\ .\end{aligned}$$

In the last equality, the integral interchange is justified by Fubini Theorem. Indeed, by (3.6) and (3.7) we have

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} \int_H \big|W_\phi\big(h^{-1} \cdot (x,\xi)\big)\, W_f(x,\xi)\big|\, dh\, dx\, d\xi \\ &\quad = \int_{\mathbb{R}^{2d}} \left(\int_H \big|W_\phi\big(h^{-1} \cdot (x,\xi)\big)\big|\, dh\right) |W_f(x,\xi)|\, dx\, d\xi \\ &\quad \leq M \int_{\mathbb{R}^{2d}} |W_f(x,\xi)|\, dx\, d\xi < \infty\ .\end{aligned}$$

Further, Moyal's identity gives

$$\int_{\mathbb{R}^{2d}} W_\phi\big(h^{-1} \cdot (x,\xi)\big)\ W_f(x,\xi)\, dx\, d\xi = \langle W_{\mu_e(h)\phi}, W_f\rangle = \langle f, \mu_e(h)\phi\rangle\overline{\langle f, \mu_e(h)\phi\rangle}\ ,$$

hence, the equality

$$\|f\|_2^2 = \int_H |\langle f, \mu_e(h)\phi\rangle|^2\, dh, \ \text{ for all } f \in \mathcal{S}\big(\mathbb{R}^d\big)\ .$$

Necessity. Conversely, assume (3.1) true and let f be in $\mathcal{S}(\mathbb{R}^d)$. Moyal's identity gives

$$\|f\|_2^2 = \int_{\mathbb{R}^{2d}} \left(\int_H W_\phi\big(h^{-1} \cdot (x,\xi)\big)\, dh\right) W_f(x,\xi)\, dx\, d\xi\ . \tag{3.9}$$

Using again (v) of Proposition 1, equality (3.9) may be recast as

$$\int_{\mathbb{R}^{2d}} \left(\int_H W_\phi\big(h^{-1} \cdot (x, \xi)\big)\, dh - 1 \right) W_f(x, \xi)\, dx\, d\xi = 0\,.$$

The function

$$R(x, \xi) = \int_H W_\phi\big(h^{-1} \cdot (x, \xi)\big)\, dh - 1$$

is real by (ii) of Proposition 1. Hence, (3.8) follows applying Lemma 2 to it. □

Motivated by Theorem 1, we give the following definition.

Definition 2. We say that a connected Lie subgroup H of $G = \mathbb{R}^{2d} \rtimes Sp(d, \mathbb{R})$ is an admissible group for μ_e if there exists a function $\phi \in L^2(\mathbb{R}^d)$ such that

$$\int_H W_\phi\big(h^{-1} \cdot (x, \xi)\big)\, dh = 1, \quad \text{for a.e. } (x, \xi) \in \mathbb{R}^{2d}\,. \tag{3.10}$$

Any $\phi \in L^2(\mathbb{R}^d)$ for which (3.10) holds is called an admissible function.

It is clear that we now dispose of two different tools for checking whether a subgroup H of $G = \mathbb{R}^{2d} \rtimes Sp(d, \mathbb{R})$ is reproducing or not. Either we find a window function ϕ for which (3.1) holds or else we check the admissibility of the subgroup H and use Theorem 1. The latter method is used in the next section, while the former is applied in Section 5. We stress that Theorem 1 admits other useful applications [6, 7].

4. The Reproducing Group $TDS(2)$

Throughout this section $d = 2$. We prove that the four-dimensional triangular group

$$TDS(2) = \left\{ A_{t,\ell,y} := \begin{bmatrix} t^{-1/2} S_{\ell/2} & 0 \\ t^{-1/2} B_y S_{\ell/2} & t^{1/2}\, {}^t S_{-\ell/2} \end{bmatrix} : t > 0,\ \ell \in \mathbb{R},\ y \in \mathbb{R}^2 \right\} \tag{4.1}$$

is a reproducing subgroup of $Sp(2, \mathbb{R})$, where

$$B_y = \begin{bmatrix} 0 & y_1 \\ y_1 & y_2 \end{bmatrix}, \quad y = (y_1, y_2) \in \mathbb{R}^2; \quad S_\ell = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix}, \quad \ell \in \mathbb{R}\,. \tag{4.2}$$

The matrix S_ℓ is called *shearing* matrix. We use the letters TDS because the restriction of the metaplectic representation to it gives rise to translation, dilation, and shearing operators. This fact will be discussed in Section 5.

The main idea of the proof is to reduce the two-dimensional condition (3.8) to the one-dimensional analogue that arises from a reproducing subgroup of $\mathbb{R}^2 \rtimes SL(2, \mathbb{R})$ and to a reproducing condition for a window function of another reproducing subgroup of $\mathbb{R}^2 \rtimes SL(2, \mathbb{R})$. It was proven in [8, Theorem 2.1] that, up to conjugation, there are exactly five reproducing subgroups of $\mathbb{R}^2 \rtimes SL(2, \mathbb{R})$. We are interested in the following two:

$$H_0 = \left\{ \left(\begin{bmatrix} q \\ p \end{bmatrix}, I \right),\ p, q \in \mathbb{R} \right\}$$

$$H_1 = \left\{ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} a^{-1/2} & 0 \\ 0 & a^{1/2} \end{bmatrix} \right),\ a > 0, b \in \mathbb{R} \right\}\,.$$

A function ϕ_0 is reproducing for H_0 if

$$\phi_0 \in L^2(\mathbb{R}), \quad \text{and} \quad \|\phi_0\|_2 = 1 , \tag{4.3}$$

while a function ϕ_1 is reproducing for H_1 if and only if $\phi_1 \in L^2(\mathbb{R})$ and

$$\int_0^\infty |\phi_1(x)|^2 \frac{dx}{x^2} = \int_0^\infty |\phi_1(-x)|^2 \frac{dx}{x^2} = \frac{1}{2}, \quad \int_0^\infty \phi_1(x)\overline{\phi_1(-x)}\frac{dx}{x^2} = 0 . \tag{4.4}$$

Clearly, $H_0 \simeq \mathbb{R}^2$ so that its Haar measure is the Lebesgue measure $dq\,dp$. The group H_1 is the only reproducing subgroup that lies entirely inside $Sp(1,\mathbb{R}) = SL(2,\mathbb{R})$, and it is isomorphic to the "$ax+b$" group.

Observe that (3.8) can be rewritten in terms of the right Haar measure $d_r h$ as

$$\int_H W_\phi\big(h^{-1}\cdot(x,\xi)\big)\,dh = \int_H W_\phi(h\cdot(x,\xi))\,d_r h = 1 ,$$

leading to the following alternative formulation that H_0 is admissible

$$\int_{\mathbb{R}^2} W_{\phi_0}(x+q,\xi+p)\,dq\,dp = 1, \quad \text{for a.e. } (x,\xi)\in\mathbb{R}^2 . \tag{4.5}$$

We can finally show that $TDS(2)$ is reproducing.

Theorem 2. *Let $\phi_0, \phi_1 \in \mathcal{S}(\mathbb{R})$ be reproducing functions for the subgroups H_0 and H_1, respectively. Then, the window function ϕ defined by*

$$\phi(x,\xi) = \frac{1}{2}\big(\phi_0 \otimes \widetilde{\phi_1}\big)(x,\xi), \quad (x,\xi)\in\mathbb{R}^2 , \tag{4.6}$$

where $\widetilde{\phi_1}(y) = y\phi_1(y)$, is a reproducing function for $TDS(2)$, i.e., $TDS(2)$ is a reproducing subgroup.

