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Wavelet Transforms for Semidirect Product Groups with Not Necessarily Commutative Normal Subgroups

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ABSTRACT. Let G be the semidirect product group of a separable locally compact unimodular group N of type I with a closed subgroup H of Aut(N)*. The group N is not necessarily commutative. We consider irreducible subrepresentations of the unitary representation of G realized naturally on* $L^2(N)$*, and investigate the wavelet transforms associated to them. Furthermore, the irreducible subspaces are characterized by certain singular integrals on N analogous to the Cauchy-Szegö integral.*

1. Introduction

It is well known that the theory of continuous wavelet transform is reduced to study of square-integrable representations of (not necessarily unimodular) locally compact groups G [13, I]. The most typical example is the case that G is the '$ax + b$'-group and the representation is realized on $L^2(\mathbb{R})$ [13, II]. The results can be naturally generalized to the case that G is the semidirect product group of a vector group V with a linear group H on V (see [5, 8] and [10]), and further extensions of the theory have been developed in various directions. For example, wavelet transforms associated to nonirreducible representations are considered recently by [12] and [19], while wavelets for vector-valued functions associated to induced representations are studied by [2] and [3] (see also [1, Chapter 10]). Discretizations of the theory are considered by many authors (see [4, 15, 16] for example). Another direction of generalization is to the case that the normal subgroup V is not necessarily commutative.

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Actually, some works discuss the Heisenberg group case [14, 17, 20, 24], while there seem to be few works treating a general semidirect product (confer [11, Section 5]). Inspired by [14], we shall study square-integrable representations and associated continuous wavelet transforms in such a general setting.

Let N be a separable locally compact unimodular group of type I, H a closed subgroup of the automorphism group $\mathrm{Aut}(N)$ of N, and G the semidirect product group $N \rtimes H$. The group $N \lhd G$ is not necessarily commutative. We write the action of $h \in H$ to $n \in N$ as $h \cdot n$. Let $d\nu$ be a Haar measure on N, and $\delta(h)$ $(h \in H)$ the positive number for which $d\nu(h \cdot n) = \delta(h)\, d\nu(n)$ $(n \in N)$. Clearly, $\delta : H \to \mathbb{R}_+$ is a representation of H. If N is a vector group, then δ is the absolute value of the determinant. We define a unitary representation L of G on the Hilbert space $L^2(N)$ by

$$\begin{aligned} L(h) f(n_0) &:= \delta(h)^{-1/2} f\big(h^{-1} \cdot n_0\big)\,, \\ L(n) f(n_0) &:= f\big(n^{-1} n_0\big) \qquad \big(f \in L^2(N),\ h \in H,\ n, n_0 \in N\big)\,. \end{aligned}$$

It is easy to see that the representation $(L, L^2(N))$ is equivalent to the induced representation $\mathrm{Ind}_H^G \mathbf{1}$, where $\mathbf{1}$ stands for the trivial representation of H.

We investigate the representation L via the Plancherel formula for the unimodular group N (confer [18]). It is known that the Plancherel measure μ on the unitary dual $\hat{N}$ is uniquely determined by the abstract Plancherel formula [6]:

$$\int_N |f(n)|^2\, d\nu(n) = \int_{\hat{N}} \|\pi_\lambda(f)\|_{\mathrm{HS}}^2\, d\mu(\lambda) \qquad \big(f \in L^1(N) \cap L^2(N)\big)\,, \tag{1.1}$$

where π_λ is a realization of each $\lambda \in \hat{N}$, and $\|\cdot\|_{\mathrm{HS}}$ stands for the Hilbert-Schmidt norm of an operator. For $\lambda \in \hat{N}$ and $h \in H$, let $h \cdot \lambda$ be the element of $\hat{N}$ for which the unitary representation $\pi_{h\cdot\lambda}$ of N is equivalent to $\pi_\lambda \circ h^{-1}$. The group H acts on $\hat{N}$ in this way. We denote by $\mathcal{O}_\lambda^*$ the H-orbit through $\lambda \in \hat{N}$. Let us assume the following:

(A1) There exist elements λ_k $(k \in K)$ of $\hat{N}$, indexed by some set K, such that $\mu(\mathcal{O}_{\lambda_k}^*) > 0$ and $\mathcal{O}_{\lambda_k}^* \cap \mathcal{O}_{\lambda_{k'}}^* = \emptyset$ $(k \neq k')$.

(A2) The stabilizer $H_k := \{\, h \in H\,;\ h \cdot \lambda_k = \lambda_k \,\}$ at each $\lambda_k \in \hat{N}$ is compact.

(A3) For $k \in K$, the map $H/H_k \ni hH_k \mapsto h \cdot \lambda_k \in \mathcal{O}_{\lambda_k}^*$ is a homeomorphism, where the topology on $\mathcal{O}_{\lambda_k}^*$ is induced from the Fell topology on $\hat{N}$.

These conditions are rather natural in the context of wavelet analysis (confer [1, Chapter 9], [10, Section IV], and [8, Section 3]. In the latter part of [8], some cases with noncompact stabilizers are also discussed).

Under the assumptions, we shall construct irreducible subspaces of $L^2(N)$ associated to the orbits $\mathcal{O}_{\lambda_k}^*$, and show that each subrepresentation is square-integrable (Theorem 2). Furthermore, if $\mu(\hat{N} \setminus \bigsqcup_{k \in K} \mathcal{O}_{\lambda_k}^*) = 0$, the unitary representation $(L, L^2(N))$ is decomposed into the direct sum of such subrepresentations (Proposition 5). When ϕ is an element of the irreducible subspace, one has $\|\phi\|^2 = \int_{\mathcal{O}_{\lambda_k}^*} \|\pi_\lambda(\phi)\|_{\mathrm{HS}}^2\, d\mu(\lambda)$ for some k. Then ϕ is admissible if and only if the integral

$$\int_{\mathcal{O}_{\lambda_k}^*} \|\pi_\lambda(\phi)\|_{\mathrm{HS}}^2 \Delta_G(h)^{-1}\, d\mu(\lambda) \qquad (\lambda = h \cdot \lambda_k) \tag{1.2}$$

is finite, where Δ_G is the Haar modulus of G [see (3.12)]. We give a Mackey-type description of our subrepresentations (Proposition 4), which indicates the connection between the present work and a general theory [18] on the Plancherel formula for group extensions (see also [11, Section 3.6]). In the last section, we describe the orthogonal projections onto the irreducible subspaces by singular integrals analogous to the Cauchy-Szegö integral.

Let us fix our notation used in what follows. We write $\mathbb{T}$ for the set of complex numbers with absolute value 1. For a Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ (resp. $\mathcal{B}_{\mathrm{HS}}(\mathcal{H})$, $\mathcal{B}_{\mathrm{Tr}}(\mathcal{H})$) be the space of bounded (resp. Hilbert-Schmidt, Trace class) operators on $\mathcal{H}$. The trace norm of $A \in \mathcal{B}_{\mathrm{Tr}}(\mathcal{H})$ is denoted by $\|A\|_{\mathrm{Tr}}$. We write $\mathrm{U}(\mathcal{H})$ for the group of unitary operators on $\mathcal{H}$.

2. Covariant Functions on the Orbits $\mathcal{O}^*_{\lambda_k}$

In this section, we introduce some covariant functions on the H-orbit $\mathcal{O}^*_{\lambda_k} \subset \hat{N}$ to get an integral formula (Proposition 3) playing a significant role later.

For each $\lambda \in \hat{N}$, we fix a realization $(\pi_\lambda, \mathcal{H}_\lambda)$ of λ. Extending the operator-valued Fourier transform $\mathbf{F} : L^1(N) \cap L^2(N) \to \int^{\oplus}_{\hat{N}} \mathcal{B}_{\mathrm{HS}}(\mathcal{H}_\lambda)\, d\mu(\lambda)$ given by $\mathbf{F}f(\lambda) := \pi_\lambda(f)$, we define the unitary isomorphism $\mathbf{F} : L^2(N) \xrightarrow{\sim} \int^{\oplus}_{\hat{N}} \mathcal{B}_{\mathrm{HS}}(\mathcal{H}_\lambda)\, d\mu(\lambda)$. Here we recall the inversion formula of the Fourier transform $\mathbf{F}$.