Proof. Notice that the assumptions (3.6) and (3.7) are trivially satisfied. Hence, it remains to verify the admissibility condition (3.8), i.e.,

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_0^\infty W_\phi\big((z,\zeta)\,{}^tA_{t,\ell,y}\big)\frac{dt}{t}\,d\ell\,dy_1\,dy_2 = 1, \quad \text{a.e. } (z,\zeta)\in\mathbb{R}^4 , \tag{4.7}$$

where $t^{-1}\,dt\,d\ell\,dy_1\,dy_2$ is the right Haar measure of $TDS(2)$. Observe that the action of the $TDS(2)$ group on $\mathbb{R}^4$ has only two orbits with nonzero Lebesgue measure on $\mathbb{R}^4$, namely $\mathcal{O}_{(0,1,0,0)}$ and $\mathcal{O}_{(0,-1,0,0)}$. Then (4.7) is equivalent to

$$\int_{\mathbb{R}^3}\int_0^\infty W_\phi\big((0,\pm1,0,0)\,{}^tA_{t,\ell,y}\big)\frac{dt}{t}\,d\ell\,dy_1\,dy_2 = 1 , \tag{4.8}$$

where

$$(0,\pm1,0,0)\,{}^tA_{t,\ell,y} = \pm t^{-1/2}\left(\frac{\ell}{2}, 1, y_1, y_1\frac{\ell}{2}+y_2\right) .$$

We only compute the integral related to the orbit $\mathcal{O}_{(0,1,0,0)}$ because the computation related to $\mathcal{O}_{(0,-1,0,0)}$ is analogous. Performing the change of variables

$$t^{-1/2} = \alpha,\ t^{-1/2}\ell/2 = u_1,\ t^{-1/2}y_1 = u_2,\ t^{-1/2}\left(y_1\frac{\ell}{2}+y_2\right) = v ,$$

with $dt\,d\ell\,dy_1\,dy_2 = 4\,d\alpha\,du_1\,du_2\,dv/\alpha^6$, we can write

$$\begin{aligned}\int_{\mathbb{R}^3}\int_0^\infty W_\phi\left(t^{-1/2}\left(\frac{\ell}{2}, 1, y_1, y_1\frac{\ell}{2}+y_2\right)\right)\frac{dt}{t}\,d\ell\,dy_1\,dy_2 \\ = 4\int_{\mathbb{R}^3}\int_0^\infty W_\phi(u_1,\alpha,u_2,v)\frac{d\alpha}{\alpha^4}\,du_1\,du_2\,dv\,. \qquad (4.9)\end{aligned}$$

In the following computations, we shall use Lemma 3, the Wigner marginal property (3.3), and the assumption $\widetilde{\phi_1}(y) = y\phi_1(y)$.

$$\begin{aligned}4\int_{\mathbb{R}^3}\int_0^\infty W_\phi(u_1,\alpha,u_2,v)\frac{d\alpha}{\alpha^4}\,du_1\,du_2\,dv &= 2\int_{\mathbb{R}^2} W_{\phi_0}(u_1,u_2)\,du_1\,du_2 \\ &\quad\times\int_0^\infty\int_{\mathbb{R}} W_{\widetilde{\phi_1}}(\alpha,v)\,dv\frac{d\alpha}{\alpha^4} \\ &= 2\int_0^\infty\int_{\mathbb{R}} W_{\widetilde{\phi_1}}(\alpha,v)\,dv\frac{d\alpha}{\alpha^4} \\ &= 2\int_0^\infty\left|\widetilde{\phi_1}(\alpha)\right|^2\frac{d\alpha}{\alpha^4} \\ &= 2\int_0^\infty|\phi_1(\alpha)|^2\frac{d\alpha}{\alpha^2}\,.\end{aligned}$$

Finally, the expression on the right-hand side is equal to 1, for ϕ_1 is a reproducing function of the subgroup H_1 and, consequently, fulfills the first reproducing condition in (4.4). □

Remark 1. The assumptions $\phi_0, \phi_1 \in \mathcal{S}(\mathbb{R})$ are actually only technical ones. Any pair of functions $\phi_0, \phi_1 \in L^2(\mathbb{R})$, reproducing for H_0 and H_1 respectively, defines a function $\phi \in L^2(\mathbb{R}^2)$ in (4.6) for which (5.17) (with constant $c_\phi = 1/2$) and (5.18) of Theorem 5 hold true.

5. Connections with Wavelet Theory

We now come closer to the group theory that lies behind the construction of two-dimensional wavelets. The analysis of oriented features in images requires more flexible objects than the wavelets arising from the tensor product of the usual one-dimensional wavelets. Answers to this problem, in the context of signal processing, have been provided by frame systems of directional functions with excellent angular selectivity, the frames of *curvelets* and *contourlets* [4, 10]. Both make use of translation and dilation operations, and while the curvelet approach obtains directional selectivity by a construction that requires a rotation operation, the contourlet setup uses a shearing operation. Although, our results do not have direct implications in either curvelet or contourlet analysis, we point out that they appear to be connected from the point of view of group theory, that is, by looking at the restriction of the metaplectic representation to two admissible subgroups of $Sp(2,\mathbb{R})$, namely $SIM(2)$ and $TDS(2)$.

The main results in this section are Theorem 3 and Theorem 4.

5.1 The Group $SIM(2)$ and its Natural Representation

In this section we prove Theorem 3. The similitude group $SIM(2)$ of the plane $\mathbb{R}^2$ is the group generated by translations, rotations, and dilations (a survey on the topic and the related two-dimensional directional wavelets is in [1]). More precisely, for a real angle θ put

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} , \tag{5.1}$$

the standard 2×2 rotation matrix. Then $SIM(2)$ consists of all the 3×3 matrices

$$T(t, y, \theta) = \begin{bmatrix} tR_{-\theta} & y \\ 0 & 1 \end{bmatrix} ,$$

where $t > 0$, y is a column vector in $\mathbb{R}^2$ and $\theta \in [0, 2\pi)$. The product in $SIM(2)$ is just matrix product and a simple calculation yields

$$T(t, y, \theta)T(s, z, \phi) = T(ts, y + tR_{-\theta}z, (\theta + \phi) \bmod 2\pi), \quad 0 \le \theta, \phi < 2\pi . \tag{5.2}$$

Formally, the action of $SIM(2)$ on $\mathbb{R}^2$ is obtained by viewing $\mathbb{R}^2$ as one of the affine charts in $\mathbb{RP}^2$, namely

$$\mathbb{R}^2 \simeq \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} : x \in \mathbb{R}^2 \right\} \subset \mathbb{RP}^2 .$$

In other words, $SIM(2)$ acts on $\mathbb{RP}^2$ preserving this affine chart:

$$T(t, y, \theta) \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} tR_{-\theta} & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} tR_{-\theta}x + y \\ 1 \end{bmatrix} .$$

The wavelet representation ν of $SIM(2)$ on $L^2(\mathbb{R}^2)$ is defined as follows:

$$\nu(t, y, \theta) f(x) = t^{-1} f\left(t^{-1}\left(R_\theta(x - y)\right)\right) ,$$

where $\nu(t, y, \theta)$ stands for $\nu(T(t, y, \theta))$. Notice that if we transpose rotations, dilations and translations to functions by

$$(R_\theta f)(x) = f(R_\theta x), \quad (D_t f)(x) = t^{-1} f\left(t^{-1}x\right), \quad \left(T_y f\right)(x) = f(x - y) ,$$

then $\nu(t, y, \theta) f = \left(T_y R_\theta D_t\right) f$. The representation ν is known to be irreducible on $L^2(\mathbb{R}^2)$ and it gives rise to a reproducing formula. A wavelet ϕ is reproducing if

$$\int_{\mathbb{R}^2} \frac{\left|\hat{\phi}(\xi)\right|^2}{|\xi|^2} d\xi < \infty .$$

For our purposes however, it is convenient to view ν in the frequency domain, that is, to compose it with the Fourier transform $\mathcal{F}$. We shall therefore write

$$\pi(t, y, \theta) f(u) = \left(\mathcal{F} \circ \nu(t, y, \theta) \circ \mathcal{F}^{-1} f\right)(u) = te^{-2\pi i \langle y, u\rangle} f(tR_\theta u) . \tag{5.3}$$

The Group $SIM(2)$ and the Action on the Lagrange Manifold

We adopt the following notation. If $y = (y_1, y_2) \in \mathbb{R}^2$, we put

$$\Sigma_y = \begin{bmatrix} y_1 & y_2 \\ y_2 & -y_1 \end{bmatrix} , \tag{5.4}$$

a 2×2 symmetric and traceless matrix. Consider the two subgroups of $Sp(2, \mathbb{R})$:

$$G_0 = \left\{ g_0(t, y) = \begin{bmatrix} t^{-1/2} & 0 \\ t^{-1/2}\Sigma_y & t^{1/2} \end{bmatrix} : t > 0,\, y \in \mathbb{R}^2 \right\} \tag{5.5}$$

$$K = \left\{ k(\theta) = \begin{bmatrix} R_{-\theta/2} & 0 \\ 0 & R_{-\theta/2} \end{bmatrix} : \theta \in [0, 4\pi) \right\} . \tag{5.6}$$

It is straightforward to check that:

$$\begin{aligned} g_0(t, y)g_0(s, z) &= g_0(ts, y + tz) \\ k(\theta)g_0(s, z)k(\theta)^{-1} &= g_0(s, R_{-\theta}, z) \end{aligned} \tag{5.7}$$

the latter being immediate from

$$R_{-\theta/2}\Sigma_x R_{\theta/2} = \Sigma_{R_{-\theta}x} . \tag{5.8}$$

The equality (5.7) shows that K normalizes G_0 and hence that G_0K inherits the structure of a semidirect product, where the product law is given by

$$\begin{aligned} g_0(t, y)k(\theta) \cdot g_0(s, z)k(\phi) &= g_0(t, y)\big[k(\theta)g_0(s, z)k(\theta)^{-1}\big]k(\theta)k(\phi) \\ &= g_0(t, y)g_0(s, R_{-\theta}z)\, k(\theta + \phi) \\ &= g_0(ts, y + tR_{-\theta}z)\, k(\theta + \phi) . \end{aligned}$$

Of course, $G_0 \rtimes K$ is a subgroup of $Sp(2, \mathbb{R})$. Further, G_0 is normal in $G_0 \rtimes K$ and obviously $G_0 \rtimes K/G_0 \simeq K$. We shall write

$$\begin{aligned} g(t, y, \theta) = g_0(t, y)k(\theta) &= \begin{bmatrix} t^{-1/2} & 0 \\ t^{-1/2}\Sigma_y & t^{1/2} \end{bmatrix} \begin{bmatrix} R_{-\theta/2} & 0 \\ 0 & R_{-\theta/2} \end{bmatrix} \\ &= \begin{bmatrix} t^{-1/2}R_{-\theta/2} & 0 \\ t^{-1/2}\Sigma_y R_{-\theta/2} & t^{1/2}R_{-\theta/2} \end{bmatrix} . \end{aligned}$$

Therefore

$$g(t, y, \theta)g(s, z, \phi) = g(ts, y + tR_{-\theta}z, (\theta + \phi) \bmod 4\pi), \quad 0 \le \theta, \phi < 4\pi .$$

The mapping

$$g(t, y, \theta) \mapsto T(t, y, \theta \bmod 2\pi)$$

is a homomorphism of $G_0 \rtimes K$ onto $SIM(2)$ [see (5.2)] with kernel given by $N = \{(1, 0, 0), (1, 0, 2\pi)\}$, that is, $G_0 \rtimes K$ is a covering group of $SIM(2)$. Consequently, the metaplectic representation can not be viewed as a representation of $SIM(2)$ on $L^2(\mathbb{R}^d)$ in the proper sense, however, when restricted to $L^2_{\text{even}}(\mathbb{R}^d)$, it becomes a proper representation. This fact will be used in Theorem 3.

Next, we identify the action of $SIM(2)$ on $\mathbb{R}^2$ with the action of $G_0 \rtimes K$ on a suitable two-dimensional cell $\mathcal{C}$ of the Lagrange manifold $L(\mathbb{R}^4)$. The Lagrange manifold is defined as the set of maximal isotropic planes in $\mathbb{R}^4$, namely the two-dimensional linear subspaces of $\mathbb{R}^4$ that enjoy the following property: If $x, y \in \mathcal{L}$, then $\omega(x, y) = 0$. This set inherits the manifold structure of a three-dimensional homogeneous space of $Sp(2, \mathbb{R})$. Indeed, let us represent planes in $\mathbb{R}^4$ as 4×2 matrices via

$$\mathcal{L}(A, B) = \operatorname{span} \begin{bmatrix} A \\ B \end{bmatrix}, \qquad A, B \in M_2(\mathbb{R}),\ \operatorname{rank} \begin{bmatrix} A \\ B \end{bmatrix} = 2 . \tag{5.9}$$

Under the identification (5.9), two 4×2 full-rank matrices identify the same plane if and only if they differ by right multiplication by some $g \in GL(2, \mathbb{R})$. Such a 4×2 full-rank matrix represents a Lagrangian plane if and only if ${}^tAB$ is symmetric. Also, its columns form an orthonormal set if and only if ${}^tAA + {}^tBB = I$. In this case,

$$g(A, B) = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \in Sp(2, \mathbb{R})$$

and $g(A, B)$ carries the "base" Lagrangian plane $\mathcal{L}_0 = \mathcal{L}(I, 0)$ onto $\mathcal{L}(A, B)$. In general, $Sp(2, \mathbb{R})$ acts on Lagrangian spaces from the left, by matrix multiplication on the spanning column vectors. Since we know that $g(A, B) \cdot \mathcal{L}_0 = \mathcal{L}(A, B)$, the $Sp(2, \mathbb{R})$-action is transitive on $L(\mathbb{R}^4)$ and the stabilizer at $\mathcal{L}_0$ is the subgroup

$$U = \left\{ \begin{bmatrix} A & A\Sigma \\ 0 & {}^tA^{-1} \end{bmatrix} : A \in GL(2, \mathbb{R}),\ \Sigma \text{ symmetric} \right\} \subset Sp(2, \mathbb{R}) \ .$$

Thus, $L(\mathbb{R}^4) \simeq Sp(2, \mathbb{R})/U$. An open set in $L(\mathbb{R}^4)$ that contains the base point $\mathcal{L}_0$ is $L_0 = \{\mathcal{L}(\Sigma) := \mathcal{L}(I, \Sigma) : \Sigma \text{ symmetric }\}$, and is diffeomorphic to $\mathbb{R}^3$ under the identification $\Sigma \leftrightarrow \mathcal{L}(\Sigma)$. We put

$$\mathcal{C} = \left\{ \mathcal{L}(x) = \mathcal{L}(\Sigma_x) : x \in \mathbb{R}^2 \right\} \ ,$$

the two-dimensional slice inside L_0 identified with the traceless symmetric matrices.

Proposition 2. *The action of $SIM(2)$ on $\mathbb{R}^2$ corresponds to the natural action of $G_0 \rtimes K$ on $\mathcal{C}$ inside the Lagrange manifold $L(\mathbb{R}^4)$.*

Proof. Allowing right multiplication by $t^{1/2}R_{\theta/2}$ and using (5.8), we compute

$$\begin{aligned} g(t, y, \theta) \cdot \mathcal{L}(x) &= \operatorname{span}\left(\begin{bmatrix} t^{-1/2}R_{-\theta/2} & 0 \\ t^{-1/2}\Sigma_y R_{-\theta/2} & t^{1/2}R_{-\theta/2} \end{bmatrix} \begin{bmatrix} I \\ \Sigma_x \end{bmatrix} \begin{bmatrix} t^{1/2}R_{\theta/2} \end{bmatrix} \right) \\ &= \operatorname{span} \begin{bmatrix} I \\ \Sigma_y + tR_{-\theta/2}\Sigma_x R_{\theta/2} \end{bmatrix} \\ &= \operatorname{span} \begin{bmatrix} I \\ \Sigma_y + t\Sigma_{R_{-\theta}x} \end{bmatrix} \\ &= \operatorname{span} \begin{bmatrix} I \\ \Sigma_{y+tR_{-\theta}x} \end{bmatrix} \\ &= \mathcal{L}(y + tR_{-\theta}x) \ . \end{aligned}$$

Therefore, under the canonical homomorphism of $G_0 \rtimes K$ onto $SIM(2)$, the action of $SIM(2)$ on $\mathbb{R}^2$ corresponds to the natural action of $G_0 \rtimes K$ on $\mathcal{C}$. □