Proposition 1 ([11, Theorem 4.15]). *Let $(A(\lambda))_{\lambda \in \hat{N}}$ be an element of the direct integral $\int^{\oplus}_{\hat{N}} \mathcal{B}_{\mathrm{Tr}}(\mathcal{H}_\lambda)\, d\mu(\lambda)$ of the Banach spaces $\mathcal{B}_{\mathrm{Tr}}(\mathcal{H}_\lambda)$, and f a function on N defined by*

$$f(n) := \int_{\hat{N}} \operatorname{tr} A(\lambda)\pi_\lambda(n)^*\, d\mu(\lambda) \qquad (n \in N) .$$

Then f belongs to $L^2(N)$ if and only if $(A(\lambda))_{\lambda \in \hat{N}} \in \int^{\oplus}_{\hat{N}} \mathcal{B}_{\mathrm{HS}}(\mathcal{H}_\lambda)\, d\mu(\lambda)$. In that case, one has $\mathbf{F}f(\lambda) = A(\lambda)$ (a. a. $\lambda \in \hat{N}$).

For $h \in H$ and $\lambda \in \hat{N}$, we take a unitary intertwining operator $C(h, \lambda) : \mathcal{H}_\lambda \to \mathcal{H}_{h\cdot\lambda}$ between $\pi_\lambda \circ h^{-1}$ and $\pi_{h\cdot\lambda}$, so that

$$C(h, \lambda) \circ \pi_\lambda\big(h^{-1} \cdot n\big) = \pi_{h\cdot\lambda}(n) \circ C(h, \lambda) \qquad (n \in N) . \tag{2.1}$$

Note that this $C(h, \lambda)$ is determined up to multiple by elements of $\mathbb{T}$ owing to Schur's lemma. Thus, the map $D(h, \lambda) : \mathcal{B}(\mathcal{H}_\lambda) \to \mathcal{B}(\mathcal{H}_{h\cdot\lambda})$ given by $D(h, \lambda)T := C(h, \lambda) \circ T \circ C(h, \lambda)^*$ is uniquely determined. For operators $S, T \in \mathcal{B}(\mathcal{H}_\lambda)$, we have

$$D(h, \lambda)(S \circ T) = D(h, \lambda)S \circ D(h, \lambda)T . \tag{2.2}$$

We see from (2.1) that

$$\pi_{h\cdot\lambda}(n) = D(h, \lambda)\pi_\lambda\big(h^{-1} \cdot n\big) . \tag{2.3}$$

For $h, h' \in H$ and $\lambda \in \hat{N}$, we have the chain rule

$$C\big(h, h' \cdot \lambda\big) \circ C\big(h', \lambda\big) = s_{h,h',\lambda} C\big(hh', \lambda\big) , \tag{2.4}$$

where $s_{h,h',\lambda}$ is an element of $\mathbb{T}$, so that we get

$$D\big(h, h' \cdot \lambda\big) \circ D\big(h', \lambda\big) = D\big(hh', \lambda\big) . \tag{2.5}$$

Proposition 2. *If $f \in L^2(N)$, $h \in H$ and $n \in N$, one has*

$$\mathbf{F}[L(h)f](\lambda) = \delta(h)^{1/2} D\big(h, h^{-1}\cdot\lambda\big)\mathbf{F}f\big(h^{-1}\cdot\lambda\big)\,, \tag{2.6}$$

$$\mathbf{F}[L(n)f](\lambda) = \pi_\lambda(n)\mathbf{F}f(\lambda) \tag{2.7}$$

for almost all $\lambda \in \hat{N}$ with respect to the measure μ.

Proof. It is sufficient to show the case $f \in L^1(N) \cap L^2(N)$. We observe that

$$\begin{aligned}\mathbf{F}[L(h)f](\lambda) &= \pi_\lambda(L(h)f)\\ &= \int_N \delta(h)^{-1/2} f\big(h^{-1}\cdot n\big)\pi_\lambda(n)\,d\nu(n)\\ &= \delta(h)^{1/2}\int_N f\big(n'\big)\pi_\lambda\big(h\cdot n'\big)\,d\nu\big(n'\big) \qquad \big(n' = h^{-1}\cdot n\big)\,.\end{aligned}$$

By (2.3), the last term equals

$$\delta(h)^{1/2}\int_N f\big(n'\big)D\big(h, h^{-1}\cdot\lambda\big)\pi_{h^{-1}\cdot\lambda}\big(n'\big)\,d\nu\big(n'\big) = \delta(h)^{1/2} D\big(h, h^{-1}\cdot\lambda\big)\mathbf{F}f\big(h^{-1}\cdot\lambda\big)\,.$$

Therefore (2.6) holds. As for (2.7), we see that

$$\begin{aligned}\mathbf{F}[L(n)f](\lambda) &= \int_N f\big(n^{-1}n_1\big)\pi_\lambda(n_1)\,d\nu(n_1)\\ &= \int_N f(n_2)\pi_\lambda(nn_2)\,d\nu(n_2) \qquad \big(n_2 = n^{-1}n_1\big)\\ &= \pi_\lambda(n)\int_N f(n_2)\pi_\lambda(n_2)\,d\nu(n_2)\\ &= \pi_\lambda(n)\mathbf{F}f(\lambda)\,.\end{aligned}$$

Hence, Proposition 2 is proved. □

Although the following lemma and succeeding discussions are found in [18, II, 107–108], we present them for completeness.

Lemma 1. *For $h \in H$, one has $d\mu(h\cdot\lambda) = \delta(h)^{-1}\,d\mu(\lambda)$.*

Proof. For a function $f \in L^2(N)$, we have by (1.1) and (2.6)

$$\begin{aligned}\|L(h)f\|^2 &= \int_{\hat{N}} \|\mathbf{F}[L(h)f](\lambda)\|_{\mathrm{HS}}^2\,d\mu(\lambda)\\ &= \delta(h)\int_{\hat{N}} \big\|D\big(h, h^{-1}\cdot\lambda\big)\mathbf{F}f\big(h^{-1}\cdot\lambda\big)\big\|_{\mathrm{HS}}^2\,d\mu(\lambda)\\ &= \delta(h)\int_{\hat{N}} \big\|\mathbf{F}f\big(\lambda'\big)\big\|_{\mathrm{HS}}^2\,d\mu\big(h\cdot\lambda'\big) \qquad \big(\lambda' = h^{-1}\cdot\lambda\big)\,.\end{aligned}$$

On the other hand,

$$\|L(h)f\|^2 = \|f\|^2 = \int_{\hat{N}} \|\mathbf{F}f(\lambda)\|_{\mathrm{HS}}^2\,d\mu(\lambda)\,.$$