We now compute the metaplectic representation μ on $G_0 \rtimes K$. We start from a simple observation. Every $g(t, \theta, y) \in G_0 \rtimes K$ decomposes as the product of a block-diagonal matrix and a block-lower triangular matrix, both in $Sp(2, \mathbb{R})$, as follows:

$$\begin{bmatrix} t^{-1/2}R_{-\theta/2} & 0 \\ t^{-1/2}\Sigma_y R_{-\theta/2} & t^{1/2}R_{-\theta/2} \end{bmatrix} = \begin{bmatrix} t^{-1/2}R_{-\theta/2} & 0 \\ 0 & t^{1/2}R_{-\theta/2} \end{bmatrix} \begin{bmatrix} I & 0 \\ t^{-1}R_{\theta/2}\Sigma_y R_{-\theta/2} & I \end{bmatrix} .$$

We rewrite this as $g(t, y, \theta) = D(t, \theta)\, L(t, y, \theta)$. Owing to (2.3) and (2.4), we have

$$\begin{aligned}
\mu(g(t, y, \theta)) f(x) &= \mu(D(t, \theta)\, L(t, y, \theta)) f(x) \\
&= \det\left(t^{-1/2} R_{-\theta/2}\right)^{-1/2} \mu(L(t, y, \theta)) f\left(t^{1/2} R_{\theta/2} x\right) \\
&= t^{1/2} \exp\left(i\pi \left\langle \left[t^{-1} R_{\theta/2} \Sigma_y R_{-\theta/2}\right] t^{1/2}\ R_{\theta/2} x, t^{1/2} R_{\theta/2} x \right\rangle\right) \\
&\quad \times f\left(t^{1/2} R_{\theta/2} x\right) \\
&= t^{1/2} \exp\left(i\pi \left\langle t^{-1/2} R_{\theta/2} \Sigma_y x, t^{1/2} R_{\theta/2} x \right\rangle\right) f\left(t^{1/2} R_{\theta/2} x\right) \\
&= t^{1/2} e^{i\pi\langle \Sigma_y x, x\rangle} f\left(t^{1/2} R_{\theta/2} x\right)\ ,
\end{aligned}$$

that is

$$\mu(g(t, y, \theta)) f(x) = t^{1/2} e^{i\pi\langle \Sigma_y x, x\rangle} f\big(t^{1/2} R_{\theta/2} x\big), \quad 0 \le \theta < 4\pi\ . \tag{5.10}$$

We observe that the metaplectic representation of $SIM(2)$ is

$$\mu(T(t, y, \theta)) f(x) = t^{1/2} e^{i\pi\langle \Sigma_y x, x\rangle} f\left(t^{1/2} R_{\theta/2} x\right), \quad 0 \le \theta < 2\pi\ ,$$

and is not a group homomorphism. Note that, when restricted to $L^2_{\text{even}}(\mathbb{R}^d)$, it becomes a proper representation.

The Intertwining Operator and the Equivalence

Consider the mapping

$$\Phi : \dot{\mathbb{R}} \times \mathbb{R}_+ \to \mathbb{R} \times \dot{\mathbb{R}}, \qquad x \mapsto \left(\frac{x_2^2 - x_1^2}{2}, -x_1 x_2\right)\ . \tag{5.11}$$

Its properties are described in the next proposition.

Proposition 3. *Let f be an even function defined on $\mathbb{R}^2$. The mapping* (5.11) *is a diffeomorphism that satisfies:*

(a) *The Jacobian of Φ at $x \in \dot{\mathbb{R}} \times \mathbb{R}_+$ is $J_\Phi(x) = \|x\|^2$;*
(b) *the Jacobian of Φ^{-1} at $u \in \mathbb{R} \times \dot{\mathbb{R}}$ is $J_{\Phi^{-1}}(u) = \|x(u)\|^{-2} = 1/(2\|u\|)$;*
(c) *$f(\Phi(aR_\theta x)) = f(a^2 R_{2\theta} \Phi(x))$ for every $a \in \mathbb{R}$, every $x \in \mathbb{R}^2$;*
(d) *$f(\Phi^{-1}(tR_\theta u)) = f(t^{1/2} R_{\theta/2} \Phi^{-1}(u))$, for every $t > 0$, every $u \in \mathbb{R} \times \dot{\mathbb{R}}$;*
(e) *$\langle \Sigma_y x, x\rangle = -2\langle y, \Phi(x)\rangle$ for every $x \in \dot{\mathbb{R}} \times \mathbb{R}_+$ and every $y \in \mathbb{R}^2$.*

Proof. First we show that Φ defines a bijective mapping of $\mathbb{R}_+ \times \mathbb{R}_+$ onto $\mathbb{R} \times \mathbb{R}_-$. Indeed, for $(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $(u_1, u_2) \in \mathbb{R} \times \mathbb{R}_+$

$$\begin{cases} \dfrac{x_2^2 - x_1^2}{2} = u_1 \\[2ex] -x_1 x_2 = u_2 \end{cases} \iff \begin{cases} x_2^2 - \dfrac{u_2^2}{x_2^2} = 2u_1 \\[2ex] x_2 = -\dfrac{u_2}{x_1}\ . \end{cases}$$

For fixed $u_2 \in \mathbb{R}_-$, the map $h(t) = t - u_2^2/t$ defined in $\mathbb{R}_+$ is increasing since $h'(t) = 1 + u_2^2/t^2 > 0$. Further, $h(t) \to -\infty$ for $t \to 0^+$ and $h(t) \to +\infty$ for $t \to +\infty$. Therefore, for any given $u_1 \in \mathbb{R}$ there is exactly one value of x_2^2 such that $h(x_2^2) = 2u_1$. Hence, for any given $u_1 \in \mathbb{R}$ and $u_2 < 0$ there is a unique $(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $\Phi(x_1, x_2) = (u_1, u_2)$. This shows that Φ is bijective from $\mathbb{R}_+ \times \mathbb{R}_+$ onto $\mathbb{R} \times \mathbb{R}_-$. Similarly, it is bijective from $\mathbb{R}_- \times \mathbb{R}_+$ onto $\mathbb{R} \times \mathbb{R}_+$ and hence from $\dot{\mathbb{R}} \times \mathbb{R}_+$ onto $\mathbb{R} \times \dot{\mathbb{R}}$. It is clearly smooth and its regularity follows from

$$J_\Phi(x) = \det \begin{bmatrix} -x_1 & x_2 \\ -x_2 & -x_1 \end{bmatrix} = x_1^2 + x_2^2 \; .$$

This establishes that Φ is a diffeomorphisms and proves (a). As for (b), it follows from (a) and the observation that

$$u_1^2 + u_2^2 = \left(\frac{x_2^2 - x_1^2}{2} \right)^2 + x_1^2 x_2^2 = \frac{1}{4} \left(x_1^2 + x_2^2 \right)^2 \; ,$$

so that $2\|u\| = \|x\|^2$.

(c) Here we compute

$$\begin{aligned} \Phi(aR_\theta x) &= \Phi\left(a(\cos\theta x_1 + \sin\theta x_2), a(-\sin\theta x_1 + \cos\theta x_2)\right) \\ &= \left(\frac{a^2}{2} \left[\cos 2\theta \left(x_2^2 - x_1^2 \right) - 2\sin 2\theta (x_1 x_2) \right] , \right. \\ &\qquad \left. \frac{a^2}{2} \left[\sin 2\theta \left(x_1^2 - x_2^2 \right) - 2\cos 2\theta (x_1 x_2) \right] \right) \\ &= a^2 \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} \frac{x_2^2 - x_1^2}{2} \\ -x_1 x_2 \end{bmatrix} \\ &= a^2 R_{2\theta} \Phi(x) \; . \end{aligned}$$

(d) Put $a = t^{1/2}$ and $\psi = 2\theta$ in (c) to get $\Phi(t^{1/2} R_{\psi/2} x) = t R_\psi \Phi(x)$. Put next $\Phi(x) = u$ and take Φ^{-1} from both sides. This yields $t^{1/2} R_{\psi/2} \Phi^{-1}(u) = \Phi^{-1}(t R_\psi u)$.

(e) From the definition of Σ_y and of Φ, we obtain

$$\langle \Sigma_y x, x \rangle = \left\langle \begin{bmatrix} y_1 x_1 + y_2 x_2 \\ y_2 x_1 - y_1 x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle = y_1 \left(x_1^2 - x_2^2 \right) + y_2 (2 x_1 x_2) = -2 \langle y, \Phi(x) \rangle \; ,$$

as desired to conclude the proof. □

Theorem 3. *The mapping*

$$\mathcal{U} f(u) = \|u\|^{-1/2} f\left(\Phi^{-1}(u)\right), \qquad u \in \dot{\mathbb{R}} \times \mathbb{R}_+$$

defines an isometry of $L^2_{\text{even}}(\mathbb{R}^2)$ onto $L^2(\mathbb{R}^2)$ that intertwines π and μ: $\pi(g) \circ \mathcal{U} = \mathcal{U} \circ \mu(g)$ for every $g \in SIM(2)$.

Proof. Let $f \in L^2_{\text{even}}(\mathbb{R}^2)$. Then, by (b) in Proposition 3

$$\begin{aligned} \|\mathcal{U} f\|_2^2 &= \int_{\mathbb{R}^2} |\mathcal{U} f(u)|^2 \; du = \int_{\mathbb{R}^2} \frac{1}{\|u\|} \left| f\left(\Phi^{-1}(u)\right) \right|^2 \; du \\ &= 2 \int_0^{+\infty} \int_{-\infty}^{+\infty} |f(x)|^2 \; dx = \int_{\mathbb{R}^2} |f(x)|^2 \; dx \\ &= \|f\|_2^2 \; . \end{aligned}$$

Thus, $\mathcal{U}$ is an isometry. By (5.3) and (d) in Proposition 3

$$\begin{aligned}\pi(t,y,\theta)\,(\mathcal{U}f)\,(u) &= te^{-2\pi i\langle y,u\rangle}\,\mathcal{U}f\,(tR_\theta u)\\ &= \frac{t}{\|tR_\theta u\|^{1/2}}e^{-2\pi i\langle y,u\rangle}\,f\left(\Phi^{-1}\,(tR_\theta u)\right)\\ &= \frac{t^{1/2}}{\|u\|^{1/2}}e^{-2\pi i\langle y,u\rangle}\,f\left(t^{1/2}R_{\theta/2}\Phi^{-1}(u)\right)\ .\end{aligned}$$

Finally, by (5.10) and (e) in Proposition 3

$$\begin{aligned}\mathcal{U}\,(\mu(t,y,\theta)f)\,(u) &= \frac{1}{\|u\|^{1/2}}\,(\mu(t,\theta,y)f)\left(\Phi^{-1}(u)\right)\\ &= \frac{1}{\|u\|^{1/2}}t^{1/2}e^{i\pi\langle\Sigma_y\Phi^{-1}(u),\Phi^{-1}(u)\rangle}f\left(t^{1/2}R_{\theta/2}\Phi^{-1}(u)\right)\\ &= \frac{t^{1/2}}{\|u\|^{1/2}}e^{-i\pi 2\langle y,u\rangle}f\left(t^{1/2}R_{\theta/2}\Phi^{-1}(u)\right)\ ,\end{aligned}$$

as desired. □

5.2 The Group $TDS(2)$, the Contourlet Point of View

In this section we prove Theorem 4. We first explain the connection between $TDS(2)$ and the two-dimensional wavelet theory that leads to the contourlet construction introduced in [10]. The point is that $TDS(2)$ is isomorphic to the group of mappings of (functions on) the plane generated by translations, dilations, and shearing, where the shearing operator is given by

$$(S_\ell f)\,(x) = f\left({}^tS_\ell x\right), \qquad f\in L^2(\mathbb{R})\ ,$$

and the matrix S_ℓ is defined in (4.2). These are the ingredients of the contourlet frames [10]. Just as for curvelets, one allows dilation and translation operations, but the angular selectivity is achieved by a shearing operation rather than a rotation.

Let L denote the two-dimensional subgroup of $GL(2,\mathbb{R})$ given by

$$L = \left\{\begin{bmatrix} t & 0\\ -\ell t & t\end{bmatrix} : t>0,\ \ell\in\mathbb{R}\right\}\ .$$

The affine action that it induces on $\mathbb{R}^2$ leads to the semidirect product $H = \mathbb{R}^2 \rtimes L$. This action has two open orbits $\mathcal{O}_+$ and $\mathcal{O}_-$ in $\mathbb{R}^2$, where $\mathcal{O}_+ = \{(x_1,x_2)\,;\ x_2>0\}$ and $\mathcal{O}_- = \{(x_1,x_2)\,;\ x_2<0\}$. The wavelet representation ν of H is

$$\nu(t,y,\ell)f = \left(T_yD_tS_\ell\right)f, \qquad f\in L^2\left(\mathbb{R}^2\right)\ , \tag{5.12}$$

but it is more convenient to view ν in the frequency domain, namely

$$\pi(t,y,\ell)f(u) = \left(\mathcal{F}\circ\nu(t,y,\ell)\circ\mathcal{F}^{-1}f\right)(u) = e^{-2\pi i\langle y,u\rangle}D_{-t}\,{}^tS_{-\ell}f\,(u)\ . \tag{5.13}$$

We have $\pi = \pi_{\mathcal{O}_+}\oplus\pi_{\mathcal{O}_-}$, where $\pi_{\mathcal{O}_+}$ and $\pi_{\mathcal{O}_-}$ are the subrepresentations of π obtained by restriction to $L^2(\mathcal{O}_+)$ and $L^2(\mathcal{O}_-)$, respectively. A wavelet ϕ such that $\hat\phi\in L^2(\mathcal{O}_+)$ is reproducing for $\pi_{\mathcal{O}_+}$ if

$$\int_0^\infty\int_{\mathbb{R}}\left|\frac{\hat\phi(\xi_1,\xi_2)}{\xi_2}\right|^2 d\xi_1\,d\xi_2<\infty$$

and similarly for $\pi_{\mathcal{O}_-}$ (see [3] for more details).

If $y = (y_1, y_2) \in \mathbb{R}^2$, we put $B_y = \left[\begin{smallmatrix} 0 & y_1 \\ y_1 & y_2 \end{smallmatrix}\right]$, a 2×2 symmetric matrix. Then, we check that $TDS(2) = G_0 \rtimes G_1$, where

$$G_0 = \left\{ g_0(t, y) = \begin{bmatrix} t^{-1/2} & 0 \\ t^{-1/2} B_y & t^{1/2} \end{bmatrix} : t > 0, y \in \mathbb{R}^2 \right\}$$
$$G_1 = \left\{ g_1(\ell) = \begin{bmatrix} S_{\ell/2} & 0 \\ 0 & {}^t S_{-\ell/2} \end{bmatrix} : \ell \in \mathbb{R} \right\} .$$

Indeed, for $y = (y_1, y_2) \in \mathbb{R}^2$, it is easy to see that, using the parametrization in (4.1)

$$g_0(t, y) g_1(\ell) = A_{t,\ell,(y_1, y_2 + \ell y_1)} ,$$

so that $TDS(2) = G_0 G_1$ set-theoretically. Furthermore,

$$g_1(\ell) g_0(t, (y_1, y_2)) g_1(\ell)^{-1} = g_0(t, (y_1, y_2 - \ell y_1)) . \tag{5.14}$$

This means that G_1 normalizes G_0, so that $TDS(2)$ is the semidirect product $G_0 \rtimes G_1$. Since G_0 is normal in $TDS(2)$, obviously $TDS(2)/G_0 \simeq G_1$. Finally, the product is

$$g(t, (y_1, y_2), \ell) g(r, (z_1, z_2), s) = g(tr, (y_1 + t z_1, y_2 + t z_2 - \ell t z_1), s + \ell) ,$$

which implicitly shows the isomorphism $TDS(2) \simeq H$, as one checks by computing the product in $H = \mathbb{R}^2 \rtimes L$. Observe that the decomposition of $TDS(2)$ as a semidirect product is similar to the decomposition of $SIM(2)$ as far as the normal factors are concerned. The basic difference consists in the structure of their quotients: It is compact for $SIM(2)$ and noncompact for $TDS(2)$.