Namely, we have

$$\int_{\hat{N}} \|\mathbf{F}f(\lambda)\|_{\mathrm{HS}}^2\,\delta(h)\,d\mu(h\cdot\lambda) = \int_{\hat{N}} \|\mathbf{F}f(\lambda)\|_{\mathrm{HS}}^2\,d\mu(\lambda)$$

for any $f \in L^2(N)$, whence Lemma 1 follows. □

It is easy to see that the measure $d_G(g) := \delta(h)^{-1}\,d\nu(n)\,dh$ $(g = (n,h) \in G)$ is a left Haar measure on G, where dh is a left Haar measure on H. Then we observe for $h_1 \in H$

$$\Delta_G(h_1)d_G(g) = d_G(gh_1) = \delta(hh_1)^{-1}\,d\nu(n)d(hh_1)\,,$$

since $gh_1 = (n, hh_1)$. On the other hand, the last term equals

$$\delta(h_1)^{-1}\Delta_H(h_1)\delta(h)^{-1}\,d\nu(n)\,dh = \delta(h_1)^{-1}\Delta_H(h_1)d_G(g)\,,$$

where Δ_H denotes the Haar modulus of H. It follows that

$$\delta(h_1) = \Delta_H(h_1)/\Delta_G(h_1)\,. \tag{2.8}$$

Since the stabilizer H_k at $\lambda_k \in \hat{N}$ is compact, we have $\delta(H_k) = \{1\}$, so that we can define a positive function u_k on $\mathcal{O}^*_{\lambda_k}$ by

$$u_k(h \cdot \lambda_k) := \delta(h) \qquad (h \in H)\,. \tag{2.9}$$

Then we see from Lemma 1 that $u_k\,d\mu$ is an H-invariant measure on the orbit $\mathcal{O}^*_{\lambda_k}$. Thus, there exists a positive constant c_k such that

$$\int_H p(h \cdot \lambda_k)\,dh = c_k \int_{\mathcal{O}^*_{\lambda_k}} p(\lambda)u_k(\lambda)\,d\mu(\lambda) \tag{2.10}$$

for positive μ-measurable functions p on $\mathcal{O}^*_{\lambda_k}$. Noting that $\Delta_G(H_k) = \{1\}$ owing to the compactness of H_k, we define a function D_k on $\mathcal{O}^*_{\lambda}$ by

$$D_k(h \cdot \lambda_k) := c_k\Delta_G(h)^{-1} \qquad (h \in H)\,.$$

Then we see from (2.8) and (2.9) that

$$D_k(h \cdot \lambda_k) = c_k\Delta_H(h)^{-1}u_k(h \cdot \lambda_k) \qquad (h \in H_k)\,.$$

Thus, we get by (2.10)

$$\int_{\mathcal{O}^*_{\lambda_k}} p(\lambda)D_k(\lambda)\,d\mu(\lambda) = \int_H p(h \cdot \lambda_k)\Delta_H(h)^{-1}\,dh\,. \tag{2.11}$$

Proposition 3. *For a positive μ-measurable function p on the orbit $\mathcal{O}^*_{\lambda_k}$, the integral $\int_H p(h^{-1} \cdot \lambda)\,dh$ does not depend on $\lambda \in \mathcal{O}^*_{\lambda_k}$, and equals $\int_{\mathcal{O}^*_{\lambda_k}} p(\lambda)D_k(\lambda)\,d\mu(\lambda)$.*

Proof. Writing $\lambda = \tilde{h} \cdot \lambda_k$ with $\tilde{h} \in H$, we have

$$\begin{aligned}\int_H p\big(h^{-1} \cdot \lambda\big)\,dh &= \int_H p\big((\tilde{h}^{-1}h)^{-1} \cdot \lambda_k\big)\,dh = \int_H p\big(h^{-1} \cdot \lambda_k\big)\,dh \\ &= \int_H p(h \cdot \lambda_k)\Delta_H(h)^{-1}\,dh\,.\end{aligned}$$

Therefore Proposition 3 follows from (2.11). □

3. Subrepresentations of L

In this section, we construct irreducible subrepresentations of $(L, L^2(N))$, and consider the wavelet transforms associated to them. The representations are described by unitary inductions in Proposition 4, and classified in Theorem 4.

Owing to (2.4), the map $\tau_k : H_k \ni h \mapsto C(h, \lambda_k) \in \mathrm{U}(\mathcal{H}_{\lambda_k})$ is a projective representation on $\mathcal{H}_{\lambda_k}$. Since H_k is compact, we have an irreducible decomposition $\mathcal{H}_{\lambda_k} = \sum_{\alpha\in A_k}^{\oplus} \mathcal{H}_{\lambda_k,\alpha}$, where A_k is an at most countable index set. Note that each $\mathcal{H}_{\lambda_k,\alpha}$ is finite dimensional. For $\lambda = \tilde{h}\cdot\lambda_k \in \mathcal{O}^*_{\lambda_k}$, we see that the stabilizer $H_\lambda := \{ h \in H \,;\, h\cdot\lambda = \lambda \}$ equals $\tilde{h} H_k \tilde{h}^{-1}$. Similarly to the above, we define a projective representation τ_λ of H_λ on $\mathcal{H}_\lambda$ by $\tau_\lambda(h) := C(h,\lambda) \in \mathrm{U}(\mathcal{H}_\lambda) \quad (h \in H_\lambda)$. We see from (2.4) that $\tau_\lambda(h) \circ C(\tilde{h}, \lambda_k) = s_h C(\tilde{h}, \lambda_k) \circ \tau_k(\tilde{h}^{-1} h \tilde{h}) \quad (h \in H_\lambda)$, where s_h is an element of $\mathbb{T}$. Therefore, putting $\mathcal{H}_{\lambda,\alpha} := C(\tilde{h}, \lambda_k)\mathcal{H}_{\lambda_k,\alpha}$ for $\alpha \in A_k$ (note that the right-hand side is independent of the choice of $\tilde{h}$ for which $\lambda = \tilde{h}\cdot\lambda_k$), we get $\mathcal{H}_\lambda = \sum_{\alpha\in A_k}^{\oplus} \mathcal{H}_{\lambda,\alpha}$, which gives an irreducible decomposition of $(\tau_\lambda, \mathcal{H}_\lambda)$. By (2.4), we have

$$C(h,\lambda)\mathcal{H}_{\lambda,\alpha} = \mathcal{H}_{h\cdot\lambda,\alpha}\,. \tag{3.1}$$

Let $P_{\lambda,\alpha}$ be the orthogonal projection onto $\mathcal{H}_{\lambda,\alpha}$, and $\mathcal{B}_{\lambda,\alpha}$ the closed subspace of $\mathcal{B}_{\mathrm{HS}}(\mathcal{H}_\lambda)$ given by

$$\mathcal{B}_{\lambda,\alpha} := \left\{ T \in \mathcal{B}_{\mathrm{HS}}(\mathcal{H}_\lambda)\,;\, T P_{\lambda,\alpha} = T \right\}\,. \tag{3.2}$$

We see from (3.1) that $D(h,\lambda)P_{\lambda,\alpha} = P_{h\cdot\lambda,\alpha}$, which leads us to

$$D(h,\lambda)\mathcal{B}_{\lambda,\alpha} = \mathcal{B}_{h\cdot\lambda,\alpha}\,. \tag{3.3}$$

If we identify the Hilbert space $\mathcal{B}_{\mathrm{HS}}(\mathcal{H}_\lambda)$ with the tensor product $\mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\lambda$, the subspace $\mathcal{B}_{\lambda,\alpha}$ equals $\mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_{\lambda,\alpha}$. Thus, we have an orthogonal decomposition

$$\mathcal{B}_{\mathrm{HS}}(\mathcal{H}_\lambda) = \sum_{\alpha\in A_k}^{\oplus} \mathcal{B}_{\lambda,\alpha}\,. \tag{3.4}$$

Keeping (3.2) in mind, we define the subspace $L_{k,\alpha}(N)$ of $L^2(N)$ by

$$\begin{aligned} L_{k,\alpha}(N) &:= \mathbf{F}^{-1}\left(\int_{\mathcal{O}^*_{\lambda_k}}^{\oplus} \mathcal{B}_{\lambda,\alpha}\, d\mu(\lambda) \right) \\ &= \left\{ f \in L^2(N)\,;\; \begin{array}{ll} \mathbf{F}f(\lambda) = \mathbf{F}f(\lambda)P_{\lambda,\alpha} & \left(\text{if}\quad \lambda \in \mathcal{O}^*_{\lambda_k}\right) \\ \mathbf{F}f(\lambda) = 0 & \left(\text{if}\quad \lambda \notin \mathcal{O}^*_{\lambda_k}\right) \end{array} \right\}. \end{aligned} \tag{3.5}$$

Thanks to Proposition 2 and (3.3), each $L_{k,\alpha}(N)$ is G-invariant.