In order to compute the metaplectic representation on $TDS(2)$, we observe first that the matrix $A_{t,y,\ell}$ in (4.1) can be written as the product of a diagonal matrix $D_{t,\ell}$ and a lower triangular matrix $L_{t,y,\ell}$ as follows

$$A_{t,y,\ell} = D_{t,\ell} L_{t,y,\ell} = \begin{bmatrix} t^{-1/2} S_{\ell/2} & 0 \\ 0 & t^{1/2}\, {}^t S_{-\ell/2} \end{bmatrix} \begin{bmatrix} I & 0 \\ t^{-1}\, {}^t S_{\ell/2} B_y S_{\ell/2} & I \end{bmatrix} .$$

We then use the fact that μ is a representation and formulae (2.3) and (2.4) to obtain that for $f \in L^2(\mathbb{R}^2)$

$$\begin{aligned} \mu(A_{t,y,\ell}) f(x) = \mu(D_{t,\ell} L_{t,y,\ell}) f(x) &= t^{1/2} (L_{t,y,\ell} f) \left(t^{1/2} S_{-\ell/2} x \right) \\ &= t^{1/2} e^{i\pi \langle {}^t S_{\ell/2} B_y x, S_{-\ell/2} x \rangle} f\left(t^{1/2} S_{-\ell/2} x \right) \\ &= t^{1/2} e^{i\pi \langle B_y x, x \rangle} f\left(t^{1/2} S_{-\ell/2} x \right) . \end{aligned}$$

The Intertwining Operator and the Equivalence for $TDS(2)$

We shall be concerned with the mapping

$$\Psi : \dot{\mathbb{R}} \times \mathbb{R}_+ \to \dot{\mathbb{R}} \times \mathbb{R}_-, \qquad x \mapsto \left(-x_1 x_2, -\frac{x_2^2}{2} \right) , \tag{5.15}$$

whose properties are described in the following elementary proposition. Its proof is analogous to that of Proposition 3 and is therefore omitted.

Proposition 4. *The mapping* (5.15) *defines diffeomorphisms from* $\dot{\mathbb{R}} \times \mathbb{R}_+$ *or from* $\dot{\mathbb{R}} \times \mathbb{R}_-$ *onto* $\dot{\mathbb{R}} \times \mathbb{R}_-$ *and is such that* $\Psi(-x) = \Psi(x)$. *Further, it satisfies:*

(a) *The Jacobian of* Ψ *at* $x = (x_1, x_2) \in \dot{\mathbb{R}}\times\mathbb{R}_+$ $(x = (x_1, x_2) \in \dot{\mathbb{R}}\times\mathbb{R}_-$, *respectively*) *is* $J_\Phi(x) = x_2^2$*;*
(b) *the Jacobian of* Ψ^{-1} *at* $u = (u_1, u_2) \in \dot{\mathbb{R}} \times \mathbb{R}_-$ *is* $J_{\Psi^{-1}}(u) = 1/(2u_2)$*;*
(c) $\Psi^{-1}(t^2 S_{2\ell} u) = t S_\ell \Psi^{-1}(u)$ *for every* $t > 0$ *and every* $u \in \dot{\mathbb{R}} \times \mathbb{R}_-$*;*
(d) $\langle B_y x, x\rangle = -2\langle y, \Psi(x)\rangle$ *for every* $x \in \dot{\mathbb{R}} \times \mathbb{R}_+$ $(x = (x_1, x_2) \in \dot{\mathbb{R}} \times \mathbb{R}_-$, *respectively*) *and every* $y \in \mathbb{R}^2$.

The proof of the following theorem is analogous to the proof of Theorem 3 and its details are given in [7].

Theorem 4. *The mapping obtained by extending*

$$\mathcal{Q}f(u) = |2u_2|^{-1/2} f\big(\Psi^{-1}(u_1, u_2)\big), \qquad u \in \dot{\mathbb{R}} \times \mathbb{R}_+$$

to $\dot{\mathbb{R}} \times \dot{\mathbb{R}}$ *as an even function defines an isometry of* $L^2_{\mathrm{even}}(\mathbb{R}^2)$ *onto itself that intertwines the representations* π *and* μ*, that is* $\pi(g) \circ \mathcal{Q} = \mathcal{Q} \circ \mu(g)$ *for every* $g \in TDS(2)$.

Admissible Functions for $TDS(2)$

The reproducibility of $TDS(2)$ follows either by the admissibility condition (3.8) (Section 4) or directly by the same techniques as in Theorem 6, with the admissibility conditions stated below.

Theorem 5. *Let* $H = TDS(2)$*. The identity*

$$\int_H |\langle f, \mu(h)\phi\rangle|^2 \, dh = c_\phi \, \|f\|_2^2 \,, \tag{5.16}$$

holds for every $f \in L^2(\mathbb{R}^2)$ *if and only if the function* ϕ *satisfies the following two admissibility conditions:*

$$c_\phi = 4 \int_{\mathbb{R}} \int_0^\infty |\phi(x)|^2 \frac{dx_2}{x_2^4} \, dx_1 = 4 \int_{\mathbb{R}} \int_0^\infty |\phi(-x)|^2 \frac{dx_2}{x_2^4} \, dx_1 \tag{5.17}$$

and

$$\int_{\mathbb{R}} \int_0^\infty \phi(x)\overline{\phi(-x)} \, \frac{dx_2}{x_2^4} \, dx_1 = 0 \,. \tag{5.18}$$

Theorem 5 is proved in [7], where examples of admissible wavelets for $TDS(2)$ are also given.

6. A Class of Reproducing Groups Including $SIM(2)$

The (double cover of) $SIM(2)$ is one in a family of reproducing groups parametrized by $\mathbb{R}$. For any parameter[1] pair $(\alpha, \beta) \neq (0, 0)$, consider the three-dimensional subgroup of $Sp(2, \mathbb{R})$

$$H_{\alpha,\beta} = \left\{ h_{\alpha,\beta}(t, y) := \begin{bmatrix} e^{-\alpha t/2} R_{\beta t/2} & 0 \\ \Sigma_y e^{-\alpha t/2} R_{\beta t/2} & e^{\alpha t/2} R_{\beta t/2} \end{bmatrix} : t \in \mathbb{R}, \; y \in \mathbb{R}^2 \right\}$$

[1] The reason for assuming $(\alpha, \beta) \neq (0, 0)$ will be discussed below.

where the rotation matrix R_θ is defined in (5.1) and the matrix Σ_y in (5.4). Clearly,

$$\begin{aligned} h_{\alpha,\beta}(t,y) &= \begin{bmatrix} I & 0 \\ \Sigma_y & I \end{bmatrix} \begin{bmatrix} e^{-\alpha t/2} R_{\beta t/2} & 0 \\ 0 & e^{\alpha t/2} R_{\beta t/2} \end{bmatrix} \\ &= \exp \begin{bmatrix} 0 & 0 \\ \Sigma_y & 0 \end{bmatrix} \exp \left(-\frac{t}{2} \begin{bmatrix} \alpha I - \beta J & 0 \\ 0 & -\alpha I - \beta J \end{bmatrix} \right) , \end{aligned}$$

where as usual $J = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Furthermore, $\Sigma_y = y_1 H + y_2 L$, where $H = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $L = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Thus, the Lie algebra $\mathfrak{h}_{\alpha,\beta}$ of $H_{\alpha,\beta}$ is spanned by

$$X = \begin{bmatrix} 0 & 0 \\ H & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix}, \qquad Z = \frac{1}{2} \begin{bmatrix} \alpha I - \beta J & 0 \\ 0 & -\alpha I - \beta J \end{bmatrix} .$$

Because of the brackets $[H, J] = 2L$ and $[L, J] = -2H$ one sees immediately that

$$\begin{cases} [X, Y] & = 0 \\ [X, Z] & = \alpha X - \beta Y \\ [Y, Z] & = \beta X + \alpha Y . \end{cases}$$