Let us consider the square-integrability of matrix coefficients of the subrepresentation $(L, L_{k,\alpha}(N))$ of G. For $f, \phi \in L_{k,\alpha}(N)$, we have by (1.1) and (2.7)

$$\begin{aligned} (f|L(n)L(h)\phi) &= \int_{\mathcal{O}^*_{\lambda_k}} \operatorname{tr} \mathbf{F}f(\lambda)\mathbf{F}[L(n)L(h)\phi](\lambda)^*\, d\mu(\lambda) \\ &= \int_{\mathcal{O}^*_{\lambda_k}} \operatorname{tr} \left(\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi](\lambda)^*\right)\pi_\lambda(n)^*\, d\mu(\lambda)\,. \end{aligned} \tag{3.6}$$

We note that each $\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi](\lambda)^*$ is a trace class operator on $\mathcal{H}_\lambda$ since both $\mathbf{F}f(\lambda)$ and $\mathbf{F}[L(h)\phi](\lambda)$ are Hilbert-Schmidt operators. Indeed, $(\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi]\,(\lambda)^*)_{\lambda\in\mathcal{O}^*_{\lambda_k}}$ belongs to $\int^{\oplus}_{\mathcal{O}^*_{\lambda_k}} \mathcal{B}_{\mathrm{Tr}}(\mathcal{H}_\lambda)\,d\mu(\lambda)$ because

$$\begin{aligned}\int_{\mathcal{O}^*_{\lambda_k}} &\left\|\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi](\lambda)^*\right\|_{\mathrm{Tr}}\,d\mu(\lambda)\\ &\le \int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}f(\lambda)\|_{\mathrm{HS}}\,\|\mathbf{F}[L(h)\phi](\lambda)\|_{\mathrm{HS}}\,d\mu(\lambda)\\ &\le \left\{\int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}f(\lambda)\|^2_{\mathrm{HS}}\,d\mu(\lambda)\right\}^{1/2} \left\{\int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}[L(h)\phi](\lambda)\|^2_{\mathrm{HS}}\,d\mu(\lambda)\right\}^{1/2}\ ,\end{aligned} \tag{3.7}$$

where the last term equals $\|f\|\,\|L(h)\phi\| = \|f\|\,\|\phi\|$ by (1.1).

Now we assume that

$$\int_G |(f|L(g)\phi)|^2\,d_G(g) = \int_H\int_N |(f|L(n)L(h)\phi)|^2\delta(h)^{-1}\,d\nu(n)\,dh < +\infty\ .$$

Then $\int_N |(f|L(n)L(h)\phi)|^2\,d\nu(n)$ is finite for almost all $h\in H$. Thus, we see from (3.6), Proposition 1 and (1.1) that

$$\int_N |(f|L(n)L(h)\phi)|^2\,d\nu(n) = \int_{\mathcal{O}^*_{\lambda_k}} \left\|\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi](\lambda)^*\right\|^2_{\mathrm{HS}}\,d\mu(\lambda)\ . \tag{3.8}$$

Therefore the integral $\int_G |(f|L(g)\phi)|^2\,d_G(g)$ is equal to

$$\int_H\int_{\mathcal{O}^*_{\lambda_k}} \left\|\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi](\lambda)^*\right\|^2_{\mathrm{HS}}\,\delta(h)^{-1}\,d\mu(\lambda)\,dh\ . \tag{3.9}$$

Now we observe that

$$\begin{aligned}\int_H &\left\|\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi](\lambda)^*\right\|^2_{\mathrm{HS}}\,\delta(h)^{-1}\,dh\\ &=\int_H\int_{H_\lambda}\Big(\mathrm{tr}\ \mathbf{F}f(\lambda)\mathbf{F}[L(h_1h)\phi](\lambda)^*\mathbf{F}[L(h_1h)\phi](\lambda)\mathbf{F}f(\lambda)^*\Big)\,\delta(h_1h)^{-1}\,dh_1\,dh\ .\end{aligned} \tag{3.10}$$

Since $L(h_1h)\phi$ belongs to $L_{k,\alpha}(N)$, we have by (3.5)

$$P_{\lambda,\alpha}\mathbf{F}[L(h_1h)\phi](\lambda)^*\mathbf{F}[L(h_1h)\phi](\lambda)P_{\lambda,\alpha} = \mathbf{F}[L(h_1h)\phi](\lambda)^*\mathbf{F}[L(h_1h)\phi](\lambda)\ ,$$

which means that we can regard $\mathbf{F}[L(h_1h)\phi](\lambda)^*\mathbf{F}[L(h_1h)\phi](\lambda)$ as a linear operator on the finite dimensional vector space $\mathcal{H}_{\lambda,\alpha}$. Let us consider the integral

$$\mathcal{I} := \int_{H_\lambda} \delta(h_1h)^{-1}\mathbf{F}[L(h_1h)\phi](\lambda)^*\mathbf{F}[L(h_1h)\phi](\lambda)\,dh_1 \in \mathrm{End}(\mathcal{H}_{\lambda,\alpha})\ .$$

We see from (2.6) that

$$\begin{aligned}\mathbf{F}[L(h_1h)\phi](\lambda) &= \delta(h_1)^{1/2}D(h_1,\lambda)\mathbf{F}[L(h)\phi](\lambda)\\ &= \delta(h_1)^{1/2}\tau_\lambda(h_1)\circ\mathbf{F}[L(h)\phi](\lambda)\circ\tau_\lambda(h_1)^{-1}\ ,\end{aligned}$$

so that

$$\mathbf{F}[L(h_1h)\phi](\lambda)^*\mathbf{F}[L(h_1h)\phi](\lambda) = \delta(h_1)\tau_\lambda(h_1) \circ \mathbf{F}[L(h)\phi](\lambda)^*\mathbf{F}[L(h)\phi](\lambda) \circ \tau_\lambda(h_1)^{-1}.$$

Thus, we have

$$\mathcal{I} = \int_{H_\lambda} \tau_\lambda(h_1) \circ \Big(\delta(h)^{-1}\mathbf{F}[L(h)\phi](\lambda)^*\mathbf{F}[L(h)\phi](\lambda)\Big) \circ \tau_\lambda(h_1)^{-1}\, dh_1 \,. \tag{3.11}$$

Therefore Schur's lemma tells us that $\mathcal{I}$ is a scalar operator on $\mathcal{H}_{\lambda,\alpha}$, that is, we can write $\mathcal{I} = cP_{\lambda,\alpha}$ with some constant $c \in \mathbb{R}$. Then

$$c\, n_{k,\alpha} = \operatorname{tr} \mathcal{I} = \delta(h)^{-1}\|\mathbf{F}[L(h)\phi](\lambda)\|_{\mathrm{HS}}^2 \,,$$

where $n_{k,\alpha} := \dim \mathcal{H}_{\lambda_k,\alpha} = \dim \mathcal{H}_{\lambda,\alpha}$. It follows that

$$\mathcal{I} = n_{k,\alpha}^{-1}\delta(h)^{-1}\|\mathbf{F}[L(h)\phi](\lambda)\|_{\mathrm{HS}}^2 P_{\lambda,\alpha} \,.$$

Thus, we see from (3.10) and (3.5) that the integral (3.9) equals

$$\begin{aligned}
&\frac{1}{n_{k,\alpha}}\int_{\mathcal{O}^*_{\lambda_k}}\int_H \Big(\operatorname{tr} \mathbf{F}f(\lambda)P_{\lambda,\alpha}\mathbf{F}f(\lambda)^*\Big)\|\mathbf{F}[L(h)\phi](\lambda)\|_{\mathrm{HS}}^2\delta(h)^{-1}\, dh\, d\mu(\lambda)\\
&\quad = \frac{1}{n_{k,\alpha}}\int_{\mathcal{O}^*_{\lambda_k}}\int_H \|\mathbf{F}f(\lambda)\|_{\mathrm{HS}}^2\|\mathbf{F}[L(h)\phi](\lambda)\|_{\mathrm{HS}}^2\delta(h)^{-1}\, dh\, d\mu(\lambda) \,.
\end{aligned}$$

By (2.6), the right-hand side is rewritten as

$$\frac{1}{n_{k,\alpha}}\int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}f(\lambda)\|_{\mathrm{HS}}^2\left(\int_H \left\|\mathbf{F}\phi\big(h^{-1}\cdot\lambda\big)\right\|_{\mathrm{HS}}^2 dh\right) d\mu(\lambda) \,,$$

which is equal to

$$\begin{aligned}
&\frac{1}{n_{k,\alpha}}\int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}f(\lambda)\|_{\mathrm{HS}}^2\, d\mu(\lambda)\int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}\phi(\lambda)\|_{\mathrm{HS}}^2 D_k(\lambda)\, d\mu(\lambda)\\
&\quad = \|f\|^2 \cdot \frac{1}{n_{k,\alpha}}\int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}\phi(\lambda)\|_{\mathrm{HS}}^2 D_k(\lambda)\, d\mu(\lambda)
\end{aligned}$$

by Proposition 3 and (1.1). Hence, under the condition that $(f|L(g)\phi)$ is a square-integrable function on G with $f \neq 0$, we have shown

$$C_\phi := \frac{1}{n_{k,\alpha}}\int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}\phi(\lambda)\|_{\mathrm{HS}}^2 D_k(\lambda)\, d\mu(\lambda) < +\infty \,, \tag{3.12}$$

and

$$\int_G |(f|L(g)\phi)|^2\, d_G(g) = C_\phi\|f\|^2 \,. \tag{3.13}$$

Conversely, if $\phi \in L_{k,\alpha}(N)$ satisfies the condition (3.12), the calculations above tell us that the integral (3.9) converges for any $f \in L_{k,\alpha}(N)$. Thus, the right-hand side of (3.8) is finite for almost all $h \in H$. Then Proposition 1 implies the equality (3.8), so that we get (3.13) again.

Theorem 1. *The unitary representation $(L, L_{k,\alpha}(N))$ of G is irreducible.*

Proof. Let $\mathcal{L}$ be a nonzero invariant subspace of $L_{k,\alpha}(N)$, and $\mathcal{L}^{\perp}$ its orthogonal complement. We take a nonzero $\phi \in \mathcal{L}$. Then for any $f \in \mathcal{L}^{\perp}$ we have by (3.13)

$$0 = \int_G |(f|L(g)\phi)|^2 \, d_G(g) = C_\phi \|f\|^2 ,$$

which implies $f = 0$. Indeed, this argument is valid even if C_ϕ is not finite. Therefore $\mathcal{L}^{\perp} = \{0\}$, and Theorem 1 is verified. □

From (3.12) and (3.13) we also obtain

Theorem 2. *The representation $(L, L_{k,\alpha}(N))$ is square-integrable. The formal degree (the Duflo-Moore operator [7]) $K_{k,\alpha} : L_{k,\alpha}(N) \to L_{k,\alpha}(N)$ of the representation is described as*

$$\mathbf{F}[K_{k,\alpha} f](\lambda) = n_{k,\alpha} D_k(\lambda)^{-1} \mathbf{F} f(\lambda) \qquad \left(f \in L_{k,\alpha}(N),\ \lambda \in \mathcal{O}^*_{\lambda_k}\right) .$$

Applying the general arguments in [13, I] to our setting, we obtain the following results for the continuous wavelet transform associated to the representation $(L, L_{k,\alpha}(N))$.

Theorem 3. *For $\phi \in L_{k,\alpha}(N)$ satisfying the admissible condition* (3.12)*, the wavelet transform $W_\phi : L_{k,\alpha}(N) \to L^2(G)$ given by*

$$W_\phi f(g) := C_\phi^{-1/2} (f|L(g)\phi) \qquad (f \in L_{k,\alpha}(N))$$

is an isometric intertwining operator from L into the left regular representation of G. The range of W_ϕ is characterized by the reproducing kernel R_ϕ defined by $R_\phi(g_1, g_2) := C_\phi^{-1} (\phi|L(g_2^{-1} g_1)\phi)$ $(g_1, g_2 \in G)$. The inverse formula of W_ϕ is given by

$$f = C_\phi^{-1/2} \int_G W_\phi f(g) L(g)\phi \, d_G(g) ,$$

where the integral is taken in the weak sense.

For an element h of the stabilizer H_k at λ_k, the operator $D(h, \lambda_k)$ maps $\mathcal{B}_{\lambda_k,\alpha}$ onto itself because of (3.3). Furthermore, for $h \in H_k$, $n \in N$ and $T \in \mathcal{B}_{\lambda_k,\alpha}$, we see from (2.2), (2.3), and (2.5) that

$$\begin{aligned} D(h, \lambda_k)\left[\pi_{\lambda_k}(n) \circ D\left(h^{-1}, \lambda_k\right)T\right] &= D(h, \lambda_k)\pi_{\lambda_k}(n) \circ D(h, \lambda_k)\left(D\left(h^{-1}, \lambda_k\right)T\right) \\ &= \pi_{\lambda_k}(h \cdot n) \circ T \ . \end{aligned}$$

Therefore we can define a unitary representation $(l_{k,\alpha}, \mathcal{B}_{\lambda_k,\alpha})$ of the semidirect product group $G_k := N \rtimes H_k$ by

$$l_{k,\alpha}(h)T := D(h, \lambda_k)T, \qquad l_{k,\alpha}(n)T := \pi_{\lambda_k}(n) \circ T \ \ (T \in \mathcal{B}_{\lambda_k,\alpha},\ h \in H_k,\ n \in N) .$$

Proposition 4. *The representation $(L, L_{k,\alpha}(N))$ of G is equivalent to the induced representation $\rho_{k,\alpha} := \mathrm{Ind}_{G_k}^{G} l_{k,\alpha}$.*

Proof. Let $L^2(G, \mathcal{B}_{\lambda_k,\alpha}; l_{k,\alpha})$ be the Hilbert space of equivalence classes of measurable $\mathcal{B}_{\lambda_k,\alpha}$-valued functions φ on G such that
(a) $\varphi(gg') = l_{k,\alpha}(g')^{-1}\varphi(g)$ for $g \in G$ and $g' \in G_k$,
(b) $\|\varphi\|^2 := \int_H \|\varphi(h)\|^2_{\mathrm{HS}}\, dh < +\infty$.
Note that the invariant integral over the quotient space G/G_k is given by the integral over H because G/G_k is isomorphic to H/H_k and H_k is compact. We realize $\rho_{k,\alpha}$ on $L^2(G, \mathcal{B}_{\lambda_k,\alpha}; l_{k,\alpha})$ by $\rho_{k,\alpha}(g)\varphi(g_0) := \varphi(g^{-1}g_0)$. We shall show that the map $\Phi_{k,\alpha} : L_{k,\alpha}(N) \to L^2(G, \mathcal{B}_{\lambda_k,\alpha}; l_{k,\alpha})$ defined by

$$\begin{aligned}\Phi_{k,\alpha} f(g) := c_k^{-1/2}\delta(h)^{-1/2} D\big(h^{-1}, h\cdot\lambda_k\big)\big[\pi_{h\cdot\lambda_k}(n)^{-1} \circ \mathbf{F}f(h\cdot\lambda_k)\big] \\ (f \in L_{k,\alpha}(N),\ g = (n,h) \in G)\end{aligned} \tag{3.14}$$

gives a unitary equivalence between the two representations $(L, L_{k,\alpha}(N))$ and $(\rho_{k,\alpha}, L^2(G, \mathcal{B}_{\lambda_k,\alpha}; l_{k,\alpha}))$. First of all, we verify that $\Phi_{k,\alpha} f$ belongs to $L^2(G, \mathcal{B}_{\lambda_k,\alpha}; l_{k,\alpha})$. For $h' \in H_k$ we have $gh' = (n, hh')$ and $\delta(h') = 1$, so that we get by (2.5)

$$\begin{aligned}&\Phi_{k,\alpha} f\big(gh'\big)\\ &= c_k^{-1/2}\delta\big(hh'\big)^{-1/2} D\big(\big(hh'\big)^{-1}, hh'\cdot\lambda_k\big)\big[\pi_{hh'\cdot\lambda_k}(n)^{-1} \circ \mathbf{F}f\big(hh'\cdot\lambda_k\big)\big]\\ &= D\big(\big(h'\big)^{-1}, \lambda_k\big)\big[c_k^{-1/2}\delta(h)^{-1/2} D\big(h^{-1}, h\cdot\lambda_k\big)\big[\pi_{h\cdot\lambda_k}(n)^{-1} \circ \mathbf{F}f(h\cdot\lambda_k)\big]\big]\\ &= l_{k,\alpha}\big(h'\big)^{-1}\Phi_{k,\alpha} f(g)\ .\end{aligned}$$