According to the classification of three-dimensional Lie algebras [17], all $\mathfrak{h}_{\alpha,\beta}$ fall in the class $\mathcal{A} = \{\mathfrak{g}_\Gamma : \Gamma \in GL(2, \mathbb{R})\}$, where $\mathfrak{g}_\Gamma = \operatorname{span}\{X, Y, Z\}$ has bracket table

$$\begin{cases} [X, Y] & = 0 \\ [X, Z] & = aX + bY \\ [Y, Z] & = cX + dY \end{cases} \qquad \Gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{R}) .$$

In our case

$$\Gamma = \Gamma_{\alpha,\beta} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

is nonsingular since $\det \Gamma = \alpha^2 + \beta^2 \neq 0$ because $(\alpha, \beta) \neq (0, 0)$. The isomorphism classes in $\mathcal{A}$ are described by Γ, as we now explain. First of all, two multiple matrices Γ and $\lambda\Gamma$ give rise to the same algebra if $\lambda \neq 0$, for if $\mathfrak{g}_\Gamma = \operatorname{span}\{X, Y, Z\}$, then the basis $\{X, Y, \lambda Z\}$ yields the bracket table that corresponds to $\lambda\Gamma$ and generates the same Lie algebra. Thus, $\mathfrak{g}_\Gamma = \mathfrak{g}_{\lambda\Gamma}$ if $\lambda \neq 0$. The isomorphism classes within $\mathcal{A}$ are in one-to-one correspondence with the conjugacy classes in $PGL(2, \mathbb{R}) = GL(2, \mathbb{R})/(\dot{\mathbb{R}} \cdot \mathrm{id})$. In other words, two nonmultiple matrices Γ and Γ' correspond to isomorphic Lie algebras if and only if they are conjugate in $GL(2, \mathbb{R})$. It is however an elementary exercise to check that a matrix $g \in GL(2, \mathbb{R})$ conjugates $\Gamma_{\alpha,\beta}$ into a matrix $\Gamma_{\gamma,\delta}$ of the same type if and only if g is of the form $g = cR_\theta$ or $g = cKR_\theta$, where c is a scalar and $K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. In this case, $g(\Gamma_{\alpha,\beta})g^{-1} = \Gamma_{\alpha,\beta}$ or $g(\Gamma_{\alpha,\beta})g^{-1} = \Gamma_{\alpha,-\beta}$, respectively. Therefore, to each point in $\{(1, \beta), \beta \geq 0\}$ there corresponds an isomorphism class in the subclass $\mathcal{H} = \{\mathfrak{h}_{\alpha,\beta} : (\alpha, \beta) \neq (0, 0)\}$ of $\mathcal{A}$. There is another issue that must be discussed, in the light of Theorem 1. One of its consequences is that an admissible subgroup of $Sp(d, \mathbb{R})$ cannot be unimodular. This fact is proved in [6] and is really a straightforward adaptation of a theorem proved in [18]. This explains why we have chosen $(\alpha, \beta) \neq (0, 0)$ from the start. Indeed, if $(\alpha, \beta) = (0, 0)$, then $H_{0,0}$ is (two-dimensional and) nilpotent, hence unimodular, and the constructions that follow cannot possibly lead to admissible groups. Furthermore, we must exclude from our parameters all those that correspond to unimodular groups. The modular function Δ on $H_{\alpha,\beta}$ is, as for any Lie group, $\Delta(x) = \det(\mathrm{Ad}(x^{-1}))$. If $v = \exp V$ is close to the identity,

$$\Delta(v) = \det\Big(\mathrm{Ad}\left(v^{-1}\right)\Big) = \det(\mathrm{Ad}(\exp(-V)) = \det\left(e^{-\,\mathrm{ad}\,V}\right) = e^{-\,\mathrm{tr}(\mathrm{ad}\,V)}$$

shows that $\Delta(v) = 1$ if and only if $\mathrm{tr}(\mathrm{ad}\,V) = 0$. Thus, $H_{\alpha,\beta}$ is unimodular if and only if this is true for every $V \in \mathfrak{h}_{\alpha,\beta}$. From the bracket table we see that $\mathrm{tr}(\mathrm{ad}\,Z) = 0$ and $\mathrm{tr}(\mathrm{ad}\,X) = \mathrm{tr}(\mathrm{ad}\,Y) = \alpha$. Hence, $H_{\alpha,\beta}$ is unimodular if and only if $\alpha = 0$. We summarize this discussion, and other elementary facts, in the proposition that follows.

Proposition 5. *The subgroups $H_{\alpha,\beta}$ of $Sp(2,\mathbb{R})$ satisfy the following properties:*

(a) *The product law in $H_{\alpha,\beta}$ is explicitly given by:*

$$h_{\alpha,\beta}(t,y)h_{\alpha,\beta}(s,z) = h_{\alpha,\beta}\big(t+s,\, y + e^{\alpha t}R_{\beta t}z\big), \quad t,s \in \mathbb{R},\ y,z \in \mathbb{R}^2\ .$$

(b) *The left Haar measure on $H_{\alpha,\beta}$ is $dh_{\alpha,\beta}(s,z) = e^{-2\alpha s}\,ds\,dz$.*

(c) *$H_{\alpha,\beta}$ is unimodular if and only if $\alpha = 0$.*

(d) *$H_{\alpha,\beta}$ and $H_{\gamma,\delta}$ are conjugate within $Sp(2,\mathbb{R})$ if and only if they are equal, if and only if $(\alpha,\beta) = \lambda(\gamma,\delta)$, for some $\lambda \neq 0$.*

(e) *Each $H_{\alpha,\beta}$ is normalized by the natural copy of $SO(2)$ inside $Sp(2,\mathbb{R})$.*

(f) *The semidirect product $H_{1,0} \rtimes SO(2)$ is (canonically) isomorphic to $G_0 \rtimes K$.*

(g) *The restriction of the metaplectic representation to $H_{\alpha,\beta}$ is given by:*

$$\mu(h_{\alpha,\beta}(t,y))f(x) = e^{\alpha t/2}e^{\pi i\langle \Sigma_y x, x\rangle}\, f\big(e^{\alpha t/2}R_{-\beta t/2}x\big)\ . \tag{6.1}$$

Proof. The statements follow either from the above discussion or from straightforward computations. We content ourselves with a couple of comments. By the natural copy of $SO(2)$ inside $Sp(2,\mathbb{R})$ we mean of course

$$SO(2) \simeq \left\{k_\theta = \begin{bmatrix} R_\theta & 0 \\ 0 & R_\theta \end{bmatrix} : \theta \in [0,2\pi)\right\} \tag{6.2}$$

and by (5.8) one computes immediately $k_\theta\, h_{\alpha,\beta}(t,y)\, k_\theta^{-1} = h_{\alpha,\beta}\,(t, R_{2\theta}y)$, which is the conjugation referred to in (e). As for (f), notice that when $\alpha = 1$, $\beta = 0$ and $\tau = e^t$, the matrix $h_{1,0}(\tau, y)$ is the G_0-component of an element in $G_0 \rtimes K$. □

By (c) and (d), we may assume $\alpha = 1$, and by (e) we may define the family of groups

$$G_\beta = H_{1,\beta} \rtimes SO(2) \qquad \beta \geq 0\ .$$

The elements of G_β will be denoted $g_\beta = h_\beta k$, where $k \in SO(2)$. Also, the left Haar measure is $dg_\beta = dh_\beta\, dk$. In the sequel, we shall parametrize $K = SO(2)$ with the angular parameter θ, as in (6.2). We prove next that the groups G_β are all reproducing.

Theorem 6. *The identity*

$$\int_{G_\beta} |\langle f, \mu(g_\beta)\phi\rangle|^2\, dg_\beta = c_\phi\, \|f\|_2^2\ , \tag{6.3}$$

holds for every $f \in L^2(\mathbb{R}^2)$ if and only if the function ϕ satisfies the following two admissibility conditions:

$$c_\phi = 2\int_{\mathbb{R}^2} \frac{|\phi(x)|^2}{\|x\|^4}\, dx < \infty\,; \tag{6.4}$$

$$\int_{\mathbb{R}^2} \phi(x)\overline{\phi(-x)}\, \frac{dx}{\|x\|^4} = 0\,. \tag{6.5}$$

First, we prove an identity of Plancherel type.