On the other hand, since $gn' = (n(h\cdot n'), h)$ for $n' \in N$, we have

$$\begin{aligned}&\Phi_{k,\alpha} f\big(gn'\big)\\ &= c_k^{-1/2}\delta(h)^{-1/2} D\big(h^{-1}, h\cdot\lambda_k\big)\big[\pi_{h\cdot\lambda_k}\big(n\big(h\cdot n'\big)\big)^{-1} \circ \mathbf{F}f(h\cdot\lambda_k)\big]\\ &= c_k^{-1/2}\delta(h)^{-1/2} D\big(h^{-1}, h\cdot\lambda_k\big)\big[\pi_{h\cdot\lambda_k}\big(h\cdot n'\big)^{-1} \circ \pi_{h\cdot\lambda_k}(n)^{-1} \circ \mathbf{F}f(h\cdot\lambda_k)\big]\\ &= \pi_{\lambda_k}\big(n'\big)^{-1} \circ c_k^{-1/2}\delta(h)^{-1/2} D\big(h^{-1}, h\cdot\lambda_k\big)\big[\pi_{h\cdot\lambda_k}(n)^{-1} \circ \mathbf{F}f(h\cdot\lambda_k)\big]\\ &= l_{k,\alpha}\big(n'\big)^{-1}\Phi_{k,\alpha} f(g)\ ,\end{aligned}$$

where we use (2.2) and (2.3) for the third equality. Thus, the condition (a) holds for $\varphi = \Phi_{k,\alpha} f$. For the condition (b), we observe from (2.9), (2.10), and (1.1) that

$$\begin{aligned}\int_H \|\Phi_{k,\alpha} f(h)\|^2_{\mathrm{HS}}\, dh &= c_k^{-1}\int_H \|\mathbf{F}f(h\cdot\lambda_k)\|^2_{\mathrm{HS}}\delta(h)^{-1}\, dh\\ &= \int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}f(\lambda)\|^2_{\mathrm{HS}}\, d\mu(\lambda)\\ &= \|f\|^2\ .\end{aligned}$$

Therefore $\Phi_{k,\alpha} f \in L^2(G, \mathcal{B}_{\lambda_k,\alpha}; l_{k,\alpha})$, and $\Phi_{k,\alpha}$ is an isometry. Conversely, for $\varphi \in L^2(G, \mathcal{B}_{\lambda_k,\alpha}; l_{k,\alpha})$ we can take a unique element $f \in L_{k,\alpha}(N)$ such that

$$\mathbf{F}f(\lambda) = c_k^{1/2}\delta\big(\tilde{h}\big)^{1/2} D\big(\tilde{h}, \lambda_k\big)\varphi\big(\tilde{h}\big) \qquad \big(\lambda = \tilde{h}\cdot\lambda_k \in \mathcal{O}^*_{\lambda_k}\big)\ , \tag{3.15}$$

because the right-hand side is independent of the choice of $\tilde{h}$ for which $\lambda = \tilde{h}\cdot\lambda_k$. Indeed, if $\tilde{h}_1\cdot\lambda_k = \lambda = \tilde{h}\cdot\lambda_k$, we can write $\tilde{h}_1 = \tilde{h}h'$ with $h' \in H_k$. Then we see from (2.5) and the condition (a) that

$$\begin{aligned}c_k^{1/2}\delta(\tilde{h}_1)^{1/2}D(\tilde{h}_1,\lambda_k)\varphi(\tilde{h}_1) &= c_k^{1/2}\delta(\tilde{h}h')^{1/2}D(\tilde{h}h',\lambda_k)\varphi(\tilde{h}h')\\ &= c_k^{1/2}\delta(\tilde{h})^{1/2}D(\tilde{h},\lambda_k)D(h',\lambda_k)D\big((h')^{-1},\lambda_k\big)\varphi(\tilde{h})\\ &= c_k^{1/2}\delta(\tilde{h})^{1/2}D(\tilde{h},\lambda_k)\varphi(\tilde{h})\ .\end{aligned}$$

Comparing (3.15) with (3.14), we can easily check that $\varphi = \Phi_{k,\alpha}f$, so that $\Phi_{k,\alpha}$ is surjective. Let us investigate the equivalence of the representations. For $g_0 = (n_0, h_0)$ and $h \in H$, we have by (2.6)

$$\begin{aligned}&\Phi_{k,\alpha}[L(h)f](g_0)\\ &= c_k^{-1/2}\delta(h_0)^{-1/2}D\big(h_0^{-1}, h_0\cdot\lambda_k\big)\big[\pi_{h_0\cdot\lambda_k}(n_0)^{-1}\circ \mathbf{F}[L(h)f](h_0\cdot\lambda_k)\big]\\ &= c_k^{-1/2}\delta(h_0)^{-1/2}\\ &\quad\times D\big(h_0^{-1}, h_0\cdot\lambda_k\big)\big[\pi_{h_0\cdot\lambda_k}(n_0)^{-1}\circ\delta(h)^{1/2}D\big(h, h^{-1}h_0\cdot\lambda_k\big)\mathbf{F}f\big(h^{-1}h_0\cdot\lambda_k\big)\big]\ .\end{aligned}$$

Using (2.2) and (2.3), we rewrite the last term as

$$\begin{aligned}&c_k^{-1/2}\delta\big(h^{-1}h_0\big)^{-1/2}\\ &\quad\times D\big(h_0^{-1}, h_0\cdot\lambda_k\big)D\big(h, h^{-1}h_0\cdot\lambda_k\big)\big[\pi_{h^{-1}h_0\cdot\lambda_k}\big(h^{-1}\cdot n_0\big)^{-1}\circ\mathbf{F}f\big(h^{-1}h_0\cdot\lambda_k\big)\big]\ ,\end{aligned}$$

which equals

$$\begin{aligned}&c_k^{-1/2}\delta\big(h^{-1}h_0\big)^{-1/2}\\ &\quad\times D\big(\big(h^{-1}h_0\big)^{-1}, h^{-1}h_0\cdot\lambda_k\big)\big[\pi_{h^{-1}h_0\cdot\lambda_k}\big(h^{-1}\cdot n_0\big)^{-1}\circ\mathbf{F}f\big(h^{-1}h_0\cdot\lambda_k\big)\big]\end{aligned}$$

by (2.5). The above is nothing but $\Phi_{k,\alpha}f(h^{-1}g_0)$ since $h^{-1}g_0 = (h^{-1}\cdot n_0, h^{-1}h_0)$. Namely we obtain

$$\Phi_{k,\alpha}[L(h)f](g_0) = \rho_{k,\alpha}(h)\Phi_{k,\alpha}f(g_0)\ .$$

For $n \in N$, we observe from (2.7)

$$\begin{aligned}&\Phi_{k,\alpha}[L(n)f](g_0)\\ &= c_k^{-1/2}\delta(h_0)^{-1/2}D\big(h_0^{-1}, h_0\cdot\lambda_k\big)\big[\pi_{h_0\cdot\lambda_k}(n_0)^{-1}\circ\mathbf{F}[L(n)f](h_0\cdot\lambda_k)\big]\\ &= c_k^{-1/2}\delta(h_0)^{-1/2}D\big(h_0^{-1}, h_0\cdot\lambda_k\big)\big[\pi_{h_0\cdot\lambda_k}(n_0)^{-1}\circ\pi_{h_0\cdot\lambda_k}(n)\circ\mathbf{F}f(h_0\cdot\lambda_k)\big]\\ &= c_k^{-1/2}\delta(h_0)^{-1/2}D\big(h_0^{-1}, h_0\cdot\lambda_k\big)\big[\pi_{h_0\cdot\lambda_k}\big(n^{-1}n_0\big)^{-1}\circ\mathbf{F}f(h_0\cdot\lambda_k)\big]\ ,\end{aligned}$$

and the last term equals $\Phi_{k,\alpha}f(n^{-1}g_0) = \rho_{k,\alpha}(n)\Phi_{k,\alpha}f(g_0)$ since $n^{-1}g_0 = (n^{-1}n_0, h_0)$. Hence, Proposition 4 is proved. □

Identifying $\mathcal{B}_{\lambda_k,\alpha}$ with $\mathcal{H}_{\lambda_k}\otimes\overline{\mathcal{H}}_{\lambda_k,\alpha}$, we can interpret Proposition 4 as the description of $(L, L_{k,\alpha}(N))$ by the method of Kleppner and Lipsman [18]. Then the square-integrability of $(L, L_{k,\alpha}(N))$ can be shown by [18, I, Corollary 11.1]. We can also deduce the following statement from [18, I, Lemma 9.7]. However, we give a direct proof for its own interest.

Theorem 4. *The representations $(L, L_{k,\alpha}(N))$ and $(L, L_{k',\alpha'}(N))$ of G are equivalent if and only if $k = k'$ and the projective representations $(\tau_k, \mathcal{H}_{k,\alpha})$ and $(\tau_k, \mathcal{H}_{k,\alpha'})$ of H_k are* *equivalent, that is, there exists an isometry* $A : \mathcal{H}_{k,\alpha} \to \mathcal{H}_{k,\alpha'}$ *such that* $\tau_k(h) \circ A = A \circ \tau_k(h)$ *for all* $h \in H_k$*.*

Proof. We first prove the "if" part. It is easy to check that the map $R_A : \mathcal{B}_{\lambda_k,\alpha'} \ni T \mapsto T \circ A \circ P_{\lambda_k,\alpha} \in \mathcal{B}_{\lambda_k,\alpha}$ gives a unitary intertwining operator from $(l_{k,\alpha'}, \mathcal{B}_{\lambda_k,\alpha'})$ onto $(l_{k,\alpha}, \mathcal{B}_{\lambda_k,\alpha})$. Therefore we obtain $(L, L_{k,\alpha'}(N)) \sim (L, L_{k,\alpha}(N))$ by Proposition 4. Next we shall show the "only if" part. Let $\Psi : L_{k,\alpha}(N) \to L_{k',\alpha'}(N)$ be an intertwining operator between representations $(L, L_{k,\alpha}(N))$ and $(L, L_{k',\alpha'}(N))$ of G. Take $f, \phi \in L_{k,\alpha}(N) \setminus \{0\}$ with $C_\phi < +\infty$, and put $f' := \Psi(f)$, $\phi' := \Psi(\phi) \in L_{k',\alpha'}(N)$. Then $(f|L(g)\phi) = (f'|L(g)\phi')$ for $g \in G$. On the other hand, recalling (3.6), we have for $g = (n, h) \in G$

$$(f|L(g)\phi) = \int_{\mathcal{O}^*_{\lambda_k}} \mathrm{tr}\,\big(\mathbf{F}f(\lambda)\mathbf{F}[L(h)\phi](\lambda)^*\big)\pi_\lambda(n)^*\,d\mu(\lambda)\,,$$

$$\big(f'|L(g)\phi'\big) = \int_{\mathcal{O}^*_{\lambda_{k'}}} \mathrm{tr}\,\big(\mathbf{F}f'(\lambda)\mathbf{F}\big[L(h)\phi'\big](\lambda)^*\big)\pi_\lambda(n)^*\,d\mu(\lambda)\,.$$

Thus, we see from Proposition 1 and (1.1) that

$$\begin{aligned} 0 &< \int_G |(f|L(g)\phi)|^2 d_G(g) \\ &= \int_G (f|L(g)\phi)\overline{\big(f'|L(g)\phi'\big)}\,d_G(g) \\ &= \int_{\mathcal{O}^*_{\lambda_k}\cap\mathcal{O}^*_{\lambda_{k'}}} \int_H \int_{H_\lambda} \Big(\mathrm{tr}\,\mathbf{F}f(\lambda)\mathbf{F}\big[L(h_1h)\phi](\lambda)^*\mathbf{F}[L(h_1h)\phi'\big](\lambda)\mathbf{F}f'(\lambda)^*\Big) \\ &\qquad \times \delta(h_1h)^{-1}\,dh_1\,dh\,d\mu(\lambda)\,. \end{aligned} \tag{3.16}$$

Therefore $\mathcal{O}^*_{\lambda_k} \cap \mathcal{O}^*_{\lambda_{k'}} \neq \emptyset$, so that $k = k'$ by (A1). On the other hand, similarly to (3.11), we have

$$\begin{aligned} &\int_{H_\lambda} \delta(h_1h)^{-1}\mathbf{F}\big[L(h_1h)\phi\big](\lambda)^*\mathbf{F}\big[L(h_1h)\phi'\big](\lambda)\,dh_1 \\ &\quad = \int_{H_\lambda} \tau_\lambda(h_1) \circ \Big(\delta(h)^{-1}\mathbf{F}[L(h)\phi](\lambda)^*\mathbf{F}\big[L(h)\phi'\big](\lambda)\Big) \circ \tau_\lambda(h_1)^{-1}\,dh_1 \\ &\quad \in \mathrm{Hom}_{H_\lambda}(\mathcal{H}_{\lambda,\alpha'}, \mathcal{H}_{\lambda,\alpha})\,. \end{aligned}$$

Since the left-hand side is nonzero by (3.16), we get $\mathrm{Hom}_{H_\lambda}(\mathcal{H}_{\lambda,\alpha'}, \mathcal{H}_{\lambda,\alpha}) \neq \{0\}$, which means that $(\tau_k, \mathcal{H}_{k,\alpha})$ and $(\tau_k, \mathcal{H}_{k,\alpha'})$ are equivalent. □

We conclude this section by presenting the following result, which is easily seen from (3.4) and the Plancherel formula (1.1).

Proposition 5. *If* $\mu(\hat{N} \setminus \bigsqcup_{k\in K} \mathcal{O}^*_{\lambda_k}) = 0$*, the representation* $(L, L^2(N))$ *is decomposed into the direct sum of irreducibles as*

$$L^2(N) = \sum_{k\in K}^{\oplus} \sum_{\alpha\in A_k}^{\oplus} L_{k,\alpha}(N)\,.$$

4. Szegö-type kernels

For a function $f \in L^2(N)$, we denote by $f_{k,\alpha}$ $(k \in K,\ \alpha \in A_k)$ the image of f by the orthogonal projection onto $L_{k,\alpha}(N)$. Then $f_{k,\alpha}$ is characterized as an element of $L^2(N)$ such that

$$\mathbf{F} f_{k,\alpha}(\lambda) = \begin{cases} \mathbf{F} f(\lambda) P_{\lambda,\alpha} & \left(\lambda \in \mathcal{O}^*_{\lambda_k}\right), \\ 0 & \left(\lambda \notin \mathcal{O}^*_{\lambda_k}\right), \end{cases} \tag{4.1}$$

in view of (3.5). In this section, we shall express $f_{k,\alpha}$ by a certain singular integral analogous to the Cauchy-Szegö integral.