Lemma 4. *Let Φ be the mapping defined in* (5.11) *and $h \in L^2(\mathbb{R}^2)$ be a function which vanishes outside some annulus $c < \|x\| < C$, with $0 < c < C < \infty$. Then*

$$\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} h(x) e^{2\pi i \langle y, \Phi(x)\rangle}\, dx \right|^2 dy = \int_0^\infty \int_{\mathbb{R}} |h(x) + h(-x)|^2 \frac{dx}{\|x\|^2}\,.$$

Proof. We make the change of variables $\Phi(x) = u$. By (b) in Proposition 3:

$$\begin{aligned}\int_{\mathbb{R}^2} h(x) e^{2\pi i\langle y,\Phi(x)\rangle}\, dx &= \int_0^\infty \int_{\mathbb{R}} (h(x) + h(-x)) e^{2\pi i \langle y, \Phi(x)\rangle}\, dx \\ &= \int_{\mathbb{R}^2} \left(h\left(\Phi^{-1}(u)\right) + h\left(-\Phi^{-1}(u)\right)\right) e^{2\pi i\langle y,u\rangle}\, \frac{du}{\|x(u)\|^2}\,.\end{aligned}$$

By the Plancherel formula we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \left(h\left(\Phi^{-1}(u)\right) + h\left(-\Phi^{-1}(u)\right)\right) e^{2\pi i\langle y,u\rangle}\, \frac{du}{\|x(u)\|^2}\right|^2 dy \\ &\quad = \int_{\mathbb{R}^2} \left|\left(h\left(\Phi^{-1}(u)\right) + h\left(-\Phi^{-1}(u)\right)\right)\right|^2 \frac{du}{\|x(u)\|^4} \\ &\quad = \int_0^\infty \int_{\mathbb{R}} |h(x) + h(-x)|^2 \frac{dx}{\|x\|^2}\,,\end{aligned}$$

as desired. □

Corollary 1. *Let h be as in Lemma 4. Then*

$$\begin{aligned}&\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} h(x) e^{-\pi i\langle \Sigma_y x, x\rangle}\, dx\right|^2 dy \\ &\quad = \int_0^\infty \int_{\mathbb{R}} \left(|h(x)|^2 + |h(-x)|^2 + 2\mathrm{Re}\, h(x)\overline{h(-x)}\right) \frac{dx}{\|x\|^2}\,.\end{aligned}$$

Proof of Theorem 6. By (6.1), we must evaluate

$$\begin{aligned}&\int_{H_{1,\beta}\rtimes K} |\langle f, \mu(h_\beta k)\phi\rangle|^2\, dh_\beta\, dk \\ &\quad = \int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} f(x) e^{t/2} e^{-\pi i\langle \Sigma_y x, x\rangle} \overline{\phi\left(e^{t/2} R_{-(\beta t/2+\theta)} x\right)}\, dx\right|^2 dy e^{-2t}\, dt\, d\theta\,.\end{aligned} \tag{6.6}$$

Take f as in Lemma 4 and apply Corollary 1 to the right-hand side of (6.6)

$$\begin{aligned}
&\int_{H_{1,\beta}\rtimes K} |\langle f, \mu(h_\beta k)\phi\rangle|^2 \, dh_\beta \, dk \\
&= \int_0^{2\pi}\int_{\mathbb{R}}\int_0^{\infty}\int_{\mathbb{R}} \Big[|f(x)|^2 e^t \big|\phi\big(e^{t/2}R_{-(\beta t/2+\theta)}x\big)\big|^2 \\
&+ |f(-x)|^2 e^t \big|\phi\big(-e^{t/2}R_{-(\beta t/2+\theta)}x\big)\big|^2 \\
&+ 2\mathrm{Re}\, f(x)\overline{f(-x)} e^t \phi\big(-e^{t/2}R_{-(\beta t/2+\theta)}x\big)\overline{\phi\big(e^{t/2}R_{-(\beta t/2+\theta)}x\big)}\Big] \\
&\times \frac{dx}{\|x\|^2} e^{-2t} \, dt \, d\theta \, .
\end{aligned} \tag{6.7}$$

Suppose at first that f satisfies the additional property: $f(x_1, x_2) = 0$ for $x_2 < 0$. Perform now the change of variable given by the mapping

$$(t, \theta) \mapsto e^{t/2} R_{-(\beta t/2+\theta)} x = y \, , \tag{6.8}$$

a well-defined diffeomorphism. One checks that $dt\, d\theta = 2e^{-t}\|x\|^{-2}\, dy$ and hence

$$\begin{aligned}
\int_{H_{1,\beta}\rtimes K} |\langle f, \mu(h_\beta k)\phi\rangle|^2 \, dh_\beta \, dk &= \int_0^{\infty}\int_{\mathbb{R}} |f(x)|^2 \left(\int_{\mathbb{R}^2} |\phi(y)|^2 \frac{2}{\|y\|^4}\, dy\right) dx \\
&= \|f\|_2^2 \left(2\int_{\mathbb{R}^2} \frac{|\phi(y)|^2}{\|y\|^4}\, dy\right) \, .
\end{aligned}$$

If $f(x_1, x_2) = 0$, for $x_2 > 0$, the same relation holds. This argument shows that if the reproducing formula (6.3) works for all $f \in L^2(\mathbb{R}^d)$, then it works for f vanishing in a half-plane and outside an annulus, so that ϕ must fulfil (6.4). Take now a bounded function f supported in some annulus $c < \|x\| < C$. Then

$$G(\theta, t, x) := 2\mathrm{Re}\, f(x)\overline{f(-x)} e^t \phi\big(-e^{t/2}R_{-(\beta t/2+\theta)}x\big)\overline{\phi\big(e^{t/2}R_{-(\beta t/2+\theta)}x\big)}\frac{1}{\|x\|^2}$$

is integrable with respect to the measure $dx e^{-2t}\, dt\, d\theta$. By performing again the change of variable (6.8), and using the established value of c_ϕ, (6.7) becomes

$$\int_{H_{1,\beta}\rtimes K} |\langle f, \mu(h_\beta k)\phi\rangle|^2 \, dh_\beta \, dk = c_\phi \|f\|_2^2 + \int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}\times\mathbb{R}_+} G(\theta, t, x)\, dx e^{-2t}\, dt\, d\theta \, .$$

The reproducing formula (6.3) implies that the integral of $G(\theta, t, x)$ vanishes. On the other hand, using once more the change of variable (6.8)

$$\begin{aligned}
&\int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}\times\mathbb{R}_+} G(\theta, t, x)\, dx e^{-2t}\, dt\, d\theta \\
&\quad = 4\int_{\mathbb{R}^2}\int_{\mathbb{R}\times\mathbb{R}_+} \mathrm{Re}\left\{ f(x)\overline{f(-x)}\phi(-y)\overline{\phi(y)}\right\} dx \frac{dy}{\|y\|^4} \, .
\end{aligned}$$

Choosing f such that $f(x)\overline{f(-x)}$ is real valued, respectively purely imaginary valued, one obtains that

$$\int_{\mathbb{R}^2} \mathrm{Re}\left\{\phi(-y)\overline{\phi(y)}\right\} \frac{dy}{\|y\|^4} = 0 \quad \text{and} \quad \int_{\mathbb{R}^2} \mathrm{Im}\left\{\phi(-y)\overline{\phi(y)}\right\} \frac{dy}{\|y\|^4} = 0 \, ,$$

so that (6.5) must be true as well.

Conversely, assume that (6.4) and (6.5) are satisfied. If f is a function as in Lemma 4, then all the terms (6.6) are integrable and (6.3) holds for f. We conclude by showing that it actually works for all $f \in L^2(\mathbb{R}^2)$. To see this, take $f \in L^2(\mathbb{R}^2)$ and let f_n be a sequence of functions as in Lemma 4 which tends to f in the L^2-norm. Then $F(f_n) = \langle f_n, \mu(g_\beta)\phi\rangle$ is a Cauchy sequence on $L^2(G_\beta, dg_\beta)$ which tends pointwise to $F(f) = \langle f, \mu(g_\beta)\phi\rangle$. Since (6.3) holds for all f_n, it follows that it also holds for f. □

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