Let $\{\chi_k^{(t)}\}_{t>0}$ be a family of L^1-functions on the orbit $\mathcal{O}^*_{\lambda_k}$ with respect to $d\mu$ such that (i) $0 \le \chi_k^{(t)}(\lambda) \le 1$, and (ii) $\lim_{t\to+0} \chi_k^{(t)}(\lambda) = 1$ for almost all $\lambda \in \mathcal{O}^*_{\lambda_k}$. Then we have $\chi_k^{(t)} \in L^2(\mathcal{O}^*_{\lambda_k}, d\mu)$ because $\|\chi_k^{(t)}\|^2_{L^2} \le \|\chi_k^{(t)}\|_{L^1}$ by (i). Using $\{\chi_k^{(t)}\}_{t>0}$, we define a family of functions $\{S^{(t)}_{k,\alpha}\}_{t>0}$ on N by

$$S^{(t)}_{k,\alpha}(n) := \int_{\mathcal{O}^*_{\lambda_k}} \chi_k^{(t)}(\lambda)\, \mathrm{tr}\, P_{\lambda,\alpha}\pi_\lambda(n)^*\, d\mu(\lambda) \qquad (n \in N), \tag{4.2}$$

and call $S^{(t)}_{k,\alpha}$ *the Szegö-type kernel.* Since $\int_{\mathcal{O}^*_{\lambda_k}} \|\chi_k^{(t)}(\lambda) P_{\lambda,\alpha}\|_{\mathrm{Tr}}\, d\mu(\lambda) = n_{k,\alpha}\|\chi_k^{(t)}\|_{L^1}$ and $\int_{\mathcal{O}^*_{\lambda_k}} \|\chi_k^{(t)}(\lambda) P_{\lambda,\alpha}\|^2_{\mathrm{HS}}\, d\mu(\lambda) = n_{k,\alpha}\|\chi_k^{(t)}\|^2_{L^2}$, Proposition 1 tells us that $S^{(t)}_{k,\alpha} \in L^2(N)$ with

$$\mathbf{F} S^{(t)}_{k,\alpha}(\lambda) = \begin{cases} \chi_k^{(t)}(\lambda) P_{\lambda,\alpha} & \left(\lambda \in \mathcal{O}^*_{\lambda_k}\right), \\ 0 & \left(\lambda \notin \mathcal{O}^*_{\lambda_k}\right). \end{cases} \tag{4.3}$$

Theorem 5. *For $f \in L^2(N)$, the convolution product $f * S^{(t)}_{k,\alpha}$ belongs to $L_{k,\alpha}(N)$ for all $t > 0$. Furthermore, one has*

$$\lim_{t\to+0} f * S^{(t)}_{k,\alpha} = f_{k,\alpha}\,.$$

Proof. By (4.2), we have $S^{(t)}_{k,\alpha}(n^{-1}) = \overline{S^{(t)}_{k,\alpha}(n)}$ $(n \in N)$, so that

$$\begin{aligned} f * S^{(t)}_{k,\alpha}(n) &= \int_N f(n_0) S^{(t)}_{k,\alpha}\big(n_0^{-1} n\big)\, d\nu(n_0) = \int_N f(n_0) \overline{S^{(t)}_{k,\alpha}\big(n^{-1} n_0\big)}\, d\nu(n_0) \\ &= \big(f \big| L(n) S^{(t)}_{k,\alpha}\big)\,. \end{aligned}$$

Then (1.1) together with (4.3) leads us to

$$\begin{aligned} f * S^{(t)}_{k,\alpha}(n) &= \int_{\mathcal{O}^*_{\lambda_k}} \big(\mathbf{F} f(\lambda) \big| \mathbf{F}\big[L(n) S^{(t)}_{k,\alpha}\big](\lambda)\big)_{\mathrm{HS}}\, d\mu(\lambda) \\ &= \int_{\mathcal{O}^*_{\lambda_k}} \mathrm{tr}\big(\chi_k^{(t)}(\lambda) \mathbf{F} f(\lambda) P_{\lambda,\alpha}\pi_\lambda(n)^*\big)\, d\mu(\lambda)\,. \end{aligned}$$

Similarly to (3.7), we observe

$$\begin{aligned}
&\int_{\mathcal{O}^*_{\lambda_k}} \left\| \chi_k^{(t)}(\lambda)\mathbf{F}f(\lambda)P_{\lambda,\alpha}\right\|_{\mathrm{Tr}}\, d\mu(\lambda)\\
&\le \left\{ \int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}f(\lambda)\|^2_{\mathrm{HS}}\, d\mu(\lambda)\right\}^{1/2} \left\{ \int_{\mathcal{O}^*_{\lambda_k}} \left|\chi_k^{(t)}(\lambda)\right|^2 \|P_{\lambda,\alpha}\|^2_{\mathrm{HS}}\, d\mu(\lambda)\right\}^{1/2}\\
&= n_{k,\alpha}^{1/2}\|f\| \, \left\|\chi_k^{(t)}\right\|_{L^2} .
\end{aligned}$$

We have also

$$\int_{\mathcal{O}^*_{\lambda_k}} \left\| \chi_k^{(t)}(\lambda)\mathbf{F}f(\lambda)P_{\lambda,\alpha}\right\|^2_{\mathrm{HS}}\, d\mu(\lambda) \le \int_{\mathcal{O}^*_{\lambda_k}} \|\mathbf{F}f(\lambda)\|^2_{\mathrm{HS}}\, d\mu(\lambda) = \|f\|^2 .$$

Thus, we see from Proposition 1 that $f * S^{(t)}_{k,\alpha} \in L^2(N)$ and that

$$\mathbf{F}\big[f * S^{(t)}_{k,\alpha}\big](\lambda) = \begin{cases} \chi_k^{(t)}(\lambda)\mathbf{F}f(\lambda)P_{\lambda,\alpha} & \big(\lambda \in \mathcal{O}^*_{\lambda_k}\big) ,\\ 0 & \big(\lambda \notin \mathcal{O}^*_{\lambda_k}\big) . \end{cases} \tag{4.4}$$

Therefore $f * S^{(t)}_{k,\alpha}$ belongs to $L_{k,\alpha}(N)$ by (3.5). We have by (4.1), (4.4), and (1.1)

$$\|f_{k,\alpha} - f * S^{(t)}_{k,\alpha}\|^2_{L^2} = \int_{\mathcal{O}^*_{\lambda_k}} \left|1 - \chi_k^{(t)}(\lambda)\right|^2 \|\mathbf{F}f(\lambda)P_{\lambda,\alpha}\|^2_{\mathrm{HS}}\, d\mu(\lambda) .$$

The dominated convergence theorem tells us that the right-hand side converges to 0 as $t \to +0$. □

Corollary 1. *One has*

$$L_{k,\alpha}(N) = \left\{ f \in L^2(N)\,;\ f = \lim_{t\to+0} f * S^{(t)}_{k,\alpha} \right\} .$$

As an example, we consider the case that G is the '$ax+b$'-group with $N = \mathbb{R}$ and $H = \mathbb{R}_+$. We identify $\hat{N}$ with $\mathbb{R}$ by $\pi_\lambda(x) := e^{-i\lambda x}$ $(x \in N,\ \lambda \in \mathbb{R})$, so that the Plancherel measure $d\mu(\lambda)$ equals $(2\pi)^{-1}d\lambda$. The representation $(L, L^2(\mathbb{R}))$ of G is described as

$$L(b,a)f(x) = a^{-1/2}f((x-b)/a) \qquad (b, x \in N,\ a \in H) ,$$

and the irreducible decomposition is given by $L^2(\mathbb{R}) = L_+(\mathbb{R}) \oplus L_-(\mathbb{R})$ with

$$L_\pm(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R})\,;\ \mathbf{F}f(\lambda) = 0 \quad \text{if} \quad \lambda \notin \mathcal{O}^*_\pm \right\} ,$$

where $\mathcal{O}^*_\pm := \{\lambda \in \mathbb{R}\,;\ \pm\lambda > 0\}$. Note that the projection $P_{\lambda,\alpha}$ is trivial in this case. Putting $\chi^{(t)}_\pm(\lambda) := e^{-t|\lambda|}$, we have

$$S^{(t)}_\pm(x) = \int_{\mathcal{O}^*_\pm} \chi^{(t)}_\pm(\lambda)\overline{\pi_\lambda(x)}\, d\mu(\lambda) = -\frac{1}{2\pi i(x \pm it)} ,$$

so that the convolution $f * S^{(t)}_\pm(x)$ is nothing but the classical Cauchy-Szegö integral [22, Chapter 3, Section 3].

On the other hand, when the argument is applied to the Heisenberg group case discussed in [14], we obtain the singular integral investigated by Strichartz [23].

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